Nonholonomic constraints: a comparison of mechanical and variational systems

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Aim

- To discuss systems subject to nonholonomic constraints
- To overview two methods of obtaining equations via: illustrative examples, discussing differences, and providing geometric interpretations
- A large (but non-smooth) variety of contributions: Bloch, Lewis, Bullo, Marsden, Ratiu, Koon, Crouch, Montgomery, Koiler, Vershik, Gershkovich, Fadeev, Bates, Sniatycki, Cardin, Favretti, Marmo, Tulczyjew, Cantrijn, Carinena, Cortes, de Leon, de Diego, Martinez, vdSchaft, Maschke, Marle, Kupka, Oliva.
- Pictures are from A.M. Bloch, Nonholonomic Mechanics and Control, Springer, 2003.

Plan

- Nonholonomic constraints
- Mechanical nonholonomic equations MNH
- Variational nonholonomic equations VNH
- Examples (rolling disk, knife edge, Chaplygin sleigh)
- Poisson geometry of MNH systems
- Projected connection for MNH
- VNH systems and sub-Riemannian geometry
- Conclusions

Notations

Q	- a smooth n-dimensional configuration manifold
$L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$	- lagrangian
$C^2(q_1, q_2, [T_0, T_1])$	- the space of C^2 – curves with the end points q_1 and q_2
$C^2(q_1, q_2, [T_0, T_1], \mathcal{D})$	- the space of C^2 – horizontal curves (i.e., tangent to $\mathcal{D}$, with the end points q_1 and q_2
$J : C^2(q_1, q_2, [T_0, T_1]) \longrightarrow \mathbb{R}$	- functional

$c \mapsto J(c) = \int_{T_0}^{T_1} L(c(t), \dot{c}(t), t)dt.$

Hamilton's principle

A curve $c \in C^2(q_1, q_2, [T_0, T_1])$ describes a motion if it is a critical point of J, i.e.,

$$\mathrm{d}J(c) \cdot u = 0$$

for every $u \in T_c C^2(q_1, q_2, [T_0, T_1])$, the tangent space at c, consisting of

- $u : [T_0, T_1] \longrightarrow TQ$
- $\pi_Q(u) = c$
- $\pi(T_0) = \pi(T_1) = 0$

Equivalently, $c = q(t)$ satisfies

$$(EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \ (= F_{ext}).$$

Constraints

Problem: How does the picture change if a system is subject to constraints?

- General constraints (on velocities) are represented by $\mathcal{D} \subset TQ$ (subset, submanifold);
- Throughout the talk: $\mathcal{D}$ is a distribution, i.e., a subbundle of TQ of constant rank m.
- The annihilator of $\mathcal{D}$ is a co-distribution, of rank $k = n - m$,

$$I = \operatorname{span}\{\omega^1, \ldots, \omega^k\}$$

- The system moves such that along any its trajectory c, the velocity $\dot{c}$ remains in $\mathcal{D}$, i.e., $\dot{c}(t) \in \mathcal{D}(c(t))$; c is horizontal.
- Equivalently $\omega^a(\dot{q}) = 0$, for $1 \leq a \leq k$.

Example: unicycle, knife edge

The instantaneous velocity of the point of contact is parallel to the unicycle, i.e.,

$$\omega = \sin\theta\,\mathrm{d}x - \cos\theta\mathrm{d}y$$

annihilates the velocity $(\dot{x}, \dot{y}, \dot{\theta})^T$, that is,

$$\sin\theta\,\dot{x} = \cos\theta\,\dot{y}.$$

The configuration manifold $Q = \mathbb{R}^2 \times S^1$ is of dimension 3, the space $\mathcal{D}(q)$ of admissible velocities at each $q \in Q$ is of dimension 2.

Holonomic and nonholonomic constraints

- Constraints are called holonomic if, locally, there exists a $\mathbb{R}^k$-valued function $h = (h^1, \ldots, h^k)$ such that $\omega^a(\dot{q}) = 0$ is equivalent to

$$\frac{\partial h}{\partial q} \cdot \dot{q} = 0$$

(which foliates Q into the integral leaves of $\mathcal{D}$);

- Otherwise, the constraints are called nonholonomic;
- The distinction of the two categories and the names were proposed by Herz in 1894:

  ó$\lambda o \xi$ (whole, integral) $\nu o \mu o \xi$ (law, principle)

How to derive equations?

