What are constraints.

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1. Introduction.

Constraints are an essential ingredient of every variational principle and must be discussed in connection with such principles. A variational principle for a static system studies equilibrium configurations of a physical system in its configuration space. The configuration space is a differential manifold. Variational formulations of dynamics deal with motions in a configuration space. The space of motions is not a manifold although it has enough structure to permit introduction of mappings differentiable in a certain sense.

A. Static systems with constraints.

2. Processes.

The *configuration space* of a static system is a differential manifold Q. Quasistatic processes are represented as oriented embedded arcs in the configuration space. An *embedded arc* is a subset $\boldsymbol{c} \subset Q$ which is the image of an embedding $\gamma : [0,a] \to Q$ of a closed interval $[0,a] \subset \mathbb{R}$ defined as the restriction to $[0,a]$ of an embedding $\widetilde{\gamma} : \mathbb{R} \to Q$. The embedding γ is a representative of an embedded arc called its *parameterization*. Different embeddings may be parameterizations of the same arc. A parameterization $\gamma : [0,a] \to Q$ of an arc $\boldsymbol{c}$ induces an *orientation* of the arc. The arc is oriented from $\gamma(0)$ to $\gamma(a)$ if the two points are distinct. The case $\gamma(0) = \gamma(a)$ is of no interest for developing criteria of equilibrium. The *boundary* $\partial\boldsymbol{c}$ of an arc $\boldsymbol{c}$ is a set of two points q and q' such that $\gamma(0) = q$ and $\gamma(a) = q'$ for some parameterization $\gamma : [0,a] \to Q$ of $\boldsymbol{c}$. The designation of one of the boundary points as the *initial configuration* of the process represented by an arc specifies an orientation. An *oriented embedded arc* is defined as a pair $(\boldsymbol{c}, q_0)$, where $\boldsymbol{c}$ is an embedded arc and q_0 is a point in the boundary $\partial\boldsymbol{c}$ designated as the initial configuration. The remaining boundary point is the *terminal configuration* of $(\boldsymbol{c}, q_0)$. A parameterization $\gamma : [0,a] \to Q$ of an arc $\boldsymbol{c}$ such that $\gamma(0) = q_0$ is said to be *compatible* with the orientation of the oriented arc $(\boldsymbol{c}, q_0)$. Compatible parameterizations will be used for oriented arcs.

Let $(\boldsymbol{c}, q_0)$ be an oriented arc. We refer to points in $\boldsymbol{c}$ as elements of $(\boldsymbol{c}, q_0)$. Elements of an oriented arc $(\boldsymbol{c}, q_0)$ are ordered by the relation $\leqslant$ defined in terms of any parameterization γ compatible with the orientation. The relation $\gamma(s') \leqslant \gamma(s)$ is equivalent to $s' \leqslant s$. Relations $<$, $\geqslant$, and $>$ are defined in a similar way. They reflect the corresponding relations between the values of the parameter.

Let $(\boldsymbol{c}, q_0)$ and $(\boldsymbol{c}', q'_0)$ be oriented arcs such that $\boldsymbol{c}' \subset \boldsymbol{c}$. There are two ordering relations in $(\boldsymbol{c}', q'_0)$ since elements of $(\boldsymbol{c}', q'_0)$ are also elements of $(\boldsymbol{c}, q_0)$. The arc $(\boldsymbol{c}', q'_0)$ is said to be *included* in the arc $(\boldsymbol{c}, q_0)$ if the two ordering relations coincide. The inclusion relation is denoted by $(\boldsymbol{c}', q'_0) \subset (\boldsymbol{c}, q_0)$. The process represented by $(\boldsymbol{c}', q'_0)$ is a *subprocess* of the process represented by $(\boldsymbol{c}, q_0)$ if $(\boldsymbol{c}', q'_0) \subset (\boldsymbol{c}, q_0)$. If $(\boldsymbol{c}^1, q^1_0)$ and $(\boldsymbol{c}^2, q^2_0)$ are oriented arcs and if one of the pairs $(\boldsymbol{c}^1 \cup \boldsymbol{c}^2, q^1_0)$ or $(\boldsymbol{c}^1 \cup \boldsymbol{c}^2, q^2_0)$ is an oriented arc, then it is considered the *union* of $(\boldsymbol{c}^1, q^1_0)$ and $(\boldsymbol{c}^2, q^2_0)$ and is denoted by $(\boldsymbol{c}^1, q^1_0) \cup (\boldsymbol{c}^2, q^2_0)$.

Let $(\boldsymbol{c}, q_0)$ be an oriented arc. A function $a : \boldsymbol{c} \to \mathbb{R}$ is considered a function on $(\boldsymbol{c}, q_0)$. This function is said to be *differentiable* if it the restriction to $\boldsymbol{c}$ of a differentiable function on Q. The orientation of $(\boldsymbol{c}, q_0)$ makes it possible to single out increasing functions. A function $h \in \mathsf{A}\boldsymbol{c}$ is *increasing* if $q' > q$ implies $h(q') > h(q)$.

An oriented arc represents a quasi static process. We use these two terms interchangeably. We denote by $\mathsf{P}Q$ the set of all processes in Q.

3. Admissible processes.

An essential part of the characterization of a static system is the specification of a set $\boldsymbol{C} \subset \mathsf{P}Q$ of *admissible processes* in the configuration space Q. Admissible processes are the processes that can be actually induced using the control devices at our disposal. The following conditions are satisfied.

(1) A subprocess of an admissible process is admissible.
(2) The union of admissible processes is admissible.

(3) If a process can be approximated with admissible processes, then it is admissible.
The last condition is made more precise by the following statement. Let

$$q^{\sharp}{}_1, q^{\sharp}{}_2, \ldots, q^{\sharp}{}_i, \ldots \tag{1}$$

be a sequence of interior configurations in a process $(\boldsymbol{c}, q_0)$ converging to the terminal configuration q_1. If processes $(\boldsymbol{c}_{q^{\sharp}{}_i|}, q_0)$ are admissible, then $(\boldsymbol{c}, q_0)$ is admissible. The symbol $(\boldsymbol{c}_{q|}, q_0)$ denotes the subprocess of $(\boldsymbol{c}, q_0)$ with $q \in \boldsymbol{c}$ as the terminal configuration. It is based on the arc

$$\boldsymbol{c}_{q|} = \{q' \in \boldsymbol{c};\ q' \leqslant q\}\,. \tag{2}$$

One speaks of *constraints* if the set $\boldsymbol{C}$ is not the set of all arcs in the configuration space. If admissible processes are all arcs in a subset $C^0 \subset Q$, then constraints are considered *holonomic*, in other cases constraints are *non holonomic*.

4. The work function.

Another object characterizing a static system is a function

$$W : \boldsymbol{C} \to \mathbb{R} \tag{3}$$

associating with each admissible process $(\boldsymbol{c}, q_0)$ the *work* $W(\boldsymbol{c}, q_0)$ of the process. Since subprocesses of admissible processes are admissible we can associate with each admissible process $(\boldsymbol{c}, q_0)$ the function

$$w_{(\boldsymbol{c},q_0)} : \boldsymbol{c} \to \mathbb{R} : q \mapsto \begin{cases} W(\boldsymbol{c}_{q|}, q_0) & \text{if } q \neq q_0 \\ 0 & \text{if } q = q_0. \end{cases} \tag{4}$$

The work function will be assumed to satisfy the following conditions.

