Exterior Differential Systems and the Inverse Problem

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EDS and the Inverse Problem

joint with

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1. The inverse problem

2. Geometric formulation

3. EDS and the inverse problem

(EDS - Exterior Differential Systems)

1. The Inverse Problem in the Calculus of Variations

"When are the solutions of

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b) a, b = 1, \ldots, n$$

the solutions of

$$\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b}\ddot{x}^b + \frac{\partial^2 L}{\partial x^b \partial \dot{x}^a}\dot{x}^b + \frac{\partial^2 L}{\partial t \partial \dot{x}^a} = \frac{\partial L}{\partial x^a}$$

for some $L(t, x^a, \ldots x^a)$?"

So, find regular g_{ab} (and L) so that

$$g_{ab}(\ddot{x}^b - F^b) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial \dot{x}^a}$$

- The multiplier problem.

Helmholtz conditions (Douglas 1941, Sarlet 1982)

necessary and sufficient conditions on g_{ab}:

$$g_{ab} = g_{ab}, \Gamma(g_{ab}) = g_{ac}\Gamma^c_b + g_{bc}\Gamma^c_a$$
$$g_{ac}\Phi^c_b = g_{bc}\Phi^c_a, \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b}$$

where

$$\Gamma^a_b := \frac{-1}{2}\frac{\partial F^a}{\partial \dot{x}^b}, \Phi^a_b := -\frac{\partial F^a}{\partial x^b} - \Gamma^c_b\Gamma^a_c - \Gamma(\Gamma^a_b)$$
$$\Gamma := \frac{\partial}{\partial t} + u^a\frac{\partial}{\partial x^a} + F^a\frac{\partial}{\partial u^a}$$

Helmholtz conditions: 1st order linear algebraic/differential equations for g_{ab} with data $f^a, \Gamma^a_b, \Phi^a_b$.

Two approaches/problems:

- For given F^a find all g's

e.g.

$$\ddot{x} + \dot{y} = 0$$
$$\ddot{y} + y = 0$$

admits no multipliers.

- For given n classify 2nd order ode's according to existence and multiplicity of solutions of the Helmholtz condition.

Done by Douglas for $n = 2$.
Still not done for $n = 3$.

Fields' medallist Jesse Douglas (1941):

"the problem ...... is one of the most important hitherto unsolved problems of the calculus of variations" .

Douglas solved the problem for the $n = 2$: he does this by a painstaking study of four main cases and many subcases. In most of the cases Douglas decides the existence question and gives all the possible Lagrangians; in the remaining cases the problem becomes a question of the closure of a certain 1-form.

Douglas's method is not easily generalizable to higher dimension and for this reason the problem has not been solved in Douglas's sense for $n = 3$ or higher.

Example 1. Non-existence.

$$\ddot{x} = \dot{y}, \qquad \ddot{y} = y.$$

The only non-zero components of Γ^a_b and Φ^a_b are:

$$\Gamma^1_2 = -1, \qquad \Phi^2_2 = -1.$$

The third of the Helmholtz conditions gives only $g_{12} = 0$ and the first then gives $g_{21} = 0$. Now the second gives

$$\begin{aligned} \Gamma(g_{11}) &= 0 = \Gamma(g_{22}), \\ \Gamma(g_{12}) &= -g_{11} = \Gamma(g_{21}). \end{aligned}$$

$$\Gamma = \frac{\partial}{\partial t} + \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}\frac{\partial}{\partial \dot{x}} + y\frac{\partial}{\partial \dot{y}}$$

Hence $g_{11} = 0$ and the multiplier matrix must be singular if it exists in violation of the initial assumption.