- How to describe motions of a system subject to nonholonomic constraints, in other words, how to modify (E-L)?
- Two basic methods leading to, respectively, Mechanical Nonholonomic equations MNH and Variational Nonholonomic equations VNH
- MNH based on d'Alambert principle of virtual work: Constraint forces do not work on all motions allowed by the constraints. It follows that the constraint force $F_{constr} = \lambda_a \omega^a$, for some functions $\lambda_a = \lambda_a(t)$.

Mechanical nonholonomic problem: formulation

Definition 1 A horizontal curve $c \in C^2(q_1, q_2, [T_0, T_1], \mathcal{D})$ solves the mechanical nonholonomic problem MNH if

$$\mathrm{d}J(c) \cdot u = 0$$

for every $u \in X_c(q_1, q_2, [T_0, T_1], \mathcal{D})$, where X_c consists of

- $u : [T_0, T_1] \longrightarrow TQ$
- $\pi_Q(u) = c$
- $\pi(T_0) = \pi(T_1) = 0$
- $u(t) \in \mathcal{D}(c(t))$

i.e., u is an element of the tangent space $T_c C^2(q_1, q_2, [T_0, T_1])$ and is horizontal.

Mechanical nonholonomic problem: characterization

Proposition 1 *The following conditions are equivalent:*

(i) *a curve c solves the MNH problem;*

(ii) *c satisfies*

$$\left[\frac{\mathrm{d}}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial l}{\partial q}\right] u = 0,$$

for every $u \in X_c(q_1, q_2, [T_0, T_1], \mathcal{D})$*;*

(iii) *c satisfies*

$$(MNH - EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \lambda_a \omega^a,$$

where $\mathcal{D} = \ker\{\omega^1(q), \ldots, \omega^{n-k}(q)\}$.

- We apply the constraints *after* making J stationary.

Variational nonholonomic problem: formulation

VNH method based on the following minimization problem:

minimize J on the space $c \in C^2(q_1, q_2, [T_0, T_1], \mathcal{D})$.

Definition 2 A horizontal curve $c \in C^2(q_1, q_2, [T_0, T_1], \mathcal{D})$ solves the variational nonholonomic problem VNH if c is a critical point of the restriction $J \mid_{C^2(q_1, q_2, [T_0, T_1], \mathcal{D})}$.

The method of Lagrange multipliers: put

$$\mathcal{L}(q, \dot{q}, t) = L(q, \dot{q}, t) - \mu_a \omega^a(\dot{q}).$$

Like in minimizing $F : Q \longrightarrow \mathbb{R}$ subject to $q \in M = \{g = 0\}$, M a submanifold, we form $\mathcal{F} = F - \mu_a g^a$.

Variational nonholonomic problem: characterization

Proposition 2 *The following conditions are equivalent:*

(i) *a curve c solves the VNH problem;*

(ii) *c satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0;$$

(iii) *c satisfies*

$$(VNH-EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \dot{\mu}_a \omega^a - \mu_a(\dot{q} \lrcorner \mathrm{d}\omega^a).$$

- We apply the constraints *before* making J stationary.

Problems

- Do (MNH-EL) and (VNH-EL) give the same solutions?

  It does not seem to be case: (VNH-EL) involves the derivatives $\dot{\mu}_a$ of the multipliers.

- If not, which does describe physical systems?

- What are geometric interpretations of (MNH-EL) and (VNH-EL)?

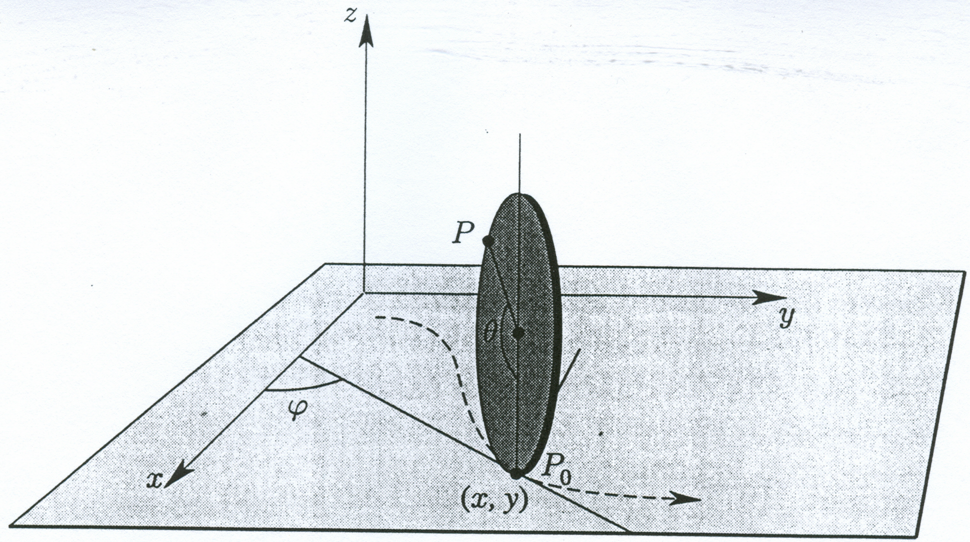


FIGURE 1.4.1. The geometry of the rolling disk.

