Cohomological Interpretation of Transversality Conditions

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XXII IWDGMTM
Będlewo, August 19–26, 2007

Introduction

We start from a very simple variational problem with free boundary and write down the well–known necessary and sufficient conditions for its extrema. The necessary conditions are the Euler–Lagrange equations. The sufficient conditions are obtained by complementing the formers with the so–called transversality conditions.

Introduction

By using the Secondary Calculus machinery we will discover the following

Main result:

After a suitable geometric interpretation of the variational problems with free boundary in the framework of the infinite jet spaces, and after studying the consequences of such an interpretation on the $\mathcal{C}$–spectral sequence, we will find out that the Euler–Lagrange equations and the transversality conditions are actually the components of a new object, which we called the relative Euler–Lagrange equation, produced by pure cohomological constructions. So the TC are not "complementary to the EL equations" but are instead the "second component of the relative Euler–Lagrange equations".

Introduction

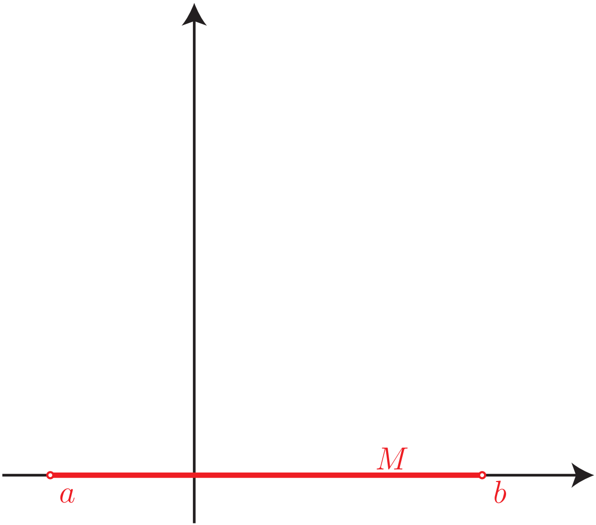
Of course such a result is absolutely general:

- no restrictions on the topology of the independent variables manifold M,
- no triviality requirements for the bundle of dependent variables π,
- and no limits for the order of the Lagrangian L!

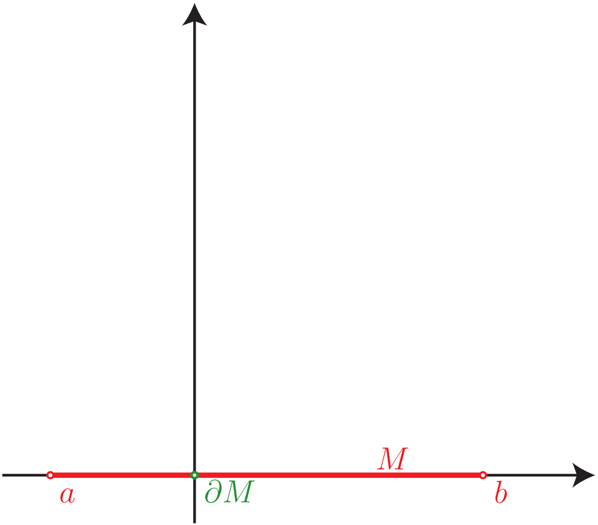
Outline

Analytical formulation

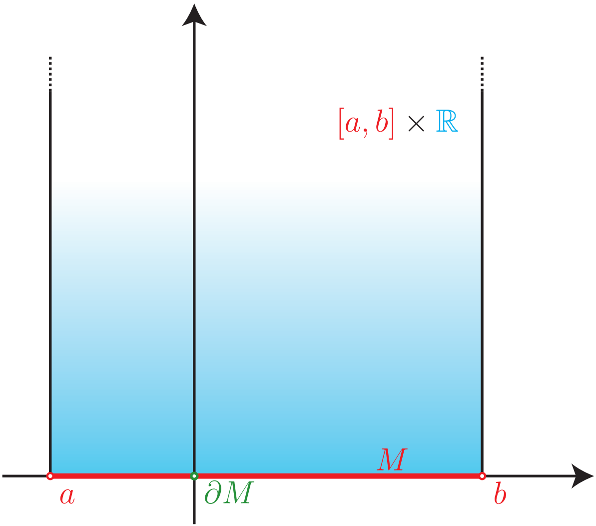
We present here the simplest instance of a well–known variational problem: that of minimizing the length of the curves connecting two fixed parallel lines in the plane.



Let $M = [a, b]$ be the manifold of the independent variables.



Let the symbol ∂M denote the submanifold $\{0\}$ of M.



Let $\pi : M \times \mathbb{R} \to M$ be the bundle of dependent variables.

Denote by x and by u the base and the fiber coordinate of π, respectively. Consider now the Lagrangian density $f = \sqrt{1 + (u')^2}$ to which corresponds the action

$$\Gamma(\pi) \ni u \longmapsto \int_a^b \sqrt{1 + (u'(x))^2} \mathrm{d}x \in \mathbb{R}. \tag{1}$$

Correspondingly, one can compute the

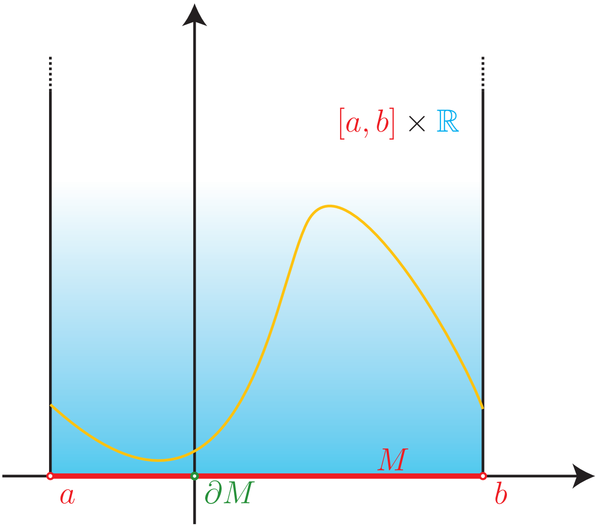
Euler–Lagrange (EL) equations

$$\widehat{\ell}_f(1) = \frac{u''}{f^3}, \tag{2}$$

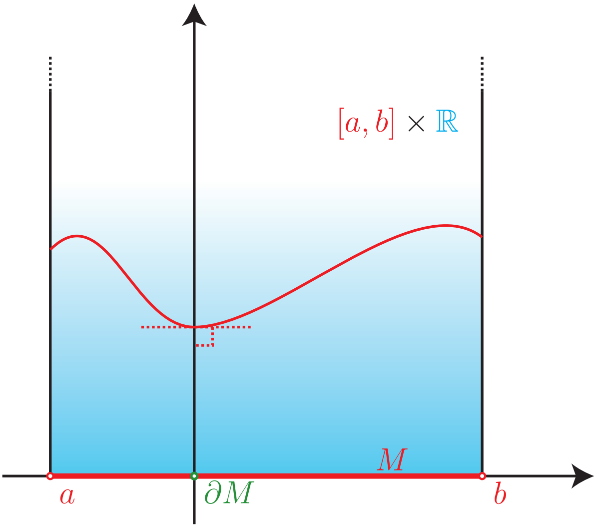
and the

transversality conditions (TC)

