Natural and invariant quantizations

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Outline

Tensor Densities

Let M be a smooth manifold of dim n
In coordinates, a density F of weight λ writes

$$F(x) = f(x^1, \cdots, x^n)\,|dx^1 \wedge \ldots \wedge dx^n|^\lambda,$$

where f is a function.
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The bundle of λ-densities is denoted by $F_\lambda(M)$, while $\mathcal{F}_\lambda(M)$ is the space of sections.

- $F_\lambda(M) = P^1M \times_{\rho_\lambda} \Delta^\lambda(\mathbb{R}^n)$.
  where P^1M is the linear frame bundle of M and

$$\begin{cases} \dim \Delta^\lambda(\mathbb{R}^n) = 1 \\ \rho_\lambda(A)e = |detA|^{-\lambda}e, \quad \forall A \in GL(n,\mathbb{R}) \end{cases}$$

- $\mathcal{F}_\lambda(M) : C^\infty(P^1M, \Delta^\lambda(R^n))_{GL(n,\mathbb{R})}$
- $\forall f \in \mathcal{F}_\lambda(M)$,

$$\rho_\lambda(\varphi)(F)(x) = F \circ \varphi^{-1}(x)|\det D_x\varphi|^{-\lambda}.$$

Differential operators

- $\mathcal{D}_{\lambda\mu}(M)$: linear diff. operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$.
- There is a filtration

$$\mathcal{D}_{\lambda\mu} = \bigcup_{k\in\mathbb{N}} \mathcal{D}^k_{\lambda\mu}.$$

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$$\mathcal{D}_{\lambda\mu} = \bigcup_{k\in\mathbb{N}} \mathcal{D}^k_{\lambda\mu}.$$

- In coordinates, if $D \in \mathcal{D}^k_{\lambda\mu}$:

$$D = \sum_{|\alpha|=0}^{k} A_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha, \quad A_\alpha \in \mathcal{F}_{\mu-\lambda}.$$

- Action of $Diff(M)$:

$$\rho_{\lambda\mu}(\varphi)D = \rho_\mu(\varphi) \circ D \circ \rho_\lambda(\varphi^{-1}).$$

Symbols

The space of symbols

We set $S^l_\delta(\mathbb{R}^n) = S^l\mathbb{R}^n \otimes \Delta^\delta(\mathbb{R}^n)$.
We set

$$S^l_\delta(M) \to M := P^1M \times_\rho S^l_\delta(\mathbb{R}^n) \to M,$$

$$\mathcal{S}^l_\delta(M) = C^\infty(P^1M, S^l_\delta(\mathbb{R}^n))_{GL(n,\mathbb{R})}$$

and

$$\mathcal{S}_\delta(M) = \oplus_{l=0}^{\infty}\mathcal{S}^l_\delta(M).$$

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Remark

As symmetric tensor fields identify with functions on T^*M that reduce to polynomials along the fibers, we identify symbols with such functions (with values in spaces of densities)

The principal symbol operator

Definition

If $\delta = \mu - \lambda$ we set $\sigma_{(l)} : \mathcal{D}^l_{\lambda,\mu}(M) \to \mathcal{S}^l_\delta(M)$:

$$D = \sum_{|\alpha|=0}^{l} A_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha \mapsto \sigma(D)(x,\xi) = \sum_{|\alpha|=l} A_\alpha(x)\xi^\alpha.$$

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Properties

The operator σ commutes with the actions of local diffeomorphisms. It induces a bijection from the graded space associated to differential operators to the space of symbols.

Projective structures

- Denote by $\mathcal{C}_M$ the space of torsion-free linear connections on M.
- Two such connections are *Projectively equivalent* if they define the same geodesics up to parametrization.
- An equivalent condition : $\nabla \sim \nabla' \Leftrightarrow \exists \alpha \in \Omega^1(M)$:

$$\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X.$$

  (This formulation was given by H. Weyl).
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- A class of equivalent torsion free linear connections defines a projective structure on M.
- To a projective structure on M is associated :
  - A Thomas-Whitehead connection $\tilde{\nabla}$ on $\tilde{M}$
  - A Cartan connection ω on a Cartan bundle $P \to M$.

