Reduction of Lie bialgebroids

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Outline

(Hamiltonian) Poisson reduction

(M, π) Poisson manifold $\qquad$ G (connected) Lie group

$G \times M \to M$ action with infinitesimal $\psi : \mathfrak{g} \to \mathfrak{X}(M)$

Poisson action $\iff$ $\mathcal{L}_{\psi(\xi)}\pi = 0, \qquad (\xi \in \mathfrak{g})$

$\Rightarrow$ M/G Poisson manifold

$$\{f_1, f_2\}_{M/G} \circ p = \{f_1 \circ p, f_2 \circ p\}, \qquad (f_1, f_2 \in C^\infty(M/G) \cong C^\infty(M)^G)$$

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$J : M \to \mathfrak{g}^*$ equivariant and

$$\pi^\sharp(d\widehat{J}_\xi) = \psi(\xi), \qquad (\xi \in \mathfrak{g}),$$

$(\widehat{J}_\xi \in C^\infty(M)$, $m \mapsto \langle J(m), \xi \rangle)$

$\lambda \in \mathfrak{g}^*$ regular value of J

$$i : J^{-1}(\lambda) \to M \qquad p : J^{-1}(\lambda) \to J^{-1}(\lambda)/G_\lambda$$

$G_\lambda = \{g \in G / Ad^*_{g^{-1}}\lambda = \lambda\}$ (isotropy group)

$\Rightarrow J^{-1}(\lambda)/G_\lambda$ Poisson manifold.

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Lie groupoids

$\mathcal{G} \rightrightarrows M$

- $\mathrm{s} : \mathcal{G} \to M$ y $\mathrm{t} : \mathcal{G} \to M$ source and target map

$$\mathcal{G}_2 = \{(x, y) \in \mathcal{G} \times \mathcal{G} \mid \mathrm{s}(x) = \mathrm{t}(y)\}$$

- $m : \mathcal{G}_2 \to \mathcal{G}$ multiplication
  - $\mathrm{s}(xy) = \mathrm{s}(y)$ and $\mathrm{t}(xy) = \mathrm{t}(x)$, for any $(x, y) \in \mathcal{G}_2$,
  - $x(yz) = (xy)z$, for any $x, y, z \in \mathcal{G}$
- $\epsilon : M \to \mathcal{G}$ identity section
  - $x\epsilon(\mathrm{s}(x)) = x$ and $\epsilon(\mathrm{t}(x))x = x$, for any $x \in \mathcal{G}$.
- $\iota : \mathcal{G} \to \mathcal{G}$ inversion
  - $xx^{-1} = \epsilon(\mathrm{t}(x))$ and $x^{-1}x = \epsilon(\mathrm{s}(x))$, for any $x \in \mathcal{G}$.

Examples of Lie groupoids

1.- *Lie groups* (Lie groupoid over a single point)

2.- *Banal groupoid*

M smooth manifold $\Rightarrow M \times M \rightrightarrows M$ Lie groupoid

$$\begin{array}{lcl} \mathrm{s} : M \times M \to M & ; & (x,y) \mapsto y \\ \mathrm{t} : M \times M \to M & ; & (x,y) \mapsto x \\ m : (M \times M)_2 \to M \times M & ; & ((x,y),(y,z)) \mapsto (x,z) \end{array}$$

3.- *Atiyah groupoid* $(Q \times Q)/G \to M$

$p : Q \to M$, principal G-bundle with action $\Phi : G \times Q \to Q$

$$\begin{array}{lcl} \tilde{\mathrm{s}} : (Q \times Q)/G \to M & ; & [(x,y)] \mapsto p(y) \\ \tilde{\mathrm{t}} : (Q \times Q)/G \to M & ; & [(x,y)] \mapsto p(x) \\ \tilde{m} : ((Q \times Q)/G)_2 \to (Q \times Q)/G & ; & ([(x,y)],[(gy,z)]) \mapsto [(gx,z)] \end{array}$$

