NEW GEOMETRIC APPROACHES IN THE STUDY OF ERMAKOV SYSTEMS

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Superposition rules & Lie systems

Given a first-order differential equation in $\mathbb{R}^n$ given by:

$$\frac{dx^i}{dt} = Y^i(t,x)\,, \qquad i = 1,\ldots,n\,, \quad (1)$$

a superposition rule for this differential equation is certain map $\Phi : \mathbb{R}^{n(m+1)} \longrightarrow \mathbb{R}^n$, i.e.

$$x = \Phi(x_{(1)}, ..., x_{(m)}; k_1, ..., k_n).$$

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This superposition map verifies that the general solution can be written, at least for sufficiently small t, as

$$x(t) = \Phi(x_{(1)}(t),...,x_{(m)}(t);k_1,...,k_n),$$

with $\{x_{(a)}(t)|\, a = 1,...,m\}$ being a fundamental set of particular solutions of the system and $k = (k_1,...,k_n)$ a set of n arbitrary constants associated with each particular solution.

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- The differential equations (1) which admit this superposition rule are called Lie systems.
- Lie systems are characterized by the Lie Theorem.

Lie's Theorem: *(1) admits a superposition rule if and only if the t-dependent vector field $Y(t,x)$ can be written*

$$Y(t,x)=\sum_{\alpha=1}^{r} b_\alpha(t)\,X_\alpha(x)$$

where the vector fields $\{X_\alpha,\,\alpha=1,\ldots,r\}$, close a r-dimensional real Lie algebra V, i.e. there exist r^3 real numbers $c_{\alpha\beta}{}^{\gamma}$ such that

$$[X_\alpha,X_\beta]=\sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma}\,X_\gamma\,,\qquad \forall\alpha,\beta=1,\ldots,r\,.$$

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Consider the abstract Lie algebra $\mathfrak{g}$ isomorphic to V and a connected Lie group G with this Lie algebra. Then, there exists an effective action $\Phi : G\times\mathbb{R}^n \to \mathbb{R}^n$ with fundamental vector fields those of V. Let $\{\mathrm{a}_\alpha\}$ be a basis of $\mathfrak{g}$ with associated fundamental vector fields $\{X_\alpha\}$ thus if the solution of the equation

$$R_{g^{-1}*g}\dot{g} = -\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{a}_\alpha, \quad g(0) = e.$$

verifies that the integral curve of Y in $x_0 \in \mathbb{R}^n$ is $x(t) = \Phi(g(t), x_0)$.

Consider now a set of t-dependent vector fields $\{Y^{(\mu)} \mid \mu = 1, \ldots, m\}$ over a set of manifolds $N^{(\mu)}$ such that

$$Y^{(\mu)}(t, x^{(\mu)}) = \sum_{\alpha=1}^{r} b_\alpha(t) Y_\alpha^{(\mu)}(x^\mu)$$

where the $Y_\alpha^{(\mu)}$ close the same commutation relations that the Y_α.

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Thus, we can consider the integral curves of the t-dependent vector field

$$\tilde{Y}(t, \tilde{x}) = Y(t, x) + \sum_{\mu=1}^{m} Y^{(\mu)}(t, x^{(\mu)}) = \sum_{\alpha=1}^{r} b_\alpha(t) \left(Y_\alpha(x) + \sum_{\mu=1}^{m} Y_\alpha^\mu(x^\mu) \right)$$

with $\tilde{x} \in N \times N^{(1)} \times \ldots \times N^{(m)}$ and

$$\tilde{Y}_\alpha(\tilde{x}) = Y_\alpha + \sum_{\mu=1}^{m} Y_\alpha^{(\mu)}(x^{(\mu)})$$

As for $\mu, \mu' = 1, \ldots, m$

$$[Y_\alpha^{(\mu)}, Y_\beta^{(\mu)}] = c_{\alpha\beta}{}^\gamma Y_\gamma^{(\mu)}$$

and

$$[Y_\alpha^{(\mu)}, Y_\beta^{(\mu')}] = 0 \quad [Y_\alpha, Y_\beta^{(\mu')}] = 0$$

then

$$[\tilde{Y}_\alpha, \tilde{Y}_\beta] = c_{\alpha\beta}{}^\gamma \tilde{Y}_\gamma$$