(1) Work is additive in the sense that if $(\boldsymbol{c}, q_0)$ is the union of admissible processes

$$(\boldsymbol{c}^0, q_0^0), (\boldsymbol{c}^1, q_0^1), \ldots, (\boldsymbol{c}^n, q_0^n) \tag{5}$$

such that for $i = 0, 1, \ldots, n-1$ the terminal configuration of $(\boldsymbol{c}^i, q_0^i)$ is the initial configuration of $(\boldsymbol{c}^{i+1}, q_0^{i+1})$, then

$$W(\boldsymbol{c}, q_0) = \sum_{i=0}^{n} W(\boldsymbol{c}^i, q_0^i). \tag{6}$$

(2) For each admissible process $(\boldsymbol{c}, q_0)$ the function $w_{(\boldsymbol{c},q_0)}$ is differentiable.

5. Stable local equilibrium configurations.

Let $C^0 \subset Q$ be the set of initial configurations of all admissible processes for a static system. A point $q_0 \in C^0$ is called a *stable local equilibrium configuration* of the system if for each admissible process $(\boldsymbol{c}, q_0)$ initiating at q_0 the function $w_{(\boldsymbol{c},q_0)}$ has a local minimum at q_0.

We are excluding constant processes represented by constant arcs. In a complete discussion of equilibrium configurations *isolated admissible configurations* and *admissible constant processes* should be considered. Only constant admissible configurations can initiate at an isolated admissible configuration. Such configurations are obviously stable local equilibrium configurations. No further discussion is necessary.

6. Composition of static systems and control.

Static systems can be composed if they share the same control configuration space. The equality of configuration spaces is usually the result of a suitable choice. Let two systems be characterized by work functions $W_1 : \boldsymbol{C}_1 \to \mathbb{R}$ and $W_2 : \boldsymbol{C}_2 \to \mathbb{R}$ defined on sets $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ of admissible processes in the same configuration manifold Q. The set $\boldsymbol{C} = \boldsymbol{C}_1 \cap \boldsymbol{C}_2$ is the set of admissible processes of the composed system and the work is the function $W = W_1|\boldsymbol{C} + W_2|\boldsymbol{C}$. Coupling the system to other systems with the same configuration space is a form of control. The function W is used to find the equilibrium configurations of the controlled system.

Composition of constrained systems may encounter difficulties.

7. Realistic constraints.

7.1. Jets of processes

Let q be a point in a differential manifold Q. In the algebra $\mathsf{A}Q$ of differentiable functions on Q we introduce a sequence of ideals

$$\mathsf{I}_0(Q,q),\ \mathsf{I}_1(Q,q),\ \ldots,\ \mathsf{I}_k(Q,q),\ \ldots. \tag{7}$$

The ideal

$$\mathsf{I}_0(Q,q) = \{f \in \mathsf{A}Q;\ f(q) = 0\} \tag{8}$$

associated with q is maximal in the sense that it is not a proper subset of any ideal except the trivial ideal $\mathsf{A}Q$. For $k \in \mathbb{N}$, the ideal $\mathsf{I}_k(Q,q)$ is the power $(\mathsf{I}_0(Q,q))^{k+1}$ of the ideal $\mathsf{I}_0(Q,q)$. Inclusion relations

$$\mathsf{I}_k(Q,q) \subset \mathsf{I}_{k'}(Q,q) \tag{9}$$

hold for all k' and k in $\mathbb{N}$ such that $k' \leqslant k$.

For each $k \in \mathbb{N}$ we introduce an equivalence relation in the set $\mathbf{P}Q$ of oriented embedded arcs in Q. Arcs $(\boldsymbol{c}, q_0)$ and $(\boldsymbol{c}', q_0')$ are equivalent if

$$\mathsf{I}_k(Q,q_0') + \mathsf{I}_0(Q,\boldsymbol{c}') = \mathsf{I}_k(Q,q_0) + \mathsf{I}_0(Q,\boldsymbol{c}). \tag{10}$$

The symbol $\mathsf{I}_0(Q,\boldsymbol{c})$ denotes the ideal

$$\mathsf{I}_0(Q,S) = \left\{f \in \mathsf{A}Q;\ \forall_{q\in S}\ f(q) = 0\right\}. \tag{11}$$

Equivalence classes are called *k-jets* of processes. The set of k-jets will be denoted by P^kQ. The k-jet of a process $(\boldsymbol{c}, q_0)$ will be denoted by $\mathsf{j}^k(\boldsymbol{c}, q_0)$. Inclusion relations (9) imply the existence of projections

$$\pi^{k'}{}_{k\,Q} : \mathsf{P}^kQ \to \mathsf{P}^{k'}Q : \mathsf{j}^k(\boldsymbol{c}, q_0) \mapsto \mathsf{j}^{k'}(\boldsymbol{c}, q_0) \tag{12}$$

for $k' \leqslant k$ in addition to

$$\pi_{k\,Q} : \mathsf{P}^kQ \to Q : \mathsf{j}^k(\boldsymbol{c}, q_0) \mapsto q_0. \tag{13}$$

7.2. Constraints determined by differential equations.

From the set $\boldsymbol{C} \subset \mathbf{P}Q$ of admissible processes of a static system we extract the sequence of sets

$$C^0 \subset Q,\ C^1 \subset \mathsf{P}^1Q, \ldots,\ C^k \subset \mathsf{P}^kQ, \ldots \tag{14}$$

The k-jet of a process is in C^k if it has a representative in $\boldsymbol{C}$. Relations

$$\pi^{k'}{}_{k\,Q}(C^k) = C^{k'} \tag{15}$$

hold for $k' \leqslant k$ and

$$\pi_{k\,Q}(C^k) = C^0 \tag{16}$$

for $k > 0$.

Let $(\boldsymbol{c}, q_0)$ be a process with the property that for each configuration $q \in (\boldsymbol{c}, q_0)$ the jet $\mathsf{j}^k(\boldsymbol{c}_{|q}, q)$ of the process $(\boldsymbol{c}_{|q}, q)$ based on the arc

$$\boldsymbol{c}_{|q} = \{q' \in \boldsymbol{c};\ q' \geqslant q\}. \tag{17}$$

is in C^k. We denote by $\boldsymbol{C}^k$ the class of all such processes. If a process $(\boldsymbol{c}, q_0)$ with terminal point q_1 is in $\boldsymbol{C}^k$, then $\mathsf{j}^k(\boldsymbol{c}_{|q}, q) \in C^k$ for each $q \in \boldsymbol{c}$. It follows that $\mathsf{j}^{k'}(\boldsymbol{c}_{|q}, q) = \pi^{k'}{}_{k\,Q}(\mathsf{j}^k(\boldsymbol{c}_{|q}, q)) \in C^{k'}$ for each $q \in \boldsymbol{c}$. Hence, $(\boldsymbol{c}, q_0) \in \boldsymbol{C}^{k'}$. We have established the inclusion

$$\boldsymbol{C}^{k'} \supset \boldsymbol{C}^k \tag{18}$$

for $k' \leqslant k$.

It may happen that the equality $\boldsymbol{C} = \boldsymbol{C}^k$ holds for $k \in \mathbb{N}$. If it holds for k and not for $k-1$, then the set $\boldsymbol{C}$ is said to represent *constraints of order* k. The set C^k is interpreted as a differential equation of order k. Admissible processes are obtained by solving this equation. Constraints of order 0 are the constraints already recognized as *holonomic*. Constraints of order higher than 0 are said to be *non holonomic*. Constraints of order higher than 1 are not usually discussed since their presence is not apparent if only first order criteria of equilibrium are considered.