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Morphism of Lie groupoids

$\mathcal{G} \rightrightarrows M \quad \mathcal{G}' \to M'$ Lie groupoids

$\Phi : \mathcal{G} \to \mathcal{G}'$ smooth map over $\Phi_0 : M \to M'$

$$\mathrm{s}'(\Phi(x)) = \Phi_0(\mathrm{s}(x)), \qquad x \in \mathcal{G},$$
$$\mathrm{t}'(\Phi(x)) = \Phi_0(\mathrm{s}(x)), \qquad x \in \mathcal{G},$$
$$\Phi(xy) = \Phi(x)\Phi(y), \qquad (x, y) \in \mathcal{G}_2.$$

Lie algebroids

$A \to M$ vector bundle

$$\begin{cases} (\Gamma(A), [\![\ ,\ ]\!]) \text{ Lie algebra} \\ \rho\colon A \to TM \text{ anchor map} \end{cases}$$

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y,$$

for $X, Y \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$.

Coordinate Expressions

$(x^1, \dots, x^m)$ local coord. on M, $\{e_\alpha\}$ local basis of sections of A

$$\begin{cases} \rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \\ [\![e_\alpha, e_\beta]\!] = C^\gamma_{\alpha\beta} e_\gamma \end{cases}$$

Lie algebroid associated with Lie groupoid $\mathcal{G}$

$\mathcal{G} \rightrightarrows M$ Lie groupoid $\Rightarrow \tau : A\mathcal{G} \to M$ Lie algebroid

$$(A\mathcal{G})_m := \mathrm{Ker}\, \mathrm{d}_{1_m}\mathrm{s}$$

$$\Gamma(A\mathcal{G}) \cong \mathfrak{X}^R(\mathcal{G})$$

$$Y \in \Gamma(A\mathcal{G}) \Rightarrow \overrightarrow{Y}_x = \mathrm{d}_{\mathrm{t}(x)} R_x(Y_{\mathrm{t}(x)}), \text{ for any } x \in \mathcal{G}$$

$([\![\cdot,\cdot]\!], \rho)$ Lie algebroid on $A\mathcal{G} \to M$

$$\begin{cases} \overrightarrow{[\![Y_1, Y_2]\!]} = [\overrightarrow{Y}_1, \overrightarrow{Y}_2] \\ \rho(Y)(m) = \mathrm{d}_{1_m}\mathrm{t}(Y_m) \end{cases}$$

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Examples of Lie algebroids

1.- *Lie algebras of finite dimension*

2.- *Tangent bundle* $TM \to M$

$$\rho = Id : TM \to TM$$
$$[\![\,,\,]\!] = [\,,\,]$$

3.- *Atiyah algebroid* $TQ/G \to M$

$p : Q \to M$, principal G-bundle with action $\Phi : G \times Q \to Q$

$\tau_Q|G : TQ/G \to M = Q/G$ Atiyah algebroid

$$\Gamma(TQ/G) \cong \mathfrak{X}(M)^G$$

$$\rho(X) = Tp(X)$$
$$[\![X, Y]\!] = [X, Y]$$

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Morphism of Lie algebroids

Exterior Differential: $\mathrm{d}\colon \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$

$$\begin{aligned} \mathrm{d}\alpha(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i \rho(X_i)(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([\![X_i,X_j]\!],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k), \end{aligned}$$

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$\tau : A \to M \quad \tau : A' \to M'$ Lie algebroids

$\Phi : A \to A'$ vector bundle morphism over $\Phi_0 : M \to M'$.

$$\Phi^*\mathrm{d}'\theta = \mathrm{d}\Phi^*\theta, \qquad \text{for every } \theta \in \Gamma(\wedge^k(A')^*)$$

$$(\Phi^*\theta)_{(m)}(a_1,\ldots,a_k) = \theta_{(\Phi_0(m))}(\Phi(a_1),\ldots,\Phi(a_k)), \text{ for } a_1,\ldots,a_k \in A_m.$$

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Actions of Lie groups on groupoids

Definition An *action of a Lie group G on a Lie groupoid* $\mathcal{G}$ is a smooth action $\Psi : G \times \mathcal{G} \to \mathcal{G}$ such that for each $g \in G$ the map:

$$\Psi_g : \mathcal{G} \to \mathcal{G}, \; x \mapsto gx,$$

is an automorphisms of $\mathcal{G}$.