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Also, at any point $\tilde{x} \in \tilde{N}$ we obtain

$$\tilde{Y}(t, \tilde{x}) \in \mathcal{V}(\tilde{x}) \equiv \langle \tilde{Y}_1(\tilde{x}), \ldots, \tilde{Y}_r(\tilde{x}) \rangle$$

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2. Thus, let p be the higher rank of the distribution $\mathcal{V}$ over an open of $\tilde{N}$ and $\tilde{n} \equiv \dim \tilde{N}$ we known that there exists $\tilde{n} - p$ time-independent integrals of motion $\{k_1, \ldots k_{\tilde{n}-p}\}$ of $\tilde{Y}$.

Consider the map $\Psi : \tilde{x} \in \tilde{N} \to (k_1(\tilde{x}), \ldots, k_{\tilde{n}-p}(\tilde{x})) \in \mathbb{R}^{\tilde{n}-p}$ such that

$$\Psi(x(t), x^{(1)}(t), \ldots, x^{(m)}(t)) = (k_1, \ldots, k_{\tilde{n}-p})$$

The constants of motion stablish relations between the coordinates of the integral curves of $\tilde{Y}$, these are

$$(x(t), x^{(1)}(t), \ldots, x^{(m)}(t)).$$

Thus $\{k_1, \ldots, k_{\tilde{n}-p}\}$ allow to relate the integral curves of Y and the $Y^{(\mu)}$.

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Suppose that we obtain as many integrals of motion (we must have at least n) as to obtain the initial n coordinates in terms of the other coordinates $\{x^{(1)}, \ldots, x^{(m)}\}$ and a certain set of constants $\{k_1, \ldots, k_n\}$. Also, if we fix the coordinates in $N^{(e)} \equiv N^{(1)} \times \ldots \times N^{(m)}$ there is a diffeomorphism

$$\Psi(x^{(1)}, \ldots, x^{(m)}) : x \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n \rightarrow k \equiv (k_1, \ldots, k_n) \in \mathbb{R}^n$$

Thus, we can obtain a map Φ such that

$$\Phi(x^{(1)}(t), \ldots, x^{(m)}(t), k_1, \ldots, k_n) = x(t)$$

Thus, we have generalized a superposition rule to mix different solutions of different Lie systems.

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The level sets of Ψ corresponding to regular values define a n-codimensional foliation $\mathcal{F}$ on an open dense subset $U \subset \tilde{N}$ and the family $\{\tilde{Y}(t), t \in \mathbb{R}\}$ of vector fields in $\tilde{N}$ consists of vector fields tangent to the leaves of $\mathcal{F}$.

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As the level sets $\mathcal{F}_k$ corresponding to $k\in\mathbb{R}^n$, given $(x_{(1)},\ldots,x_{(m)})\in N^{(e)}$ there is a unique point $x_{(0)}$ such that $(x_{(0)},x_{(1)},\ldots,x_{(m)})\in\mathcal{F}_k$. Therefore the projection

$$pr:(x_{(0)},x_{(1)},...,x_{(m)})\in\tilde{N}\to(x_{(1)},\ldots,x_{(m)})\in N^{(e)}$$

induces diffeomorphisms on the leaves $\mathcal{F}_k$ of $\mathcal{F}$.

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SODE Lie systems

A system of second-order differential equations

$$\ddot{x}^i = f^i(t, x, \dot{x}) \,, \qquad i = 1, \ldots, n,$$

can be studied through the system of first-order differential equations

$$\left\{ \begin{array}{rcl} \dfrac{dx^i}{dt} & = & v^i \\ \dfrac{dv^i}{dt} & = & f^i(t, x, v) \end{array} \right.$$

with associated t-dependent vector field

$$X = v^i \frac{\partial}{\partial x^i} + f^i(t, x, v) \frac{\partial}{\partial v^i} \,.$$

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We call SODE Lie systems those for which X is a Lie system, i.e. it can be written as a linear combination with t-dependent coefficients of vector fields closing a finite-dimensional real Lie algebra.

