Vakonomic Mechanics on Lie algebroids

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D. Iglesias, J.C. Marrero, D. Martín de Diego and D. Sosa: Singular Lagrangian systems and variational constrained mechanics on Lie algebroids, Preprint arXiv:0706.2789

Motivation

- A. Weinstein
  Lagrangian Mechanics and groupoids
  Fields Inst. Comm. **7** (1996), 207-231.

- E. Martínez
  Lagrangian Mechanics on Lie algebroids
  Acta Appl. Math. **67** (2001), 295-320.

- M. de León, J.C. Marrero and E. Martínez
  Lagrangian submanifolds and dynamics on Lie algebroids
  J.Phys.A: Math.Gen. **38** (2005), 241-308.

- K. Grabowska, P. Urbański, J. Grabowski
  Geometrical mechanics on algebroids
  Int. J. Geom. Methods Mod. Phys. **3** (3) (2006), 559-575.

- J. Cortés, M. de León, J.C. Marrero and E. Martínez
  Nonholonomic Lagrangian systems on Lie algebroids
  Preprint *arXiv:math-ph/0512003* (2005).

Scheme of the talk

1. Lie algebroids
2. The prolongation of a Lie algebroid over a fibration
3. Lagrangian mechanics on Lie algebroids
4. Constraint algorithm for presymplectic Lie algebroids
5. Vakonomic mechanics on Lie algebroids
    1. Vakonomic equations and vakonomic bracket
    2. The variational point of view

Definition

E vector bundle of rank n over Q, $\quad dimQ = m$
$\tau : E \to Q$ the vector bundle projection

A *Lie algebroid structure* on E:

$[\![\cdot,\cdot]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ Lie bracket

$\rho : E \to TQ$ bundle map, the *anchor map*

$(\rho : \Gamma(E) \to TQ$ homomorphism of $C^\infty(Q)$-modules)

such that

$$[\![X, fY]\!] = f[\![X, Y]\!] + \rho(X)(f)Y$$

for $X, Y \in \Gamma(E)$ and $\quad f \in C^\infty(Q)$

$(E, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid over $Q \Rightarrow \rho$ is a homomorphism between the Lie algebras $(\Gamma(E), [\![\cdot,\cdot]\!])$ and $(\mathcal{X}(Q), [\cdot,\cdot])$

Examples

1. $\tau_Q : TQ \to Q$, Q a differentiable manifold $\Rightarrow (TQ, [\cdot,\cdot], Id)$
2. A real Lie algebra $\mathfrak{g}$ of finite dimension $\Rightarrow (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, 0)$
3. $\pi : Q \to M = Q/G$ principal bundle with structural group G

   - TQ/G is a vector bundle over $M = Q/G$
     $\tau_Q|G : TQ/G \to Q/G$ the vector bundle projection
     $\Gamma(TQ/G) \equiv \{X \in \mathcal{X}(Q) | \, X \text{ is } G-\text{invariant}\}$
   - $\tau_Q|G : TQ/G \to Q/G$ is a Lie algebroid
     $X, Y \in \Gamma(TQ/G) \Rightarrow [X, Y] \in \Gamma(TQ/G)$
     $X \in \Gamma(TQ/G) \Rightarrow X$ is π-projectable
     $\rho : \Gamma(TQ/G) \to \mathcal{X}(M)$

$\tau_Q|G : TQ/G \to Q/G$ *the Atiyah Lie algebroid associated with* π

$\triangleright$ $(E, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid

• *the differential of E* $\quad d^E : \Gamma(\wedge^k E^*) \longrightarrow \Gamma(\wedge^{k+1} E^*)$

$$\begin{aligned}(d^E\mu)(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i\rho(X_i)(\mu(X_0,\dots,\widehat{X}_i,\dots,X_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\mu([\![X_i,X_j]\!],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_k)\end{aligned}$$

$\mu \in \Gamma(\wedge^k E^*)$, $X_0,\dots,X_k \in \Gamma(E)$ $\qquad (d^E)^2 = 0$

• $X \in \Gamma(E)$, *the Lie derivate with respect to X*

$$\mathcal{L}^E_X : \Gamma(\wedge^k E^*) \longrightarrow \Gamma(\wedge^k E^*)$$

$$\mathcal{L}^E_X = i_X \circ d^E + d^E \circ i_X$$

- E^* admits a *linear Poisson structure*

$$\{\cdot,\cdot\}_{E^*} : C^\infty(E^*) \times C^\infty(E^*) \to C^\infty(E^*) \quad \mathbb{R}\text{-bilinear map}$$

i) Skew-symmetry: $\{F, G\}_{E^*} = -\{G, F\}_{E^*}$

ii) The Leibniz rule: $\{FF', G\}_{E^*} = F\{F', G\}_{E^*} + F'\{F, G\}_{E^*}$

iii) The Jacobi identity:

$$\{F, \{G, H\}_{E^*}\}_{E^*} + \{G, \{H, F\}_{E^*}\}_{E^*} + \{H, \{F, G\}_{E^*}\}_{E^*} = 0$$

for $F, F', G, H \in C^\infty(E^*)$ and, in addition,

P, P' linear functions on $E^* \Rightarrow \{P, P'\}_{E^*}$ linear function on E^*

- (x^i) local coordinates on Q

$\{e_\alpha\}$ local basis of $\Gamma(E)$

$\Downarrow$

(x^i, y^α) local coordinates on E

the *structure functions of the Lie algebroid* $\mathcal{C}^\gamma_{\alpha\beta},\ \rho^i_\alpha \in C^\infty(Q)$

$$[\![e_\alpha, e_\beta]\!] = \mathcal{C}^\gamma_{\alpha\beta} e_\gamma \qquad \rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$$

satisfy *the structure equations*

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma \mathcal{C}^\gamma_{\alpha\beta}$$

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\rho^i_\alpha \frac{\partial \mathcal{C}^\nu_{\beta\gamma}}{\partial x^i} + \mathcal{C}^\nu_{\alpha\mu} \mathcal{C}^\mu_{\beta\gamma} \right) = 0$$