Quantizations

Definition

A *quantization* on M is a linear bijection $Q_M : \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$ s.t.

$$\sigma(Q_M(S)) = S, \quad \forall S \in \mathcal{S}^k_\delta(M),\ \forall k \in \mathbb{N}.$$

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A Trivial example : The affine quantization map

Take $M = \mathbb{R}^n$.
If $S(x,\xi) = \sum_{|\alpha|=k} C_\alpha(x)\xi^\alpha$, then set

$$Q_{Aff}(S) = \sum_{|\alpha|=k} C_\alpha(x) \circ (\frac{\partial}{\partial x})^\alpha.$$

Q_{Aff} is the so-called *Standard ordering*.

Natural and projectively invariant quantizations

Natural quantizations

A natural quantization is a collection of maps (defined for every manifold M)

$$Q_M : \mathcal{C}_M \times \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$$

such that

- For all ∇ in $\mathcal{C}_M$, $Q_M(\nabla)$ is a quantization,
- If $\phi : M \to N$ is a local diffeomorphism, one has

$$Q_M(\phi^*\nabla)(\phi^*S) = \phi^*(Q_N(\nabla)(S)), \quad \forall\nabla \in \mathcal{C}_N, \forall S \in \mathcal{S}_\delta(N).$$

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Projective invariance

A quantization Q_M is *projectively invariant* if one has $Q_M(\nabla) = Q_M(\nabla')$ whenever ∇ and ∇' are projectively equivalent torsion-free linear connections on M.

Natural and conformally equivariant quantizations

Natural quantizations (conformal case)

In the conformal sense, a *natural quantization* is a collection of quantizations Q_M depending on a pseudo-Riemannian metric such that

- For all pseudo-Riemannian metric g on M, $Q_M(g)$ is a quantization,
- If ϕ is a local diffeomorphism from M to N, then one has

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for all pseudo-Riemannian metrics g on N, and all $S \in \mathcal{S}_\delta(N)$.

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Conformal invariance

A quantization Q_M is *conformally invariant* if one has $Q_M(g) = Q_M(g')$ whenever g and g' are conformally equivalent.

$\mathfrak{sl}(n+1)$-equivariant quantizations

A remark

Suppose $\{Q_M\}$ is natural and projectively invariant. Then $Q_{\mathbb{R}^n}(\nabla_0)$ is a quantization over $\mathbb{R}^n$ and commutes with transformations of $\mathbb{R}^n$ that take ∇_0 into a projectively equivalent connection.
At the infinitesimal level, $Q_{\mathbb{R}^n}(\nabla_0)$ is an $\mathfrak{sl}(n+1,\mathbb{R})$-equivariant quantization : it commutes with the action of

$$\mathfrak{sl}(n+1,\mathbb{R}) = span\{\partial_k, x^j\,\partial_k, x^j\sum_{k=1}^{m} x^k\partial_k\}.$$

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The explicit formula (Duval Ovsienko, 2001)

$$Q_{proj}(T) = Q_{Aff}(\sum_{l=0}^{k} \frac{(\lambda+\frac{k-1}{n+1})\cdots(\lambda+\frac{k-l}{n+1})}{\gamma_{2k-1}\cdots\gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix} Div^l\,T),$$

where $\gamma_r = \frac{n+r-(n+1)\delta}{n+1}$

History of the subject

The flat situation

- Existence and uniqueness of $\mathfrak{sl}(n+1,\mathbb{R})$-equivariant quantization for $\delta = 0$: Lecomte, Ovsienko (1999)
- Existence, explicit formulae for arbitrary δ : Lecomte (2000) and Duval, Ovsienko (2001)
- The conformal case, arbitrary δ : Duval, Lecomte, Ovsienko (2000)
- Other algebras : Boniver, M. (2001,2006)
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Natural quantizations

- Bouarroudj wrote Q_{proj} using connections (2001)
- Lecomte : The exact setting (2001)
- Bordemann : Existence (2002) and Hansoul (2004)
- M. and Radoux : Existence using Cartan connections (2005)

Cartan connections in general

The ingredients

We need a manifold M, a group G, a closed subgoup H s.t. $dim M = dim G/H$ and $Q \to M$: a principal H-bundle.

