NEW DEVELOPMENTS IN HAMILTON–JACOBI THEORY

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PLAN OF THE TALK

1. Classical Hamilton-Jacobi theory (geometric version) and Motivation
2. Geometric Hamilton-Jacobi Theory on almost Lie algebroids
3. An application to nonholonomic mechanical systems

Classical Hamilton-Jacobi theory (geometric version)

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^A)$ (called the principal function) such that

$$\frac{\partial S}{\partial t} + h(q^A, \frac{\partial S}{\partial q^A}) = 0. \qquad (1)$$

If we put $S(t, q^A) = W(q^A) - tE$, where E is a constant, then W satisfies

$$h(q^A, \frac{\partial W}{\partial q^A}) = E; \qquad (2)$$

W **is called the characteristic function.**

Equations (1) and (2) are indistinctly referred as the Hamilton-Jacobi equation.

R. Abraham, J.E. Marsden: *Foundations of Mechanics* **(2nd edition). Benjamin-Cumming, Reading, 1978.**

J.F. Carinena, X. Gracia, G. Marmo, E. Martinez, M. Munoz-Lecanda, N. Román-Roy: Geometric Hamilton-Jacobi theory. Int. J. Geom. Meth. Mod. Phys. 3 (7) (2006), 1417-1458.

Let M be the configuration manifold, and T^*M its cotangent bundle equipped with the canonical symplectic form

$$\omega_M = dq^A \wedge dp_A$$

where (q^A) are coordinates in M and (q^A, p_A) are the induced ones in T^*M.

Let $h : T^*M \longrightarrow \mathbb{R}$ a hamiltonian function and X_h the corresponding hamiltonian vector field:

$$i_{X_h} \omega_M = dh$$

The integral curves of X_h, $(q^A(t), p_A(t))$, satisfy the Hamilton equations:

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}, \; \frac{dp_A}{dt} = -\frac{\partial h}{\partial q^A}$$

Let λ be a closed 1-form on M, say $d\lambda = 0$; (then, locally $\lambda = dW$)

Hamilton-Jacobi Theorem

The following conditions are equivalent:

(i) If $\sigma : I \to M$ satisfies the equation

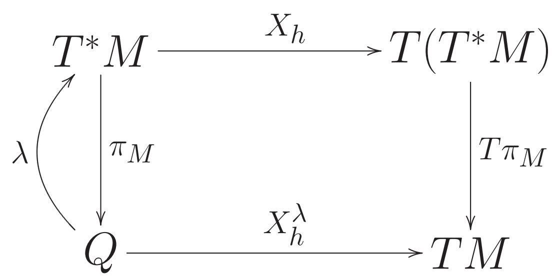
$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(ii) $d(h \circ \lambda) = 0$

Define a vector field on M:

$$X_h^\lambda = T\pi_M \circ X_h \circ \lambda$$



The following conditions are equivalent:

(i) If $\sigma : I \to M$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(i)' If $\sigma : I \to M$ is an integral curve of X_h^λ, then $\lambda \circ \sigma$ is an integral curve of X_h;

(i)" X_h and X_h^λ are λ-related, i.e.

$$T\lambda(X_h^\lambda) = X_h \circ \lambda$$

Hamilton-Jacobi Theorem

Let λ be a closed 1-form on M. Then the following conditions are equivalent:

(i) X_h^λ **and** X_h **are** λ**-related;**

(ii) $d(h \circ \lambda) = 0$

If

$$\lambda = \lambda_A(q)\, dq^A$$

then the Hamilton-Jacobi equation becomes

$$h(q^A, \lambda_A(q^B)) = const.$$

and we recover the classical formulation when

$$\lambda_A = \frac{\partial W}{\partial q^A}$$

Let $L : TM \longrightarrow \mathbb{R}$ a lagrangian subject to linear constraints given by a distribution D on M. Denote by $\bar{D} \subset T^*M$ the image of $D \subset TM$ by the Legendre transformation, and by h the corresponding hamiltonian function on T^*M. In that case, we have proved the following result:

Hamilton-Jacobi Theorem

Let λ be a 1-form on M taking values into $\bar{D}$ and satisfying $d\lambda \in \mathcal{I}(D^o)$. Then the following conditions are equivalent:

(i) X^{λ}_{nh} **and** X_{nh} **are** λ**-related;**

(ii) $d(h \circ \lambda) \in D^o$

Here, X_{nh} **is the nonholonomic dynamics.**

D. Iglesias, M. de León, D. Martín de Diego: Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems, ***Preprint*** **(2007).**