Rolling disk

- $Q = SE(2) \times S^1 = \mathbb{R}^2 \times S^1 \times S^1$ the configuration manifold Q is the group of planar rigid motions times the circle
- $L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2$, where m is the mass, I and J are inertia momenta.
- nonholonomic constraints:

$$\dot{x} = R(\cos\phi)\dot{\theta}$$

$$\dot{y} = R(\sin\phi)\dot{\theta}.$$

Nonholonomic constraints

The nonholonomic constraints:

$$\dot{x} = R(\cos\phi)\dot{\theta}$$

$$\dot{y} = R(\sin\phi)\dot{\theta}$$

define, respectively, the differential 1-forms

$$\omega^1 = \mathrm{d}x - R\cos\phi \; \mathrm{d}\theta$$

$$\omega^2 = \mathrm{d}y - R\sin\phi \; \mathrm{d}\theta,$$

yielding the constraint distribution

$$\mathcal{D} = \mathrm{span}\,\{\frac{\partial}{\partial\theta} + R\cos\phi\frac{\partial}{\partial x} + R\sin\phi\frac{\partial}{\partial y}, \frac{\partial}{\partial\phi}\}.$$

Rolling disk MNH-equations

$$(MNH - EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \lambda_a \omega^a,$$

gives, via eliminating λ_1 and λ_2, the following equations

$$\begin{aligned}
\dot{\phi} &= v_\phi \\
J\dot{v}_\phi &= 0 \ (= u_\phi) \\
\dot{\theta} &= v_\theta \\
(I + mR^2)\dot{v}_\theta &= 0 \ (= u_\theta) \\
\dot{x} &= R\cos\phi \ \dot{\theta} \\
\dot{y} &= R\cos\phi \ \dot{\theta},
\end{aligned}$$

Rolling disk MNH-solutions

Solutions of

$$J\ddot{\phi} = 0$$

$$(I + mR^2)\ddot{\theta} = 0$$

$$\dot{x} = R\cos\phi\ \dot{\theta}$$

$$\dot{y} = R\cos\phi\ \dot{\theta},$$

are

$$\phi = \omega t + \phi_0$$

$$\theta = \Omega t + \theta_0$$

$$x = \frac{\Omega}{\omega} R\sin(\omega t + \phi_0) + x_0$$

$$y = \frac{\Omega}{\omega} R\cos(\omega t + \phi_0) + y_0.$$

Rolling disk: VNH-equations

$$(VNH - EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \mu_a \omega^a - \mu_a(\dot{q} \lrcorner \mathrm{d}\omega^a).$$

applied to the lagrangian

$$\mathcal{L} = L + \mu_1(\dot{x} - R\dot{\theta}\cos\phi) + \mu_2(\dot{y} - R\dot{\theta}\sin\phi)$$

gives the following equations

$$m\ddot{x} = -\dot{\mu}_1 \ (\Rightarrow \mu_1 = -mR\cos\phi + A)$$

$$m\ddot{y} = -\dot{\mu}_2 \ (\Rightarrow \mu_2 = -mR\sin\phi + B)$$

$$J\ddot{\phi} = R\dot{\theta}(A\sin\phi - B\cos\phi)$$

$$(I + mR^2)\ddot{\theta} = R\dot{\phi}(-A\sin\phi + B\cos\phi).$$

A comparison of solutions

$$VNH$$

$$J\ddot{\phi} = R\dot{\theta}(A\sin\phi - B\cos\phi)$$

$$(I + mR^2)\ddot{\theta} = R\dot{\phi}(-A\sin\phi + B\cos\phi),$$

$$MNH$$

$$J\ddot{\phi} = 0$$

$$(I + mR^2)\ddot{\theta} = 0,$$

constraints (the same for both)

$$\dot{x} = R\cos\phi\ \dot{\theta}$$

$$\dot{y} = R\cos\phi\ \dot{\theta}.$$

A comparison of solutions - cont.