$\triangleright$ $f \in C^\infty(Q)$: $\quad d^E f = \dfrac{\partial f}{\partial x^i}\rho^i_\alpha e^\alpha$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$

$\triangleright$ $\theta = \theta_\gamma e^\gamma \in \Gamma(E^*)$: $\quad d^E\theta = (\dfrac{\partial \theta_\gamma}{\partial x^i}\rho^i_\beta - \dfrac{1}{2}\theta_\alpha \mathcal{C}^\alpha_{\beta\gamma})e^\beta \wedge e^\gamma$

$\triangleright$ $F, G \in C^\infty(E^*)$: $\quad (x^i, y_\alpha)$ local coordinates on E^*

$$\{F, G\}_{E^*} = \rho^i_\alpha \left(\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial y_\alpha} - \frac{\partial F}{\partial y_\alpha}\frac{\partial G}{\partial x^i} \right) - \mathcal{C}^\gamma_{\alpha\beta} y_\gamma \frac{\partial F}{\partial y_\alpha}\frac{\partial G}{\partial y_\beta}$$

- $(E, [\![\cdot,\cdot]\!], \rho)$ $(E', [\![\cdot,\cdot]\!]', \rho')$ Lie algebroids over Q and Q'

  (F, f) morphism of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \tau \downarrow & & \downarrow \tau' \\ Q & \xrightarrow{f} & Q' \end{array}$$

$$\phi' \in \Gamma(\wedge^k (E')^*) \Rightarrow (F, f)^* \phi' \in \Gamma(\wedge^k E^*)$$

$$((F, f)^* \phi')_x (a_1, \ldots, a_k) = \phi'_{f(x)}(F(a_1), \ldots, F(a_k))$$

$$x \in Q,\ a_1, \ldots, a_k \in E_x$$

(F, f) is a *Lie algebroid morphism* iff

$$d^E((F, f)^* \phi') = (F, f)^* (d^{E'} \phi'), \quad \phi' \in \Gamma(\wedge^k (E')^*)$$

- (F, f) Lie algebroid morphism, f injective immersion and $F_{|E_x} : E_x \to E'_{f(x)}$ injective

$$\Downarrow$$

$(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ is said to be a *Lie subalgebroid* of $(E', [\![\cdot,\cdot]\!]_{E'}, \rho_{E'})$

The prolongation of a Lie algebroid over a fibration

$(E, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid, $\tau : E \to Q$ $\quad$ *rank* $E = n$, $\quad$ *dim* $Q = m$

$\pi : P \to Q$ fibration, $\quad$ *dim* $P = m'$

$$\mathcal{T}^E P = \bigcup_{p \in P} \mathcal{T}^E_p P \subset E \times TP:$$

$$\mathcal{T}^E_p P = \{(b, v_p) \in E_{\pi(p)} \times T_p P \mid \rho(b) = (T_p \pi)(v_p)\}$$

where $T\pi : TP \to TQ$ is the tangent map to π

$$\tau^\pi : \mathcal{T}^E P \to P, \quad \tau^\pi(b, v_p) = \tau_P(v_p) = p$$

$\tau_P : TP \to P$ being the canonical projection

$$\Downarrow$$

$\mathcal{T}^E P$ is a vector bundle over P of rank $n + m' - m$ with vector bundle projection $\tau^\pi : \mathcal{T}^E P \to P$

- $\tilde{X} \in \Gamma(\mathcal{T}^E P)$ is said to be *projectable* if

  $\exists\, X \in \Gamma(E)$, $\exists\, U \in \mathcal{X}(P)$ π-projectable to $\rho(X)$ s.t.

$$\tilde{X}(p) = (X(\pi(p)), U(p)), \quad \forall p \in P$$

  $\tilde{X} \equiv (X, U)$

$\tau^\pi : \mathcal{T}^E P \to P$ admits a Lie algebroid structure $([\![\cdot,\cdot]\!]^\pi, \rho^\pi)$:

$$[\![(X_1, U_1), (X_2, U_2)]\!]^\pi = ([\![X_1, X_2]\!], [U_1, U_2])$$

$$\rho^\pi(X_1, U_1) = U_1$$

$(\mathcal{T}^E P, [\![\cdot,\cdot]\!]^\pi, \rho^\pi)$ is called *the prolongation of E over* π or *the E-tangent bundle to P*

PARTICULAR CASE 1: $(E, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid

$\tau : E \to Q$ the vector bundle projection

$$\Downarrow$$

the *E*-tangent bundle to *E*:

$$\mathcal{T}^E E = \{(e, v) \in E \times TE \mid \rho(e) = (T\tau)(v)\}$$

$(\mathcal{T}^E E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ Lie algebroid over E of rank $2n$

- (x^i) local coordinates on Q and $\{e_\alpha\}$ local basis of $\Gamma(E)$

$$\Downarrow$$

$\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ local basis of $\tau^\tau : \mathcal{T}^E E \to E$:

$$\mathcal{X}_\alpha(e) = (e_\alpha(\tau(e)), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|e}) \qquad \mathcal{V}_\alpha(e) = (0, \frac{\partial}{\partial y^\alpha}_{|e})$$

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^\tau = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}_\gamma \qquad [\![\mathcal{X}_\alpha, \mathcal{V}_\beta]\!]^\tau = [\![\mathcal{V}_\alpha, \mathcal{V}_\beta]\!]^\tau = 0$$

$$\rho^\tau(\mathcal{X}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \rho^\tau(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha}$$

- Two canonical objects on $\mathcal{T}^E E$:

- *the Euler section* $\Delta \in \Gamma(\mathcal{T}^E E)$: $\quad \Delta = y^\alpha \mathcal{V}_\alpha$
- *the vertical endomorphism* $S \in \Gamma((\mathcal{T}^E E) \otimes (\mathcal{T}^E E)^*)$:

$$S = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$$

$\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$

- $\xi \in \Gamma(\mathcal{T}^E E)$ is said to be a *second order differential equation* (SODE) on E if

$$S(\xi) = \Delta$$

PARTICULAR CASE 2: $(E, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid

$\tau^* : E^* \to Q$ the dual vector bundle projection

$$\Downarrow$$

the *E*-tangent bundle to E^*:

$$\mathcal{T}^E E^* = \{(e, v) \in E \times TE^* \mid \rho(e) = (T\tau^*)(v)\}$$

$$(\mathcal{T}^E E^*, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*}) \text{ Lie algebroid over } E^* \text{ of rank } 2n$$