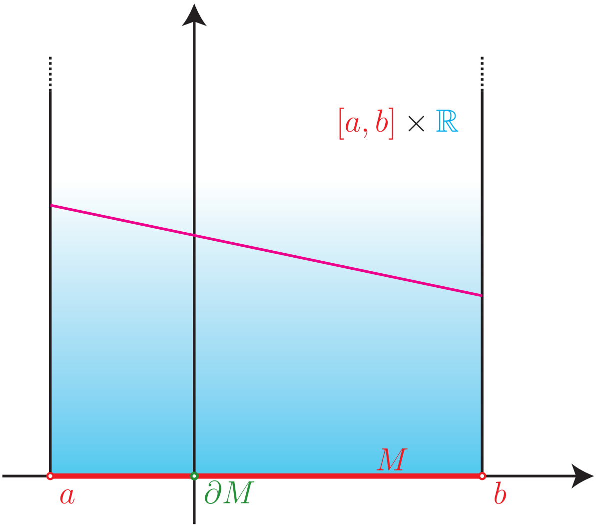
$$\ell'_f = \left. \frac{u'}{f} \right|_{\partial M}. \tag{3}$$



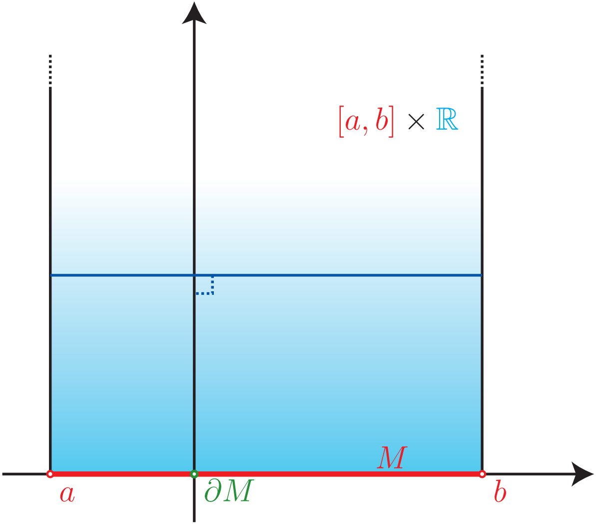
This u does not fulfill neither the EL equations nor the TC.



This u fulfills the TC but not the EL equations.



This u fulfills the EL equations but not the TC.



This u fulfills both the EL equations and the TC.

We can easily draw the following

Observation

The u's that are extrema for the action

$$\Gamma(\pi) \ni u \longmapsto \int_a^b \sqrt{1 + (u'(x))^2}\mathrm{d}x \in \mathbb{R}.$$

are exactly those that fulfill both the EL equations and the TC.

Re–formulation in terms of Secondary Calculus

What is a Secondary Manifold?

The analog of a smooth manifold in Secondary Calculus is called diffiety (from differential variety) and it is not the space $\boldsymbol{M}$ of solutions of a given PDE. More precisely, a diffiety is a couple $(\mathcal{O}, \mathcal{C})$ were $\mathcal{O}$ is the geometrical object corresponding to a filtered smooth algebra, and $\mathcal{C}$ is a finite–dimensional completely integrable distribution on it. Leaves of $\mathcal{C}$ are called the secondary points of the diffiety, and their totality can be denoted by $\boldsymbol{M}$.

WARNING

The space $\boldsymbol{M}$ never plays any active role in Secondary Calculus! It is only used for didactical purposes for its concreteness (so that students are not scared away). For instance, when $\mathcal{O}$ is a fiber bundle and $\mathcal{C}$ is the vertical distribution on it, $\boldsymbol{M}$ is just the base of the bundle (i.e., the manifold of all the fibers)!

J^∞: the UNIVERSAL RECEPTACLE for non–linear PDEs – Part I

If $\pi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is the trivial bundle, then the space $\boldsymbol{M} = \Gamma(\pi)$ of its smooth sections is the space of all smooth vector functions in n variables with m components. Let the x_i's be the coordinates on $\mathbb{R}^n$ and the u^j's be the coordinates on $\mathbb{R}^m$. Construct an extension π_k of the bundle π by enlarging the fibers of the latter with new coordinates u^j_σ, σ being a multi–index of length $|\sigma| \leq k$. Any section $f \in \boldsymbol{M}$ of π can be prolonged to a section $j_k(f)$ of π_k just by putting

$$u^j_\sigma(j_k(f)(\boldsymbol{x})) = \left.\frac{\partial^{|\sigma|} f^j}{\partial x^\sigma}\right|_{\boldsymbol{x}}, \quad \forall \boldsymbol{x} \in \mathbb{R}^n.$$

The total space E_{π_k} is denoted by $J^k(\pi)$ and is called the k–th jet space of the bundle π. $j_k(f)$ is called the k–th jet prolongation of f. Obviously all these constructions can be carried over for arbitrary bundles.

J^∞: the UNIVERSAL RECEPTACLE for non–linear PDEs – part II

Intuitively, the sections of π_k encode all the information about the derivatives up to order k of all the sections of π. Denote by $\mathcal{F}_k$ the algebra $C^\infty(J^k(\pi))$. The zero locus $\mathcal{E}$ of an element $\varphi \in \mathcal{F}_k$ is then a non–linear partial differential equation of order k in m unknown functions of n independent variables. Solutions of $\mathcal{E}$ are exactly those elements $f \in \boldsymbol{M}$ the graph of whose k–th jet prolongation lies in $\mathcal{E}$.

J^∞: the UNIVERSAL RECEPTACLE for non–linear PDEs – part III

Since each manifold $J^k(\pi)$ projects over $J^{k-1}(\pi)$, we have the corresponding embedding of smooth algebras $\mathcal{F}_{k-1} \subseteq \mathcal{F}_k$. The geometrical object $J^\infty(\pi)$ corresponding to the filtered smooth algebra $\mathcal{F} = \lim \mathcal{F}_k$ is called the infinite jets space of the bundle π. The projection π_∞ is defined in the obvious way, and any $f \in \boldsymbol{M}$ can now be prolonged to a section $j_\infty(f)$ of π_∞, called the infinite jet prolongation of f. Unlike the J^k's, the Cartan distribution $\mathcal{C}$ on $J^\infty(\pi)$ is completely integrable and n–dimensional! So $(J^k(\pi), \mathcal{C})$ is a diffiety, whose secondary points are exactly the sections of π!

J^∞: the UNIVERSAL RECEPTACLE for non–linear PDEs – part IV

To an element $\varphi \in \mathcal{F}_k$ is associated its zero locus $\mathcal{E}$ in $J^\infty(\pi)$. So it would be nice to be able to interpret the solutions of $\mathcal{E}$ as the secondary points of the diffiety $(\mathcal{E}, \mathcal{C}|_\mathcal{E})$. But the restricted distribution $\mathcal{C}|_\mathcal{E}$ is not completely integrable, since, in general, $\mathcal{C}$ is not tangent to $\mathcal{E}$! The biggest subvariety of $\mathcal{E}$ to which $\mathcal{C}$ is tangent is called the infinite prolongation of $\mathcal{E}$ and denoted by $\mathcal{E}_\infty$. Algebraically the latter is obtained by the former by adding to φ all its differential consequences (i.e., the total derivatives). So we can conclude that the solution space of our PDE is the set of secondary points of the diffiety $(\mathcal{E}_\infty, \mathcal{C}|_{\mathcal{E}_\infty})$.