Example 2. Non-uniqueness.

$$\ddot{x} = y, \quad \ddot{y} = 0.$$

$$\Gamma = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + y\frac{\partial}{\partial \dot{y}}$$

$$\Gamma^a_b = 0; \;\; \Phi^1_2 = -1; \;\; \Phi^a_b = 0, \;\; a \neq 1, \;\; b \neq 2.$$

The symmetry condition $g_{ac}\Phi^c_b = g_{bc}\Phi^c_a$ leads immediately to $g_{11} = 0$ and the last of the Helmholtz conditions yields

$$0 = \frac{\partial g_{11}}{\partial \dot{y}} = \frac{\partial g_{12}}{\partial \dot{x}}, \;\; \frac{\partial g_{21}}{\partial \dot{y}} = \frac{\partial g_{22}}{\partial \dot{x}}.$$

Using the first of these in the second condition, $\Gamma(g_{12}) = 0$:

$$\frac{\partial g_{12}}{\partial t} + \dot{x}\frac{\partial g_{12}}{\partial x} + \dot{y}\frac{\partial g_{12}}{\partial y} = 0,$$

and $\dfrac{\partial g_{12}}{\partial \dot{x}} = 0$ implies

$$\frac{\partial g_{12}}{\partial t} + \dot{y}\frac{\partial g_{12}}{\partial y} = 0, \;\; \frac{\partial g_{12}}{\partial x} = 0.$$

Hence

$$g_{12} = G_1(\dot{y}, y - \dot{y}t).$$

More work with $u := y - \dot{y}t,\ v := \dot{y}:$

$$g_{11} = 0, \quad g_{12} = g_{21} = G_1(u,v),$$

$$\begin{aligned} g_{22} = & \left(\frac{\partial G_1}{\partial v} - t\frac{\partial G_1}{\partial u}\right)\dot{x} \\ & + \frac{\partial G_1}{\partial u}x - \frac{\partial G_1}{\partial v}ut + \frac{t^2}{2}\left(v\frac{\partial G_1}{\partial v} - u\frac{\partial G_1}{\partial u}\right) \\ & - \frac{t^3}{3}v\frac{\partial G_1}{\partial u} + G_4(u,v). \end{aligned}$$

So

$$\ddot{x} = y, \quad \ddot{y} = 0$$

is variational and (g_{ab}) is determined up to 2 arbitrary functions of two variables. To find all possible regular Lagrangians, integrate

$$g_{ab} = \frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b}.$$

For example

$$L_1 := xy\dot{y} + \frac{1}{2}(x - \frac{t^2}{2}y)\dot{y}^2 - \frac{1}{2}t\dot{x}\dot{y}^2 + \frac{1}{36}t^3\dot{y}^3$$

$(G_1 := u,\ G_4 := 0)$

$$L_2 := \frac{1}{2}\dot{x}\dot{y}^2 - \frac{1}{2}ty\dot{y}^2 + \frac{1}{4}t^2\dot{y}^3$$

$(G_1 := v,\ G_4 := 0)$

Significance

- Non-existence.

  No Lagrangian $\rightarrow$ No Hamiltonian $\rightarrow$ No Quantisation.

  No Lagrangian $\rightarrow$ No Mechanics Theorems $\rightarrow$ Hard Work.

- Non-uniqueness.

  Too many Lagrangians $\rightarrow$ Too many Hamiltonians $\rightarrow$ Ambiguous Quantum Mechanics.

2. Geometric Formulation

$$
\begin{array}{ccc}
E := \mathbb{R} \times TM & & (t, x^a, u^a) \\
\swarrow t \quad \downarrow \pi & & dim(E) = 2n + 1 \\
\mathbb{R} \underset{t}{\longleftarrow} \mathbb{R} \times M & & (t, x^a)
\end{array}
$$

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b)$$

$$\to \qquad \Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a} \in \mathfrak{X}(E)$$

E is equipped with

- vertical dist'n $V(E) = sp\{V_a := \frac{\partial}{\partial u^a}\}$
- contact dist'n
  $\Theta(E) = sp\{\theta^a := dx^a - u^a dt\}$
- vertical endomorphism $S = V_a \otimes \theta^a$

