FROM POINCARÉ DUALITY TO EVANS-LU-WEINSTEIN COHOMOLOGY PAIRING

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Abstract. The Evans-Lu-Weinstein representation (Q_A, D) for a Lie algebroid A on a manifold M is studied in the transitive case. To consider at the same time non-oriented manifolds as well, this representation is slightly modified to $\left(Q_A^{or}, D^{or}\right)$ by tensoring by orientation flat line bundle, $Q_A^{or} = Q_A \otimes or\,(M)$ and $D^{or} = D \otimes \partial_A^{or}$. It is shown that the induced cohomology pairing is non-degenerate and that the representation $\left(Q_A^{or}, D^{or}\right)$ is the unique (up to isomorphy) line representation for which the top group of compactly supported cohomology is nontrivial. In the case of trivial Lie algebroid $A = TM$ the theorem reduce to the following: the orientation flat bundle $(or\,(M), \partial^{or})$ is the unique (up to isomorphy) flat line bundle (ξ, ∇) for which the twisted de Rham complex of compactly supported differential forms on M with values in ξ possesses the nontrivial cohomology group in the top dimension. Finally it is obtained the characterization of transitive Lie algebroids for which the Lie algebroid cohomology with trivial coefficients (or with coefficients in the orientation flat line bundle) gives Poincaré duality. In proofs of these theorems for Lie algebroids it is used the Hochschild-Serre spectral sequence and it is shown the general fact concerning pairings between graded filtered differential $\mathbb{R}$-vector spaces: assuming that the second terms live in the finite rectangular, nondegeneration of the pairing for the second terms (which can be infinite dimensional) implies the same for cohomology spaces.

1. Basic results

The cohomology pairing coming from Evans-Lu-Weinstein representation of a Lie algebroid [E-L-W] is very important in many applications of Lie algebroids (Poisson geometry [W], [E-L-W], intrinsic characteristic classes [C], [F]). This pairing generalizes the well known pairings that give Poincaré duality for Lie algebra cohomology and de Rham cohomology of a manifold and real cohomology of transitive invariantly oriented Lie algebroids [K2]. For a Poisson manifold, this pairing agree with the pairing on the Poisson homology. The authors of [E-L-W] give an example of a nontransitive Lie algebroid for which the pairing is not necessarily non-degenerate and post the problem of when it is non-degenerate. This paper gives the positive answer for the case of any transitive Lie algebroids and prove the capacity of this representation: it is the one (up to isomorphy) for which the top group of compactly supported cohomology is nontrivial.

More detailed, this paper is devoted to prove two cycles of theorems, mutually overcoming.

Key words and phrases. twisted cohomology, cohomology of Lie algebras, Poincaré duality, Lie algebroid, modular class, cohomology pairing, Evans-Lu-Weinstein pairing, Hochschild-Serre spectral sequence .

FIRST CYCLE concerns non-degenerate cohomology pairings for manifolds, Lie algebras and Lie algebroids.

Theorem 1.1. *Assume that M is a connected m-dimensional manifold (oriented or not) and ξ_1, ξ_2 are two flat vector bundles with flat covariant derivatives ∇_1 and ∇_2 respectively. Denote by $or\,(M)$ the orientation bundle with canonical flat structure ∂^{or}. If $F:\xi_1\times\xi_2\to or\,(M)$ is a pairing (i.e. 2-linear homomorphism) of vector bundles compatible with the flat structures $(\nabla_1,\nabla_2,\partial^{or})$ and non-degenerate at least at one point (therefore, at every) then the induced pairing in cohomology*

$$H^j_{\nabla_1}(M,\xi_1)\times H^{m-j}_{\nabla_2,c}(M,\xi_2)\xrightarrow{F_\#}H^m_{\partial^{or},c}(M,or\,(M))\xrightarrow{\int_M^{or,\#}}\mathbb{R}$$

is non-degenerate in the sense that

$$H^j_{\nabla_1}(M,\xi_1)\overset{\cong}{\to}\left(H^{m-j}_{\nabla_2,c}(M,\xi_2)\right)^*.$$

The index "c" means that the compactly supported cohomology are considered. This theorem generalizes the classical Poincaré duality [B-T] as well as the one for d^ω-cohomologies [G-L], [H-R].