Now we can go back to the re–formulation of our variational problem in terms of Secondary Calculus.

What are the u's?

The u's are the secondary points of the diffiety $J^\infty(\pi)$.

What is the action determined by f?

The action determined by f is the secondary function on the diffiety $J^\infty(\pi)$ determined by the horizontal cohomology class $L = [f\overline{\mathrm{d}}x] \in \overline{H}^1(J^\infty(\pi))$.

What are the EL equations associated with f?

The left–hand side of the EL equations associated with f is obtained by applying the d_1 differential of the $\mathcal{C}$–spectral sequence to the secondary function L.

If Secondary Calculus really follows the same logic of the Standard Calculus, then the words "extrema" and "vanishing differential" must be synonymous.
The existence of the TC is itself a proof that the d_1 differential is not the right secondary analog of the standard differential in this particular setting. So the gap to be filled up is the following:

What are the TC associated with f?

We will discover that in Secondary Calculus there is a "relative" version of the d_1 differential, denoted by $d_{1,\mathrm{rel}}$, which, applied to the secondary function L, gives a two–component object, the first component being the left–hand side of the EL equations and the second component being the left–hand side of the TC.

Before going on, it might be interesting to answer to the following question:

What do NOT are the transversality conditions?

Just by looking at them

$$\ell_f' = \left.\frac{u'}{f}\right|_{\partial M},$$

it is brightly evident that they are not PDEs imposed on the sections of the restricted bundle $\pi|_{\partial M}$! In fact, any equation in $J^\infty(\pi|_{\partial M})$ would establish conditions only on the derivatives along the boundary ∂M (i.e., boundary conditions) but no conditions on the normal derivatives with respect to the boundary (i.e., the transversal ones).

In the next section we will discover that the TC are in fact non–linear PDEs, but imposed on the section of another bundle over ∂M, an extension obtained from $\pi|_{\partial M}$ by adding to it the information about all the normal with respect to ∂M partial derivatives.

Outline

Geometrical portrait of the free boundary variational problems

SETTINGS

Since we are about to construct a general theory, π will be from now on an arbitrary bundle of basis M and total space E. Basis and fiber coordinates will be $x_1, \ldots, x_n$ and $u^1, \ldots, u^m$, respectively. The symbol ∂M will now denote the boundary of M (even though the theory can be developed for any submanifold of codimension 1).

NOTATION

Denote by B the diffiety $J^\infty(\pi)$ and let u^i_σ be the fiber coordinates of the projection $\pi_\infty : J^\infty(\pi) \to M$. Denote by Λ the differential form algebra of B and by $\mathcal{C}$ the ideal of the vanishing on the Cartan distribution differential forms. Let E_r be the corresponding $\mathcal{C}$–spectral sequence.

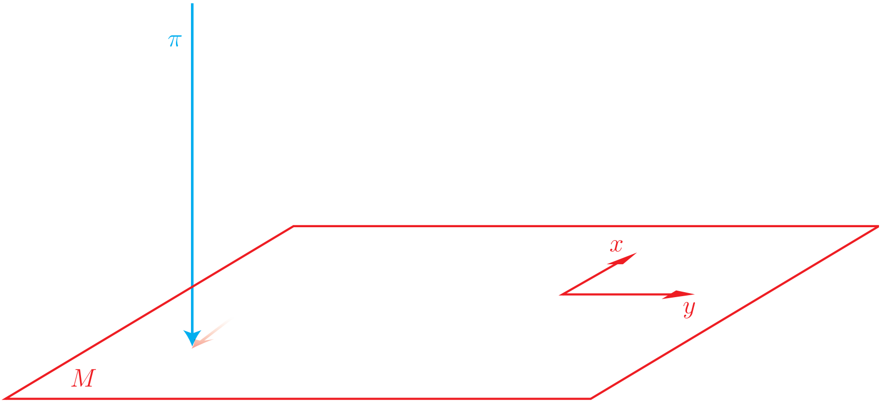
How to interpret the "normal" derivatives to ∂M?

IDEA

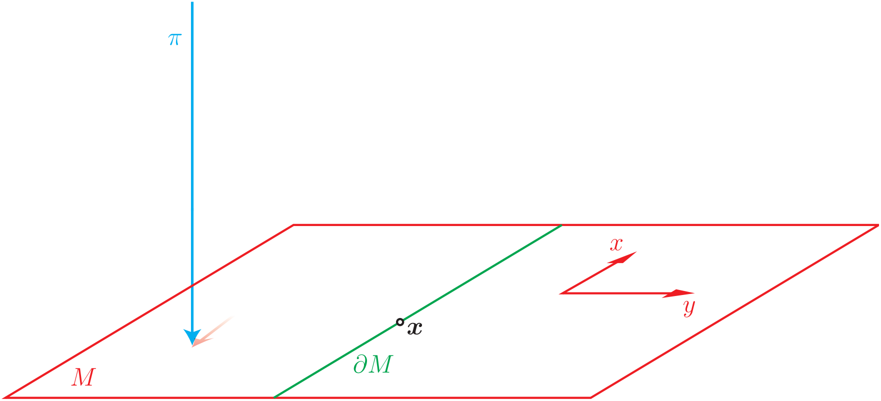
The construction of $J^\infty(\pi)$ can be somehow factorized. In fact, the infinite jet prolongation $j_\infty(f)$ of a section $f \in \Gamma(\pi)$ is obtained by adding to the components of f all their partial derivatives. Such a process, at least on the intuitive level, can be carried out in two steps: first adding the "normal" to ∂M partial derivatives, and then the "tangent" ones.

To perform such a trick one must assume that ∂M is the leaf of a completely integrable distribution Δ on M possessing a complementary (and completely integrable, being 1–dimensional) distribution ∇.

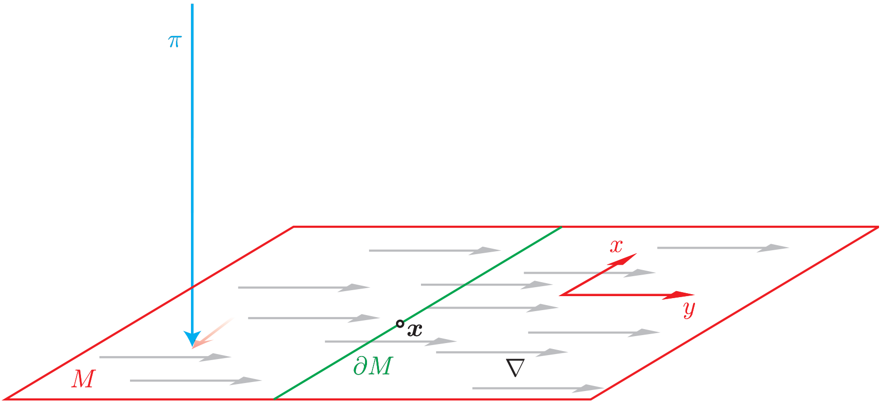
Start from the manifold M, assumed here to be 2–dimensional.