• (x^i) local coordinates on Q, $\{e_\alpha\}$ local basis of $\Gamma(E)$ and $\{e^\alpha\}$ dual basis $\Rightarrow$ (x^i, y_α) local coordinates on E^*

$$\Downarrow$$

$\{\mathcal{Y}_\alpha, \mathcal{U}_\alpha\}$ local basis of $\tau^{\tau^*} : \mathcal{T}^E E^* \to E^*$:

$$\mathcal{Y}_\alpha(e^*) = (e_\alpha(\tau^*(e^*)), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|e^*}) \qquad \mathcal{U}_\alpha(e^*) = (0, \frac{\partial}{\partial y_\alpha}_{|e^*})$$

$$[\![\mathcal{Y}_\alpha, \mathcal{Y}_\beta]\!]^{\tau^*} = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{Y}_\gamma \qquad [\![\mathcal{Y}_\alpha, \mathcal{U}_\beta]\!]^{\tau^*} = [\![\mathcal{U}_\alpha, \mathcal{U}_\beta]\!]^{\tau^*} = 0$$

$$\rho^{\tau^*}(\mathcal{Y}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \rho^{\tau^*}(\mathcal{U}_\alpha) = \frac{\partial}{\partial y_\alpha}$$

- Two canonical objects on $\mathcal{T}^E E^*$:

- *the Liouville section* $\lambda_E \in \Gamma((\mathcal{T}^E E^*)^*)$: $\lambda_E(e^*)(\tilde{e}, v) = <e^*, \tilde{e}>$

$$\lambda_E(x^i, y_\alpha) = y_\alpha \mathcal{Y}^\alpha$$

- *the canonical symplectic section* $\Omega_E \in \Gamma(\wedge^2(\mathcal{T}^E E^*)^*)$:

$$\Omega_E = -d^{\mathcal{T}^E E^*}\lambda_E$$

$$\Omega_E = \mathcal{Y}^\alpha \wedge \mathcal{U}^\alpha + \frac{1}{2}\mathcal{C}^\gamma_{\alpha\beta} y_\gamma \mathcal{Y}^\alpha \wedge \mathcal{Y}^\beta$$

$\{\mathcal{Y}^\alpha, \mathcal{U}^\alpha\}$ the dual basis of $\{\mathcal{Y}_\alpha, \mathcal{U}_\alpha\}$

$(E, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid, $\tau : E \to Q$ the bundle projection
$(\mathcal{T}^E E, [\![\cdot,\cdot]\!]^\tau, \rho^\tau)$ the E-tangent bundle to E
$L : E \to \mathbb{R}$ a Lagrangian function

- *the Cartan 1-section* $\theta_L \in \Gamma((\mathcal{T}^E E)^*)$: $\theta_L = S^*(d^{\mathcal{T}^E E} L)$

$$\theta_L = \frac{\partial L}{\partial y^\alpha} \mathcal{X}^\alpha$$

- *the Cartan 2-section* $\omega_L \in \Gamma(\wedge^2 (\mathcal{T}^E E)^*)$: $\omega_L = -d^{\mathcal{T}^E E} \theta_L$

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha} \rho^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta} \rho^i_\alpha + \frac{\partial L}{\partial y^\gamma} \mathcal{C}^\gamma_{\alpha\beta} \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta$$

- *the Lagrangian energy* $E_L \in C^\infty(E)$: $E_L = \mathcal{L}^{\mathcal{T}^E E}_\Delta L - L$

$$E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L$$

- A curve $t \to c(t)$ on E is a solution of the *Euler-Lagrange equations* for L if
  - c is admissible (i.e., $(c(t), \dot{c}(t)) \in \mathcal{T}^E_{c(t)}E$, for all t)
  - $i_{(c(t),\dot{c}(t))}\omega_L(c(t)) - d^{\mathcal{T}^E E}E_L(c(t)) = 0$, for all t.

or, locally, if $c(t) = (x^i(t), y^\alpha(t))$

$$\begin{cases} \dot{x}^i = \rho^i_\alpha y^\alpha \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^\alpha}\right) + \dfrac{\partial L}{\partial y^\gamma}\mathcal{C}^\gamma_{\alpha\beta}y^\beta - \rho^i_\alpha \dfrac{\partial L}{\partial x^i} = 0 \end{cases}$$

- L is said to be *regular* if ω_L is a symplectic section

$$\Downarrow$$

there exists a unique solution Γ_L verifying

$$i_{\Gamma_L}\omega_L - d^{\mathcal{T}^E E} E_L = 0$$

$$\Downarrow$$

Γ_L is a SODE section

Thus, the integral curves of Γ_L (that is, the integral curves of the vector field $\rho^\tau(\Gamma_L)$) are solutions of the Euler-Lagrange equations for L. Γ_L is called the *Euler-Lagrange section* associated with L.

Locally,

$$L \text{ is regular} \Leftrightarrow (W_{\alpha\beta}) = \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right) \text{ is regular}$$

$$\Downarrow$$

The local expression of Γ_L is

$$\Gamma_L = y^\alpha \mathcal{X}_\alpha + W^{\alpha\beta}\left(\rho^i_\beta \frac{\partial L}{\partial x^i} - \rho^i_\gamma y^\gamma \frac{\partial^2 L}{\partial x^i \partial y^\beta} + y^\gamma \mathcal{C}^\nu_{\gamma\beta} \frac{\partial L}{\partial y^\nu}\right)\mathcal{V}_\alpha$$

$(W^{\alpha\beta})$ being the inverse matrix of $(W_{\alpha\beta})$

- If ω_L is not symplectic (i.e. $(W_{\alpha\beta}) = \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right)$ is non regular) the Lagrangian is called *singular* or *degenerate Lagrangian*