8. Jets of processes with volume, the work integral

A *k-jet of a process with volume* is a pair $(\mathsf{j}^k(\boldsymbol{c}, q_0), v)$, where $v \in \mathsf{T}_{q_0}Q$ is a vector represented by one of the parameterizations of $\boldsymbol{c}$ compatible with the orientation of $(\boldsymbol{c}, q_0)$. We denote by V^kQ the space of k-jets with volume. Spaces V^0Q and V^1Q are identified with the bundle $\overset{\circ}{\mathsf{T}}Q \subset \mathsf{T}Q$ of tangent vectors with the zero vectors removed. This is possible since in the pair $(\mathsf{j}^0(\boldsymbol{c}, q_0), v)$ or $(\mathsf{j}^1(\boldsymbol{c}, q_0), v)$ the vector v already contains the complete information about the jet. We denote by $\boldsymbol{V}^k$ the k-jets with volume associated with admissible processes. Note that if $(\mathsf{j}^k(\boldsymbol{c}, q_0), v) \in \boldsymbol{V}^k$, then $(\mathsf{j}^k(\boldsymbol{c}, q_0), \lambda v) \in \boldsymbol{V}^k$ for each $\lambda > 0$.

The work of a realistic static system is defined in terms of a *work form*

$$\vartheta \colon \boldsymbol{V}^k \to \mathbb{R}. \tag{19}$$

The work form is positive homogeneous in its vector argument: if $\lambda > 0$, then $\vartheta(\mathsf{j}^k(\boldsymbol{c}, q_0), \lambda v) = \lambda\vartheta(\mathsf{j}^k(\boldsymbol{c}, q_0), v)$. The work form is used to define the work function $w_{(\boldsymbol{c},q_0)}$ along an admissible processes. Let q_1 be the terminal point of the process $(\boldsymbol{c}, q_0)$. For each configuration $q \in \boldsymbol{c} \setminus \{q_1\}$ the value of the work function is the integral

$$w_{(\boldsymbol{c},q_0)}(q) = \int_{(\boldsymbol{c}|q,q_0)} \vartheta \tag{20}$$

of the work form defined as the Riemann integral

$$\int_0^{\gamma^{-1}(q)} \vartheta(\mathsf{j}^k(\boldsymbol{c}_{|\gamma(s)}, \gamma(s)), \mathsf{t}\gamma(s))\mathrm{d}s \tag{21}$$

in terms of a parameterization $\gamma \colon [0, a] \to Q$. Homogeneity of ϑ makes the integral independent of the parameterization. The work of the process is the integral

$$W(\boldsymbol{c}, q_0) = \int_0^{\gamma^{-1}(q_1)} \vartheta(\mathsf{j}^k(\boldsymbol{c}_{|\gamma(s)}, \gamma(s)), \mathsf{t}\gamma(s))\mathrm{d}s. \tag{22}$$

The work form of a system of the usually considered type is a positive homogeneous function

$$\vartheta \colon \boldsymbol{V}^0 \to \mathbb{R} \tag{23}$$

or

$$\vartheta \colon \boldsymbol{V}^1 \to \mathbb{R}. \tag{24}$$

9. Stable local equilibria defined in terms of germs.

The function $w_{(\boldsymbol{c},q_0)}$ will have a local minimum at q_0 if there is a point $q \in \boldsymbol{c}$ such that the function $w_{(\boldsymbol{c},q_0)}$ restricted to the arc

$$\boldsymbol{c}_{q|} = \{q' \in \boldsymbol{c};\ q' \leqslant q\} \tag{25}$$

is increasing. This observation implies that equilibrium at q_0 is a property of the germ of $w_{(\boldsymbol{c},q_0)}$ at q_0. We will state the definition of equilibrium in terms of germs.

With a process $(\boldsymbol{c}, q_0)$ we associate an ideal $\mathsf{I}_\mathfrak{c}(\boldsymbol{c}, q_0)$ in the algebra $\mathsf{A}(\boldsymbol{c}, q_0)$ of differentiable functions on $(\boldsymbol{c}, q_0)$. A function h is in this ideal if there is a neighbourhood $V \in \boldsymbol{c}$ of the initial configuration q_0 such that $h|V = 0$. The quotient algebra

$$\mathsf{A}^\mathfrak{c}(Q, q_0) = \mathsf{A}(\boldsymbol{c}, q_0)/\mathsf{I}_\mathfrak{c}(Q, q_0) \tag{26}$$

is the algebra of *germs* of functions on $(\boldsymbol{c}, q_0)$ at q_0. An element of $\mathsf{A}^\mathfrak{c}(\boldsymbol{c}, q_0)$ is said to be *increasing* if it has an increasing representative in $\mathsf{A}(\boldsymbol{c}, q_0)$.

A point $q_0 \in C^0$ is a *stable local equilibrium configuration* if for each process $(\boldsymbol{c}, q_0)$ the germ $\mathsf{j}^\mathfrak{c} w_{(\boldsymbol{c}, q_0)}$ is increasing.

10. Jets of functions on $[0, a] \subset \mathbb{R}$

Some elementary results concerning the algebra $\mathsf{A}[0, a]$ of differentiable functions on $[0, a] \subset \mathbb{R}$, its ideals

$$\mathsf{I}_0([0, a], 0),\ \mathsf{I}_1([0, a], 0),\ \ldots,\ \mathsf{I}_k([0, a], 0),\ \ldots \tag{27}$$

and the quotient algebras

$$\begin{aligned} \mathsf{A}^0([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_0([0, a], 0),\ \mathsf{A}^1([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_1([0, a], 0), \ldots \\ \ldots,\ \mathsf{A}^k([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_k([0, a], 0),\ \ldots \end{aligned} \tag{28}$$

are needed for developing differential conditions of equilibrium. Elements of the algebra $\mathsf{A}^k([0, a], 0)$ are k-jets of functions on $[0, a]$. A differentiable function on $[0, a]$ is the restriction to $[0, a]$ of a differentiable function on $\mathbb{R}$. A function is in an ideal $\mathsf{I}_k([0, a], 0)$ if and only if it is the restriction to $[0, a]$ of a function in $\mathsf{I}_k(\mathbb{R}, 0)$. As a consequence the quotient algebra $\mathsf{A}^k([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_k([0, a], 0)$ is isomorphic to $\mathsf{A}^k(\mathbb{R}, 0) = \mathsf{A}\mathbb{R}/\mathsf{I}_k(\mathbb{R}, 0)$.

In each algebra $\mathsf{A}^k([0, a], 0)$ there is a sequence of ideals

$$\begin{aligned} {\mathsf{I}^k}_0([0, a], 0) = \mathsf{I}_0([0, a], 0)/\mathsf{I}_k([0, a], 0),\ {\mathsf{I}^k}_1([0, a], 0) = \mathsf{I}_1([0, a], 0)/\mathsf{I}_k([0, a], 0), \ldots \\ \ldots,\ {\mathsf{I}^k}_k([0, a], 0) = \mathsf{I}_k([0, a], 0)/\mathsf{I}_k([0, a], 0). \end{aligned} \tag{29}$$

A function $g \in \mathsf{A}[0, a]$ is said to be *increasing* if the inequality $s' > s$ implies $q(s') > g(s)$. An element of a quotient algebra $\mathsf{A}^\mathfrak{c}([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_\mathfrak{c}([0, a], 0)$ is said to be *increasing* if it has an increasing representative in $\mathsf{A}[0, a]$. A function $g \in \mathsf{A}[0, a]$ is in $\mathsf{I}^\mathfrak{c}([0, a], 0)$ if and only if there is a real number $\delta > 0$ such that $g(s) = 0$ for $s \leqslant \delta$. It follows that the germ $\mathsf{j}^\mathfrak{c} g(0)$ of a function $g \in \mathsf{A}[0, a]$ is increasing if there is a number $\delta > 0$ such that g is increasing in $[0, \delta]$.