$\psi(\xi) : \mathcal{G} \to T\mathcal{G}$ is a groupoid morphims from $\mathcal{G} \rightrightarrows M$ to $T\mathcal{G} \rightrightarrows TM$

$$\psi(\xi)(xy) = \mathrm{d}_{(x,y)} m(\psi(\xi)(x), \psi(\xi)(y)),$$

Such a vector field is called *multiplicative*.

Consequence

$$[\psi(\xi), \overrightarrow{X}] \text{ is right-invariant, for all } \xi \in \mathfrak{g}, X \in \Gamma(A).$$

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$$\Psi_g : A \to A,\ a \mapsto ga,$$

is a Lie algebroid automorphism of A.

$\psi(\xi) : A \to TA$ is a Lie algebroid derivation

$$D([\![X, Y]\!]) = [\![D(X), Y]\!] + [\![X, D(Y)]\!], \quad X, Y \in \Gamma(A).$$

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Inner Actions

Definition A Lie group action $\Psi : G \times \mathcal{G} \to \mathcal{G}$ on a Lie groupoid $\mathcal{G}$ is called an *inner action* if there exists a map $\Phi : G \times M \to \mathcal{G}$ such that:

$$\Psi(g, x) = \Phi(g, \mathrm{t}(x)) \cdot x \cdot \Phi(g, \mathrm{s}(x))^{-1}.$$

Definition An infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}_{\mathrm{mult}}(\mathcal{G})$ of a Lie algebra $\mathfrak{g}$ on a Lie groupoid $\mathcal{G}$ is called an *inner action* if there exists a Lie algebra morphism $\phi : \mathfrak{g} \to \Gamma(A)$ such that $\psi(\xi) = \overrightarrow{\phi(\xi)} - \overleftarrow{\phi(\xi)}$.

$$D_\xi(X) = [\![\phi(\xi), X]\!], \qquad \xi \in \mathfrak{g},\ X \in \Gamma(A).$$

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Theorem Let $\Phi : G \times A \to A$ be an inner action of the Lie group G over the Lie algebroid A with respect to $\phi : \mathfrak{g} \to \Gamma(A)$. Then,

$$\Phi^T : (G \times \mathfrak{g}) \times A \to A, \quad \Phi^T(g, \xi)(a_x) = \Phi_g(a_x) + \Phi_g(\phi(\xi)_x)$$

defines an affine action of $TG \cong G \times \mathfrak{g}$ over A.

Poisson groupoids

Definition
A *Poisson groupoid* $(\mathcal{G}, \Pi)$ is a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a Poisson structure Π which is multiplicative, that is, the graph of the groupoid multiplication

$$\Lambda = \{(x, y, xy) \,|\, \mathrm{s}(x) = \mathrm{t}(y)\}$$

is a coisotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$

Proposition If $(\mathcal{G}, \Pi)$ is a Poisson groupoid, then it induces a Poisson bivector π on M, such that $\mathrm{s} : \mathcal{G} \to M$ is a Poisson map and $\mathrm{t} : \mathcal{G} \to M$ is anti-Poisson.

$(\mathcal{G}, \Pi)$ Poisson groupoid

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Examples of Poisson groupoids

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$(\mathcal{G}, \Pi = \omega^{-1})$ Poisson groupoid $\Rightarrow (\mathcal{G}, \omega)$ symplectic groupoid

2.- Poisson Lie groups

(G, Π) with Λ coisotropic $\iff m : G \times G \to G$ Poisson map

- Particular situation: $\mathfrak{g}$ Lie algebra $\Rightarrow \mathfrak{g}^*$ Poisson Lie group

3.- Exact Poisson groupoids

$\mathcal{G} \rightrightarrows M$ Lie groupoid $\qquad \pi_A \in \Gamma(\wedge^2 A) \qquad [\![\pi_A, \pi_A]\!] = 0$

$\Rightarrow (\mathcal{G}, \overrightarrow{\pi_A} - \overleftarrow{\pi_A})$ Poisson groupoid

(M, π) Poisson manifold $\Rightarrow (M \times M, \pi \oplus -\pi)$ Poisson groupoid

