Isoparallel problems as generalized nonholonomic systems

Sebastián J. Ferraro

Departamento de Matemáticas, IMAFF – CSIC, Madrid

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Constrained mechanical system

- $L\colon TQ \to \mathbb{R}$
- $C \subseteq TQ$ (distribution)

Principle of virtual work: No work is done by the constraint forces in any virtual displacement consistent with the constraints

$$F \in C^{\circ} \subset T^*Q$$

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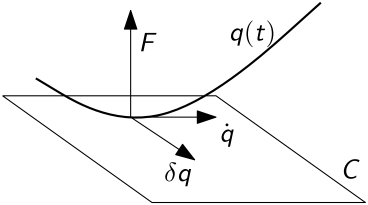

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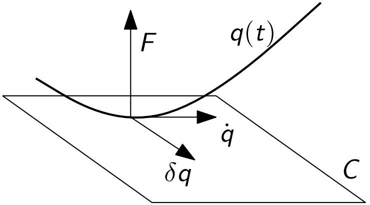

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Lagrange–d'Alembert's principle

In a Lagrangian system,

$$F = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

so $F \in C^\circ$ means

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right) \cdot \delta q = 0 \quad \text{for } \delta q \in C$$

(in addition, $\dot{q} \in C$)

$F \in C^\circ \quad \Rightarrow \quad F \cdot \dot{q} = 0$ (power) $\quad \Rightarrow \quad$ preservation of energy

Constraints on motion + pple. of virtual work

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constraints on variations

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Some advantages:

- It is convenient
- Applies to a wide range of problems
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- Reduction, discretization...
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Generalization

Suitable generalization of Lagrange–d'Alembert's principle

- elastic rolling bodies, pneumatic tires
- dissipative systems
- servomechanisms, control strategies
- isoparallel problems

Maintain advantages, such as covariance and reduction theory

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Generalized nonholonomic systems

Generalized nonholonomic system (GNHS) on Q:

- $L\colon TQ \to \mathbb{R}$
- $C_K \subseteq TQ$ (kinematic distribution)
- $C_V \subseteq TQ$ (variational distribution)

A trajectory of a GNHS is a curve $q(t)$ such that

- $\dot{q} \in C_K$
- $\delta \int L(q, \dot{q})\, dt = 0$ with respect to $\delta q \in C_V$, which implies

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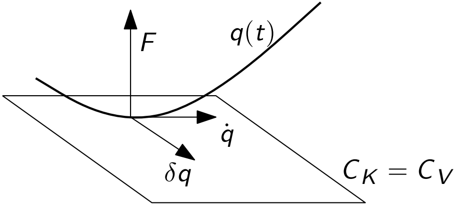

Nonholonomic
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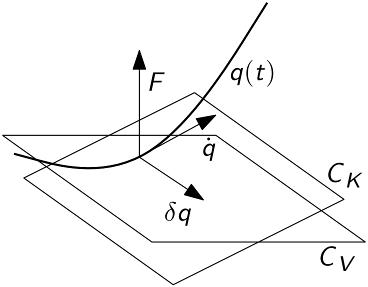

Generalized nonholonomic

Energy is not preserved in GNHS

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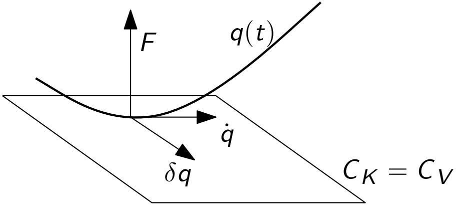

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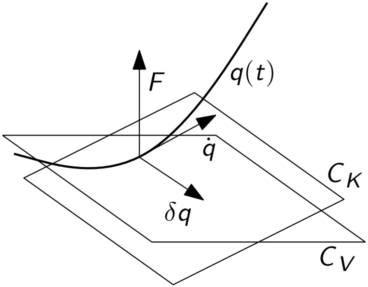

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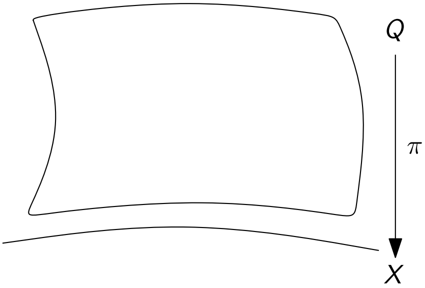
Isoparallel problems

- Principal bundle Q, with structure group G
- Principal connection $A\colon TQ \to \mathfrak{g}$



- For each curve $x(t)$ joining x_0 to x_1, there is a *parallel transport operator* mapping $\pi^{-1}(x_0)$ into $\pi^{-1}(x_1)$
- Many curves might share the same parallel transport operator

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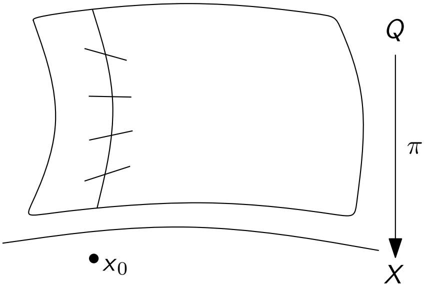