Proposition 1. *For $k \in \mathbb{N}$ a function $g \in \mathsf{A}[0, a]$ is in $\mathsf{I}_k([0, a], 0)$ if and only if $\mathrm{D}^i g(0) = 0$ for each $i \leqslant k$.*

Proof: The derivatives of a function $g \in \mathsf{A}[0, a]$ at 0 are well defined and the function can be represented by the Taylor formula

$$g = e_0(g) + e_1(g)s + \ldots + e_k(g)s^k + rs^{k+1}, \tag{30}$$

where

$$e_i(g) = \frac{1}{k!}\mathrm{D}^i g(0), \tag{31}$$

$s\colon [0, a] \to \mathbb{R}$ is the canonical injection, and r is a differentiable function on $[0, a]$. The function s is in $\mathsf{I}_0([0, a], 0)$ and the power s^{k+1} is in $\mathsf{I}_k([0, a], 0)$. If $\mathrm{D}^i g(0) = 0$ for each $i \leqslant k$, then $g = rs^{k+1}$ is in $\mathsf{I}_k([0, a], 0)$.

A function $g \in \mathsf{I}_l([0, a], 0)$ is a combination of products $g_0 g_1 \cdots g_l$ of elements of $\mathsf{I}_0([0, a], 0)$. The derivative $\mathrm{D}g$ is a combination of products of functions with each product containing at least l factors in $\mathsf{I}_0([0, a], 0)$. It follows that the derivative $\mathrm{D}g$ of a function $g \in \mathsf{I}_l([0, a], 0)$ is in $\mathsf{I}_{l-1}([0, a], 0)$. If

$g \in \mathsf{I}_k([0,a],0)$, then $\mathrm{D}^0 g = g \in \mathsf{I}_k([0,a],0)$, $\mathrm{D}^1 g = \mathrm{D}g \in \mathsf{I}_{k-1}([0,a],0)$, $\mathrm{D}^2 g \in \mathsf{I}_{k-2}([0,a],0)$, $\ldots$, $\mathrm{D}^k g \in \mathsf{I}_0([0,a],0)$. Hence, $\mathrm{D}^i g(0) = 0$ for each $i \leqslant k$. ∎

It follows from Proposition 1 that the jet $\mathsf{j}^k g(0) \in \mathsf{A}^k([0,a],0)$ is fully represented by the sequence

$$e_0(g),\ e_1(g), \ldots,\ e_k(g) \tag{32}$$

of derivatives

$$e_i(g) = \frac{1}{k!}\mathrm{D}^i g(0) \tag{33}$$

of its representative $g \in \mathsf{A}[0,a]$. The polynomial

$$e_0(g) + e_1(g)s + \ldots + e_k(g)s^k \tag{34}$$

represents the jet $\mathsf{j}^k g(0)$ and the product of two jets is represented by the product of the corresponding polynomials truncated after the first $k+1$ terms.

An element of the ideal $\mathsf{I}^k{}_0([0,a],0) \subset \mathsf{A}^k([0,a],0)$ is represented by the sequence $e_0,\ e_1, \ldots,\ e_k$ with $e_0 = 0$. This element is said to be *positive* if the first non zero element in the sequence is positive. The element is said to be *negative* if the first non zero element in the sequence is negative. Each element of the ideal $\mathsf{I}^k{}_0([0,a],0)$ is either positive or negative if it is not zero. There are obvious relations $>$, $<$, $\geqslant$, and $\leqslant$ between elements of $\mathsf{I}^k{}_0([0,a],0)$.

PROPOSITION 2. *If a jet $\mathsf{j}^k g(0)$ in the ideal $\mathsf{I}^k{}_0([0,a],0)$ is positive, then the germ $\mathsf{j}^{\mathsf{c}} g(0)$ is increasing. If the jet $\mathsf{j}^k g(0)$ is negative, then the germ $\mathsf{j}^{\mathsf{c}} g(0)$ is decreasing.*

PROOF: is represented by Taylor formula

$$g(s) = e_l(g)s^l + r(s)s^{l+1}. \tag{35}$$

From

$$\lim_{s\to 0}\left((l+1)r(s)s + \mathrm{D}r(s)s^2\right) = 0 \tag{36}$$

it follows that there is a number $\delta > 0$ such that

$$|(l+1)r(s)s + \mathrm{D}r(s)s^2| < |le_l(g)| \tag{37}$$

for $|s| < \delta$. If $e_l(g) > 0$, then the function g is increasing in the interval $[0,\delta]$ since the derivative

$$\begin{aligned} \mathrm{D}g &= le_l s^{l-1} + (l+1)r(s)s^l + \mathrm{D}r(s)s^{l+1} \\ &= \left(le_l + (l+1)r(s)s + \mathrm{D}r(s)s^2\right)s^{l-1} \end{aligned} \tag{38}$$

is positive for $0 < s < \delta$. This is a consequence of the Lagrange mean value theorem. It follows that the germ $\mathsf{j}^{\mathsf{c}} g(0)$ is increasing. It is shown in a similar way that if e_l is negative, then the germ is decreasing. ∎

It follows from the proposition that if the germ $\mathsf{j}^{\mathsf{c}} g(0)$ of a function g is increasing, then the jet $\mathsf{j}^k g(0)$ is non negative for each $k \in \mathbb{N}$.

11. Differential criteria of local equilibrium.

Let h be a function in the algebra $\mathsf{A}(\boldsymbol{c}, q_0)$ of differentiable functions on $\boldsymbol{c}$ and let $\gamma\colon[0,a] \to Q$ be a parameterization of the process $(\boldsymbol{c}, q_0)$. The parameterization induces the mapping $\boldsymbol{c}|\gamma\colon[0,a] \to \boldsymbol{c}\colon s \mapsto \gamma(s)$. The composition $h \circ \boldsymbol{c}|\gamma$ is a function on $[0,a]$. The function h is increasing if and only if $h \circ \boldsymbol{c}|\gamma$ is increasing. Let h be in the ideal $\mathsf{I}_0(\boldsymbol{c}, q_0) \subset \mathsf{A}(\boldsymbol{c}, q_0)$. For $k \in \mathbb{N}$, the jet $\mathsf{j}^k h \in \mathsf{I}^k{}_0(\boldsymbol{c}, q_0)$ is said to be *positive* if the jet $j^k(h \circ \boldsymbol{c}|\gamma)(0)$ is positive. The jet $\mathsf{j}^k h$ is said to be *negative* if the jet $j^k(h \circ \boldsymbol{c}|\gamma)(0)$ is negative. Being positive or negative is a property of the jet $\mathsf{j}^k h$ independent of the choice of the parameterization. The following proposition is an adaptation of Proposition 2.

Proposition 3. *If a jet* $\mathsf{j}^k h$ *in the ideal* $\mathsf{I}^k{}_0(\boldsymbol{c}, q_0)$ *is positive, then the germ* $\mathsf{j}^{\mathfrak{c}} h$ *is increasing. If the jet* $\mathsf{j}^k h$ *is negative, then the germ* $\mathsf{j}^{\mathfrak{c}} h$ *is decreasing.*

The proposition implies that if the germ $\mathsf{j}^{\mathfrak{c}} h$ of a function h is increasing, then the jet $\mathsf{j}^k h$ is non negative for each $k \in \mathbb{N}$.

Let q_0 be a configuration in the set C^0 an let $k \in \mathbb{N}$. If for each jet $j^k(\boldsymbol{c}, q_0)$ of an admissible process $(\boldsymbol{c}, q_0)$ the jet $\mathsf{j}^k w_{(\boldsymbol{c},q_0)}$ is positive, then the germ $\mathsf{j}^{\mathfrak{c}} w_{(\boldsymbol{c},q_0)}$. Hence, q_0 is a stable equilibrium configuration. We have obtained a sufficient condition for a point $q_0 \in C^0$ to be a stable local equilibrium configuration for each $k \in \mathbb{N} \cup \{\infty\}$.