Theorem 1.2. *Assume $\mathfrak{g}$ is an arbitrary n-dimensional Lie algebra and $\nabla_1,\nabla_2:\mathfrak{g}\to L_{\mathbb{R}}$ are two representations of $\mathfrak{g}$ in $\mathbb{R}$. Denote by $\nabla_{\text{trad}}:\mathfrak{g}\to L_{\mathbb{R}}$ the trace-representation $(\nabla_{\text{trad}})_a=\operatorname{tr}(\operatorname{ad}_a)\cdot\operatorname{id}$. Then the top group of cohomology $H^n_{\text{trad}}(\mathfrak{g})$ of $\mathfrak{g}$ with respect to ∇_{trad} is nonzero, $H^n_{\text{trad}}(\mathfrak{g})\overset{\cong}{\to}\mathbb{R}$, and if the multiplication of reals is compatible with respect to $(\nabla_1,\nabla_2,\nabla_{\text{trad}})$ then the induced pairing in cohomology*

$$H^i_{\nabla_1}(\mathfrak{g})\times H^{n-i}_{\nabla_2}(\mathfrak{g})\to H^n_{\text{trad}}(\mathfrak{g})\cong\mathbb{R}$$

is non-degenerate, i.e.

$$H^i_{\nabla_1}(\mathfrak{g})\overset{\cong}{\to}\left(H^{n-i}_{\nabla_2}(\mathfrak{g})\right)^*.$$

In particular, for $(\nabla_1,\nabla_2,\nabla_{\text{trad}}):=(0,\nabla_{\text{trad}},\nabla_{\text{trad}})$ we obtain the non-degenerate pairing

$$H^i(\mathfrak{g})\times H^{n-i}_{\text{trad}}(\mathfrak{g})\to H^n_{\text{trad}}(\mathfrak{g})\cong\mathbb{R},$$
$$H^i(\mathfrak{g})\overset{\cong}{\to}\left(H^{n-i}_{\text{trad}}(\mathfrak{g})\right)^*.$$

For unimodular Lie algebra $\mathfrak{g}$ the usual Poincaré duality is obtained in this way.

Theorem 1.3. *Let $A=(A,[\![\cdot,\cdot]\!],\#_A)$ be a Lie algebroid on M and*

$$Q_A=\Lambda^{\text{top}}A\otimes\Lambda^{\text{top}}T^*M$$

the line vector bundle with canonical Evans-Lu-Weinstein representation [E-L-W],

$$D_\gamma(Y\otimes\varphi)=L_\gamma(Y)\otimes\varphi+Y\otimes L_{\#_A(\gamma)}(\varphi).$$

To consider non-oriented manifolds we modify it into

$$Q_A^{or}=Q_A\otimes or\,(M)$$

and

$$D^{or}=D\otimes\partial^{or}_A$$

tensoring by the orientation bundle and its flat structure ($(\partial^{or}_A)_\gamma\sigma=(\partial^{or})_{\#_A(\gamma)}\sigma$, $\sigma\in\Gamma(or\,(M))$, $\#_A:A\to TM$ is the anchor of A). For transitive Lie algebroid with n-dimensional isotropy Lie algebras and multiplications by reals $(M\times\mathbb{R})\otimes Q_A^{or}\to Q_A^{or}$ the induced pairing in cohomology

$$H^j(A)\times H^{m+n-j}_{D^{or},c}(A;Q_A^{or})\to H^{m+n}_{D^{or},c}(A;Q_A^{or})\to\mathbb{R}$$

is non-degenerate, i.e. $H^{m+n}_{D^{or},c}(A; Q^{or}_A) \cong \mathbb{R}$ *and*

$$H^j(A) \cong \left(H^{m+n-j}_{D^{or},c}(A; Q^{or}_A)\right)^*.$$

SECOND CYCLE shows the uniqueness of the representation for which the top group of compactly supported cohomology is not zero.

Theorem 1.4. $H^m_{\nabla,c}(M;\xi) \neq 0$ *if and only if* (ξ,∇) *is, up to isomorphy, the orientation flat line bundle* $(or(M), \partial^{or})$. *In particular, for oriented manifold,* $H^m_{\nabla,c}(M;\xi) \neq 0$ *if and only if* (ξ,∇) *is, up to isomorphy, the trivial flat line bundle* $(M \times \mathbb{R}, \partial)$.

Theorem 1.5. *For an n-dimensional Lie algebra* $\mathfrak{g}$ *the trace-representation* $\nabla = \nabla_{\mathrm{trad}}$ *is the unique line representation for which* $H^n_\nabla(\mathfrak{g}) \neq 0$.

Theorem 1.6. *Let* A *be a transitive Lie algebroid and* ∇ *a representation of* A *in a line vector bundle* ξ. *Then* $H^{m+n}_{\nabla,c}(A;\xi) \neq 0$ *if and only if* (ξ,∇) *is, up to isomorphy, the E-L-W-representation* (Q^{or}_A, D^{or}).