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The curvature Ω is defined as usual by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

A trivial example

The flat case.

Example

- G : a Lie group
- H : a closed subgroup
- $M = G/H$ and ω : the Maurer Cartan form of G
- The curvature is zero, due to Maurer-Cartan equations.

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Normal Cartan connections

A Cartan connection is normal if its curvature fulfills some trace free conditions.

The projective setting

The group G

The group G is $PGL(n+1,\mathbb{R}) = GL(n+1,\mathbb{R})/\mathbb{R}_0 Id$
It acts on $\mathbb{R}P^n$.

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The group H

The group H is the stabilizer of $[e_{n+1}]$ in $\mathbb{R}P^n$, that is,

$$H = \{\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix} : A \in, \xi \in \mathbb{R}^{n*}, a \neq 0\}/\mathbb{R}_0\mathrm{Id}$$

Therefore $H \cong G_0 \rtimes G_1$, where G_0 is isomorphic to $GL(n,\mathbb{R})$ and G_1 is isomorphic to $\mathbb{R}^{n*}$.

The projective setting

The algebras

We have $\mathfrak{g} \cong \mathfrak{sl}(n+1,\mathbb{R}) \cong \mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R}) \oplus \mathbb{R}^{n*} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.
We have $\mathfrak{h} \cong \mathfrak{g}_0 \oplus \mathfrak{g}_1$

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We have $\mathfrak{h} \cong \mathfrak{g}_0 \oplus \mathfrak{g}_1$

$H \hookrightarrow G^2_n$

The group H acts locally on $\mathbb{R}^n$ by projective transformations that leave the origin fixed. Such a transformation is characterized by its second order jet.
We may consider reductions of P^2M to H.

Results of Kobayashi and Nagano

Theorem (Kobayashi, Nagano (1960))

There is a 1-1 correspondence between projective structures on M and reductions P of P^2M to H.
This association is natural.

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To summarize

There is a natural association $(M, [\nabla]) \longleftrightarrow (P \to M, \omega)$

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For extensions of these results, see Cap, Slovack, Soucek (1997)

Conformal connections

Consider the bilinear symmetric form g of signature $(p+1, q+1)$ on $\mathbb{R}^{n+2}$ $(p+q=n)$ defined by

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & J & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

where

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

(J represents a nondegenerate symmetric bilinear form g_0 on $\mathbb{R}^n$)
The Möbius space is the projection of the light cone associated to g on the projective space $\mathbb{R}P^{n+1}$.

Conformal connections

The group G

The group G is made of linear transformations that leave B invariant, modulo its center, that is,

$$G = \{X \in GL(n+2, \mathbb{R}) : {}^tXSX = S\}/\{\pm \mathrm{Id}\}.$$

It acts transitively on the Möbius space S^n.

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The group H

H is the stabilizer of $[e_{n+2}]$ of the Möbius space :

$$H = \{\begin{pmatrix} a^{-1} & 0 & 0 \\ a^{-1}A\xi^\flat & A & 0 \\ \frac{1}{2a}|\xi|^2 & \xi & a \end{pmatrix} : A \in O(p,q), a \in \mathbb{R}_0, \xi \in \mathbb{R}^{n*}\}/\{\pm \mathrm{Id}\}.$$

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The results about the algebras, P and ω are exactly the same.