If q_0 is a stable equilibrium configuration, then the germ $\mathsf{j}^{\mathfrak{c}} w_{(\boldsymbol{c},q_0)}$ is increasing for each germ $j^{\mathfrak{c}}(\boldsymbol{c}, q_0)$ of an admissible process $(\boldsymbol{c}, q_0)$. It follows that for each $k \in \mathbb{N}$ and each jet $j^k(\boldsymbol{c}, q_0)$ of an admissible process $(\boldsymbol{c}, q_0)$ the jet $\mathsf{j}^k w_{(\boldsymbol{c},q_0)}$ is non negative. This results in a series of necessary conditions for a point $q_0 \in C^0$ to be a stable local equilibrium configuration. Neither of these conditions is sufficient. Variational principles of classical physics are based on the necessary equilibrium condition of order $k = 1$.

12. Refinements of the criteria of equilibrium.

In the set $\mathsf{P}Q$ of oriented embedded arcs in Q we introduce an equivalence relation. Arcs $(\boldsymbol{c}, q_0)$ and $(\boldsymbol{c}', q_0')$ are equivalent

$$\mathsf{I}_{\mathfrak{c}}(Q, q_0') + \mathsf{I}_0(Q, \boldsymbol{c}') = \mathsf{I}_{\mathfrak{c}}(Q, q_0) + \mathsf{I}_0(Q, \boldsymbol{c}). \tag{39}$$

Equivalence classes are called *germs* of processes. The set of germs is denoted by $\mathsf{P}^{\mathfrak{c}}Q$. The germ of a process $(\boldsymbol{c}, q_0)$ is denoted by $\mathsf{j}^{\mathfrak{c}}(\boldsymbol{c}, q_0)$. The symbol $C^{\mathfrak{c}}$ will denote the set of germs of admissible processes.

The algebra $\mathsf{A}(\boldsymbol{c}, q_0)$ is canonically isomorphic to the quotient algebra $\mathsf{A}Q/\mathsf{I}_0(Q, \boldsymbol{c})$ and the ideal $\mathsf{I}_{\mathfrak{c}}(\boldsymbol{c}, q_0)$ is isomorphic to the quotient

$$\left(\mathsf{I}_{\mathfrak{c}}(Q, q_0) + \mathsf{I}_0(Q, \boldsymbol{c})\right)/\mathsf{I}_0(Q, \boldsymbol{c}). \tag{40}$$

We will adopt the following identification

$$\mathsf{A}^{\mathfrak{c}}(\boldsymbol{c}, q_0) = \mathsf{A}(\boldsymbol{c}, q_0)/\mathsf{I}_{\mathfrak{c}}(\boldsymbol{c}, q_0) \ \text{ is identified with } \ \mathsf{A}Q/\left(\mathsf{I}_{\mathfrak{c}}(Q, q_0) + \mathsf{I}_0(Q, \boldsymbol{c})\right). \tag{41}$$

As a consequence of this identification the algebra $\mathsf{A}^{\mathfrak{c}}(\boldsymbol{c}, q_0)$ is associated with the germ $\mathsf{j}^{\mathfrak{c}}(\boldsymbol{c}, q_0)$ rather than with the process $(\boldsymbol{c}, q_0)$ since the algebra $\mathsf{A}^{\mathfrak{c}}(\boldsymbol{c}', q_0')$ is the same as the algebra $\mathsf{A}^{\mathfrak{c}}(\boldsymbol{c}, q_0)$ if $(\boldsymbol{c}', q_0')$ and $(\boldsymbol{c}, q_0)$ are equivalent.

Each function $h \in \mathsf{A}(\boldsymbol{c}, q_0)$ has a germ $\mathsf{j}^{\mathfrak{c}} h \in \mathsf{A}^{\mathfrak{c}}(\boldsymbol{c}, q_0)$. The germ $\mathsf{j}^{\mathfrak{c}} w_{(\boldsymbol{c},q_0)}$ of the work function $w_{(\boldsymbol{c},q_0)}$ along an admissible process $(\boldsymbol{c}, q_0)$ is the same as the germ $\mathsf{j}^{\mathfrak{c}} w_{(\boldsymbol{c}',q_0')}$ if $(\boldsymbol{c}', q_0')$ and $(\boldsymbol{c}, q_0)$ are equivalent.

A reformulation of the definiion of stable local equilibrium is now possible. A point $q_0 \in C^0$ is a *stable local equilibrium configuration* if for each process $\mathsf{j}^{\mathfrak{c}}(\boldsymbol{c}, q_0)$ the germ $\mathsf{j}^{\mathfrak{c}} w_{(\boldsymbol{c},q_0)}$ is increasing.

In the algebra $\mathsf{A}(\boldsymbol{c}, q_0)$ of differentiable functions on the oriented arc $(\boldsymbol{c}, q_0)$ we have ideals

$$\mathsf{I}_0(\boldsymbol{c}, q_0),\ \mathsf{I}_1(\boldsymbol{c}, q_0),\ \ldots,\ \mathsf{I}_k(\boldsymbol{c}, q_0),\ \ldots \tag{42}$$

defined just as the ideals in the sequence (7) and the corresponding quotient algebras

$$\begin{aligned} \mathsf{A}^0(\boldsymbol{c}, q_0) = \mathsf{A}(\boldsymbol{c}, q_0)/\mathsf{I}_0(\boldsymbol{c}, q_0),\ \mathsf{A}^1(\boldsymbol{c}, q_0) = \mathsf{A}(\boldsymbol{c}, q_0)/\mathsf{I}_1(\boldsymbol{c}, q_0), \ldots \\ \ldots,\ \mathsf{A}^k(\boldsymbol{c}, q_0) = \mathsf{A}(\boldsymbol{c}, q_0)/\mathsf{I}_k(\boldsymbol{c}, q_0),\ \ldots \end{aligned} \tag{43}$$

For each $k \in \mathbb{N}$ the ideal $\mathsf{I}_k(\boldsymbol{c}, q_0)$ is isomorphic to the quotient

$$\left(\mathsf{I}_k(Q, q_0) + \mathsf{I}_0(Q, \boldsymbol{c})\right)/\mathsf{I}_0(Q, \boldsymbol{c}). \tag{44}$$

This justifies the following identification

$$\mathsf{A}^k(\boldsymbol{c}, q_0) = \mathsf{A}(\boldsymbol{c}, q_0)/\mathsf{I}_k(\boldsymbol{c}, q_0) \text{ is identified with } \mathsf{A}Q/\left(\mathsf{I}_k(Q, q_0) + \mathsf{I}_0(Q, \boldsymbol{c})\right). \tag{45}$$

The identification implies that the algebra $\mathsf{A}^k(\boldsymbol{c}, q_0)$ is associated with the k-jet $\mathsf{j}^k(\boldsymbol{c}, q_0)$ rather than with the process $(\boldsymbol{c}, q_0)$ since the algebra $\mathsf{A}^k(\boldsymbol{c}', q_0')$ is the same as the algebra $\mathsf{A}^k(\boldsymbol{c}, q_0)$ if $(\boldsymbol{c}', q_0')$ and $(\boldsymbol{c}, q_0)$ have the same k-jet.

In view of these observations it is possible to certain refinements to the differential criteria of equlibrium. Only the k-jet $\mathsf{j}^k(\boldsymbol{c}, q_0)$ of a process intervenes in equlibrium criteria of order k. We will apply such refinements to particular cases of the differential criteria.