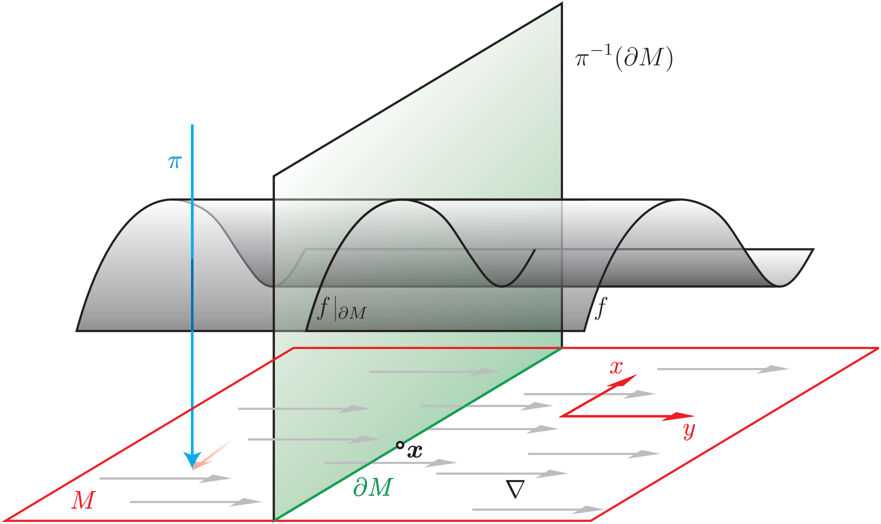
Consider the bundle π over it.



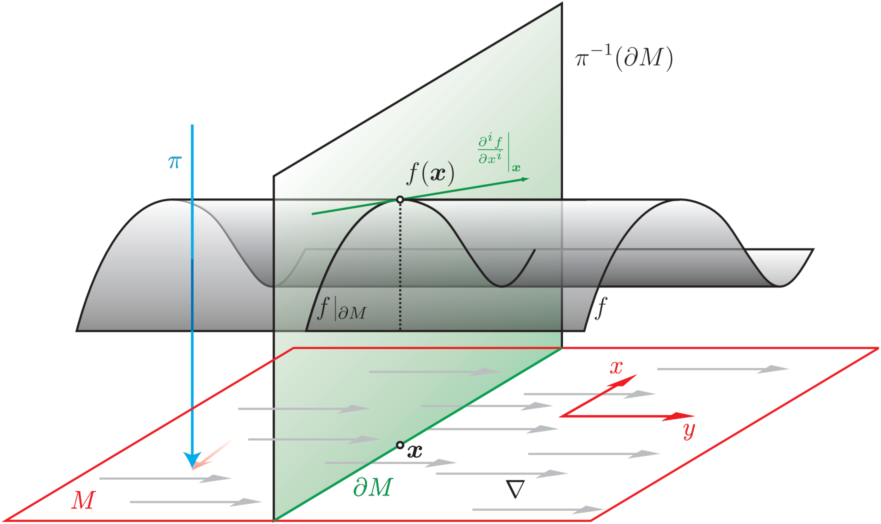
Suppose that the submanifold ∂M is the leaf of a c.i. distribution Δ...



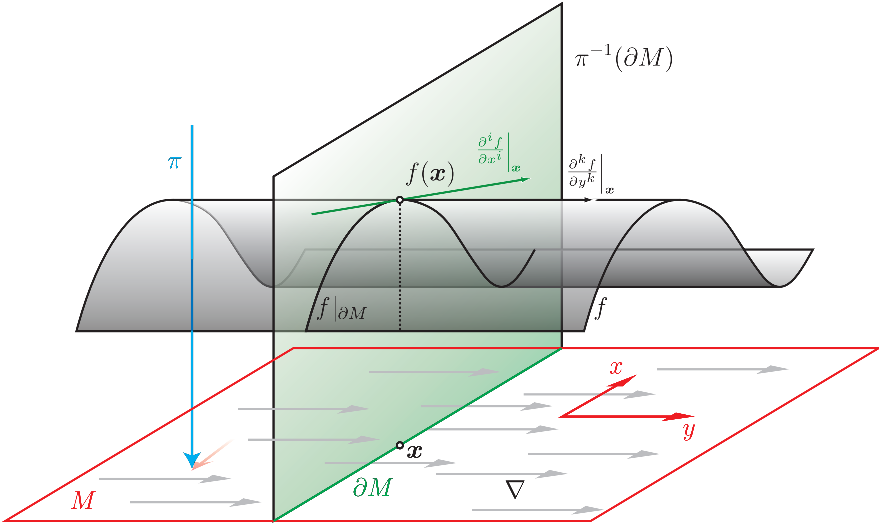
... admitting a complementary (1–dimensional) distribution ∇.



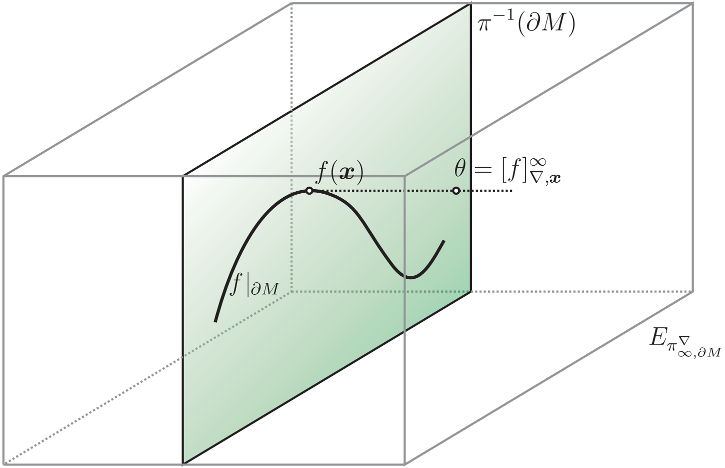
Given a section f of π...



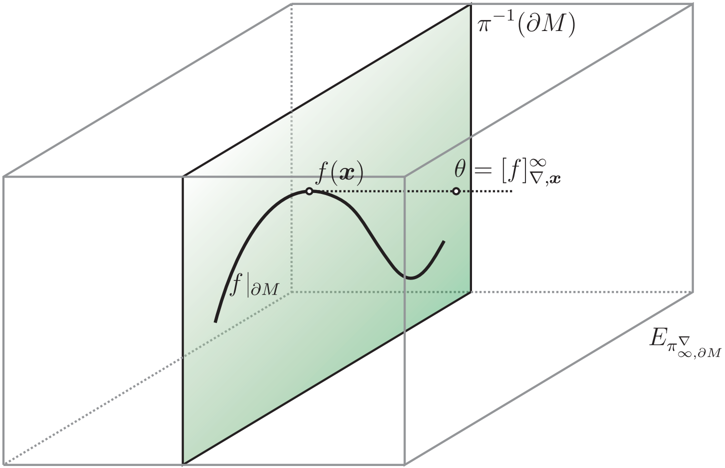
...we can compute the derivatives along Δ of f...