Invariant quantization and Cartan connections

Joint work with F. Radoux (Letters in mathematical physics, 2005)

Lift of equivariant functions

There is a projection $p : P \to P^1M$. Let (V, ρ) be a rep. of $GL(n, \mathbb{R})$. If $f \in C^\infty(P^1M, V)_{GL(n,\mathbb{R})}$, then $p^*f \in C^\infty(P, V)_H$, where the action of H is given by

$$\rho'(\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix}) = \rho(\frac{A}{a})$$

(the part G_1 does not act).

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A consequence

The map p^* is a 1-1 correspondence. Hence we can lift densities and symbols to H-equivariant functions on P.

Invariant differentiation and Q_ω

The definition (Ehresman, Cap, Slovack, Soucek)

Let (V, ρ) be a representation of H. If $f \in C^\infty(P, V)$, then the invariant differential of f with respect to ω is the function $\nabla^\omega f \in C^\infty(P, \mathbb{R}^{n*} \otimes V)$ defined by

$$\nabla^\omega f(u)(X) = L_{\omega^{-1}(X)} f(u) \quad \forall u \in P, \quad \forall X \in \mathbb{R}^n \subset \mathfrak{g}.$$

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The iterated symmetrized version

If $f \in C^\infty(P, V)$ then $(\nabla^\omega)^k f \in C^\infty(P, S^k\mathbb{R}^{n*} \otimes V)$ is defined by

$$(\nabla^\omega)^k f(u)(X_1, \dots, X_k) = \frac{1}{k!} \sum_\nu L_{\omega^{-1}(X_{\nu_1})} \circ \dots \circ L_{\omega^{-1}(X_{\nu_k})} f(u)$$

for $X_1, \dots, X_k \in \mathbb{R}^n$.

Standard ordering associated to ω

The map Q_ω

It associates to $S \in C^\infty(P, S^k\mathbb{R}^n \otimes \Delta^\delta(\mathbb{R}^n))$ and $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^n))$ $((\nabla^\omega)^k f \in C^\infty(P, S^k\mathbb{R}^{n*} \otimes \Delta^\lambda(\mathbb{R}^n)))$ the function

$$Q_\omega(S)(f) := \langle S, (\nabla^\omega)^k f\rangle.$$

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The problem

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The solution

Measure the default to G_1-equivariance and add lower degree correcting terms to S.

Default of equivariance and divergence

The G_0 and G_1 equivariances

If $f \in \mathcal{C}^\infty(P, \Delta^\lambda\mathbb{R}^n)_H$, then

- $(\nabla^\omega)^k f$ belongs to $\mathcal{C}^\infty(P, S^k\mathbb{R}^{n*} \otimes \Delta^\lambda\mathbb{R}^n)_{G_0}$,
- there holds

$$L_{h^*}(\nabla^\omega)^k f = -k((n+1)\lambda + k - 1)((\nabla^\omega)^{k-1} f \vee h),$$

for every $h \in \mathbb{R}^{n*} \cong \mathfrak{g}_1$.

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The divergence operator

We fix dual bases (e_j) in $\mathbb{R}^n$ and (ϵ^j) in $\mathbb{R}^{n*}$ and

$$div^\omega : C^\infty(P, S^k_\delta(\mathbb{R}^n)) \to C^\infty(P, S^{k-1}_\delta(\mathbb{R}^n)) : S \mapsto \sum_{j=1}^n i(\epsilon^j) \nabla^\omega_{e_j} S,$$

Results

Lie derivative of divergence

For every $S \in C^\infty(P, S^k_\delta(\mathbb{R}^n))_H$,

- the function $(div^\omega)^l S$ belongs to $C^\infty(P, S^{k-l}_\delta(\mathbb{R}^n))_{G_0}$,
- there holds, for $h \in \mathbb{R}^{n*} \cong \mathfrak{g}_1$ and $\gamma_r = \frac{n+r-(n+1)\delta}{n+1}$

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The formula

If δ is not critical, $Q_M : \mathcal{C}_M \times \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$:

$$Q_M(\nabla, S) = {p^*}^{-1} \circ Q_\omega(\sum_{l=0}^{k} C_{k,l} div^{\omega^l} p^* S) \circ p^*, \quad \forall S \in \mathcal{S}^k_\delta(M)$$

defines a projectively invariant natural quantization.