$(\tau : E \to Q, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid

Assume that:

- $\Omega \in \Gamma(\wedge^2 E^*)$ presymplectic 2-section $(d^E\Omega = 0)$
- $\alpha \in \Gamma(E^*)$ closed 1-section $(d^E\alpha = 0)$
- the kernel of Ω vector subbundle of E

$\flat_\Omega : E \to E^*$ vector bundle morphism over the identity of Q

$\flat_\Omega(e) = i(e)\Omega(x)$

F_x a subspace of E_x, with $x \in Q$,

$$F_x^\perp = \{e \in E_x \mid \Omega(x)(e,f) = 0, \forall f \in F_x\}$$

$\triangleright$ $\flat_{\Omega_x}(F_x) = (F_x^\perp)^\circ$, where $\flat_{\Omega_x} = \flat_{\Omega|E_x}$ and $(F_x^\perp)^\circ$ is the annihilator of the subspace $F_x^\perp$

- The dynamics of the presymplectic system defined by (Ω, α) is given by a section $X \in \Gamma(E)$ satisfying the dynamical equation

$$i_X \Omega = \alpha$$

$$\triangleright\ Q_1 = \{x \in Q \,|\, \exists e \in E_x :\ i(e)\Omega(x) = \alpha(x)\}$$
$$= \{x \in Q \,|\, \alpha(x)(e) = 0,\ \forall e \in Ker\Omega(x) = E_x^{\perp}\}$$

If Q_1 is an embedded submanifold of Q

$$\Downarrow$$

$\exists\ X : Q_1 \to E$ a section of $\tau : E \to Q$ along Q_1: $\quad i_X\Omega = \alpha$

But $\rho(X)$ is not, in general, tangent to Q_1

Thus, we have that to restrict to $E_1 = \rho^{-1}(TQ_1)$

If E_1 is a manifold and $\tau_1 = \tau_{|E_1} : E_1 \to Q_1$ is a vector bundle

$$\Downarrow$$

$\tau_1 : E_1 \to Q_1$ is a Lie subalgebroid of $\tau : E \to Q$

$$\triangleright\ Q_2 = \{x \in Q_1 \mid \alpha(x) \in \flat_{\Omega_x}((E_1)_x) = \flat_{\Omega_x}(\rho^{-1}(T_x Q_1))\}$$
$$= \{x \in Q_1 \mid \alpha(x)(e) = 0,\ \forall e \in (E_1)_x^{\perp} = (\rho^{-1}(T_x Q_1))^{\perp}\}$$

If Q_2 is an embedded submanifold of Q_1

$$\Downarrow$$

$\exists\ X : Q_2 \to E_1$ a section of $\tau_1 : E_1 \to Q_1$ along Q_2: $\quad i_X \Omega = \alpha$

But, $\rho(X)$ is not, in general, tangent to Q_2

Thus, we have that to restrict to $E_2 = \rho^{-1}(TQ_2)$

If E_2 is a manifold and $\tau_2 = \tau_{|E_2} : E_2 \to Q_2$ is a vector bundle

$$\Downarrow$$

$\tau_2 : E_2 \to Q_2$ is a Lie subalgeborid of $\tau_1 : E_1 \to Q_1$

If we repeat the process, we obtain a sequence of Lie subalgebroids:

$$\begin{array}{ccccccccccccc} \dots \hookrightarrow & Q_{k+1} & \hookrightarrow & Q_k & \hookrightarrow \dots \hookrightarrow & Q_2 & \hookrightarrow & Q_1 & \hookrightarrow & Q_0 = Q \\ & \uparrow \tau_{k+1} & & \uparrow \tau_k & & \uparrow \tau_2 & & \uparrow \tau_1 & & \uparrow \tau_0 = \tau \\ \dots \hookrightarrow & E_{k+1} & \hookrightarrow & E_k & \hookrightarrow \dots \hookrightarrow & E_2 & \hookrightarrow & E_1 & \hookrightarrow & E_0 = E \end{array}$$

where

$$Q_{k+1} = \{x \in Q_k \,|\, \alpha(x)(e) = 0, \; \forall e \in (\rho^{-1}(T_x Q_k))^{\perp}\}$$

$$E_{k+1} = \rho^{-1}(TQ_{k+1})$$

If $\exists\, k \in \mathbb{N}$: $Q_k = Q_{k+1} \Rightarrow$ we say that the sequence stabilizes

$$\Downarrow$$

$$Q_f = Q_{k+1} = Q_k \qquad E_f = E_{k+1} = E_k = \rho^{-1}(TQ_k)$$

$\tau_f = \tau_k : E_f \to Q_f$ is a Lie subalgebroid of $\tau : E \longrightarrow Q$

$$\Downarrow$$

$$\exists\, X \in \Gamma(E_f): \quad i_X\Omega = \alpha$$

Moreover, every arbitrary solution is of the form

$$X' = X + Y, \quad Y \in \Gamma(E_f) \text{ and } Y(x) \in \ker \Omega(x),\, x \in Q_f$$

In addition, if we denote by Ω_f and α_f the restriction of Ω and α, respectively, to the Lie algebroid $E_f \longrightarrow Q_f$, we have that:

- Ω_f is a presymplectic 2-section
- $X \in \Gamma(E_f)$: $\quad i_X\Omega = \alpha \quad \Rightarrow \quad i_X\Omega_f = \alpha_f$

but in principle, there are solutions of $\; i_X\Omega_f = \alpha_f$ which are not solutions of $i_X\Omega = \alpha$ (since $\ker \Omega \cap E_f \subset \ker \Omega_f$)

$\tau : E \to Q$ Lie algebroid of rank n, $\quad dim\, Q = m$

$L : E \to \mathbb{R}$ Lagrangian function on E

$M \subset E$ embedded submanifold, *the constraint submanifold*

$\tau_M = \tau_{|M} : M \to Q$ surjective submersion, $\quad dim\, M = n + m - \bar{m}$

$$\begin{array}{ccc} & E^* \oplus E & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ E^* & & E \end{array} \qquad \begin{array}{ccc} & W_0 = pr_2^{-1}(M) = E^* \times_Q M & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ E^* & & M \end{array}$$