- For $A \neq 0$, $B \neq 0$ the trajectories of the VNH rolling disk are not solutions of the MNH rolling disk;
- A and B are determined neither by the nonholonomic constraints nor be the initial condition (velocity and configuration) of the system: there are many trajectories issued by the same initial condition. They are determined by $\mu_1(0)$ and $\mu_2(0)$.
- We are tempted to believe that real physical systems realize the trajectories of MNH (and *not* those of VNH);
- Not always the MNH-trajectories form a proper subset of the VNH-trajectories.

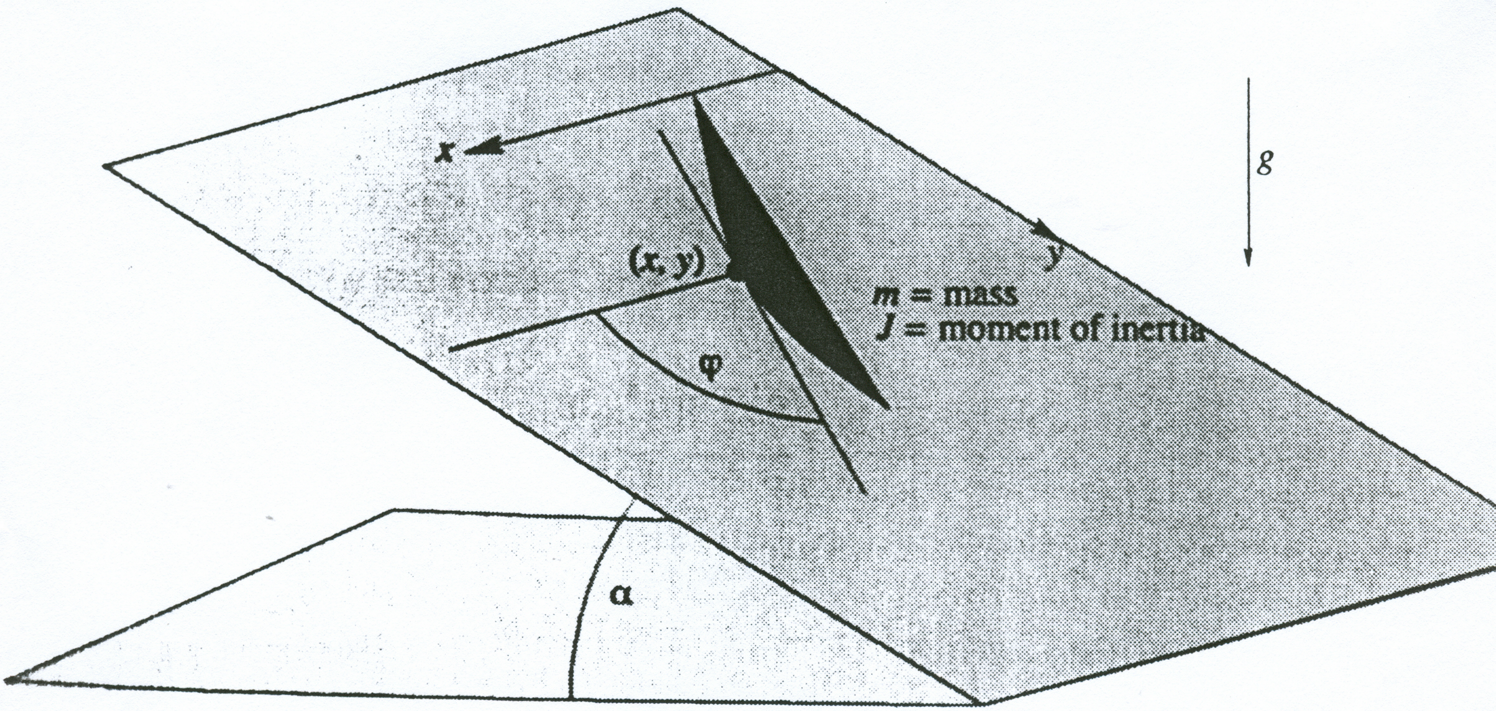


FIGURE 1.6.1. Motion of a knife edge on an inclined plane.