Basic tools in Classical Hamilton-Jacobi theory

$TM \xrightarrow{\tau_{TM}} M$ $\rightsquigarrow$ **vector bundle over a manifold** M

The canonical symplectic 2-form ω_M **in** $T^*M \simeq$ **The canonical Poisson** 2**-vector** Λ_{T^*M} **on** T^*M $\rightsquigarrow$ **a linear bivector on the dual of the vector bundle.**

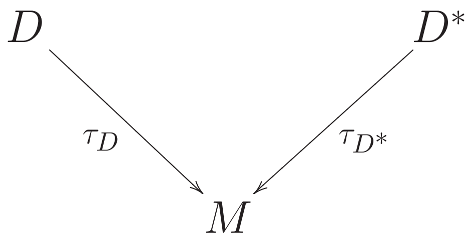
A hamiltonian function $h : T^*M \longrightarrow \mathbb{R}$ $\rightsquigarrow$ **A function** h **defined on the dual of the vector bundle**

A section $\lambda : M \longrightarrow T^*M$ **such that** $d\lambda = 0$ $\rightsquigarrow$ **A section of the dual of the vector bundle which is closed with respect to the "induced differential".**

Geometric Hamilton-Jacobi Theory

Ingredients:

- $\tau_D : D \longrightarrow M$ **a** vector bundle**, and** $\tau_{D^*} : D^* \longrightarrow M$ **its dual vector bundle.**



- **A** linear bivector[2] Λ_{D^*} **on** D^* **(not Jacobi identity is required). We denote by** $\{\ ,\ \}_{D^*}$ **the corresponding almost-Poisson bracket.**

- $h : D^* \longrightarrow \mathbb{R}$ **a** hamiltonian function**.**

This framework was used by J. Grabowski, P. Urbanski and K. Grabowska to develop a general differential calculus on vector bundles and geometric mechanics on Lie algebroids in several recent papers.

[2]linear means that the bracket of two linear functions is a linear function

Λ_{D^*} **is linear**

$\Downarrow$

Proposition 1

We have that:

(a) $\xi_1, \xi_2 \in \Gamma(\tau_D) \Rrightarrow \{\widehat{\xi_1}, \widehat{\xi_2}\}_{D^*}$ **is a linear function on** D^***,**

(b) $\xi \in \Gamma(\tau_D)$**,** $f \in C^\infty(M) \Rrightarrow \{\widehat{\xi}, f \circ \tau_{D^*}\}_{D^*}$ **is a basic function with respect to** τ_{D^*}**,**

(c) $f, g \in C^\infty(M) \Rrightarrow \{f \circ \tau_{D^*}, g \circ \tau_{D^*}\}_{D^*} = 0$

$\Downarrow$

Given local coordinates (x^μ) in the base manifold M and a local basis of sections of D, $\{e_\alpha\}$, we induce local coordinates (x^μ, y_α) on D^* and the bivector Λ_{D^*} is written as

$$\Lambda_{D^*} = \rho^\mu_\alpha \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial x^\mu} + \frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}$$

The corresponding Hamiltonian vector field is

$$X_h = \sharp_{\Lambda_{D^*}}(dh)$$

or, in coordinates,

$$X_h = \rho^\mu_\alpha \frac{\partial h}{\partial y_\alpha} \frac{\partial}{\partial x^\mu} - \left(\rho^\mu_\alpha \frac{\partial h}{\partial x^\mu} + C^\gamma_{\alpha\beta} y_\gamma \frac{\partial h}{\partial y_\beta} \right) \frac{\partial}{\partial y_\alpha}$$

Thus, the Hamilton equations are

$$\frac{dx^\mu}{dt} = \rho^\mu_\alpha \frac{\partial h}{\partial y_\alpha}, \quad \frac{dy_\alpha}{dt} = - \left(\rho^\mu_\alpha \frac{\partial h}{\partial x^\mu} + C^\gamma_{\alpha\beta} y_\gamma \frac{\partial h}{\partial y_\beta} \right)$$

Almost Lie algebroid structure on $\tau_D : D \longrightarrow M$

The linear bivector Λ_{D^*} induces the following structure on D:

- **an almost Lie bracket on the space $\Gamma(\tau_D)$**

$$\begin{array}{rcl} [\ ,\ ]_D :\ \Gamma(\tau_D)\times\Gamma(\tau_D) & \longrightarrow & \Gamma(\tau_D) \\ (\xi_1,\xi_2) & \longmapsto & [\xi_1,\xi_2]_D \end{array}$$

where $\widehat{[\xi_1,\xi_2]_D} = \{\widehat{\xi_1},\widehat{\xi_2}\}_{D^*}$ **(**$[e_\alpha, e_\beta]_D = C^\gamma_{\alpha\beta} e_\gamma$**).**