$\pi_1 = (pr_1)_{|W_0} \qquad \pi_2 = (pr_2)_{|W_0}$

$\nu : W_0 = E^* \times_Q M \to Q$ the canonical projection

Vakonomic Mechanics on Lie algebroids

$(\mathcal{T}\pi_1, \pi_1)$ Lie algebroid morphism

$\mathcal{T}\pi_1 = (Id, T\pi_1)$

$$\begin{array}{ccc} \mathcal{T}^E W_0 & \xrightarrow{\mathcal{T}\pi_1} & \mathcal{T}^E E^* \\ {\scriptstyle \tau^\nu}\downarrow & & \downarrow{\scriptstyle \tau^{\tau^*}} \\ W_0 & \xrightarrow{\pi_1} & E^* \end{array}$$

$\Omega_0 = (\mathcal{T}\pi_1, \pi_1)^*\Omega_E$ is a presymplectic section on $\mathcal{T}^E W_0$

Ω_E being the canonical symplectic section on $\mathcal{T}^E E^*$

- The Pontryagin Hamiltonian $\quad H_{W_0} : W_0 = E^* \times_Q M \to \mathbb{R}$

$$H_{W_0}(e^*, \check{e}) = \langle e^*, \check{e} \rangle - \tilde{L}(\check{e}), \quad \tilde{L} = L_{|M}$$

$$\Downarrow$$

$(\mathcal{T}^E W_0, \Omega_0, d^{\mathcal{T}^E W_0} H_{W_0})$ is a presymplectic hamiltonian system

Definition

The *vakonomic problem on Lie algebroids* is find the solutions for the equation

$$i_X \Omega_0 = d^{\mathcal{T}^E W_0} H_{W_0}$$

i.e., to solve the constraint algorithm for $(\mathcal{T}^E W_0, \Omega_0, d^{\mathcal{T}^E W_0} H_{W_0})$

- (x^i) local coordinates on an open subset U of Q

  $\{e_\alpha\}$ local basis of $\Gamma(E)$ on U

$$M \cap \tau^{-1}(U) \equiv \{(x^i, y^\alpha) \in \tau^{-1}(U) \,|\, \Phi^A(x^i, y^\alpha) = 0, A = 1, \dots, \bar{m}\}$$

where Φ^A are the local independent constraint functions for the submanifold M

$$(y^\alpha) = (y^A, y^a), \quad 1 \leqslant \alpha \leqslant n, \quad 1 \leqslant A \leqslant \bar{m}, \quad \bar{m} + 1 \leqslant a \leqslant n$$

$$\Downarrow$$

$\exists$ open subset $\widetilde{V}$ of $\tau^{-1}(U)$, open subset $W \subseteq \mathbb{R}^{m+n-\bar{m}}$ and smooth real functions $\Psi^A : W \to \mathbb{R}, \quad A = 1, \dots, \bar{m}$, s.t

$$M \cap \widetilde{V} \equiv \{(x^i, y^\alpha) \in \widetilde{V} \,|\, y^A = \Psi^A(x^i, y^a), \; A = 1, \dots, \bar{m}\}$$

$$\Downarrow$$

(x^i, y^a) are local coordinates on M

$$\Downarrow$$

(x^i, y_α, y^a) local coordinates for $W_0 = E^* \times_Q M$

$\{\mathcal{Y}_\alpha, \mathcal{U}_\alpha, \mathcal{V}_a\}$ local basis of $\Gamma(\mathcal{T}^E W_0)$:

$$\mathcal{Y}_\alpha(e^*, \check{e}) = (e_\alpha(x), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|e^*}, 0)$$

$$\mathcal{U}_\alpha(e^*, \check{e}) = (0, \frac{\partial}{\partial y_\alpha}_{|e^*}, 0) \qquad \mathcal{V}_a(e^*, \check{e}) = (0, 0, \frac{\partial}{\partial y^a}_{|\check{e}})$$

where $(e^*, \check{e}) \in W_0$ and $\nu(e^*, \check{e}) = x$

$([\![\cdot, \cdot]\!]^\nu, \rho^\nu)$ the Lie algebroid structure on $\mathcal{T}^E W_0$:

$$[\![\mathcal{Y}_\alpha, \mathcal{Y}_\beta]\!]^\nu = \mathcal{C}^\gamma_{\alpha\beta} \mathcal{Y}_\gamma$$

$$\rho^\nu(\mathcal{Y}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \rho^\nu(\mathcal{U}_\alpha) = \frac{\partial}{\partial y_\alpha} \qquad \rho^\nu(\mathcal{V}_a) = \frac{\partial}{\partial y^a}$$

$\{\mathcal{Y}^\alpha, \mathcal{U}^\alpha, \mathcal{V}^a\}$ the dual basis of $\{\mathcal{Y}_\alpha, \mathcal{U}_\alpha, \mathcal{V}_a\}$:

$$\Omega_0 = \mathcal{Y}^\alpha \wedge \mathcal{U}^\alpha + \frac{1}{2}\mathcal{C}^\gamma_{\alpha\beta} y_\gamma \mathcal{Y}^\alpha \wedge \mathcal{Y}^\beta$$

$$H_{W_0}(x^i, y_\alpha, y^a) = y_a y^a + y_A \Psi^A(x^i, y^a) - \tilde{L}(x^i, y^a)$$

$$d^{\mathcal{T}^E W_0} H_{W_0} = (y_A \frac{\partial \Psi^A}{\partial x^i} - \frac{\partial \tilde{L}}{\partial x^i})\rho^i_\alpha \mathcal{Y}^\alpha + \Psi^A \mathcal{U}^A + y^a \mathcal{U}^a + (y_a + y_A \frac{\partial \Psi^A}{\partial y^a} - \frac{\partial \tilde{L}}{\partial y^a})\mathcal{V}^a$$

$\triangleright$ If we apply the constraint algorithm

$$W_1 = \{w \in E^* \times_Q M \,|\, d^{\mathcal{T}^E W_0} H_{W_0}(w)(Y) = 0, \; \forall Y \in Ker\,\Omega_0(w)\}$$