EXAMPLES

A) 1-dim harmonic time-dependent frequency oscillator

The equation of motion is $\ddot{x} = -\omega^2(t)x$, with associated system

$$\begin{cases} \dot{x} &= v \\ \dot{v} &= -\omega^2(t)x \end{cases}$$

and t-dependent vector field

$$X = v\frac{\partial}{\partial x} - \omega^2(t)x\,\frac{\partial}{\partial v}\ ,$$

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$$X = v\frac{\partial}{\partial x} - \omega^2(t)x\,\frac{\partial}{\partial v}\ ,$$

This last vector field is a linear combination $X = X_2 - \omega^2(t)X_1$ with

$$X_1 = x\frac{\partial}{\partial v}\ , \qquad X_2 = v\,\frac{\partial}{\partial x}\ , \qquad X_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right)\ .$$

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As

$$[X_1, X_2] = 2\,X_3\ , \quad [X_1, X_3] = -X_1\ , \quad [X_2, X_3] = X_2\ ,$$

X defines a Lie system with associated Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

B) Pinney equation

The Pinney equation is the second-order non-linear differential equation:

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The corresponding system of first-order differential equations is

$$\left\{ \begin{array}{rcl} \dot{x} & = & v \\ \dot{v} & = & -\omega^2(t)x + \frac{k}{x^3} \end{array} \right.$$

and the associated t-dependent vector field

$$X = v\frac{\partial}{\partial x} + \left(-\omega^2(t)x + \frac{k}{x^3}\right)\frac{\partial}{\partial v} \,.$$

This is a Lie system because it can be determined by a time-dependent vector field which can be written as

$$X = X_2 - \omega^2(t) X_1 \,,$$

where

$$X_1 = x\frac{\partial}{\partial v}, \qquad X_2 = \frac{k}{x^3}\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}, \qquad X_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right) \,.$$

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As these vector fields are such that

$$[X_1, X_2] = 2X_3, \quad [X_3, X_2] = -X_2, \quad [X_3, X_1] = X_1$$

then the vector fields X_i with $i = 1, 2, 3$ span a three-dimensional real Lie algebra $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

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Also, the X_i in examples A and B close under commutator relations the same structure constants and their X's can be written in the same way as

$$X = X_2 - \omega^2(t) X_1$$

C) Ermakov system

Consider the system

$$\begin{cases} \dot{x} &= v_x \\ \dot{v}_x &= -\omega^2(t)x \\ \dot{y} &= v_y \\ \dot{v}_y &= -\omega^2(t)y + \frac{1}{y^3} \end{cases}$$

with associated t-dependent vector field

$$L = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} - \omega^2(t)x \frac{\partial}{\partial v_x} + \left(-\omega^2(t)y + \frac{1}{y^3}\right) \frac{\partial}{\partial v_y} \,,$$

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This is a linear combination with time-dependent coefficients, $L = L_2 - \omega^2(t) L_1$, of the vector fields

$$L_1 = x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial v_y} \,, \qquad L_2 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \frac{1}{y^3} \frac{\partial}{\partial v_y} \,.$$

that with L_3 given by

$$L_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y}\right) .$$

close on a $\mathfrak{sl}(2,\mathbb{R})$ algebra with the field L expressed in terms of the vector fields L_1, L_2 and L_3 as before.

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If we get enough integrals of motion and we get to obtain the coordinates of one of the system in terms of the other we have obtained a superposition principle.

In this case there is just one integral and we cannot obtain a superposition rule. Nevertheless, this integral is the knowm Lewis-Ermakov invariant.

$$\psi(x,y,v_x,v_y) = \left(\frac{x}{y}\right)^2 + \xi^2 = \left(\frac{x}{y}\right)^2 + (xv_y - yv_x)^2 ,$$

Pinney equation revisited

Consider the system of first-order differential equations:

$$\begin{cases} \dot{x} &=& v_x \\ \dot{y} &=& v_y \\ \dot{z} &=& v_z \\ \dot{v}_x &=& -\omega^2(t)x \\ \dot{v}_y &=& -\omega^2(t)y + \frac{k}{y^3} \\ \dot{v}_z &=& -\omega^2(t)z \end{cases}$$

which corresponds to the vector field

$$X = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + \frac{k}{y^3}\frac{\partial}{\partial v_y} - \omega^2(t)\left(x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial v_z}\right)$$