- **an anchor map** $\rho_D : \Gamma(\tau_D) \longrightarrow \mathfrak{X}(M)$

$$f \in C^\infty(M), \xi \in \Gamma(D) \Rrightarrow \rho_D(\xi)(f)\circ\tau_{D^*} = \{\widehat{\xi}, f\circ\tau_{D^*}\}_{D^*}$$

(in coordinates, $\rho_D(e_\alpha) = \rho^\mu_\alpha \frac{\partial}{\partial x^\mu}$**).**

Properties

a) $[\ ,\ ]_D$ **is antisymmetric**

b) $[\xi_1, f\xi_2]_D = f[\xi_1, \xi_2]_D + \rho_D(\xi_1)(f)\xi_2$

In general, $[\ ,\ ]_D$ **does not satisfy the Jacobi identity. In the case when it satisfies the Jacobi identity, then** $(D, [\ ,\ ]_D, \rho_D)$ **is a Lie algebroid.**

The almost differential $d^D : \Gamma(\Lambda^k D^*) \longrightarrow \Gamma(\Lambda^{k+1} D^*)$

Given $\Omega \in \Gamma(\Lambda^k D^*)$ **then** $d^D\Omega \in \Gamma(\Lambda^{k+1} D^*)$ **and**

$$\begin{aligned} d^D\Omega(\xi_0, \xi_1, \dots, \xi_k) &= \sum_{i=0}^{k} (-1)^i \rho_D(\xi_i)(\Omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)) \\ &\quad + \sum_{i<j} \Omega([\xi_i, \xi_j]_D, \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k) \end{aligned}$$

where $\xi_0, \xi_1, \dots, \xi_k \in \Gamma(\tau_D)$

From the definition, we deduce that

$$\begin{aligned} (1)\ (d^D f)(\xi) &= \rho_D(\xi)(f), \quad f \in C^\infty(M), \quad \xi \in \Gamma(\tau_D) \\ (2)\ d^D\sigma(\xi_1, \xi_2) &= \rho_D(\xi_1)(\sigma(\xi_2)) - \rho_D(\xi_2)(\sigma(\xi_1)) - \sigma[\xi_1, \xi_2]_D, \\ &\qquad \sigma \in \Gamma(\tau_{D^*}), \quad \xi_1, \xi_2 \in \Gamma(\tau_D) \\ (3)\ d^D(\Omega \wedge \Omega') &= d^D\Omega \wedge \Omega' + (-1)^k \Omega \wedge d^D\Omega', \quad \Omega \in \Gamma(\Lambda^k D^*), \Omega' \in \Gamma(\Lambda^{k'} D^*) \end{aligned}$$

In general $\boxed{(d^D)^2 \neq 0}$**.**

A linear bivector Λ_{D^*} **on** D^*

$\Downarrow$

An almost Lie algebroid structure $([\ ,\ ]_D, \rho_D)$ **on** D

$\Downarrow$

An almost differential $d^D : \Gamma(\Lambda^k D^*) \longrightarrow \Gamma(\Lambda^{k+1} D^*)$ **satisfying (1) and (2)**

The inverse process also works

An almost differential $d^D\colon \Gamma(\Lambda^k D^*) \to \Gamma(\Lambda^{k+1} D^*)$ **satisfying (1) and (2)**

$$\Downarrow$$

An almost Lie algebroid structure $([\ ,\ ]_D, \rho_D)$ **on** D

$$\rho_D(\xi)(f) = d^D(f)(\xi),$$

$$\omega([\xi,\xi']_D) = -(d^D\omega)(\xi,\xi') + d^D(\omega(\xi'))(\xi) - d^D(\omega(\xi))(\xi')$$

$$\xi, \xi' \in \Gamma(\tau_D), \quad f \in C^\infty(D), \quad \omega \in \Gamma(\tau_{D^*})$$