Knife edge (skate) on inclined plane

- $Q = \mathbb{R}^2 \times S^1$
- $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2 + mgy\sin\alpha$, where m is the mass, J the inertia moment (about the vertical axis through the point of contact)
- nonholonomic constraint: $\dot{y}\sin\phi = \dot{x}\cos\phi$ defines the differential 1-form

$$\omega = \sin\phi\,\mathrm{d}y - \cos\phi\,\mathrm{d}x$$

yielding the constraint distribution

$$\mathcal{D} = \mathrm{span}\,\{\cos\phi\frac{\partial}{\partial y} + \sin\phi\frac{\partial}{\partial x}, \frac{\partial}{\partial \phi}\}.$$

Knife edge: MNH-equations

$$(MNH - EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \lambda_a \omega^a,$$

gives the following equations

$$m\ddot{x} = -\lambda \cos\phi$$

$$m\ddot{y} = \lambda \sin\phi + yg\sin\alpha$$

$$J\ddot{\phi} = 0,$$

together with the constraint

$$\dot{y}\sin\phi = \dot{x}\cos\phi.$$

How does the contact point $x(t), y(t)$ move assuming that $\dot{x}(0) = \dot{y}(0) = \phi(0) = 0$ and $\dot{\phi}(0) = \omega$?

The solution is

$$\phi(t) = \omega t,$$
$$x(t) = \frac{g}{2\omega^2}\sin\alpha(\omega t - \frac{1}{2}\sin 2\omega t)$$
$$y(t) = \frac{g}{2\omega^2}\sin\alpha\sin^2\omega t;$$

The point of contact undergoes a *cycloid* motion, in particular, does not (in average) slide down the plane;

$$0 \leq\mid y(t) \mid\leq \frac{g}{2\omega^2}\sin\alpha.$$

Knife edge: VNH-equations

The constrained Lagrangian

$$\mathcal{L} = L - \mu\omega(\dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2 + mgy\sin\alpha - \mu(\dot{y}\sin\phi - \dot{x}\cos\phi)$$

leads, via

$$(VNH - EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \mu_a\omega^a - \mu_a(\dot{q}\lrcorner\mathrm{d}\omega^a),$$

to the equations (assuming $\dot{\phi}(0) = \omega$ and $p_x(0) = p_y(0) = 0$, where the momenta are defined by $p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}}$ and $p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}}$):

Knife edge: VNH-solutions

$$\dot{x} = (mg \sin\alpha \sin\phi \cos\phi)t$$

$$\dot{y} = (mg \sin\alpha \cos^2\phi)t$$

$$\ddot{\phi} = (\frac{m}{J} g \sin^2\alpha \sin\phi \cos\phi)t^2.$$

We can observe that $\phi(t)$ converges to $\frac{\pi}{2}$ and the point of contact slides monotonically down the plane.

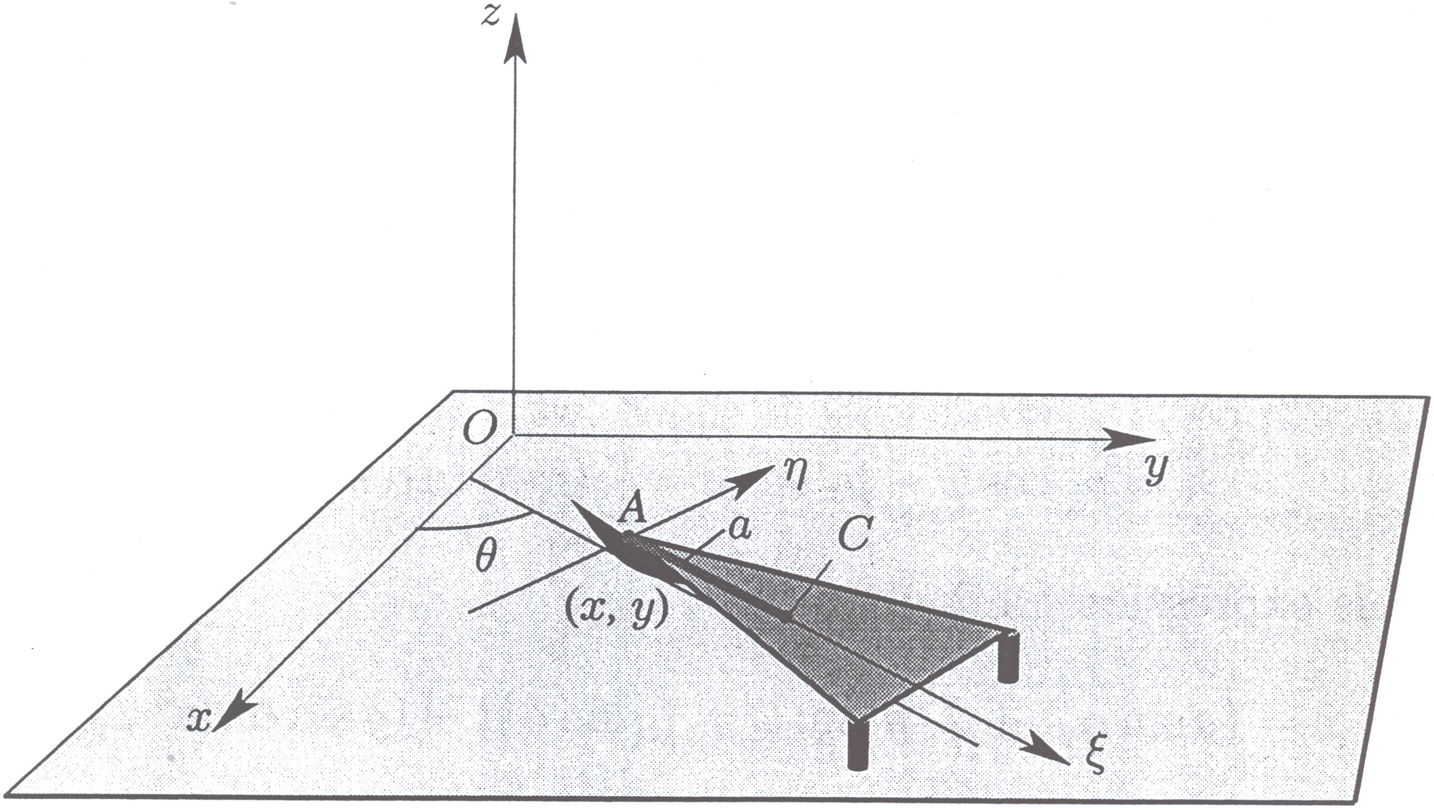
Which solutions correspond to real physically realizable trajectories?