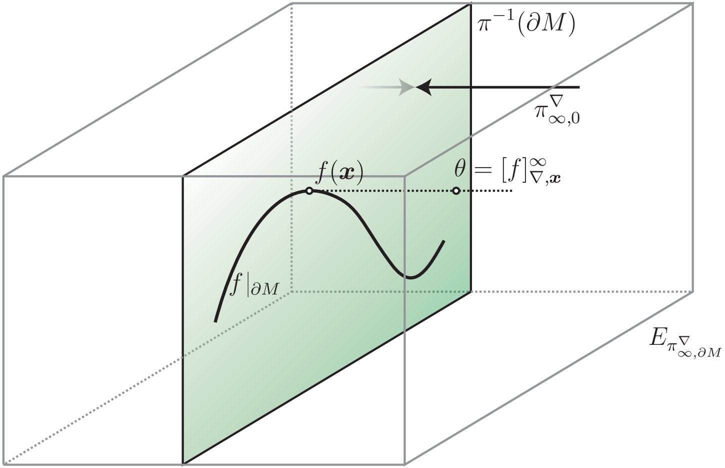
...and the derivatives along ∇ of f, i.e., the "normal" ones.



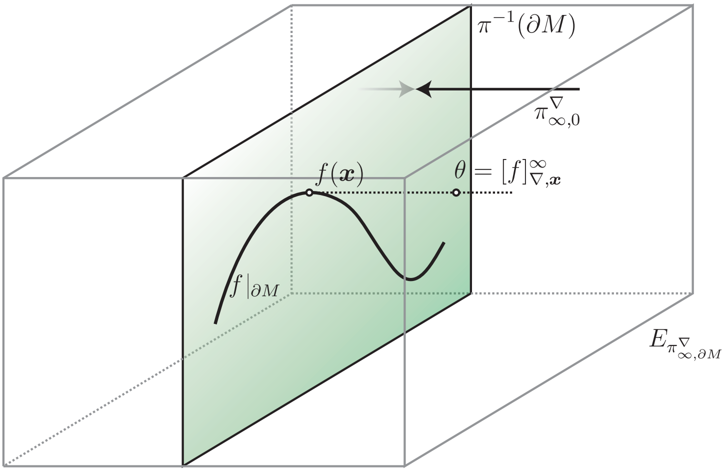
Denote by θ the collection of all the normal derivatives of f at $\boldsymbol{x} \in \partial M$.



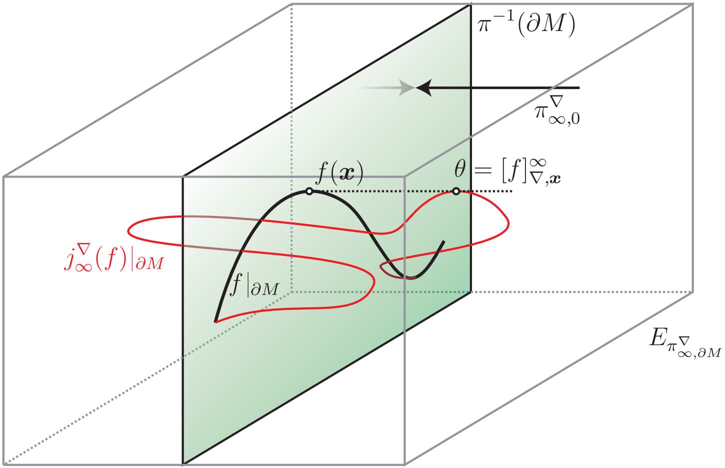
The manifold $E_{\pi^{\nabla}_{\infty,\partial \boldsymbol{M}}}$ is filled up by the θ's obtained by taking all the f's and all the $\boldsymbol{x}$'s.



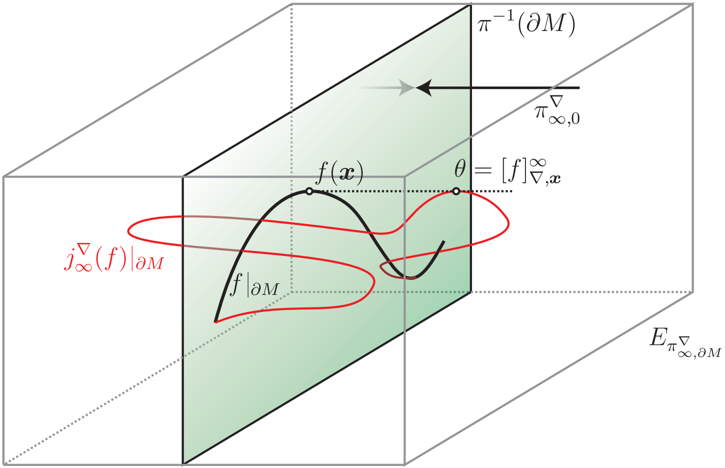
Clearly $E_{\pi^{\nabla}_{\infty,\partial M}}$ projects over $\pi^{-1}(\partial M)$, which in turn projects over ∂M.



Denote by $\pi^{\nabla}_{\infty,\partial M}$ the overall projection of $E_{\pi^{\nabla}_{\infty,\partial M}}$ over ∂M.

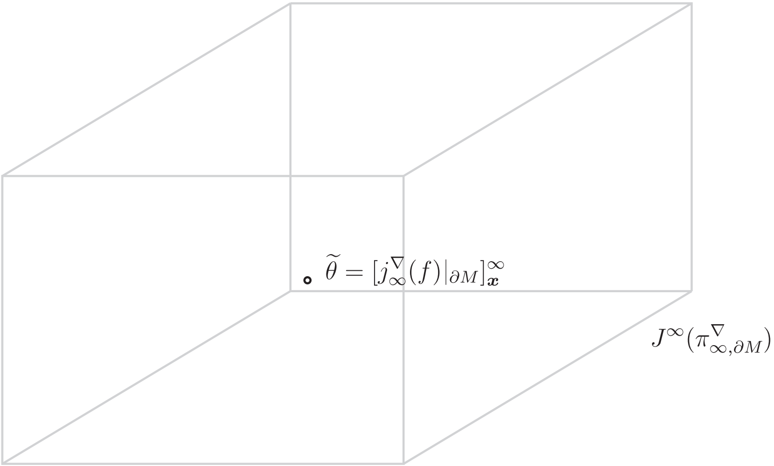


Any f can now be lifted to its infinite normal jet $j^{\nabla}_{\infty}(f)|_{\partial M}$, restricted to ∂M, which is a section of $\pi^{\nabla}_{\infty,\partial M}$.