An almost Lie algebroid structure $([\ ,\ ]_D, \rho_D)$ **on** D

$$\Downarrow$$

A linear bivector Λ_{D^*} **on** D^* **with almost Poisson bracket** $\{\ ,\ \}_{D^*}$

$$\{\widehat{\xi}, \widehat{\xi'}\}_{D^*} = \widehat{[\xi, \xi']_D}, \quad \{\widehat{\xi}, f \circ \tau_{D^*}\}_{D^*} = \rho_D(\xi)(f) \circ \tau_{D^*},$$

$$\{f \circ \tau_{D^*}, f' \circ \tau_{D^*}\}_{D^*} = 0$$

$$f, f' \in C^\infty(M), \quad \xi, \xi' \in \Gamma(\tau_D)$$

In conclusion

A linear bivector Λ_{D^*} **on** D^*

$$\Updownarrow$$

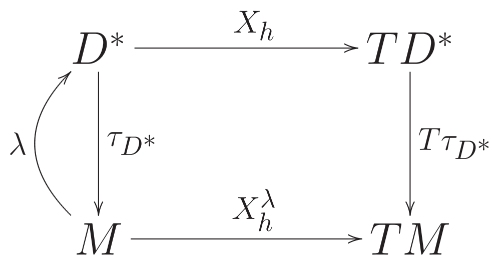
An almost Lie algebroid structure $([\ ,\ ]_D, \rho_D)$**on** D

$$\Updownarrow$$

An almost differential $d^D : \Gamma(\Lambda^k D^*) \to \Gamma(\Lambda^{k+1} D^*)$**satisfying (1) and (2)**

Hamilton-Jacobi Theorem

Let Λ_{D^*} **be a linear bivector on** D **and** $\lambda : M \longrightarrow D^*$ **be a section of** $\tau_{D^*} : D^* \longrightarrow M$



We define $X_h^\lambda = T\tau_{D^*} \circ X_h \circ \lambda$

It is easy to show that $X_h^\lambda(x) \in \rho_D(D_x)$**,** $\forall x \in M$
Indeed, look the local expressions

$$X_h^\lambda = \rho_\alpha^\mu \frac{\partial h}{\partial y_\alpha} \frac{\partial}{\partial x^\mu} = \rho(\frac{\partial h}{\partial y_\alpha} e_\alpha)$$

Hamilton-Jacobi Theorem

Assume that $d^D\lambda = 0$.

(i) $\sigma : I \longrightarrow M$ **integral curve of** $X_h^\lambda \Rightarrow \lambda \circ \sigma$ **integral curve of** X_h

$$\Updownarrow$$

$$(ii) \quad d^D(h \circ \lambda) = 0$$

For the proof, we will need the following preliminary results (Propositions 2 and 3).

Proposition 2

Let $\lambda : M \longrightarrow D^*$ be a section of τ_{D^*}. Then, λ is a 1-cocycle with respect to d^D (i.e. $d^D\lambda = 0$)

$$\Updownarrow$$

for all $x \in M$ the subspace

$$\mathcal{L}_{\lambda,D}(x) = (T_x\lambda)(\rho_D(D_x)) \subseteq T_{\lambda(x)}D^*$$

is Lagrangian with respect to Λ_{D^*}, that is,

$$\sharp_{\Lambda_{D^*}}(\mathcal{L}_{\lambda,D})^o = \mathcal{L}_{\lambda,D}$$

Remark: **Proposition 2 is the generalization of the well-known result for the particular case $D = TM$ and $\Lambda_{D^*} = \Lambda_{T^*M}$:**

> **"Let λ be a 1-form on M. Then, λ is closed if and only if $\lambda(M)$ is a Lagrangian submanifold of T^*M."**

Proposition 3

Let $\lambda : M \longrightarrow D^*$ **be a section of** τ_{D^*} **such that** $d^D\lambda = 0$**. Then**

$$(\ker \sharp_{\Lambda_{D^*}})_{\lambda(x)} \subseteq (\mathcal{L}_{\lambda,D})^o\,, \text{ for all } x \in M$$

Remark: In the particular case when $D = TM$ **this Proposition is trivial since**

$$\ker \sharp_{\Lambda_{T^*M}} = \{0\}$$

((T^*M, ω_M) **is a symplectic manifold).**

Remember that

$$\sharp_{\Lambda_{D^*}} : T^*_{\lambda(x)} D^* \longrightarrow T_{\lambda(x)} D^*$$

Proof of the Theorem

Let $\lambda : M \longrightarrow D^*$ be a section such that $d^D\lambda = 0$.

(i) $\Rightarrow$ (ii)

We assume that the integral curves of X_h^λ and X_h are λ-related, that is, X_h^λ and X_h are λ-related.

Moreover, we know that $X_h^\lambda(x) \in \rho_D(D_x)$, $\forall x \in M$.