$Ker\,\Omega_0 = span\{\mathcal{V}_a\} \Rightarrow W_1$ is locally characterized by

$$y_a = \frac{\partial \tilde{L}}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a}, \quad \bar{m} + 1 \leq a \leq n$$

A solution of the vakonomic problem is of the form

$$X = y^a \mathcal{Y}_a + \Psi^A \mathcal{Y}_A + \Big[\Big(\frac{\partial \tilde{L}}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i}\Big)\rho^i_\alpha - y^a \mathcal{C}^\beta_{\alpha a} y_\beta - \Psi^A \mathcal{C}^\beta_{\alpha A} y_\beta\Big]\mathcal{U}_\alpha + \Upsilon^a \mathcal{V}_a$$

Therefore, the vakonomic equations are

$$\begin{cases} \dot{x}^i = y^a \rho^i_a + \Psi^A \rho^i_A \\ \dot{y}_A = \Big(\frac{\partial \tilde{L}}{\partial x^i} - y_B \frac{\partial \Psi^B}{\partial x^i}\Big)\rho^i_A - y^a \mathcal{C}^\beta_{Aa} y_\beta - \Psi^B \mathcal{C}^\beta_{AB} y_\beta \\ \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a}\right) = \Big(\frac{\partial \tilde{L}}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i}\Big)\rho^i_a - y^b \mathcal{C}^\beta_{ab} y_\beta - \Psi^A \mathcal{C}^\beta_{aA} y_\beta \end{cases}$$

There exist solution sections X of $\mathcal{T}^E W_0$ along W_1, but they may not be sections of $(\rho^\nu)^{-1}(TW_1) = \mathcal{T}^E W_1$

$$\Downarrow$$

we obtain a sequence of embedded submanifolds

$$\ldots \hookrightarrow W_{k+1} \hookrightarrow W_k \hookrightarrow \ldots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 = E^* \times_Q M$$

- If the algorithm stabilizes

$$\Downarrow$$

$\exists$ a final constraint submanifold W_f

$$\exists\, X \in \Gamma(\mathcal{T}^E W_f): \quad (i_X \Omega_0 = d^{\mathcal{T}^E W_0} H_{W_0})_{|W_f}$$

▷ We analyze the case $W_f = W_1$

Denote Ω_1 the restriction of Ω_0 to $\mathcal{T}^E W_1$

Theorem

If Ω_1 is a symplectic 2-section on the Lie algebroid $\mathcal{T}^E W_1 \to W_1$ then there exists a unique section ξ_1 of $\mathcal{T}^E W_1 \to W_1$ whose integral curves are solutions of the vakonomic equations for the system (L, M). In fact, if H_{W_1} is the restriction to W_1 of the Pontryagin Hamiltonian H_{W_0} then ξ_1 is the Hamiltonian section of H_{W_1} with respect to the symplectic section Ω_1, that is,

$$i_{\xi_1}\Omega_1 = d^{\mathcal{T}^E W_1} H_{W_1}$$

Definition

The vakonomic system (L, M) on the Lie algebroid $\tau : E \to Q$ is said to be *regular* if Ω_1 is a symplectic 2-section of the Lie algebroid $\mathcal{T}^E W_1 \to W_1$.

Proposition

Ω_1 is a symplectic section of the Lie algebroid $\mathcal{T}^E W_1$ if and only if for any system of coordinates (x^i, y_α, y^a) on W_0 we have that

$$\det\left(\frac{\partial^2 \tilde{L}}{\partial y^a \partial y^b} - y_A \frac{\partial^2 \Psi^A}{\partial y^a \partial y^b}\right) \neq 0, \text{ for all point in } W_1.$$

Denote

$$\mathcal{R}_{ab} = \frac{\partial \tilde{L}}{\partial y^a \partial y^b} - y_A \frac{\partial^2 \Psi^A}{\partial y^a \partial y^b}, \quad \text{for all } a \text{ and } b$$

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- If the vakonomic system (L, M) is regular

$$\Downarrow$$

$$\det(\mathcal{R}_{ab}) \neq 0$$

$$\Downarrow$$

(x^i, y_α, y^a) are local coordinates on an open subset of W_0 s.t.

(x^i, y_α) are local coordinates on W_1 $\quad (y^a = \mu^a(x^i, y_\alpha))$

$\{\mathcal{Y}_{\alpha 1}, \mathcal{U}_{\alpha 1}\}$ is a local basis of $\Gamma(\mathcal{T}^E W_1)$

If $\nu_1 : W_1 \to Q$ is the canonical projection and $([\![\cdot,\cdot]\!]^{\nu_1}, \rho^{\nu_1})$ is the Lie algebroid structure on $\mathcal{T}^E W_1 \to W_1$:

$$[\![\mathcal{Y}_{\alpha 1}, \mathcal{Y}_{\beta 1}]\!]^{\nu_1} = \mathcal{C}^{\gamma}_{\alpha\beta}\mathcal{Y}_{\gamma 1}$$

$$\rho^{\nu_1}(\mathcal{Y}_{\alpha 1}) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \rho^{\nu_1}(\mathcal{U}_{\alpha 1}) = \frac{\partial}{\partial y_\alpha}$$

$\{\mathcal{Y}_1^\alpha, \mathcal{U}_1^\alpha\}$ the dual basis of $\{\mathcal{Y}_{\alpha 1}, \mathcal{U}_{\alpha 1}\}$:

$$\Omega_1 = \mathcal{Y}_1^\alpha \wedge \mathcal{U}_1^\alpha + \frac{1}{2}\mathcal{C}_{\alpha\beta}^\gamma y_\gamma \mathcal{Y}_1^\alpha \wedge \mathcal{Y}_1^\beta$$

$$\Downarrow$$

$$\begin{aligned} \xi_1(x^j, y_\beta) = {} & \mu^a(x^j, y_\beta)\mathcal{Y}_{a1} + \Psi^A(x^j, \mu^a(x^j, y_\beta))\mathcal{Y}_{A1} \\ & -\Big[\mathcal{C}_{\alpha a}^b y_b \mu^a(x^j, y_\beta) + \mathcal{C}_{\alpha A}^b y_b \Psi^A(x^j, \mu^a(x^j, y_\beta)) \\ & +\rho_\alpha^i\Big(y_A \frac{\partial \Psi^A}{\partial x^i}_{|(x^j,\mu^a(x^j,y_\beta))} - \frac{\partial \tilde{L}}{\partial x^i}_{|(x^j,\mu^a(x^j,y_\beta))}\Big)\Big]\mathcal{U}_{\alpha 1} \end{aligned}$$