Further results

F. Radoux : Explicit formulae on the base manifold M(Lett. Math Phys. 2007)

F. Radoux : Analysis of the uniqueness (Submitted)

Arbitrary tensors : the material

Joint work with F.Radoux (London Math Soc 2007), see also S. Hansoul.

"Tensors"

Let (V, ρ_D) be the representation of $GL(n, \mathbb{R})$ corresponding to a Young diagram Y_D of depth $m < n$. Fix $\lambda \in \mathbb{R}$ and $z \in \mathbb{Z}$ and set

$$\rho(A)u = |det(A)|^{\lambda}(det(A))^{z}\rho_D(A)u, \quad \forall A \in GL(n, \mathbb{R}), u \in V.$$

Also set

$$V(M) = P^1M \times_\rho V \quad \text{and} \quad \mathcal{V}(M) = C^\infty(P^1M, V)_{GL(n,\mathbb{R})}$$

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Differential operators

$\mathcal{D}(\mathcal{V}_1(M), \mathcal{V}_2(M))$ (or simply by $\mathcal{D}(M)$) is the space of linear differential operators from $\mathcal{V}_1(M)$ to $\mathcal{V}_2(M)$.

Symbols and quantizations

Symbols

We set $S^l_{V_1,V_2} = S^l\mathbb{R}^n \otimes V_1^* \otimes V_2$ and

$$S^l_{V_1,V_2}(M) \to M := P^1M \times_\rho S^l_{V_1,V_2} \to M.$$

Symbols of degree l belong to $\mathcal{S}^l_{V_1,V_2}(M) = C^\infty(P^1M, S^l_{V_1,V_2})_{GL(n,\mathbb{R})}$ and

$$\mathcal{S}(M) = \bigoplus_{l=0}^{\infty} \mathcal{S}^l_{V_1,V_2}(M).$$

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Natural and projectively invariant quantizations

It is a collection of maps

$$Q_M : \mathcal{C}(M) \times \mathcal{S}(M) \to \mathcal{D}(\mathcal{V}_1(M), \mathcal{V}_2(M))$$

s.t. for every ∇, $Q_M(\nabla)$ is a quantization + diffeo + invariance

$\mathfrak{sl}(n+1,\mathbb{R})$- equivariant quantizations

We look at $M = \mathbb{R}^n$.

Identifications

$$\begin{array}{rcl} \mathcal{V}_1(\mathbb{R}^n) & \cong & C^\infty(\mathbb{R}^n, V_1), \\ \mathcal{S}^k_{V_1,V_2}(\mathbb{R}^n) & \cong & C^\infty(\mathbb{R}^n, S^k_{V_1,V_2}). \end{array}$$

$$(L_X S)(x) = X.S(x) - \rho_*(D_x X)S(x) \quad X \in Vect(\mathbb{R}^n),\ S \in C^\infty(\mathbb{R}^n, S^k_{V_1,V_2})$$

$$\mathcal{L}_X D = [L_X, D] \quad \forall D \in \mathcal{D}(\mathcal{V}_1(\mathbb{R}^n), \mathcal{V}_2(\mathbb{R}^n))$$

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The map $Q_{\mathbb{R}^n}(\nabla_0)$: (if it exists)

- It is quantization over $\mathbb{R}^n$
- It is projectively equivariant i.e.

$$Q_{\mathbb{R}^n}(\nabla_0) \circ L_X = \mathcal{L}_X \circ Q_{\mathbb{R}^n}(\nabla_0) \quad \forall X \in \mathfrak{sl}(n+1,\mathbb{R}).$$

The algebra $\mathfrak{sl}(n+1,\mathbb{R})$

The decomposition

It is isomorphic to the sum of 3 subalgebras

$$\mathfrak{gl}(n+1,\mathbb{R})/\mathbb{R}\mathrm{Id} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 : \left[\begin{pmatrix} A & v \\ \xi & a \end{pmatrix}\right] \mapsto (v, A - a\,Id, \xi),$$

where $A \in gl(n,\mathbb{R})$, $a \in \mathbb{R}$, $v \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{n*}$.