Therefore, $X_h(\lambda(x)) \in (T_x\lambda)(\rho_D(D_x)) = \mathcal{L}_{\lambda,D}(x)$, for all $x \in M$.

From Proposition 1 ($\mathcal{L}_{\lambda,D}$ is lagrangian) we deduce that

$$X_h(\lambda(x)) = \sharp_{\Lambda_{D^*}}(\eta_{\lambda(x)}), \text{ for some } \eta_{\lambda(x)} \in (\mathcal{L}_{\lambda,D})^o$$

Moreover from the definition of hamiltonian vector field

$$X_h(\lambda(x)) = \sharp_{\Lambda_{D^*}}(dh(\lambda(x))$$

Thus,

$$\eta_{\lambda(x)} - dh(\lambda(x)) \in \ker \sharp_{\Lambda_{D^*}}(\lambda(x)) \subseteq \mathcal{L}_{\lambda,D}(x)^o \text{ (by Proposition 2)}$$

Then, $dh(\lambda(x)) \in \mathcal{L}_{\lambda,D}(x)^o, \forall x \in M$.

Finally, if $a_x \in D_x$, then

$$\begin{aligned} d^D(h \circ \lambda)(x)(a_x) &= \rho_D(a_x)(h \circ \lambda) = (T_x\lambda)(\rho_D(a_x))(h) \\ &= dh(\lambda(x))(T_x\lambda)(\rho_D(a_x)) = 0 \end{aligned}$$

(ii) $\Rightarrow$ (i)

The condition $d^D(h \circ \lambda) = 0$ implies that

$$dh(\lambda(x)) \in \mathcal{L}_{\lambda,D}(x)^o, \forall x \in M$$

Then

$$\begin{aligned} X_H(\lambda(x)) &= \sharp_{\Lambda_{D^*}}(dh)(\lambda(x)) \in \sharp_{\Lambda_{D^*}}(\mathcal{L}_{\lambda,D}(x)^o) = \mathcal{L}_{\lambda,D} \text{ (Proposition 2)} \\ &= T_x\lambda(\rho_D(D_x)) \end{aligned}$$

Therefore X_h^λ and X_h are λ-related and we conclude (i). $\square$

Local expression of the Hamilton-Jacobi equations

Take local coordinates (x^μ) in the base manifold M, a local basis of sections of D, $\{e_\alpha\}$, and induced coordinates (x^μ, y_α) on D^*. Then if

$$\lambda : (x^\mu) \longrightarrow (x^\mu, \lambda_\alpha(x^\mu)) \equiv (x, \lambda(x))$$

we have

$$d^D(h \circ \lambda) = 0$$

is locally written as

$$\begin{aligned} 0 &= d^D(h \circ \lambda)(e_\alpha)_x \\ &= \rho_D(x)(e_\alpha(x))(h \circ \lambda) \\ &= \rho^\mu_\alpha(x) \frac{\partial}{\partial x^\mu}(h \circ \lambda)_x \\ &= \rho^\mu_\alpha(x) \left[\frac{\partial h}{\partial x^\mu}(x, \lambda(x)) + \frac{\partial h}{\partial y_\beta}(x, \lambda(x)) \frac{\partial \lambda_\beta}{\partial x^\mu}(x) \right], \quad \forall \alpha \end{aligned}$$

The Hamilton-Jacobi Equations

$$\rho^\mu_\alpha(x) \left[\frac{\partial h}{\partial x^\mu}(x, \lambda(x)) + \frac{\partial h}{\partial y_\beta}(x, \lambda(x)) \frac{\partial \lambda_\beta}{\partial x^\mu}(x) \right] = 0$$

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J. Cortés, M. de León, J.C. Marrero, E. Martínez: Nonholonomic Lagrangian systems on on Lie algebroids, *Preprint* (2006).

Application: Mechanical systems with nonholonomic constraints

Let $\mathcal{G} : E \times_M E \to \mathbb{R}$ be a bundle metric on a Lie algebroid $(E, [\cdot\,,\cdot], \rho)$

The class of systems that were considered is that of *mechanical systems with nonholonomic constraints* determined by:

- **The Lagrangian function L:**

  $$L(a) = \frac{1}{2}\mathcal{G}(a,a) - V(\tau(a)), \quad a \in E,$$

  with V a function on M

- **The nonholonomic constraints determined by a subbundle D of E**

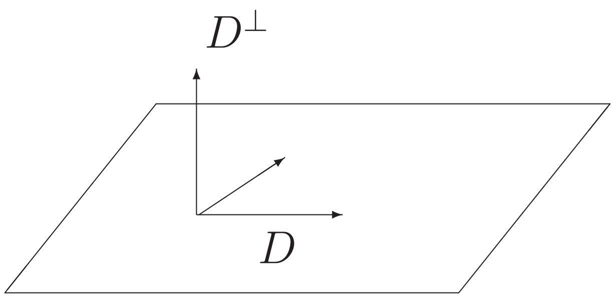
Consider the orthogonal decomposition $E = D \oplus D^{\perp}$, and the associated orthogonal projectors

$$P \; : \; E \longrightarrow D$$
$$Q \; : \; E \longrightarrow D^{\perp}$$

Take local coordinates (x^{μ}) in the base manifold M and a local basis of sections of E (moving basis), $\{e_{\alpha}\}$, adapted to the nonholonomic problem (L, D), in the sense that

(i) $\{e_{\alpha}\}$ is an orthonormal basis with respect to $\mathcal{G}$ (that is $\mathcal{G}(e_{\alpha}, e_{\beta}) = \delta_{\alpha\beta}$)

(ii) $\{e_{\alpha}\} = \{e_a, e_A\}$ where $D = \mathrm{span}\{e_a\}$, $D^{\perp} = \mathrm{span}\{e_A\}$.



Denoting by $(x^\mu, y^\alpha) = (x^\mu, y^a, y^A)$ the induced coordinates on E, the constraint equations determining D just read $\boxed{y^A = 0.}$ Therefore we choose (x^μ, y^a) as a set of coordinates on D

$$
\begin{array}{ccc}
D & \overset{i_D}{\hookrightarrow} & E \\
 & {\scriptstyle \tau_D} \searrow \quad \swarrow {\scriptstyle \tau} & \\
 & M &
\end{array}
$$

In these coordinates we have the inclusion

$$
\begin{array}{rccc}
i_D : & D & \longrightarrow & E \\
 & (x^\mu, y^a) & \longmapsto & (x^\mu, y^a, 0)
\end{array}
$$

and the dual map

$$
\begin{array}{rccc}
i_D^* : & E^* & \longrightarrow & D^* \\
 & (x^\mu, y_a, y_A) & \longmapsto & (x^\mu, y_a)
\end{array}
$$

where (x^μ, y_α) are the induced coordinates on E^* by the dual basis of $\{e_\alpha\}$.

Moreover, from the orthogonal decomposition we have that

$$\begin{aligned} P: \quad E &\longrightarrow D \\ (x^\mu, y^a, y^\alpha) &\longmapsto (x^\mu, y^a) \end{aligned}$$

and its dual map

$$\begin{aligned} P^*: \quad D^* &\longrightarrow E^* \\ (x^\mu, y_a) &\longmapsto (x^\mu, y_a, 0) \end{aligned}$$

In these coordinates, the nonholonomic system is given by

i) The Lagrangian $L(x^\mu, y^\alpha) = \frac{1}{2}\sum_\alpha (y^\alpha)^2 - V(x^\mu)$**,**

ii) The nonholonomic constraints $y^A = 0$**.**

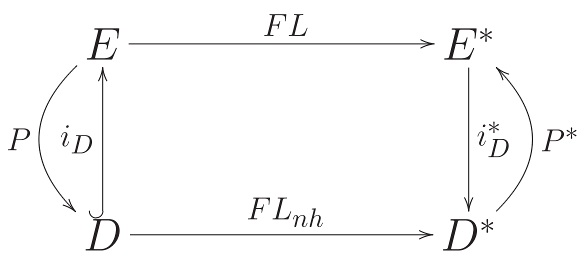
In this case, the Legendre transformation associated with L is the isomorphism $FL : E \longrightarrow E^*$ induced by the metric $\mathcal{G}$. Therefore, locally, the Legendre transformation is

$$\begin{array}{rccc} FL : & E & \longrightarrow & E^* \\ & (x^\mu, y^\alpha) & \longmapsto & (x^\mu, y_\alpha = y^\alpha) \end{array}$$

and we can define the ***nonholonomic Legendre transformation*** $FL_{nh} = i_D^* \circ FL \circ i_D : D \longrightarrow D^*$

$$\begin{array}{rccc} FL_{nh} : & D & \longrightarrow & D^* \\ & (x^\mu, y^a) & \longmapsto & (x^\mu, y_a = y^a) \end{array}$$