In conlusion we obtain a full classification of transitive Lie algebroids for which the algebra of real cohomologies with trivial coefficients satisfies the Poincaré duality.

Theorem 1.7. *The following conditions are equivalent:*

- $H^{m+n}_c(A) \neq 0$,
- $H^{m+n}_c(A) \cong \mathbb{R}$ *and* $H^j(A) \cong \left(H^{m+n-j}_c(A)\right)^*$,
- A *is orientable vector bundle and the modular class of* A *is zero,* $\theta_A = 0$.

In particular,

Theorem 1.8. *For an orientable manifold* M *we have:* $H^{m+n}_c(A) \neq 0$ *if and only if* A *is a TUIO-Lie algebroid* [K1], *i.e. the adjoint Lie Algebra Bundle* $\boldsymbol{g} = \ker \#_A$ *is oriented and there is a global nonsingular section* $\varepsilon \in \Gamma(\Lambda^n \boldsymbol{g})$ *invariant with respect to the adjoint representaion.*

The above theorem for a compact oriented manifold M and 1-rank adjoint LAB $\boldsymbol{g} = M \times \mathbb{R}$ was proved earlier in [K-K-V-W].

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The theorem 1.2 is based on a generalization of Lemma 3 from [Ch-H-S] concerning Poincaré differentiation from algebras to pairings. The asumptions on finite dimensionality is superfluous.

Lemma 1.1. *Let* $A_s = \bigoplus_{i=0}^n A^i_s$, $d_s : A_s \to A_s$, $s = 1,2,3$, *be three graded differential* $\mathbb{R}$*-vector spaces such that*

(1) $d_s\left[A^i_s\right] \subset A^{i+1}_s$,
(2) $d^2_s = 0$,
(3) $d_3\left[A^{n-1}_3\right] = 0$.
(4) $A^n_3 \cong \mathbb{R}$, $A^i_3 = 0$ *for* $i > n$.

Let

$$\cdot : A_1 \times A_2 \to A_3$$

be a pairing such that

(5) $d_3(x \cdot y) = d_1 x \cdot y + (-1)^{\deg x} x \cdot d_2 y$,

(6) $\cdot : A_1^r \times A_2^{n-r} \to A_3^n \cong \mathbb{R}$, $r = 0, 1, ..., n$ *are nondegenerate in the sense that the induced mappings*

$$i_r : A_1^r \xrightarrow{\cong} \left(A_2^{n-r}\right)^*$$

are linear isomorphisms.

Then the induced homomorphisms in cohomology

$$\cdot : H^r(A_1, d_1) \times H^{n-r}(A_2, d_2) \to H^n(A_3, d_3) \cong \mathbb{R}$$

are nondegenerate as well, i.e. the induced linear homomorphism

$$i_r' : H^r(A_1, d_1) \to \left(H^{n-r}(A_2, d_2)\right)^*$$

are linear isomorphisms.

To prove the above theorems 1.3, 1.6 for Lie algebroids we use the Hochshild-Serre spectral sequences and the following general theorem concerning a pairing

$$\cdot : {}^1A \times {}^2A \to {}^3A$$

between graded filtered differential $\mathbb{R}$-vector spaces and theirs spectral sequences $\left({}^rE_s^{j,i}, d_s^{j,i}\right)$.

Theorem 1.9. *If the second terms*

$${}^rE_2^{j,i}$$

live in the rectangular $0 \le j \le m$, $0 \le i \le n$ *and that*

$${}^3E_2^{(m+n)} = {}^3E_2^{m,n} \cong \mathbb{R}$$

and the multiplication of the second terms

$$\langle \cdot, \cdot \rangle_2 : {}^1E_2^{(j)} \times {}^2E_2^{(m+n-j)} \to {}^3E_2^{m,n} \cong \mathbb{R}$$

is nondenenerate in the sense that

$${}^1E_2^{(j)} \cong \left({}^2E_2^{(m+n-j)}\right)^*,$$

then the cohomology pairing for cohomologies of spaces is non-degenerate as well, i.e.

$${}^3H^{m+n} \cong \mathbb{R}$$

and

$${}^1H^j \xrightarrow{\cong} \left({}^2H^{m+n-j}\right)^*.$$

We pay attention that the $\mathbb{R}$-vector spaces ${}^rE_2^{j,i}$ may be infinite dimensional. This theorem generalizes the suitable theorem for graded filtered differential algebras [K-M].

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