13. The principle of virtual work.

Variational principles of classical physics are versions of what is known in statics as the *principle of virtual work*. This principe consists in applying the first necessary equilibrium condition to a static system. Constraints are represented by a set $\boldsymbol{V}^1 \in \mathsf{T}Q$.

and that the work is represented by a work form

$$\vartheta : \boldsymbol{V}^1 \to \mathbb{R}. \tag{46}$$

Elements of $\boldsymbol{V}^1$ are the *admissible virtual displacements*. The principle of virtual work states that the inequality

$$\vartheta(v) \geqslant 0 \tag{47}$$

holds for each virtual displacement v tangent to an admissible process initiating at a configuration of equilibrium.

A process is represented by first jets with volume identified with tangent vectors. This is a simplicication based on the observations of the preceding section.

14. Potential systems and the Legendre transformation.

A *potential* of a *potential static system* is a differentiable function U on the configuration space Q. A potential system is unconstrained. For each process $(\boldsymbol{c}, q_0) \in \boldsymbol{C} = \mathsf{P}Q$ with terminal point q_1 the function $w_{(\boldsymbol{c}, q_0)}$ is defined by

$$w_{(\boldsymbol{c}, q_0)}(q) = U(q) - U(q_0) \tag{48}$$

for $q \in \boldsymbol{c} \setminus \{q_1\}$ and the work $W(\boldsymbol{c}, q_0) = w_{(\boldsymbol{c}, q_0)}(q_1)$ is the limit

$$W(\boldsymbol{c}, q_0) = \lim_{q \to q_1} w_{(\boldsymbol{c}, q_0)}(q). \tag{49}$$

The set $\boldsymbol{V}^1$ of a potential system is the tangent bundle $\mathring{\mathsf{T}}Q$ and the function ϑ is the function

$$\vartheta : \mathring{\mathsf{T}}Q \to \mathbb{R} : v \mapsto \langle \mathrm{d}U, v \rangle \tag{50}$$

constructed from the differential $\mathrm{d}U$ of the potential. The principle of virtual work for a potential system states that the equality

$$\langle \mathrm{d}U, v \rangle = 0 \tag{51}$$

holds for every virtual displacement from a configuration of equilibrium. The appearance of the equality in place of the inequality is due to the *reversibility* of virtual displacements: if v is a admissible virtual displacement from a configuration of equilibrium, then so is $-v$ and $\langle \mathrm{d}U, -v \rangle = -\langle \mathrm{d}U, v \rangle$

The *Legendre transformation* associates with a static system represented by a set $\boldsymbol{V}^1 \subset \mathring{\mathsf{T}}Q$ and a function $\vartheta : \boldsymbol{V}^1 \to \mathbb{R}$ the *constitutive set* S of the system defined as

$$S = \left\{ f \in \mathsf{T}^*Q;\ \pi_Q(f) \in C^0, \vartheta(v) - \langle f, v \rangle \geqslant 0 \text{ for each } v \in V \text{ such that } \tau_Q(v) = \pi_Q(f) \right\} \tag{52}$$

The set C^0 is obtained by applying the tangent projection τ_Q to virtual displacements. If constraints are holonomic, then $\boldsymbol{V}^1 = \mathring{\mathsf{T}} C^0$. A virtual displacement is in $\mathring{\mathsf{T}} C^0$ if there is a curve $\xi : \mathbb{R} \to Q$ and a number $\varepsilon > 0$ such that $\xi([0, \varepsilon]) \subset C^0$ and $v = \mathsf{t}\xi(0) \neq 0$. The inequality in the definition of S is replaced by an equality in the case of reversibility.

The physics of the Legendre transformation is that of control of a static system by means of potential external devices. The covector $f = -\mathrm{d}U(\pi_Q(f))$ is the *external force* applied to the controlled system by the potential device with potential U. The constitutive set S characterizes the response of a static system to control by potential devices only. Under certain conditions (convexity) this characterization is complete since the set V and the function ϑ can by reconstructed from the constitutive set by the *inverse Legendre transformation.*

15. Examples of static systems and Legendre transformations.

Configuration spaces of most systems considered here are affine spaces. If Q is an affine space modeled on a vector space W, then the tangent bundle $\mathsf{T}Q$ is identified with the product $Q \times W$, the cotangent bundle is identified with $Q \times W^*$ and the canonical pairing is the mapping

$$\langle\ ,\ \rangle : (Q \times W^*) \times (Q \times W) : ((q, f), (q, w)) \mapsto \langle f, w \rangle. \tag{53}$$

We denote by $q_1 - q_0$ the vector associated with the points q_0 and q_1.

For the sake of simplicity we will not exclude zero vectors in the set V of virtual displacements.

Example 1. Let Q be the configuration manifold of a static mechanical system and let $\rho : \mathsf{T}Q \to \mathsf{T}^*Q$ be a mapping with the properties of a Euclidean metric. This means that ρ defines an isomorphism of vector bundles

$$\begin{array}{ccc} \mathsf{T}Q & \xrightarrow{\ \rho\ } & \mathsf{T}^*Q \\ {\scriptstyle \tau_Q}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Q} \\ Q & = & Q \end{array}, \tag{54}$$

the symmetry relation

$$\langle \rho(u), v \rangle = \langle \rho(v), u \rangle \tag{55}$$

holds for any pair $(u, v) \in \mathsf{T}Q \underset{(\tau_Q, \tau_Q)}{\times} \mathsf{T}Q$, the inequality

$$\langle \rho(v), v \rangle > 0 \tag{56}$$

holds for each $v \neq 0$. The function

$$\vartheta : \mathsf{T}Q \to \mathbb{R} : v \mapsto \sqrt{\langle \rho(v), v \rangle} \tag{57}$$

is positive homogeneous on fibres of the tangent fibration. It represents the virtual work of a virtual displacement v due to friction. The principle of virtual work is the inequality

$$\sqrt{\langle \rho(v), v \rangle} - \langle f, v \rangle \geqslant 0. \tag{58}$$

It is satisfied if f is in the constitutive set S and $(f, v) \in \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q$.

Let $f \in \mathsf{T}^*Q$ be in the constitutive set. By using $v = \rho^{-1}(f)$ in the principle of virtual work we arrive at the inequality

$$\langle f, \rho^{-1}(f) \rangle \leqslant \sqrt{\langle f, \rho^{-1}(f) \rangle}. \tag{59}$$

Hence,

$$\langle f, \rho^{-1}(f) \rangle \leqslant 1. \tag{60}$$

The inequality

$$\langle f, v\rangle \leqslant \sqrt{\langle f, \rho^{-1}(f)\rangle}\sqrt{\langle \rho(v), v\rangle}. \tag{61}$$

is the result of the Schwarz inequality

$$\langle \rho(u), v\rangle \leqslant \sqrt{\langle \rho(u), u\rangle}\sqrt{\langle \rho(v), v\rangle} \tag{62}$$

applied to the pair of vectors $u = \rho^{-1}(f)$ and v. If $\langle f, \rho^{-1}(f)\rangle \leqslant 1$, then

$$\langle f, v\rangle \leqslant \sqrt{\langle \rho(v), v\rangle}. \tag{63}$$

Hence, $f \in S$.