PINNEY EQUATION REVISITED

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which corresponds to the vector field

$$X = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} + \frac{k}{y^3}\frac{\partial}{\partial v_y} - \omega^2(t)\left(x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial v_z}\right)$$

The vector field X can be expressed as $X = N_2 - \omega^2(t)N_1$ where the vector fields N_1 and N_2 are:

$$N_1 = y\frac{\partial}{\partial v_y} + x\frac{\partial}{\partial v_x} + z\frac{\partial}{\partial v_z}, \quad N_2 = v_y\frac{\partial}{\partial y} + \frac{1}{y^3}\frac{\partial}{\partial v_y} + v_x\frac{\partial}{\partial x} + v_z\frac{\partial}{\partial z},$$

These vector fields generate a three-dimensinal real Lie algebra with the vector field N_3 given by

$$N_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial z} - v_z\frac{\partial}{\partial v_z}\right) .$$

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In terms of these three integrals we can obtain an explicit expression of y in terms of x, z and the integrals I_1, I_2, W:

$$y = \frac{\sqrt{2}}{W}\left(I_2 x^2 + I_1 z^2 \pm \sqrt{4I_1 I_2 - cW^2}\,xz\right)^{1/2}$$

This can be interpreted as saying that there is a superposition rule allowing us to express the general solution of the Pinney equation in terms of two independent solutions of the corresponding harmonic oscillator with time-dependent frequency

QUASI LIE SYSTEMS

Consider a non-autonomous system of first-order differential equations describing the integral curves of a t-dependent vector field

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in a manifold N where $X(t,x)$ can be written as

$$X(t,x) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x)$$

We can associate with this differential equation the $\mathbb{R}$-linear space V of linear combinations of the vector fields X_α:

$$V \equiv \{X \,|\, X = \sum_{\alpha=1}^{r} \lambda_\alpha X_\alpha,\ \lambda_\alpha \in \mathbb{R}\}$$

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Quasi-Lie system if there exists a Lie algebra $W \subset V$ **such that** $[W,V] \subset V$

- G a connected Lie group with Lie algebra isomorphic to that of W
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Let X be a t-dependent vector field with integral curves $x(t)$ we define the transformed t-dependent vector field X' by $\bar{g} \in \mathcal{G}$ as that with integral curves $x'(t) = \Phi(\bar{g}(t), x(t))$.

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The map

$$\Psi : (\bar{g}, X) \in \mathcal{G} \times \mathcal{M} \to b' = \Psi(\bar{g}, X') \in \mathcal{M}$$

is an action of the group $\mathcal{G}$ in $\mathcal{M}$.

Consider the family of differential equations

$$\ddot{x} = a(t)\dot{x} + b(t)x + c(t)\frac{1}{x^3}\,.$$

We associate with such a second-order differential equation a system of first-order differential equations by introducing a new variable $v \equiv \dot{x}$:

$$\begin{cases} \dot{v} &= a(t)v + b(t)x + c(t)\frac{1}{x^3} \\ \dot{x} &= v. \end{cases}$$

The vector fields

$$X_1 = v\frac{\partial}{\partial v}, \quad X_2 = x\frac{\partial}{\partial v}, \quad X_3 = \frac{1}{x^3}\frac{\partial}{\partial v}, \quad X_4 = v\frac{\partial}{\partial x}, \quad X_5 = x\frac{\partial}{\partial x}$$

are a basis for a $\mathbb{R}$-linear space V, which however **is not a Lie algebra because the commutator** $[X_3, X_4]$ **is not in** V.