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Lie bialgebroids

Definition
Suppose that $p : A \to M$ is a Lie algebroid such that its dual bundle $A^* \to M$ also carries a Lie algebroid structure. Then (A, A^*) is a *Lie bialgebroid* if for any $X,\ Y \in \mathfrak{X}(A)$,

$$\mathrm{d}_{A^*}[\![X, Y]\!] = \mathcal{L}_X \mathrm{d}_{A^*} Y - \mathcal{L}_Y \mathrm{d}_{A^*} X.$$

Proposition Let (A, A^*) be a Lie bialgebroid over M. Then, there exists a Poisson structure π_M on M characterized by

$$\pi_M(df_1, df_2) = \rho(\mathrm{d}_{A^*} f_1)(f_2) = \langle \mathrm{d}_{A^*} f_1, \mathrm{d}_A f_2 \rangle, \qquad (f_1, f_2 \in C^\infty(M)).$$

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$$\pi_M(df_1, df_2) = \rho(\mathrm{d}_{A^*} f_1)(f_2) = \langle \mathrm{d}_{A^*} f_1, \mathrm{d}_A f_2 \rangle, \qquad (f_1, f_2 \in C^\infty(M)).$$

Examples of Lie bialgebroids

1.- Poisson groupoids and Lie bialgebroids

$$(\mathcal{G}, \Pi) \text{ Poisson groupoid} \Rightarrow (A\mathcal{G}, A\mathcal{G}^*) \text{ Lie bialgebroid}$$

2.- Exact Lie bialgebroids

$A \to M$ Lie algebroid $\quad \pi_A \in \Gamma(\wedge^2 A) \quad [\![\pi_A, \pi_A]\!] = 0$

$\Rightarrow A^* \to M$ Lie algebroid $\quad (A, A^*)$ Lie bialgebroid

$$\begin{aligned} [\![\alpha, \beta]\!]_{A^*} &= \mathcal{L}_{\pi_A^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi_A^\sharp(\beta)}(\alpha) - \mathrm{d}_A(\pi_A(\alpha, \beta)), \\ \rho_* &= \rho \circ \pi_A^\sharp. \end{aligned}$$

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Actions on Poisson groupoids

$(\mathcal{G} \rightrightarrows M, \Pi)$ Poisson groupoid $\qquad$ G (connected) Lie group

$G \times \mathcal{G} \to \mathcal{G}$ Poisson action with infinitesimal $\psi : \mathfrak{g} \to \mathfrak{X}_{\mathrm{mult}}(\mathcal{G})$

$$\psi^* : \mathfrak{g} \to \mathrm{Der}(A\mathcal{G}^*)$$

$\mathcal{G}/G \rightrightarrows M/G$ Poisson groupoid

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$$J : \mathcal{G} \to \mathfrak{g}^* \text{ equiv. such that } \begin{cases} J(xy) = J(x) + J(y), & ((x,y) \in \mathcal{G}_2), \\ \Pi^\sharp(\mathrm{d}\widehat{J_\xi}) = \psi(\xi), & (\xi \in \mathfrak{g}). \end{cases}$$

$$j := AJ : A\mathcal{G} \to \mathfrak{g}^* \text{ such that } \begin{cases} \text{Lie algebroid morphism,} & \\ \psi^*(\xi) = [\![j^*(\xi), \cdot]\!]_{A^*}, & (\xi \in \mathfrak{g}). \end{cases}$$

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$0 \in \mathfrak{g}^*$ regular value of J

$$i : J^{-1}(0) \to \mathcal{G} \qquad p : J^{-1}(0) \to J^{-1}(0)/G$$

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Hamiltonian actions on Lie bialgebroids

Definition
Let (A, A^*) be a Lie bialgebroid over M and G be a Lie group acting on A, with infinitesimal map $\psi_* : \mathfrak{g} \to \mathrm{Der}(A)$. The action is *Hamiltonian* if there exists a Lie algebroid morphism $j : A \to \mathfrak{g}^*$, called the *momentum map*, such that

$$\psi^*(\xi) = [\![j^*(\xi), \cdot]\!]_{A^*}, \qquad (\xi \in \mathfrak{g}),$$

$$\psi^*(\xi) \circ j^* = j^* \circ \mathrm{ad}_\xi \qquad (\xi \in \mathfrak{g}).$$