MNH- or VNH-trajectories are realizable physically?

- Korteweg 1899: real mechanical systems satisfy the d'Alambert principle and thus follow the trajectories of

$$(MNH - EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \lambda_a \omega^a;$$

- paradoxically, MNH-EL means $\frac{\mathrm{d}}{ds}J(c_s)\mid_{s=0}= 0$, where the variations c_s, in general, do not satisfy $c_s(t) \in \mathcal{D}(c_s)$ (but only $u = \frac{\partial c_s}{\partial s}\mid_{s=0} \in \mathcal{D}(c)$);
- What are the VNH-trajectories modelling?
- How to interpret the MNH- and VNH-trajectories?



IGURE 1.7.1. The Chaplygin sleigh is a rigid body moving on two sliding post
ne knife edge.

Chaplygin sleigh

- $Q = \mathbb{R}^2 \times S^1$
- $L = \frac{1}{2}m(\dot{x}_{mc}^2 + \dot{y}_{mc}^2) + \frac{1}{2}I\dot{\theta}^2$, where m is the mass, (x_{mc}, y_{mc}) the mass center, I the inertia moment (about the center of mass), (x, y) is the point of contact, where $x = x_{mc} - a\cos\theta$, $y = y_{mc} - a\sin\theta$.
- nonholonomic constraint:

$$\dot{x}\sin\phi = \dot{y}\cos\phi.$$

- The angular velocity

$$\omega = \dot{\theta}$$

and the velocity in the direction of motion

$$v = \dot{x}\cos\theta + \dot{y}\sin\theta$$

satisfy the momentum equation

$$\dot{v} = a\omega^2,$$
$$\dot{\omega} = -\frac{ma}{I + ma^2} v\omega.$$

- In the absence of nonholonomic constraints, this equation would give conservation of angular momentum
- The equilibria form the curve $\{\omega = 0\}$ and the eigenvalues of the linearization around any of these equilibria are $\lambda_1 = 0$, $\lambda_2 \neq 0$ showing a "dissipative" nature of nonholonomic systems: integral curves are ellipses along which the system converges towards positive v-axis.
- Are MNH systems hamiltonian?

Hamiltonian description of MNH systems

Consider

$$(MNH-EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = \lambda_a \omega^a,$$

together with constraints

$$\omega^a(\dot q) = 0.$$

Define the hamiltonian $H : T^*Q \longrightarrow \mathbb{R}$ by $H = (p, q) - L$ and the Legendre transformation $\mathcal{LT} : TQ \longrightarrow T^*Q$ by $p = \mathcal{LT}(v_q) = \frac{\partial L}{\partial \dot q}$.