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Vector fields

$$\begin{cases} X^h_x = -h & \text{if } h \in \mathfrak{g}_{-1} \\ X^h_x = -[h,x] & \text{if } h \in \mathfrak{g}_0 \\ X^h_x = -\frac{1}{2}[[h,x],x] & \text{if } h \in \mathfrak{g}_1 \end{cases},$$

where $x \in \mathfrak{g}_{-1} \cong \mathbb{R}^n$.

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathbb{R}\mathcal{E}$$

where $\mathfrak{h}_0 \cong sl(n,\mathbb{R})$ and $ad(\mathcal{E})|_{\mathfrak{g}_{-1}} = -Id$.

The method

- Use Q_{Aff} to pull the problem back on symbols

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- Analyse the eigenvector problem for Casimir operators C and $\mathcal{C}$ associated to symbols and to differential operators : Associate to each eigenvector of the first a single eigenvector of the latter.

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Casimir operators

If $\mathfrak{g}$ is semi-simple finite dim. Lie alg., each basis (u_i) has a Killing-dual basis (u_i^*). If (V, ρ) is a rep then the Casimir operator is given by

$$\sum_i \rho(u_i) \circ \rho(u_i^*).$$

The map Q_{aff}

The affine quantization map

If $S(x,\xi) = \sum_{|\alpha|=l} C_\alpha(x)\xi^\alpha$, where α is a multi-index, $\xi \in \mathbb{R}^{n^*}$, and $C_\alpha(x) \in V_1^* \otimes V_2$, then

$$Q_{Aff}(S) = \sum_{|\alpha|=l} C_\alpha(x) \circ (\frac{\partial}{\partial x})^\alpha.$$

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Lie derivative $\mathcal{L}$ on symbols

Define

$$\mathcal{L}_X : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) : S \mapsto Q_{Aff}^{-1} \circ \mathcal{L}_X \circ Q_{Aff}(S)$$

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We seek for an equivariant map from $(\mathcal{S}(\mathbb{R}^n), L)$ to $(\mathcal{S}(\mathbb{R}^n), \mathcal{L})$.

Properties of C

Main property

The operator C is semisimple.

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As a representation of $\mathfrak{h}_0 \cong sl(n,\mathbb{R})$, we have

$$S^l_{V_1,V_2} = \oplus_{s=1}^{n_l} I_{l,s}.$$

We set $E_{l,s} = C^\infty(\mathbb{R}^n, I_{l,s})$ and define $\delta = \frac{1}{n}(a_1 - a_2)$ if $\rho_*(Id)|_{V_i} = a_i\, Id$.

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Theorem

The restriction of C to $E_{l,s}$ is equal to $\alpha_{l,s} Id_{E_{l,s}}$ where

$$\alpha_{l,s} = \frac{1}{2n}(n\delta - l)(n(\delta - 1) - l) + \frac{n}{n+1}(\mu_{I_{l,s}}, \mu_{I_{l,s}} + 2\rho_S).$$

Properties of $\mathcal{C}$

The map γ

Define

$$\gamma : \mathfrak{g} \to gl(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)) : h \mapsto \gamma(h) = \mathcal{L}_{X^h} - L_{X^h}$$

It vanishes on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ and for $h \in \mathfrak{g}_1$, $\gamma(h)$ is a diff op. of order 0 that maps $\mathcal{S}^k(\mathbb{R}^n)$ into $\mathcal{S}^{k-1}(\mathbb{R}^n)$.

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Find Q such that $(L_{X^h} + \gamma(h)) \circ Q = Q \circ L_{X^h}$.

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The relation

- We fix a basis $(e_j) \in \mathfrak{g}_{-1}$ and denote by (ϵ^j) the Killing dual basis in $\mathfrak{g}_1$.
- We set $N = 2\sum_j \gamma(\epsilon^j) L_{X^{e_j}}$.
- We hav $\mathcal{C} = C + N$.