Semispray:

$$\underbrace{\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a} \qquad S = V_a \bigotimes \theta^a}_{\mathcal{L}_\Gamma S}$$

$$TE = Sp\{\Gamma\} \bigoplus H(E) \bigoplus V(F)$$

Projectors:

$$P_\Gamma = \Gamma \bigotimes dt, \;\; P_H = H_a \bigotimes \theta^a, \;\; P_V = V_a \bigotimes \phi^a$$

$$\begin{pmatrix} H_a := \frac{\partial}{\partial x^a} - \Gamma^b_a \frac{\partial}{\partial u^b}, \phi^a := du^a - F^a dt + \Gamma^a_b \theta^b \\ [H_a, H_b] = R^d_{ab} V_d \end{pmatrix}$$

Jacobi endomorphism:

$$\Phi := P_V \circ \mathcal{L}_\Gamma P_H$$

$$\Phi = \left(-\frac{\partial F^b}{\partial x^a} - \Gamma^b_c \Gamma^c_a - \Gamma(\Gamma^b_a) \right) V_b \otimes \theta^a,$$

$$V_a(\Phi^c_b) - V_b(\Phi^c_a) = 3R^c_{ab}$$

In the autoparallel case

$$\Gamma^a_b = \Gamma^a_{bc} u^c, \quad R^c_{ab} = R^c_{dab} u^d,$$

$$\Phi^a_b = R^a_{cdb} u^c u^d$$

When $\ddot{x}^a = F^a(t, x^b, \dot{x}^b)$ are (normalized) Euler-Lagrange equations, then Γ is the unique vectors field on E s.t.

$$\Gamma \lrcorner\ d\theta_L = 0, dt(\Gamma) = 1$$

where

$$\theta_L := Ldt + dL \circ S = Ldt + \frac{\partial L}{\partial u^a}\theta^a$$

$$d\theta_L = \frac{\partial^2 L}{\partial u^a \partial u^b}\psi^a \wedge \theta^b$$

Theorem (CPT 1984) Given a SODE Γ, necessary and sufficient conditions for the existence of a Lagrangian whose $E-L$ field is Γ are that there exists $\Omega \in \Lambda^2(E)$:

1. Ω has max'l rank

2. $\Omega(V_1, V_2) = 0 \ \forall V_1, V_2 \in V(E)$

3. $\Gamma \lrcorner \Omega = 0$

4. $d\Omega = 0$

Usually begin the search for Ω by assuming 1, 2, 3 i.e.

$$\Omega = g_{ab}\psi^a \wedge \theta^b, |g_{ab}| \neq 0$$

and requiring

$$d\Omega = 0.$$

$d\Omega(X, Y, Z) = 0$ give the Helmholtz conditions e.g.

$$d\Omega(\Gamma, V_a, H_b) = 0 \Leftrightarrow \Gamma(g_{ab}) - g_{bc}\Gamma^c_a - g_{ac}\Gamma^c_b = 0$$
$$d\Omega(\Gamma, V_a, V_b) = 0 \Leftrightarrow g_{ab} = g_{ba}$$
$$d\Omega(\Gamma, H_a, H_b) = 0 \Leftrightarrow g_{ac}\Phi^c_b = g_{bc}\Phi^c_a$$

etc.

3. EDS and the Inverse Problem.

EDS reference Bryant, Chern et al 1991.

IP reference Anderson and Thompson 1992.

In EDS terms, the I.P. is

"Find all closed, maximal rank 2-forms in $\Sigma := Sp\{\psi^a \wedge \theta^b\} \subset \Lambda^2(E)$"

3 steps

1. Find the largest differential ideal generated by Σ.

2. Create a Pfaffian system from the closure condition on this ideal.

3. Apply Cartan-Kähler to determine the generality of the solution of this Pfaffian system.

The differential ideal step.