The constraints $\omega^a(\dot q) = 0$ define

$$\mathcal{M} = \{(q, p) \; : \; \omega^a(\frac{\partial H}{\partial p}) = 0\} \subset T^*Q.$$

Restrict the hamiltonian system from T^*Q to $\mathcal{M}$:

- Denote by $T_{\mathcal{M}}(T^*Q)$ the restriction of $T(T^*Q)$ to $\mathcal{M} \subset T^*Q$;
- Represent $T_{\mathcal{M}}(T^*Q) = T\mathcal{M} \bigoplus \mathcal{V}$, where $\mathcal{V}$ is the vertical bundle;
- Decompose the hamiltonian vector filed X_H on T^*Q restricted to $\mathcal{M}$ as

  $$X_H \mid_{\mathcal{M}} = X_{\mathcal{M}} + X_{\mathcal{V}},$$

  where the vector fields $X_{\mathcal{M}}$ and $X_{\mathcal{V}}$ are smooth sections of, respectively, $T\mathcal{M}$ and $\mathcal{V}$.
- Project the Poisson tensor Λ on T^*Q onto $\mathcal{M}$ and denote it $\Lambda_{\mathcal{M}}$.
- Define $\{F, G\}_{\mathcal{M}} = \Lambda_{\mathcal{M}}(\mathrm{d}F, \mathrm{d}G)$, for any smooth functions F and G on $\mathcal{M}$.

Proposition 3 *The bracket* $\{\cdot,\cdot\}_{\mathcal{M}}$

(i) *is skew symmetric;*

(ii) *satisfies the Leibniz identity;*

(iii) *satisfies the Jacobi identity if and only if the constraint distribution* $\mathcal{D}$ *is involutive.*

Moreover, $X_{\mathcal{M}} = \Lambda^{\sharp}_{\mathcal{M}}(\mathrm{d}H_{\mathcal{M}})$ *and* $H_{\mathcal{M}}$ *is its first integral.*

Theorem 1 *The MNH-EL equation is equivalent to the hamiltonian vector field* $X_{\mathcal{M}}$ *via the Legendre transformation.*

Back to Newton's law

$$(MNH - EL) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \lambda_a \omega^a,$$

can be rewritten as

$$\nabla_{\dot{q}}\dot{q} = \sum \lambda_a W_a,$$

where the vector fields W_a are defined by $< W_a, \dot{q} >= \omega^a(\dot{q}) = 0$. The constraint distribution defines a submanifold $\mathcal{N} \subset TQ$ by

$$\mathcal{N}(q) = \{f(q) \ : \ \omega^a(f) = 0\}.$$

Back to Newton's law

- Define $\tilde{\nabla}$, the projection of the covariant derivative ∇ on $\mathcal{N}$. Then $c = q(t)$ is a motion of MNH if and only if

$$\tilde{\nabla}_{\dot{q}}\dot{q} = 0$$

- Geometry of MNH: Motions of MNH are the "straightest" curves with respect to the (non-metric) connection $\tilde{\nabla}$ (Herz, 1894).
- What is a (the?) geometry of VNH?

Introducing controls

- How to parameterize $C^2(q_1, q_2, [T_0, T_1], \mathcal{D})$? Choose, locally, m vector fields f_i such that $\mathcal{D} = \text{span}\{f_1, ..., f_m\}$. Then $c = q(t)$ is horizontal, i.e., $\dot{c} \in \mathcal{D}(c(t))$ if, in coordinates,

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)),$$

where $u_i(t)$, for $1 \leq i \leq m$, are called controls.

Geometry of VNH

- Assume that $L = T$, i.e., $L(q,\dot{q}) = g(\dot{q},\dot{q})$, where g is the bilinear form on TQ given by the Riemannian metric defining T. Choose the vector fields f_i to be orthonormal with respect to g, i.e., $g(f_i, f_j) = \delta_{ij}$.

- The energy of a curve $c = q(t)$ joining $q_1 = q(T_1)$ and $q_2 = q(T_2)$ is ($I = [T_1, T_2]$)

$$E(c) = \frac{1}{2}\int_I \|\dot{c}(t)\|^2 dt = \frac{1}{2}\int_I g(\dot{q}(t), \dot{q}(t))dt = \frac{1}{2}\int_I \sum_{i=1}^{m} u_i^2(t)dt$$

- With the help of g we can also define the length of a curve c

$$l(c) = \int_I \|\dot{c}(t)\|dt = \int_I (g(\dot{q}(t), \dot{q}(t)))^{\frac{1}{2}} dt$$

- We can thus endow Q with a metric d: the sub-Riemannian distance $d(q_1, q_2)$ between two pints q_1 and q_2 is the infimum of $l(\gamma)$ over all horizontal curves joining q_1 and q_2; provided that $\mathcal{D}$, together with all its iterated Lie brackets, spans T_qQ at each q ($\mathcal{D}$ is bracket generating, completely nonholonomic, the system is controllable, Rashevsky-Chow theorem).