Reduction of Lie bialgebroids

Theorem Let (A, A^*) be a Lie bialgebroid over M, G be a Lie group and $\rho : \mathfrak{g} \to \mathfrak{X}(A)$ be Hamiltonian action with momentum map $j : A \to \mathfrak{g}^*$. Then, $A_{\mathrm{red}} := \operatorname{Ker} j/G \to M/G$ is endowed with a Lie algebroid structure. Moreover, the dual bundle A^*_{red} is also equipped with a Lie algebroid structure and $(A_{\mathrm{red}}, A^*_{\mathrm{red}})$ is a Lie bialgebroid over M/G. Moreover, the Poisson structure on M/G induced by the Lie bialgebroid coincides with the reduction of π_M.

$$\mathrm{d}_{A^*_{\mathrm{red}}} X = p(\mathrm{d}_{A^*} \widetilde{X}), \qquad (X \in \mathfrak{X}(A_{\mathrm{red}})),$$

where $\widetilde{X}$ is a G-invariant section of $\operatorname{Ker} j$ associated with X.

Examples

1.- (A, A^*, π_A) exact Lie algebroid $\quad + \quad \phi : \mathfrak{g} \to \Gamma(A)$

$$\mathcal{L}_{\phi(\xi)}\pi_A = 0$$

$j : A \to \mathfrak{g}^*$ Lie algebroid morphism

$$\pi_A^\sharp(j^*(\xi)) = \phi(\xi), \qquad (\xi \in \mathfrak{g}).$$

2.- Symplectic groupoids and reduction of Poisson manifolds

$G \times (M, \pi) \to (M, \pi)$ Poisson action with infinitesimal $\psi : \mathfrak{g} \to \mathfrak{X}(M)$

$\Rightarrow G \times T^*M \times T^*M$ Hamiltonian action on (T^*M, TM) with momentum map $j := \psi^* : T^*M \to \mathfrak{g}^*$

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Inner actions with coboundary momentum map

Definition

Let (A, A^*) be a Lie bialgebroid over M and Φ be an inner action of a Lie group G on A with associated map $\phi : \mathfrak{g} \to \Gamma(A)$. Moreover, suppose that $\mu : M \to \mathfrak{g}^*$ is a smooth equivariant map with respect to the action Φ_0 on the base space M. We say that Φ is an *inner Hamiltonian action* if

$$\phi(\xi) = \mathrm{d}_{A^*}\widehat{\mu}_\xi, \qquad (\xi \in \mathfrak{g}).$$

Remark

(ϕ, μ) Inner Hamiltonian action $\Rightarrow$ $(\Phi, d\mu \circ \rho)$ Hamiltonian action

$$
\begin{array}{rcl}
\Phi^T\colon TG \times A & \to & A \\
((g,\xi), a_x) & \mapsto & \Phi_g(a_x) + \Phi_g(\phi(\xi)_x)
\end{array}
$$

$$
\begin{array}{rcl}
\mu^T\colon A & \to & T\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}^* \\
a & \mapsto & \mu^T(a_x) = ((d\mu \circ \rho)(a_x), \mu(x))
\end{array}
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Theorem

Let (A, A^*) be a Lie bialgebroid over M and Φ be an inner Hamiltonian action of a Lie group G on A with momentum map $\mu : M \to \mathfrak{g}^*$. Then,

(i) $\widetilde{A}_\mu = (\mu^T)^{-1}(0,\lambda)/TG_\lambda \to \mu^{-1}(\lambda)/G_\lambda$ is a Lie algebroid.

(ii) The dual bundle $((\mu^T)^{-1}(0,\lambda)/TG_\lambda)^* \to \mu^{-1}(\lambda)/G_\lambda$ is endowed with a Lie algebroid structure.

(iii) $(\widetilde{A}_\lambda, \widetilde{A}^*_\lambda)$ is a Lie bialgebroid over $\mu^{-1}(\lambda)/G_\lambda$.

$$\begin{array}{rcl} \Phi^T\colon TG\times A & \rightarrow & A \\ ((g,\xi),a_x) & \mapsto & \Phi_g(a_x)+\Phi_g(\phi(\xi)_x) \end{array}$$

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