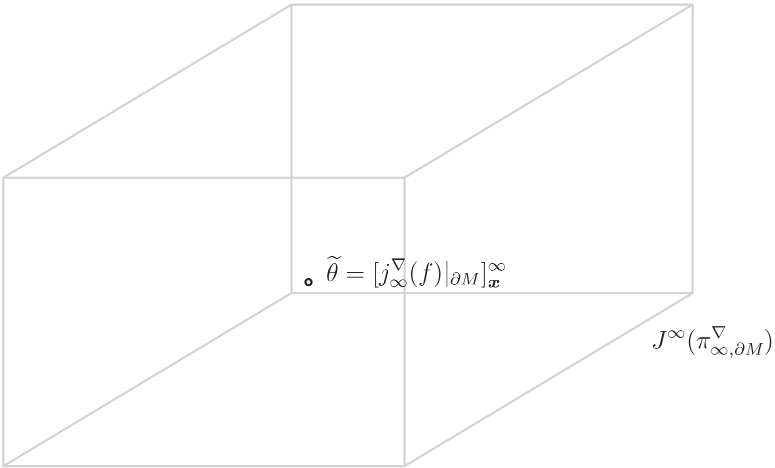


Observe that in order to define $j^{\nabla}_{\infty}(f)|_{\partial M}$ is not enough to know only $f|_{\partial M}$.

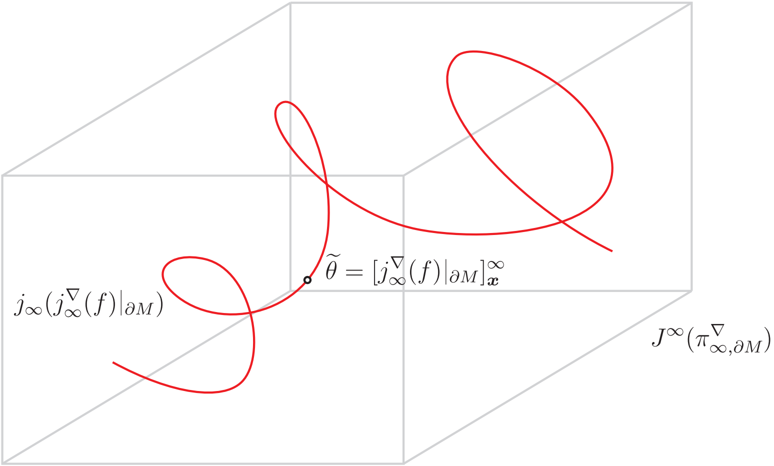
Now we can consider the infinite jets space of the bundle $\pi^{\nabla}_{\infty,\partial M}$...



...a point $\widetilde{\theta}$ of which is the set of all derivatives of a section of $\pi_{\infty,\partial M}^\nabla$...



...like, in our case, $j_\infty^\nabla(f)|_{\partial M}$. Observe that $\widetilde{\theta}$ projects over θ.



The lifting of $j^{\nabla}_{\infty}(f)|_{\partial M}$ now contains the information about all the derivatives of f at all the points of ∂M!

The above (informal) geometrical construction should be enough to motivate the following

Theorem (GM)

The diffiety $J^{\infty}(\pi^{\nabla}_{\infty,\partial M})$ is naturally identified with $\pi_{\infty}^{-1}(\partial M)$.

From now on, $\pi_{\infty}^{-1}(\partial M)$ will be denoted by ∂B.
Is it clear that the points of ∂B encode the differential information about all the derivatives (both tangent and normal to ∂M) of all the sections of π at the points of ∂M. So $\pi^{\nabla}_{\infty,\partial M}$ is the bundle we were looking for!

INTERMISSION: Rethinking the Lagrangians...

Accordingly to the Stoke's formula, if the Lagrangian density $\overline{\omega} \in \overline{\Lambda}^n$ is a relative to ∂B exact horizontal n–form, then the determined by it action

$$\Gamma(\pi) \ni \sigma \longmapsto \int_M j_\infty(\sigma)^*(\overline{\omega}) \in M$$

is constant to 0!

Then it is natural to understand the secondary functions in our particular case as the relative to ∂B horizontal n–cohomology classes. So the Lagrangians of the free boundary variational problems will be from now on the elements of the space $\overline{H}^n(B, \partial B)$.

REMARK about horizontal calculus...

The horizontal calculus on the diffiety $(J\infty(\pi), \mathcal{C})$ is made of $\mathcal{C}$–differential operators acting between horizontal modules. An horizontal module, like $\overline{\Lambda}$ is the $\mathcal{F}$–module of sections of an induced from π_∞ vector bundle. For instance, $\overline{\Lambda} = \Gamma(\pi_\infty^*(\tau))$, being τ the bundle of skew–symmetric multi–covectors on M. So an element φ of the horizontal module $P = \Gamma(\pi_\infty^*(\xi))$ gives rise to a family $\{j_\infty(\sigma)^*(\varphi)\}_{\sigma\in \boldsymbol{M}}$ of elements of $\Gamma(\xi)$ parametrized by the secondary points of $(J^\infty(\pi), \mathcal{C})$! Similarly, a $\mathcal{C}$–differential operator Δ between the horizontal modules $P = \Gamma(\pi_\infty^*(\xi))$ and $Q = \Gamma(\pi_\infty^*(\eta))$, can be interpreted as a family $\{\Delta_\sigma\}_{\sigma\in \boldsymbol{M}}$ of differential operators acting between $\Gamma(\xi)$ and $\Gamma(\eta)$.

REMARK about horizontal calculus...

Example: symmetries

An higher symmetry $\varphi \in \varkappa$, due to the identification $\varkappa \equiv \Gamma(\pi_\infty^*(\pi))$, can be interpreted as the family $\{\varphi_\sigma\}_{\sigma \in \boldsymbol{M}}$. The section φ_σ is exactly a vertical vector field along the graph of σ! In the standard terminology of the Calculus of Variation, the sum $\sigma + \varphi_\sigma$ is denoted by $\sigma + \delta\sigma$. In Secondary Calculus, if φ is a secondary vector field, then φ_σ must the secondary tangent vector obtained by evaluating it on the secondary point σ! So while the experts of Calculus of Variations say “let's take a small variation $\delta\sigma$” of σ, the experts of Secondary Calculus say instead “let's evaluate a secondary vector field on σ”... A similar conversation took place centuries ago, when Leibniz began the foundation of the modern Differential Calculus!

REMARK about horizontal calculus...

Example: the universal linearization

A function $f \in \mathcal{F}$ gives rise to the $\mathcal{C}$–differential operator $\ell_f : \varkappa \to \mathcal{F}$ defined as $\ell_f(\varphi) = \varphi(f)$. If we have a secondary vector field, i.e., "a family of small variations $\delta\sigma$", then each $(\ell_f)_\sigma$ can be applied to the corresponding element of the family. This explains why the construction of the secondary differential d_1 is based on ℓ_f. In fact, when the result $(\ell_f)_\sigma(\delta\sigma)$ is integrated over M, the only relevant part is $\langle \widehat{\ell_f}(1)_\sigma, \delta\sigma \rangle$ by the Green formula. So if the last quantity is 0 for all the small variations $\delta\sigma$, then σ is a solution to the EL equations.

Example: the $\mathcal{C}$–analogue of a standard theorem

Instead of applying the Green formula to each member $(\ell_f)_\sigma$ of the family determined by ℓ_f, there exists an analogue of the Green formula, called the Green $\mathcal{C}$–formula, which holds for ℓ_f and reduces to the standard one on the single elements of the family.