- Sub-Riemannian geometry problem: find horizontal curves minimizing the length $l(\gamma)$, i.e. find sub-Riemannian geodesics.

- As in Riemannian geometry, due to Cauchy-Schwartz inequality, the minimizers of both problems coincide. Namely, a horizontal curve c minimizes the energy E among all horizontal curves joining q_1 and q_2 in time T if and only if it minimizes the length l among all horizontal curves joining q_1 and q_2 and is parameterized to have constant speed $c = d(q_1, q_2)/T$.

What is the VNH problem?

The following are equivalent:

- Solve the VNH problem (with $L = T$).
- Solve the Sub-Riemannian geometry problem: find horizontal curves minimizing the length $l(\gamma)$, i.e. find the sub-Riemannian geodesics.
- Find horizontal curves minimizing the energy $E(\gamma)$.
- Solve the optimal control problem:

$$\text{minimize } \frac{1}{2}\int_I \sum_{i=1}^{m} u_i^2(t)dt$$

subject to

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)).$$

Sub-Riemannian problem: an example

- Consider the differential form $\alpha = \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x)$. Then $\alpha|_R = 0$ on any ray R through the origin and $\mathrm{d}\alpha = \mathrm{d}x \wedge \mathrm{d}y$
- The area A enclosed by a curve γ and a ray R is

$$A(\gamma) = \int_\gamma \alpha.$$

- The length of γ is

$$l(\gamma) = \int_I (\dot{x}^2(t) + \dot{y}^2(t))^{\frac{1}{2}} dt.$$

- **Problem:** Minimize $l(\gamma)$ subject to $A(\gamma) = a$=const.
- **Dual Problem (Dido)**: Maximize $A(\gamma)$ subject to $l(\gamma) = l$=const.

- Add z satisfying

$$\dot{z} = \frac{1}{2}(x\dot{y} - y\dot{x}).$$

- Let c be a lift of γ (all lifts being parameterized by $z(0)$). Define $l(c) = l(\pi(c)) = l(\gamma)$.

- If $z(0) = 0$, then

$$z(T) = z(T) - z(0) = \frac{1}{2}\int_\gamma (x\mathrm{d}y - y\mathrm{d}x) = A(\gamma).$$

- Define

$$f_1 = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z} \qquad f_2 = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}$$

  and

$$\mathcal{D} = \operatorname{span}\{f_1, f_2\}.$$

  f_1 and f_2 and $f_3 = [f_1, f_2]$ span the Heisenberg Lie algebra (the simplest model of a non-involutive rank 2 distribution in $\mathbb{R}^3$).

Dido meets Heisenberg

The following problems are equivalent

- Minimize $l(\gamma)$, among all curves γ joining $(0,0)$ and (x,y), subject to $A(\gamma) = a$=fixed;
- Minimize $l(c) = l(\pi(c)) = l(\gamma)$, among all curves c joining $(0,0,0)$ and (x,y,z) (where $z = a$, subject to $\dot{c}(t) = \mathcal{D}(c(t))$.
- Minimize $l(c)$, among all curves c joining $(0,0,0)$ and (x,y,z) subject to $\dot{c}(t) = \mathcal{D}(c(t)$, where f_1 and f_2 are orthonormal.

Optimal solutions

- The VNH equations give

$$\ddot{x} = 2\lambda\dot{y}$$

$$\ddot{y} = -2\lambda\dot{x}$$

$$\dot{\lambda} = 0$$

$$\dot{z} = 1/2(x\dot{y} - y\dot{x})$$

whose solutions are circles, passing through $(0,0,0)$, formed by $(x(t), y(t))$ together with

$$z(t) = \frac{ta}{T} - ta^2 \sin\frac{2\pi t}{T}.$$

The Heisenberg sphere of radius r looks like an apple!

- MNH solutions are straight lines corresponding to $\lambda = 0$.

Conclusions

MNH	VNH
d'Alembert principle	min of J on horiz. curves
describe real systems	do not
involve λ	involve μ and $\dot{\mu}$
determined	underdetermined
the straightest	the shortest