The constitutive set of the system is the set

$$S = \left\{ f \in \mathsf{T}^*Q;\ \langle f, \rho^{-1}(f)\rangle \leqslant 1 \right\}. \tag{64}$$

▲

Example 2. Let a material point with configuration q in the Euclidean affine space Q be tied with a rigid rod of length a to a point with configuration q_0. The configuration q is constrained to the sphere

$$C^0 = \{q \in Q;\ \|q - q_0\| = a\}. \tag{65}$$

This is a system with holonomic bilateral constraints. The set

$$V = \{(q, \delta q) \in Q \times W;\ \|q - q_0\| = a,\ \langle g(q - q_0), \delta q\rangle = 0\} \tag{66}$$

of admissible virtual displacements is the tangent set $\mathsf{T}C^0$ of the holonomic constraint C^0. With the work form $\vartheta = 0$ the constitutive set is the set

$$S = \left\{ (q, f) \in Q \times W^*;\ \|q - q_0\| = a,\ f = a^{-2}\langle f, q - q_0\rangle g(q - q_0) \right\}. \tag{67}$$

▲

Example 3. Let $Q = X \times D$ be the configuration space of a skate. The space X is an affine plane modeled on a vector space W and D is the projective space of directions in X. We use a Euclidean metric in X represented by a mapping $g\colon W \to W^*$ to identify the space D with the unit circle

$$D = \{\varphi \in W;\ \langle g(\varphi), \varphi\rangle = 1\}. \tag{68}$$

This is the only use we make of the metric. The tangent bundle $\mathsf{T}Q$ is identified with $X \times W \times \mathsf{T}D$, where

$$\mathsf{T}D = \{(\varphi, \delta\varphi) \in D \times W;\ \langle g(\varphi), \delta\varphi\rangle = 0\}. \tag{69}$$

The skate is a system with non holonomic constraints. The set C^0 is the entire space Q. The constraint consists in restricting virtual displacements in X to those parallel to the direction specified by an element of D. Thus

$$V = \left\{ (x, \delta x, \varphi, \delta\varphi) \in X \times W \times \mathsf{T}D;\ \exists_{k\in\mathbb{R}}\ \delta x = k\varphi \right\}. \tag{70}$$

The cotangent bundle T^*Q is the product $X \times W^* \times \mathsf{T}^*D$. For each $\varphi \in D$ the fibre $\mathsf{T}_\varphi D$ is a subset of W. Hence, the cotangent bundle T^*D can be specified as the set of pairs (φ, τ), where $\varphi \in D$ and τ is in the quotient space $W^*/\mathsf{T}^\circ_\varphi D$. The set

$$\begin{aligned} S &= \left\{ (x, f, \varphi, \tau) \in \mathsf{T}^*Q;\ \langle f, \xi\rangle + \langle \tau, \delta\varphi\rangle = 0 \text{ foreach } (x, \delta x, \varphi, \delta\varphi) \in V \right\} \\ &= \left\{ (x, f, \varphi, \tau) \in \mathsf{T}^*Q;\ \langle f, \varphi\rangle = 0, \tau = 0 \right\} \end{aligned} \tag{71}$$

is the constitutive set of the system. ▲

EXAMPLE 4. Let Q be the Euclidean affine space of Newtonian mechanics. The model space for Q is a vector space W of dimension 3. The Euclidean structure is represented by a metric tensor $g\colon W \to W^*$.

The example gives a formal description of experiments performed by Coulomb in his study of static friction. Let a material point be constrained to the set

$$C^0 = \{q \in Q;\ \langle g(k), q - q_0\rangle \geqslant 0\}\,, \tag{72}$$

where q_0 is a point in Q and $k \in W$ is a unit vector. The boundary

$$\partial C^0 = \{q \in Q;\ \langle g(k), q - q_0\rangle = 0\} \tag{73}$$

is a plane passing through q_0 and orthogonal to k. In its displacements on the boundary the point encounters friction proportional to the component of the external force pressing the point against the boundary. The system is characterized by the virtual work function $\vartheta = 0$ defined on the non holonomic constraints

$$V = \Big\{(q, \delta q) \in \mathsf{T}Q;\ \langle g(k), q - q_0\rangle \geqslant 0,\ \langle g(k), \delta q\rangle\rangle \geq \nu\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle)^2} \quad \text{if } \langle g(k), q - q_0\rangle = 0\Big\}\,. \tag{74}$$

The principle of virtual work states that (q, f) is in the constitutive set S if and only if the inequality

$$\langle f, \delta q\rangle \leqslant 0 \tag{75}$$

is satisfied for each $(q, \delta q) \in V$.

If $\langle g(k), q - q_0\rangle > 0$, then a pair $(q, f) \in \mathsf{T}^*Q$ is in the constitutive set S if and only if $f = 0$.

We consider pairs (q, f) with $\langle g(k), q - q_0\rangle = 0$. If $f = -\|f\|k$, then (q, f) is in the constitutive set and $\|f\|^2 - \langle f, k\rangle^2 = 0$. Let (q, f) be in the constitutive set and let $\|f\|^2 - \langle f, k\rangle^2 \neq 0$. The virtual displacement $(q, \delta q)$ with

$$\delta q = g^{-1}(f) - \langle f, k\rangle k + \nu\sqrt{\|f\|^2 - \langle f, k\rangle^2}k \tag{76}$$

is in V since

$$\langle g(k), \delta q\rangle = \nu\sqrt{\|f\|^2 - \langle f, k\rangle^2}. \tag{77}$$

From the principle of virtual work and

$$\langle f, \delta q\rangle = \|f\|^2 - \langle f, k\rangle^2 + \sqrt{\|f\|^2 - \langle f, k\rangle^2}\langle f, k\rangle \tag{78}$$

it follows that

$$\|f\|^2 - \langle f, k\rangle^2 + \sqrt{\|f\|^2 - \langle f, k\rangle^2}\langle f, k\rangle \leqslant 0 \tag{79}$$

and

$$\sqrt{\|f\|^2 - \langle f, k\rangle^2\rangle} + \nu\langle f, k\rangle \leqslant 0 \tag{80}$$

since $\|f\|^2 - \langle f, k\rangle^2\rangle > 0$.

The Schwarz inequality

$$\langle g(u), v\rangle - \langle g(k), u\rangle\langle g(k), v\rangle \leqslant \sqrt{\|u\|^2 - \langle g(k), u\rangle^2}\sqrt{\|v\|^2 - \langle g(k), v\rangle^2} \tag{81}$$

for the bilinear symmetric form

$$(u, v) \mapsto \langle g(u), v\rangle - \langle g(k), u\rangle\langle g(k), v\rangle \tag{82}$$

applied to the pair $(g^{-1}(f), \delta q)$ leads to the inequality

$$\langle f, \delta q\rangle - \langle f, \delta q\rangle\langle g(k), \delta q\rangle \leqslant \sqrt{\|f\|^2 - \langle f, k\rangle^2}\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2}. \tag{83}$$

If $\langle g(k), q - q_0\rangle = 0$, $\sqrt{\|f\|^2 - \langle f, k\rangle^2\rangle} + \nu\langle f, k\rangle \leqslant 0$, and $\langle g(k), \delta q\rangle\rangle \geq \nu\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle\rangle^2}$, then $\langle f, \delta q\rangle \leqslant 0$. Hence, (q, f) is in the constitutive set S.

We have shown that the set

$$\begin{aligned} S = \big\{(q, f) \in \mathsf{T}^*Q;\ \langle g(k), q - q_0\rangle \geqslant 0,\ f = 0 \text{ if } \langle g(k), q - q_0\rangle > 0 \\ \text{and } \sqrt{\|f\|^2 - \langle f, k\rangle^2\rangle} + \nu\langle f, k\rangle \leqslant 0 \text{ if } \langle g(k), q - q_0\rangle = 0\big\} \end{aligned} \tag{84}$$

is the constitutive set of the system. ▲

B. Autonomous dynamic systems with constraints.

16. Motions, processes.

A *motion* is a differentiable mapping

$$\xi \colon [t_0, t_1] \to Q \tag{85}$$

of a closed time interval $[t_0, t_1] \subset \mathbb{R}$ in the *configuration space* Q. The configuration space is a differential manifold. The space of motions defined on an interval $[t_0, t_1]$ will be denoted by $\boldsymbol{Q}_{[t_0,t_1]}$. A *process* is an oriented arc $\boldsymbol{c}$ in $\boldsymbol{Q}_{[t_0,t_1]}$ differentiable in the sense that it is the image of a mapping

$$\eta \colon [0, a] \to \boldsymbol{Q}_{[t_0,t_1]} \colon s \mapsto \chi(s, \cdot\,) \tag{86}$$

derived from a differentiable mapping

$$\chi \colon [0, a] \times [t_0, t_1] \to Q. \tag{87}$$

For each $s \in [0, a]$ the motion $\eta(s)$ is the mapping

$$\eta(s) \colon [t_0, t_1] \to Q \colon t \mapsto \chi(s, t). \tag{88}$$

A process does not change the time interval $[t_0, t_1]$. For each $t \in [t_0, t_1]$ the mapping

$$\zeta = \chi(\,\cdot\,, t) \colon [0, a] \to Q \colon s \mapsto \chi(s, t) \tag{89}$$

is an embedding. The mapping η is a parameterization of the process. Different mappings η may be parameterizations of the same process. A process has an *initial motion* ξ_0 and a *terminal motion* ξ_1 assumed to be distinct. A parameterization η is said to be *compatible* with the orientation of a process if $\xi_0 = \eta(0)$. Compatibility will be required. The symbol $(\boldsymbol{c}, \xi_0)$ wil be used to denote a process.