Q. Set $\Sigma^0 := \Sigma = Sp\{\phi^a \wedge \theta^b\}$.

Is $\langle \Sigma^0 \rangle$ closed?

A. Yes - done!

No - define $\Sigma^1 := \{\omega \in \Sigma^0 : d\omega \in \langle \Sigma^0 \rangle\}$

Q. Is $\langle \Sigma^1 \rangle$ closed?

etc.

This process terminates for some (possibly empty) Σ^{final}.

If $\Sigma^{\text{final}} \neq \phi$ go to step 2.
Otherwise go home (yours not mine).

Notes

1. The differential ideal steps

$$\Sigma^0 \to \Sigma^1 \to \ldots \to \Sigma^{\text{final}}$$

   generate hierarchies of **algebraic conditions** on the multiplier, eg if $\omega \in \Sigma^k$ then

$$\omega(X^V, Y^H) = \omega(Y^V, X^H)$$
$$\omega(\Phi(X)^V, Y^H) = \omega(\Phi(Y)^V, X^H)$$
$$\ldots$$
$$\omega((\nabla^k\Phi(X))^V, Y^H) = \omega((\nabla^k\Phi(Y))^V, X^H)$$

   There is a similar hierarchy of curvature conditions.

2. If Σ^k **is** a differential ideal then we get conditions on Φ, eg, $\Phi = \lambda I$.

3. If Σ^k is a differential ideal **and** contains closed 2 forms then we get **differential conditions** on the multiplier (EDS step 2).

Question Are there further independent algebraic conditions on the multiplier (at the differential ideal step)? For example,

$$\sum_{(XYZ)} d\Omega(\Phi(X)^V, \Phi(Y)^V, Z^H) = 0$$

gives an apparently new algebraic condition at the Σ^1 step.

More concretely:

Question Can we find necessary and sufficient algebraic conditions defining

$$\Sigma^k := \{\omega \in \Sigma^{k-1} : d\omega \in \langle \Sigma^{k-1} \rangle\}?$$

Answer Yes, if we can give necessary and sufficient conditions on the adapted basis components of three forms in $\langle \Sigma^{k-1} \rangle$. This can be done for Σ^0.

Example 3. (n=3)

$$\Sigma^1 = sp\{\omega^{11}, \omega^{22}, \omega^{33}\}$$

$$(\omega^{aa} := \psi^a \wedge \theta^b)$$

$$\Sigma^2 = sp\{\omega^1 := \omega^{11} + r^1_3 \omega^{33},\ \omega^2 := \omega^{22} + r^2_3 \omega^{33}\}$$

For $\omega = \omega^1 + p\omega^2 \in \Sigma^3$, $\exists\ \lambda_{1,2}$:

$$d\omega \in \langle \Sigma^2 \rangle \iff$$

$$d\omega^1 + pd\omega^2 = \lambda_1 \wedge \omega^1 + \lambda_2 \wedge \omega^2 \qquad (1)$$

Now we use $d\omega^1, d\omega^2 \in \langle \Sigma^1 \rangle$ and

$$\omega^1 := \omega^{11} + r^1_3 \omega^{33},\ \omega^2 := \omega^{22} + r^2_3 \omega^{33}$$

to get (1) in terms of ω^{11}, ω^{22}.

We get a linear system of 4 equations in 5 unknowns (p and 4 components of λ_1, λ_2) whose rank depends on r_3^1, r_3^2.

The differential ideal step generates all the necessary and sufficient conditions in a basis calculation.

Open Questions

1. $n = 3$

2. nature of classification for arbitrary n. Probably by Φ:

   (a) Φ diag'ble distinct e'vals: p e'spaces integrable $(n - p)$ are not integrable

   (b) Φ diag'ble repeated e'vals and integrability again.

   (c) then Jordan normal forms but possible subclassification by differential ideal step.

3. The rest of the EDS process !!