- *The vakonomic bracket* associated with the system (L, M)

$$\{\cdot,\cdot\}_{(L,M)} : C^\infty(W_1)\times C^\infty(W_1)\to C^\infty(W_1)$$
$$\{F_1, G_1\}_{(L,M)} = \Omega_1(\mathcal{H}_{F_1}^{\Omega_1}, \mathcal{H}_{G_1}^{\Omega_1}) = \rho^{\nu_1}(\mathcal{H}_{G_1}^{\Omega_1})(F_1)$$

$\mathcal{H}_{F_1}^{\Omega_1}$ being the *Hamiltonian section of* F_1 with respect to Ω_1

Theorem

The vakonomic bracket $\{\cdot,\cdot\}_{(L,M)}$ associated with a regular vakonomic system is a Poisson bracket on W_1. Moreover, if $F_1\in C^\infty(W_1)$ then the temporal evolution of F_1, $\dot{F}_1$, is given by

$$\dot{F}_1 = \{F_1, H_{W_1}\}_{(L,M)}$$

Note that $\xi_1 = \mathcal{H}_{H_{W_1}}^{\Omega_1}$

Locally,

$$\{F_1, G_1\}_{(L,M)} = \rho^i_\alpha \left(\frac{\partial F_1}{\partial x^i} \frac{\partial G_1}{\partial y_\alpha} - \frac{\partial F_1}{\partial y_\alpha} \frac{\partial G_1}{\partial x^i} \right) - \mathcal{C}^\gamma_{\alpha\beta} y_\gamma \frac{\partial F_1}{\partial y_\alpha} \frac{\partial G_1}{\partial y_\beta}$$

Corollary

If (L, M) is a regular vakonomic system on a Lie algebroid E then the restriction $(\pi_1)_{|W_1} : W_1 \to E^*$ of $\pi_1 : W_0 \to E^*$ to W_1 is a local Poisson isomorphism.
Moreover, if $\mathcal{T}(\pi_1)_{|W_1} : \mathcal{T}^E W_1 \to \mathcal{T}^E E^*$ is the corresponding prolongation then the pair $(\mathcal{T}(\pi_1)_{|W_1}, (\pi_1)_{|W_1})$ is a local symplectomorphism between the symplectic Lie algebroids $(\mathcal{T}^E W_1, \Omega_1)$ and $(\mathcal{T}^E E^*, \Omega_E)$.

$\tau : E \to Q$ a Lie algebroid and $L : E \to \mathbb{R}$ a Lagrangian function

E. Martínez, *preprint arXiv:math-ph/0603028*.

▷ The set of E-paths:

$$\mathcal{A}dm([t_0, t_1], E) = \{a : [t_0, t_1] \to E \mid \rho \circ a = \frac{d}{dt}(\tau \circ a)\}$$

• $a_0, a_1 \in \mathcal{A}dm([t_0, t_1], E)$ are E-homotopic if there exists a Lie algebroid morphism

$$\Phi : T[0, 1] \times T[t_0, t_1] \to E$$

such that if $a(s, t) = \Phi(\partial_t|_{(s,t)})$ and $b(s, t) = \Phi(\partial_s|_{(s,t)})$, then

$$a(0, t) = a_0(t), \;\; a(1, t) = a_1(t), \;\; b(s, t_0) = 0, \;\; b(s, t_1) = 0$$

$\mathcal{P}([t_0, t_1], E) \equiv \mathcal{A}dm([t_0, t_1], E)$ with the second differentiable Banach manifold structure induced by the E-homotopy classes

For $a \in \mathcal{P}([t_0, t_1], E)$:

$T_a\mathcal{P}([t_0, t_1], E) = \{\eta^c \in T_a\mathcal{A}dm([t_0, t_1], E) \mid \eta(t_0) = 0,\ \eta(t_1) = 0\}$

• If $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and η is a time-dependent section locally given by $\eta = \eta^\alpha e_\alpha$, then *the complete lift of* η

$$\eta^c = \eta^\alpha \rho^i_\alpha \frac{\partial}{\partial x^i} + (\rho^i_\beta \frac{\partial \eta^\gamma}{\partial x^i} - \eta^\alpha \mathcal{C}^\gamma_{\alpha\beta}) y^\beta \frac{\partial}{\partial y^\gamma}$$

Fix $x, y \in Q$:

$\mathcal{P}([t_0, t_1], E)^y_x = \{a \in \mathcal{P}([t_0, t_1], E) \mid \tau(a(t_0)) = x,\ \tau(a(t_1)) = y\}$

▶ The action functional $\quad \delta S : \mathcal{P}([t_0, t_1], E) \to \mathbb{R}$

$$\delta S(a) = \int_{t_0}^{t_1} L(a(t))dt$$

- (L, M) vakonomic system on the Lie algebroid $\tau : E \to Q$

$$\Downarrow$$

infinitesimal variations are complete lifts η^c tangent to M

$\Downarrow$ (M locally defined by $y^A - \Psi^A(x^i, y^a) = 0$)

$$\eta^c(y^A - \Psi^A(x^i, y^a)) = 0$$

or, equivalently,

$$\frac{d\eta^A}{dt} = \rho^i_\alpha \eta^\alpha \frac{\partial \Psi^A}{\partial x^i} + \frac{d\eta^a}{dt}\frac{\partial \Psi^A}{\partial y^a} + \mathcal{C}^a_{\beta\alpha} y^\beta \eta^\alpha \frac{\partial \Psi^A}{\partial y^a} - \mathcal{C}^A_{\beta\alpha} y^\beta \eta^\alpha$$

$$\mathcal{P}(M) = \{a : I \to M \mid a(t) = (x^i(t), y^A(t)) \text{ s.t. } \dot{x}^i(t) = \rho^i_A y^A(t)\}$$