Implications on the cohomology level

The geometrical part of our job is finished. Now we switch to the algebraic part and we will see how easy is to draw very powerful consequences with a very small effort—after all, Homological Algebra is just a game of points and arrows!

Observation

The differential algebra Λ is doubly filtered. One filtration is due to the powers of the Cartan ideal $\mathcal{C}^p$, and the other is due to the ideal $\Lambda(B, \partial B)$ of vanishing on ∂B differential forms.

Consequences

Each filtration will affect the spectral sequence determined by the other, opening the way to a very rich theory of the filtered spectral sequences and, consequently, of the spectral sequences of spectral sequences.

Remark

In our case, being the filtration $\Lambda \supseteq \Lambda(B, \partial B) \supseteq 0$ almost trivial, we will get a simple long exact sequence of spectral sequences.

Definition

The E_0 term of the relative $\mathcal{C}$–spectral sequence is defined as follows:

$$E_0^p(B,\partial B) = \frac{\mathcal{C}^p \cap \Lambda(B,\partial B) + \mathcal{C}^{p+1}}{\mathcal{C}^{p+1}} \tag{4}$$

Then we easily discover the following

Short exact sequence of E_0 terms:

$$0 \to E_0^p(B,\partial B) \xrightarrow{i} E_0^p \xrightarrow{\alpha} E_0^p(\partial B) \to 0, \tag{5}$$

to which is associated the corresponding

Long exact sequence of E_1 terms:

$$\begin{array}{ccc} E_1^p(B,\partial B) & \xrightarrow{H(i)} & E_1^p \\ & {}_{\partial}\nwarrow \quad \swarrow_{H(\alpha)} & \\ & E_1^p(\partial B). & \end{array} \tag{6}$$

Everything is quite natural and all the results are just straightforward computations!

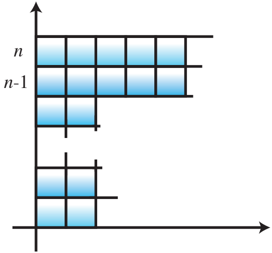
Observe that, until now, the geometrical and the algebraic theories could have been developed separately...

In a moment we will combine them and discover the true nature of the transversality conditions!

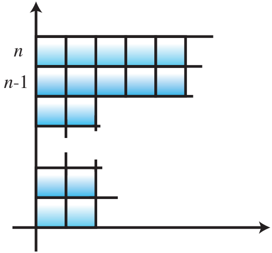
Just rewrite the exact triangle

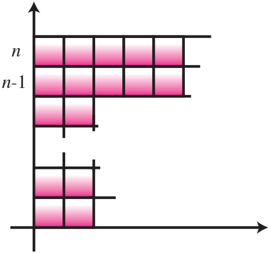
$$\begin{array}{ccc} E_1^p(B,\partial B) & \xrightarrow{H(i)} & E_1^p \\ & \nwarrow^{\partial} \quad \swarrow_{H(\alpha)} & \\ & E_1^p(\partial B). & \end{array}$$

in a more choreographic way...

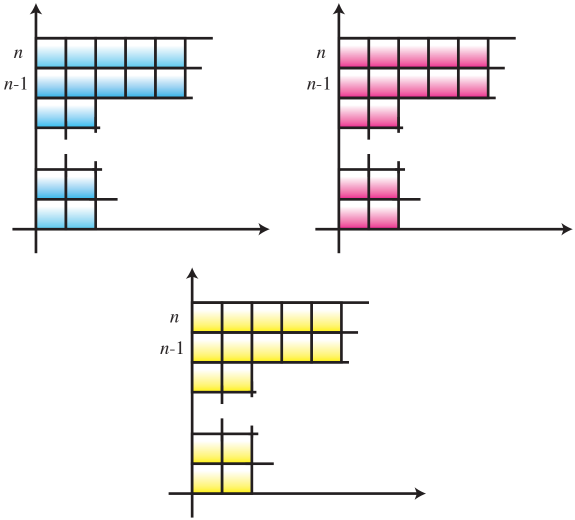


...first the relative E_1 term...

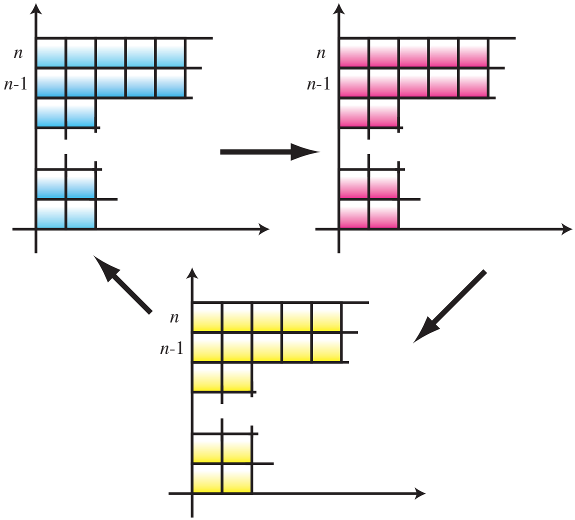




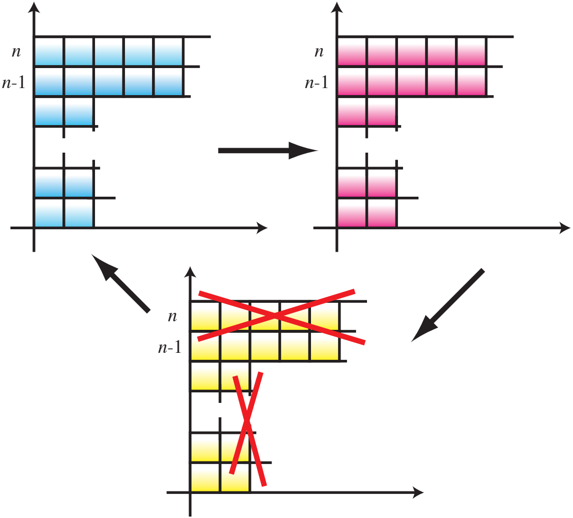
...then the absolute E_1 term...



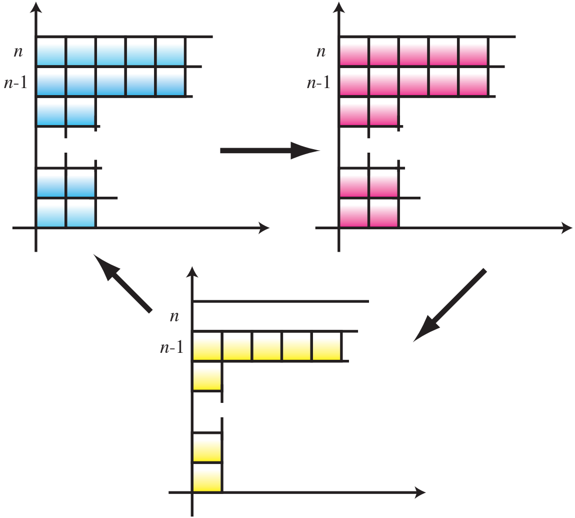
...and finally the E_1 term of ∂B.