If $p \equiv (x, v)$ denotes a point of $T\mathbb{R}$ our differential equation gives the integral curves of

$$X(p,t) = a(t)X_1(p) + b(t)X_2(p) + c(t)X_3(p) + X_4(p)\,.$$

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Consider the two-dimensional Lie algebra W generated by the vector fields

$$Y_1 = X_1 = v\frac{\partial}{\partial v}, \qquad Y_2 = X_2 = x\frac{\partial}{\partial v}\,,$$

which satisfy

$$[Y_1, Y_2] = -Y_2\,,$$

and therefore W is a 2-dimensional non-Abelian Lie algebra. The other commutation relations among elements of W and V are determined by:

$$\begin{array}{lll} [Y_1, X_3] = X_3, & [Y_1, X_4] = X_4, & [Y_1, X_5] = 0 \\ [Y_2, X_3] = 0, & [Y_2, X_4] = X_5 - X_1, & [Y_2, X_5] = -X_2 \end{array}$$

and thus $[W, V] \subset V$, which shows that X **is a Quasi-Lie system**.

The corresponding set of transformations of $T\mathbb{R}$ associated with W is given by:

$$\begin{cases} v = \alpha(t)v' + \beta(t)x' \\ x = x' \end{cases}$$

with $\alpha(t) \neq 0$. These transformations allow us to transform the Quasi-Lie system (20) into a new system of first-order differential equations in which the time-dependent vector field determining the dynamics can be written as a linear combination of the fields of V at each time. More explicitly, if $p' = (x', v')$ then

$$X'(p', t) = a'(t)X_1(p') + b'(t)X_2(p') + c'(t)X_3(p') + d'(t)X_4(p') + e'(t)X_5(p') \quad (0.1)$$

with

$$a'(t) = a(t) - \beta(t) - \frac{\dot{\alpha}(t)}{\alpha(t)} \qquad c'(t) = \frac{c(t)}{\alpha(t)}$$

$$b'(t) = \frac{b(t)}{\alpha(t)} + a(t)\frac{\beta(t)}{\alpha(t)} - \frac{\beta^2(t)}{\alpha(t)} - \frac{\dot{\beta}(t)}{\alpha(t)} \qquad d'(t) = \alpha(t)$$

$$e'(t) = \beta(t).$$

The integral curves of X' are the integral curves of:

$$
\begin{aligned}
\frac{dv'}{dt} &= \left(a(t) - \beta(t) - \frac{\dot\alpha(t)}{\alpha(t)}\right) v' + \left(\frac{b(t)}{\alpha(t)} + a(t)\frac{\beta(t)}{\alpha(t)} - \frac{\beta^2(t)}{\alpha(t)} - \frac{\dot\beta(t)}{\alpha(t)}\right) x' \\
&+ \frac{c(t)}{\alpha(t)}\frac{1}{x'^3} \\
\frac{dx'}{dt} &= \alpha(t)v' + \beta(t)x'
\end{aligned}
$$

In the most general case, once the coefficient $a(t)$ has been fixed, the coefficient $c(t)$ of a system that can be reduced to a Lie-Ermakov system:

$$
c(t) = K \exp\left(-\int^t 2a(t')dt'\right)
$$

and thus, we can we write

$$
a(t) = -\frac{p_1(t)}{p(t)}
$$

Then the most general differential equation of our type we can describe as a Quasi-Lie system is

$$p(t)\ddot{x}+p_1(t)\dot{x}+q(t)r=\frac{p(t)}{\exp(2F(t))}\frac{k^2}{x^3}$$

where

$$F(t)=\int^t\frac{p_1(t')}{p(t')}dt'$$

and we recover in this way a Lie systems. In this case, we can choose:

$$\alpha(t)=\exp(-F(t))$$

and from it we obtain

$$\begin{aligned}\frac{dv'}{dt}&=-q(t)\exp(F(t))x'+\exp(-F(t))\frac{k^2}{x'^3}\\ \frac{dx'}{dt}&=\exp(-F(t))v'\end{aligned}$$

Through the t-reparametrization

$$\tau = \int^t \exp(-F(t'))dt'$$

becomes:

$$\frac{dv'}{d\tau} = -\frac{q(t)}{p(t)} \exp(2F(t))x' + \frac{k^2}{x'^3}$$
$$\frac{dx'}{d\tau} = v'.$$

now we can consider the next integral for this Lie system:

$$I = (\bar{x}v' - \bar{v}x')^2 + k^2 \left(\frac{x'}{\bar{x}}\right)^2$$