- The action $\delta S : \mathcal{P}(M) \to \mathbb{R}$

$$a(t) \mapsto \int L(a(t))dt$$

We look for the critical points of the action δS:

$$\frac{d}{ds}_{|s=0} \int L(a_s(t))dt = 0$$

$$\Downarrow$$

$$\begin{cases} \dot{x}^i = y^a \rho^i_a + \Psi^A \rho^i_A \\ \dot{y}_A = \left(\frac{\partial \tilde{L}}{\partial x^i} - y_B \frac{\partial \Psi^B}{\partial x^i}\right)\rho^i_A - y^a \mathcal{C}^\beta_{Aa} y_\beta - \Psi^B \mathcal{C}^\beta_{AB} y_\beta \\ \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a}\right) = \left(\frac{\partial \tilde{L}}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i}\right)\rho^i_a - y^b \mathcal{C}^\beta_{ab} y_\beta - \Psi^A \mathcal{C}^\beta_{aA} y_\beta \end{cases}$$

with $y_a = \frac{\partial \tilde{L}}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a}$, that is, the vakonomic equations for the vakonomic system (L, M) on the Lie algebroid $\tau : E \to Q$

Example (The tangent bundle to a manifold)

Q a differentiable manifold

$\tau_Q : TQ \to Q$ is a Lie algebroid with the structure $([\cdot,\cdot], Id)$

$$(q^i) \text{ local coordinates on } Q$$

$$\{\frac{\partial}{\partial q^i}\} \text{ local basis of } \tau_Q : TQ \to Q$$

$$\Downarrow$$

$$\rho^i_j = \delta_{ij} \text{ and } \mathcal{C}^k_{ij} = 0$$

$$\Downarrow$$

The classical vakonomic equations

$$\left\{ \begin{array}{rcl} \dot{q}^A & = & \Psi^A(q^i, \dot{q}^a) \\ \dot{p}_A & = & \dfrac{\partial \tilde{L}}{\partial q^A} - p_B \dfrac{\partial \Psi^B}{\partial q^A} \\ \dfrac{d}{dt}\left(\dfrac{\partial \tilde{L}}{\partial \dot{q}^a} - p_A \dfrac{\partial \Psi^A}{\partial \dot{q}^a}\right) & = & \dfrac{\partial \tilde{L}}{\partial q^a} - p_B \dfrac{\partial \Psi^B}{\partial q^a} \end{array} \right.$$

Example (Lie algebras of finite dimension)

$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, 0)$, being $\mathfrak{g}$ a Lie algebra of dimension n

$\mathfrak{C}$ affine subspace of $\mathfrak{g}$ modelled over the vector space C

$$dim\, C = n - \bar{m} \text{ and } e_0 \in \mathfrak{C},\ e_0 \neq 0$$

- $\{e_\alpha\} = \{e_a, e_0, e_{\bar{a}}\} = \{e_a, e_A\}$ basis of $\mathfrak{g}$ such that

$$\{e_a\} \text{ basis of } C \text{ and } [e_\alpha, e_\beta] = \mathcal{C}_{\alpha\beta}^{\gamma} e_\gamma$$

$(y^a, y^0, y^{\bar{a}}) = (y^a, y^A)$ coordinates on $\mathfrak{g}$

$\mathfrak{C}$ given by the equations: $y^0 = 1, \quad y^{\bar{a}} = 0$

$(y_a, y_0, y_{\bar{a}}) = (y_a, y_A)$ dual coordinates on $\mathfrak{g}^*$

$L : \mathfrak{g} \to \mathbb{R}$ Lagrangian function

$\tilde{L} : \mathfrak{C} \to \mathbb{R}$ the restriction of L to $\mathfrak{C}$

Example (Lie algebras of finite dimension)

$\sigma : t \mapsto (y^a(t), y^0(t), y^{\bar{a}}(t)) = (y^a(t), 1, 0, \dots, 0)$ a curve in $\mathfrak{C}$ is a solution of the constrained system $(L, \mathfrak{C})$ if and only if

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{\partial \tilde{L}}{\partial y^a}\Big) = -\dfrac{\partial \tilde{L}}{\partial y^c}(y^b \mathcal{C}_{ab}^c + \mathcal{C}_{a0}^c) - y_B(y^b \mathcal{C}_{ab}^B + \mathcal{C}_{a0}^B) \\ \dot{y}_A = -\dfrac{\partial \tilde{L}}{\partial y^c}(y^b \mathcal{C}_{Ab}^c + \mathcal{C}_{A0}^c) - y_B(y^b \mathcal{C}_{Ab}^B + \mathcal{C}_{A0}^B) \end{cases}$$

The curve in $\mathfrak{g}^*$ $\quad \gamma : t \mapsto (\dfrac{\partial \tilde{L}}{\partial y^a}_{|\sigma(t)}, y_A(t)) = \dfrac{\partial \tilde{L}}{\partial y}(\sigma(t)) + \lambda(t)$

$$\lambda(t) = (0, y_A(t)) \in C^\circ$$

Then γ satisfies the Euler-Poincaré equations

$$\frac{d}{dt}\Big(\frac{\partial \tilde{L}}{\partial y} + \lambda\Big) = ad^*_\sigma\Big(\frac{\partial \tilde{L}}{\partial y} + \lambda\Big)$$

"*Optimization Theorem for Nonholonomic Systems on Lie groups*"
W-S. Koon, J.E. Marsden, *SIAM J. Control Optim.* **35** (1997) 901-929.

Example (Atiyah Lie algebroids)

$\pi : Q \to M$ principal bundle with structural group G

$\tau_{Q|G} : TQ/G \to M$ the associated *Atiyah Lie algebroid*

$$\Downarrow$$

"*The reduced Lagrangian Optimization Theorem for Nonholonomic Systems*"
W-S. Koon, J.E. Marsden, *SIAM J. Control Optim.* **35** (1997) 901-929.