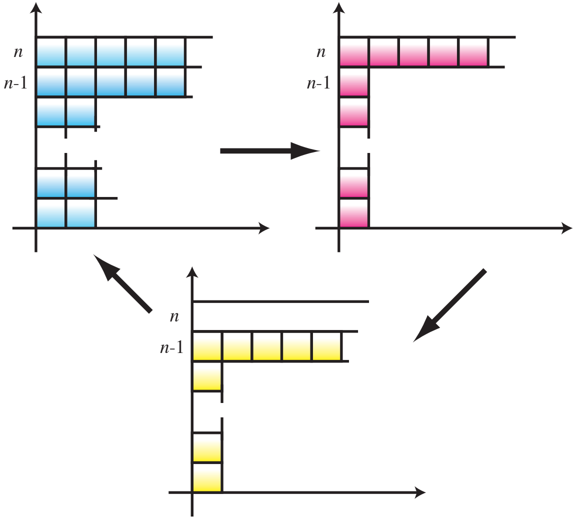
But do not forget the arrows!



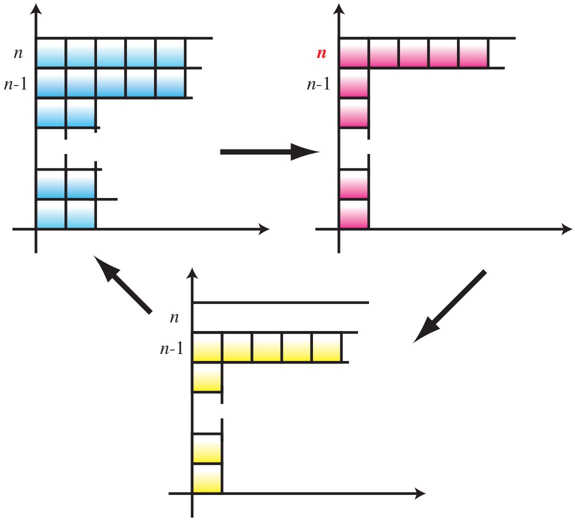
The geometric comprehension of ∂B we have just gained...



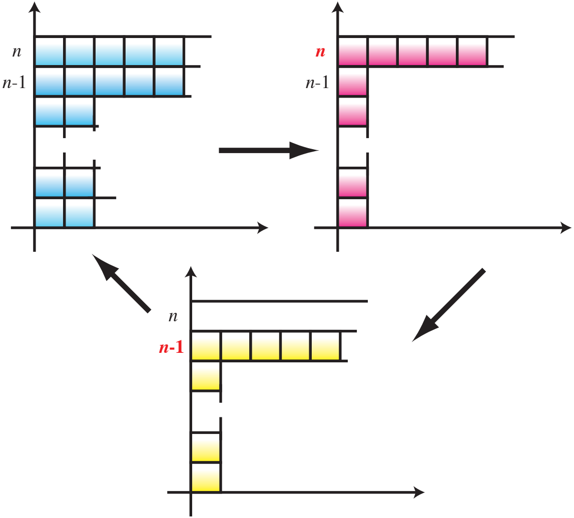
...allows us to apply the one–line theorem!



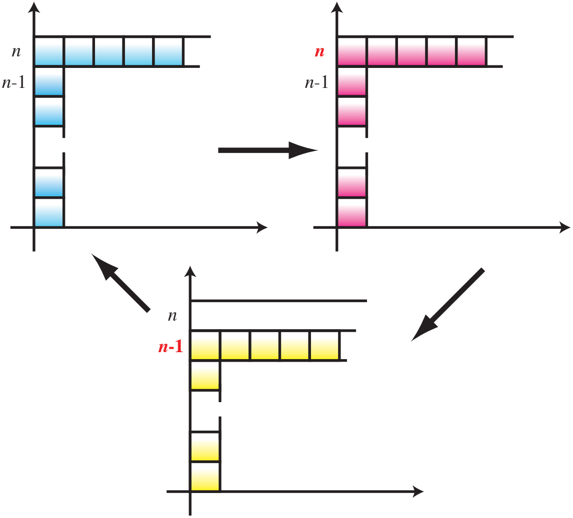
Which of course does hold also for B.



Notice that in B we have n independent variables...



...whereas this number is $n-1$ for ∂B.



So also the relative spectral sequence is forced to be one–line!

By simply playing with colored squares and arrows we have proved the following

Proposition

The exact triangle

$$\begin{array}{ccc} E_1^p(B,\partial B) & \xrightarrow{\quad H(i) \quad} & E_1^p \\ & {\scriptstyle \partial}\nwarrow \qquad \swarrow{\scriptstyle H(\alpha)} & \\ & E_1^p(\partial B) & \end{array}$$

reduces, for $p > 0$, to the short exact sequence

$$0 \longrightarrow E_1^{p,n-1}(\partial B) \xrightarrow{\partial} E_1^{p,n}(B,\partial B) \xrightarrow{H(i)} E_1^{p,n} \longrightarrow 0. \qquad (7)$$

For $p = 1$ the above sequence reads

$$0 \longrightarrow \widehat{\varkappa}(\partial B) \xrightarrow{\partial} E_1^{1,n}(B, \partial B) \xrightarrow{H(i)} \widehat{\varkappa} \longrightarrow 0. \quad (8)$$

By using the Green $\mathcal{C}$–formula, we can easily prove the following

Proposition

The sequence (8) splits.

Outline

definition

The differential $d_{1,\mathrm{rel}}^{0,n} : \overline{H}^n(B,\partial B) \to E_1^{1,n}(B,\partial B)$ is called the relative Euler operator.

By applying the relative Euler operator to a Lagrangian

$$L = [f\,\mathrm{d}^n x] \in \overline{H}^n(B,\partial B)$$

one gets the pair

$$(\widehat{\ell_f}(1), \ell'_f)$$

accordingly to the above splitting result.

FINAL RESULT

The first component represents the left–hand side of the Euler–Lagrange equations, while the second component represents the left–hand side of the Transversality Conditions.

Outline

Secondary Calculus is a rich theory, whose childhood has not yet been completely spent.

Working in Secondary Calculus is not an easy task...

Each step forward raises unexpected difficulties and forces to rethink some aspects of the theory, even the fundamental ones (e.g., that of the secondary functions).

The tight interplay between Algebra and Geometry which characterizes Secondary Calculus reveals itself even in the simplest situations, like the one herewith presented.

1st thing to do:

Prove the sufficiency of the relative Euler–Lagrange Equations for the variational problems with free boundary to have an extremum.

2nd thing to do:

Analyze the cases of Lagrangian systems with constraints and Lagrangian systems with boundary conditions.

3rd thing to do:

Many physically relevant non–linear PDEs (e.g., the evolutionary equations and the fields equations on space–time) posses a natural fibered structure. We must study the secondary analogous of the differential Leray–Serre spectral sequence which is produced by such a structure. Expectedly, this will provide an effective method for computing the $\mathcal{C}$–spectral sequence for non–linear PDEs and clarify the nature of the Hamiltonian formalism in fields theory.

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Acknowledgments

Thanks to the creator of the LaTeX Beamer Class!