Definitions of Section 2 are applied to processes in the space of motions. Motions within a process are ordered in terms of any compatible parameterization. Subprocesses and unions of processes are defined.

The space of processes will be denoted by $\boldsymbol{P}Q_{[t_0,t_1]}$.

17. Admissible processes.

Admissible processes are a subset $\boldsymbol{C}_{[t_0,t_1]} \subset \boldsymbol{P}Q_{[t_0,t_1]}$. We say that *constraints* are present if not all processes are admissible. Constraints are *holonomic* if there is submanifold $C^{(0,0)} \in Q$ and all processes within this submanifold are admissible. The following conditions are again satisfied.

(1) A subprocess of an admissible process is admissible.
(2) The union of admissible processes is admissible.
(3) If a process can be approximated with admissible processes, then it is admissible.

The symbol $(\boldsymbol{c}_{\xi|}, q_0)$ denotes the subprocess of $(\boldsymbol{c}, \xi_0)$ with $\xi \in \boldsymbol{c}$ as the terminal configuration. It is based on the arc

$$\boldsymbol{c}_{\xi|} = \{\xi' \in \boldsymbol{c};\ \xi' \leqslant \xi\}\,. \tag{90}$$

18. The action.

An autonomous dynamic system is characterized by a function

$$A_{[t_0,t_1]} \colon \boldsymbol{C}_{[t_0,t_1]} \to \mathbb{R} \tag{91}$$

associating with each admissible process $(\boldsymbol{c}, \xi_0)$ the *action* $A(\boldsymbol{c}, \xi_0)$ of the process. Since subprocesses of admissible processes are admissible we can associate with each admissible process $(\boldsymbol{c}, \xi_0)$ the function

$$a_{(\boldsymbol{c},\xi_0)} \colon \boldsymbol{c} \to \mathbb{R} \colon q \mapsto \begin{cases} A(\boldsymbol{c}_{q|}, q_0) & \text{if } \xi \neq \xi_0 \\ 0 & \text{if } \xi = \xi_0. \end{cases} \tag{92}$$

The action will be assumed to satisfy the following conditions.

(1) Action is additive in the sense that if $(\boldsymbol{c}, q_0)$ is the union of admissible processes

$$(\boldsymbol{c}^0, \xi_0^0), (\boldsymbol{c}^1, \xi_0^1), \ldots, (\boldsymbol{c}^n, \xi_0^n) \tag{93}$$

such that for $i = 0, 1, \ldots, n-1$ the terminal configuration of $(\boldsymbol{c}^i, \xi_0^i)$ is the initial configuration of $(\boldsymbol{c}^{i+1}, \xi_0^{i+1})$, then

$$A(\boldsymbol{c}, \xi_0) = \sum_{i=0}^{n} A(\boldsymbol{c}^i, \xi_0^i). \tag{94}$$

(2) For each admissible process $(\boldsymbol{c}, \xi_0)$ the function $a_{(\boldsymbol{c},\xi_0)}$ is differentiable.

19. Stable local equilibrium motions.

Let $C^{(0,0)} \subset \boldsymbol{Q}_{[t_0,t_1]}$ be the set of initial motions of all admissible processes for an autonomous dynamic system. A motion $\xi_0 \in C^{(0,0)}$ is called a *stable local equilibrium motion* of the system if for each admissible process $(\boldsymbol{c}, \xi_0)$ initiating at ξ_0 the function $a_[t_0, t_1](\boldsymbol{c}, \xi_0)$ has a local minimum at ξ_0.

20. Composition of systems and control.

Systems can be composed if they share the same space of motions. Let two systems be characterized by actions $A_1 \colon \boldsymbol{C}_1 \to \mathbb{R}$ and $A_2 \colon \boldsymbol{C}_2 \to \mathbb{R}$ defined on sets $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ of admissible processes in the same space of motions $\boldsymbol{Q}_{[t_0,t_1]}$. The set $\boldsymbol{C} = \boldsymbol{C}_1 \cap \boldsymbol{C}_2$ is the set of admissible processes of the composed system and the action is the function $A = A_1|\boldsymbol{C} + A_2|\boldsymbol{C}$. Coupling the system to other systems is a form of control. The function A is used to find the equilibrium motions of the controlled system.

21. Realistic constraints.

21.1. Jets of processes

In order to simplify the discussion we consider each process parameterized by a mapping

$$\eta \colon [0, a] \to \boldsymbol{Q}_{[t_0,t_1]} \colon s \mapsto \chi(s,\,\cdot\,) \tag{95}$$

derived from a differentiable mapping

$$\chi \colon [0, a] \times [t_0, t_1] \to Q. \tag{96}$$

The mapping χ is in a sense a parameterization of the process. With each $(s, t) \in [0, a] \times [t_0, t_1]$ we associate an element w of the iterated tangent bundle $\mathsf{T}^l\mathsf{T}^k Q$. This vector is the l-tangent vector

$$w = \mathsf{t}^l \gamma(s) \tag{97}$$

where γ is the mapping

$$\gamma \colon [0, a] \to \mathsf{T}^k Q \colon s \mapsto \mathsf{t}^k \chi(s,\,\cdot\,)(t). \tag{98}$$

The element w will be denoted by $\mathsf{t}^{(l,k)}\chi(s,t)$.

21.2. Constraints determined by differential equations.

Let $\boldsymbol{C}$ be the set of admissible processes represented by mappings (96). We introduce sets $C^{(l,k)} \subset \mathsf{T}^l\mathsf{T}^kQ$. A jet $\mathsf{t}^{(l,k)}\chi(s,t)$ is in $C^{(l,k)}$ if χ represents an admissible process. Each set $C^{(l,k)}$ is a differential equation for mappings (96) representing processes. Let $\boldsymbol{C}^{(l,k)}$ denote the set of processes represented by solutions of $C^{(l,k)}$. It may happen that $\boldsymbol{C} = \boldsymbol{C}^{(l',k')}$ holds for $(l',k') = (l,k)$ and not if $l' < l$ or $k' < k$. In this case, the set $\boldsymbol{C}$ is said to represent constraints of order (l,k).

Constraints encountered in analytical mechanics are of order $(1,1)$. The differences between *vaconomic mechanics* [1] and the *d'Alembert principle* have to be analysed in terms of the set $C^{(1,1)}$. See [2] for this analysis applied to autonomous systems with scleronomic constraints in an affine framework.

22. References.

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[2] Wlodimierz M. Tulczyjew *A note on holonomic constraints* in *Revisiting the Foundations of Relativistic Physics*, A. Ashtekar et al. (eds), Kluwer Academic Publishers.