The eigenvector problem

Theorem

If (V_1, V_2) is not critical, for every $S \in E_{l,s}$ there exists a unique eigenvector $\hat{S}$ of $\mathcal{C}$ with eigenvalue $\alpha_{l,s}$ such that

$$\begin{cases} \hat{S} = S_l + S_{l-1} + \cdots + S_0, \quad S_l = S \\ S_r \in \mathcal{T}^{l-r}(E_{l,s}) \quad \text{for all } r \leq l-1. \end{cases}$$

Indeed these conditions write

$$\begin{cases} C(S) = \alpha_{l,s} S \\ (C - \alpha_{l,s}\mathrm{Id}))S_{l-r} = -N(S_{l-r+1}) \end{cases}$$

In this case, the following map will do the job

$$Q|_{E_{l,s}}(S) = \hat{S}.$$

Main points of the proof

- Uniqueness of the association between eigenvectors.
- $(L_{X^h} + \gamma(h)) \circ Q$ and $Q \circ L_{X^h}$ are eigenvectors of $\mathcal{C}$ with the same leading term because of relations

$$\left\{ \begin{array}{lcl} [L_{X^h} + \gamma(h), \mathcal{C}] & = & 0 \\ [L_{X^h}, C] & = & 0 \end{array} \right.$$

The curved situation

Ingredients and method

- Ingredients : $(M, [\nabla]) \longleftrightarrow (P, G, H, \omega)$,
- We lift tensors and symbols to H-equivariant functions on P,
- We use the map Q_ω to turn symbols into differential operators,
- We use Casimir-like operators to produce correcting terms.

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Remind Q_ω

If $S = tA \otimes h_1 \vee \cdots \vee h_k$ for $t \in C^\infty(P)$, $A \in V_1^* \otimes V_2$ and $h_1, \cdots, h_k \in \mathbb{R}^m \cong \mathfrak{g}_{-1}$ then one has

$$Q_\omega(S)f = \frac{1}{k!} \sum_\nu tA \circ L_{\omega^{-1}(h_{\nu_1})} \circ \cdots \circ L_{\omega^{-1}(h_{\nu_k})} f,$$

Key Results

Theorem

A nice formula The relation

$$L_{h^*} Q_\omega(S)(f) - Q_\omega(S)(L_{h^*} f) = Q_\omega((L_{h^*} + \gamma(h))S)(f)$$

holds for all $f \in C^\infty(P, V_1)_{G_0}$, $h \in \mathfrak{g}_1$, *and* $T \in C^\infty(P, S^k_{V_1,V_2})$.

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Idea

Find a map Q such that

$$(L_{h^*} + \gamma(h))Q(S) = Q(L_{h^*}S),$$

for every $h \in \mathfrak{g}_1$ and every $S \in C^\infty(P, S_{V_1,V_2})_{G_0}$.
Then $Q_\omega \circ Q$ will be a solution of the problem.

Casimir-like operators

We set

$$N^\omega = -2\sum_i \gamma(\epsilon^i) L_{\omega^{-1}(e_i)}.$$

$$\left\{ \begin{array}{lcl} C^\omega(S) & = & \alpha_{k,s} S \\ \mathcal{C}^\omega(S) & = & C^\omega(S) + N^\omega(S), \end{array} \right. \quad \forall S \in C^\infty(P, I_{k,s})$$

Theorem

For every $h \in \mathfrak{g}_1$, one has

$$[L_{h^*} + \gamma(h), \mathcal{C}^\omega] = 0$$

on $C^\infty(P, S^k_{V_1,V_2})_{G_0}$.

So the eigenvector problem is the same as in $\mathbb{R}^n$, and the association defines a map Q

Conformal situation

- The nice formula is OK for densities up to the order three,
- For symbols of degree 4, there are additional terms, but we could manage them by hands,
- The general solution is not known.

Koniec

Thanks for your attention