Summarizing, we have the following diagram



The nonholonomic bracket

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$$(E, [\ ,\ ], \rho) \textbf{ is a Lie algebroid}$$

$$\Downarrow$$

$$\Lambda_{E^*} \textbf{ is a linear Poisson structure on } E^*$$

If f_1 **and** f_2 **are functions on** M**, and** ξ_1 **and** ξ_2 **are sections of** E**, then:**

$$\{f_1 \circ \tau_{E^*}, g_1 \circ \tau_{E^*}\}_{E^*} = 0, \quad \{\widehat{\xi_1}, f_1 \circ \tau_{E^*},\}_{E^*} = (\rho(\xi_1))\, f_1 \circ \tau_{E^*}, \quad \{\widehat{\xi_1}, \widehat{\xi_2}\}_{E^*} = \widehat{[\xi_1, \xi_2]}$$

In the induced coordinates (x^μ, y_α)**, the Poisson bracket relations on** E^* **are**

$$\{x^\mu, x^\eta\}_{E^*} = 0, \qquad \{y_\alpha, x^\mu,\}_{E^*} = \rho^\mu_\alpha, \qquad \{y_\alpha, y_\beta\}_{E^*} = C^\gamma_{\alpha\beta} y_\gamma$$

In other words

$$\Lambda_{E^*} = \rho^\mu_\alpha \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial x^\mu} + \frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}$$

The ***nonholonomic bracket*** **on** D^***,** $\{\ ,\ \}_{nh,D^*}$**, is defined by**

$$\boxed{\{F,G\}_{nh,D^*} = \{F\circ i_D^*, G\circ i_D^*\}_{E^*}\circ P^*}$$

for all $F, G \in C^\infty(D^*)$

The induced bivector Λ_{nh,D^*} **is**

$$\boxed{\Lambda_{nh,D^*} = \rho_a^\mu \frac{\partial}{\partial y_a}\wedge\frac{\partial}{\partial x^\mu} + \frac{1}{2}C_{ab}^c y_c \frac{\partial}{\partial y_a}\wedge\frac{\partial}{\partial y_b}}$$

That is,

$$\boxed{\{x^\mu, x^\eta\}_{nh,D^*} = 0, \qquad \{y_a, x^\mu\}_{nh,D^*} = \rho_a^\mu, \qquad \{y_a, y_b\}_{nh,D^*} = C_{ab}^c y_c}$$

Λ_{nh,D^*} **is a linear bivector on** D^***, but in general, does not satisfy Jacobi identity. So, we are in the case considered in the very beginning.**

Particular cases

1. $E = TM$. Then the linear Poisson structure on $E^* = T^*M$ is the canonical symplectic structure. Thus, D is a distribution on M and $\{\ ,\ \}_{nh,D^*}$ is the nonholonomic bracket studied by A.J. Van der Schaft, B.M. Maschke, and others.

2. $E = \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra. E is a Lie algebroid over a single point (the anchor map is the zero map). In this case, the linear Poisson structure on $E^* = \mathfrak{g}^*$ is the $\pm$ Lie-Poisson structure. Thus, if $D = \mathfrak{h}$ is a vector subspace of $\mathfrak{g}$, we obtain that the nonholonomic bracket (nonholonomic Lie-Poisson bracket) is given by

$$\{F, G\}_{nh,D^*\pm}(\mu) = \pm \left\langle \mu, P\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle$$

for $\mu \in \mathfrak{h}^*$, and $F, G \in C^\infty(\mathfrak{h}^*)$. In adapted coordinates

$$\{y_a, y_b\}_{nh,D^*\pm} = \pm C_{ab}^c y_c$$

3. $E =$ **the Atiyah algebroid associated with a principal G-bundle** $\pi : Q \longrightarrow Q/G$

$$E = TQ/G$$

The linear Poisson structure on $E^* = T^*Q/G$ is characterized by the following condition: "the canonical projection $T^*Q \longrightarrow T^*Q/G$ is a Poisson epimorphism"

"the Hamilton-Poincaré bracket on T^*Q/G"

(See J.P. Ortega and T. S. Ratiu : Momentum maps and Hamiltonian reduction, Progress in Math., 222 Birkhauser, Boston 2004)

D **a G-invariant distribution on $Q \Rrightarrow D/G$ is a vector subbundle of $E = TQ/G$**

Thus, we obtain a reduced non-holonomic bracket $\{\ ,\ \}_{nh,D^*/G}$ (the non-holonomic Hamilton-Poincaré bracket on D^*/G)

We return to the general case

Taking the hamiltonian function $H : E^* \longrightarrow \mathbb{R}$ **defined by**

$$H(x^\mu, y_\alpha) = \frac{1}{2}\sum_\alpha (y_\alpha)^2 + V(x^\mu)$$

then we induce a hamiltonian function $h : D^* \longrightarrow \mathbb{R}$ **by taking** $h = H \circ P^*$**. In coordinates,**

$$h(x^\mu, y_a) = \frac{1}{2}\sum_a (y_a)^2 + V(x^\mu)$$

The nonholonomic dynamics is determined on D^* by the linear bivector Λ_{nh,D^*} and the hamiltonian function $h : D^* \longrightarrow \mathbb{R}$, that is

$$\dot{F} = \{F, h\}_{nh,D^*}$$

or, in coordinates, by the equations

$$\begin{aligned} \dot{x}^\mu &= \rho_a^\mu \frac{\partial h}{\partial y_a} = \rho_a^\mu y_a \\ \dot{y}_a &= -C_{ab}^c y_c \frac{\partial h}{\partial y_b} - \rho_a^\mu \frac{\partial h}{\partial x^\mu} \\ &= -C_{ab}^c y_c y_b - \rho_a^\mu \frac{\partial V}{\partial x^\mu} \end{aligned}$$

$$\Downarrow$$

we can apply Hamilton-Jacobi theory to nonholonomic mechanics!

An example: The mobile robot with fixed orientation

The robot has three wheels with radius R, which turn simultaneously about independent axes, and perform a rolling without sliding over a horizontal floor.

Let (x, y) denotes the position of the centre of mass, θ the steering angle of the wheel, ψ the rotation angle of the wheels in their rolling motion over the floor. So, the configuration manifold is

$$M = S^1 \times S^1 \times \mathbb{R}^2$$

The lagrangian L is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{3}{2}J_\omega\dot{\psi}^2$$

where m is the mass, J is the moment of inertia and J_ω is the axial moment of inertia of the robot.

The constraints giving the distribution D are induced by the conditions that the wheels roll without sliding, in the direction in which they point, and that the instantaneous contact point of the wheels with the floor have no velocity component orthogonal to that direction:

$$\begin{aligned} \dot{x}\sin\theta - \dot{y}\cos\theta &= 0, \\ \dot{x}\cos\theta + \dot{y}\sin\theta - R\dot{\psi} &= 0. \end{aligned}$$

The vector fields

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{J}}\frac{\partial}{\partial\theta} \\ e_2 &= \frac{1}{\sqrt{mR^2 + 3J_\omega}}\left[R\cos\theta\frac{\partial}{\partial x} + R\sin\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial\psi}\right] \end{aligned}$$

are an orthonormal basis generating D.

Moreover,

$$[e_1, e_2] = \frac{1}{\sqrt{J(mR^2 + 3J_\omega)}} \left(-R\sin\theta \frac{\partial}{\partial x} + R\cos\theta \frac{\partial}{\partial y} \right)$$

Therefore

$$[e_1, e_2]_D = 0 \Rrightarrow C_{12}^1 = C_{12}^2 = 0$$

The linear bivector is

$$\begin{aligned} \Lambda_{nh,D^*} \;=\; & \frac{1}{\sqrt{J}} \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial \theta} + \frac{R\cos\theta}{\sqrt{mR^2 + 3J_\omega}} \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial x} \\ & + \frac{R\sin\theta}{\sqrt{mR^2 + 3J_\omega}} \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y} + \frac{1}{\sqrt{mR^2 + 3J_\omega}} \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial \psi} \end{aligned}$$

For any section $\lambda : M \longrightarrow D^*$:

$$(x,y,\theta,\psi) \longmapsto (x,y,\theta,\psi,\lambda_1(x,y,\theta,\psi),\lambda_2(x,y,\theta,\psi))$$

the condition

$$d^D\lambda = 0 \Leftrightarrow e_1(\lambda_2) - e_2(\lambda_1) = 0$$

Now it is trivial to show that taking $\lambda_1 = k_1$ and $\lambda_2 = k_2$ where k_1, k_2 are constants then

$$d^D(h \circ \lambda) = 0$$

since

$$h = \frac{1}{2}(y_1^2 + y_2^2)$$

Then, to integrate the nonholonomic problem is equivalent to integrate the vector fields on the configuration space:

$$X_h^\lambda = k_1' \frac{\partial}{\partial \theta} + k_2' \left[R\cos\theta \frac{\partial}{\partial x} + R\sin\theta \frac{\partial}{\partial y} + \frac{\partial}{\partial \psi} \right]$$

where $(k_1', k_2') \in \mathbb{R}^2$.