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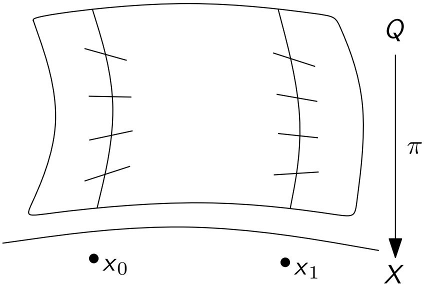

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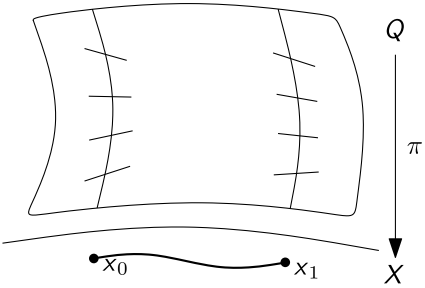

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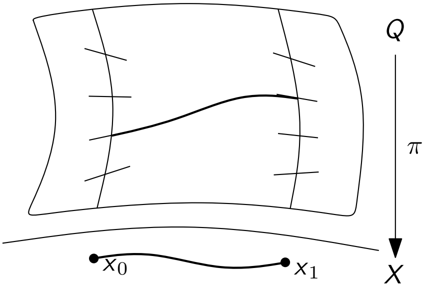

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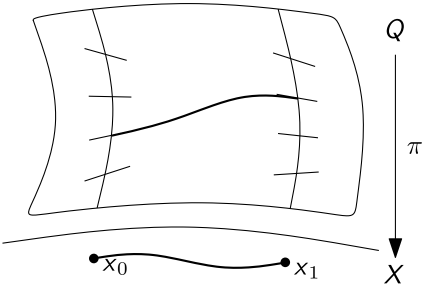

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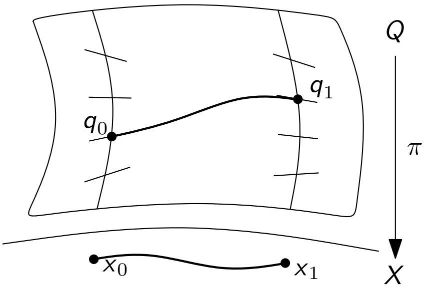


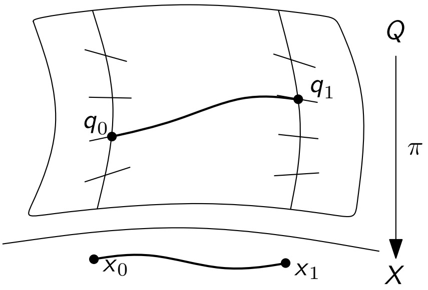
- Riemannian metric on the base

Isoparallel problem: Find a curve on X that extremizes length among those with a given parallel transport operator.

Equivalently: Among those horizontal curves joining q_0 to q_1, find one whose projection extremizes length.

When $x_0 = x_1$, this is called the *isoholonomic* problem.

Isoparallel problems



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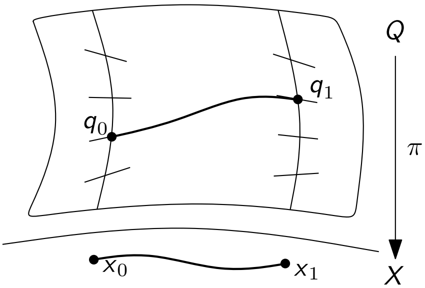

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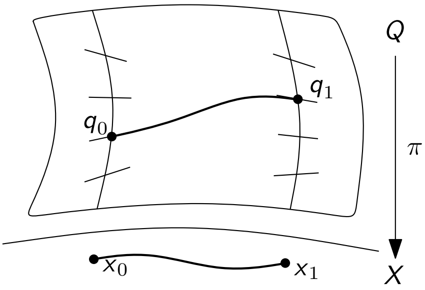

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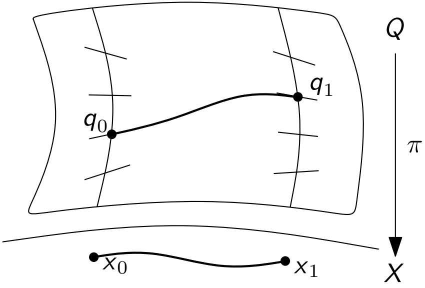

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For a given parallel transport operator...

- Is there any curve realizing it?
- Are there “enough” curves?
  - Working with C^1 curves, not always (*rigid curves*)
  - Working with absolutely continuous curves, yes (if horizontal distribution is bracket-generating)

Absolutely continuous curves provide a nice setting:

- Horizontal lifts are still well defined
- Mixed partial derivatives are equal in L^2
- Integration by parts works

We will assume the regular case

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Examples

Falling cat:

$$Q = \text{space of positions of the cat as a deformable body}$$
$$G = \mathrm{SO}(3)$$
$$X = Q/G = \text{shapes (disregarding rigid rotations)}$$

Horizontal distribution in TQ defined by “angular momentum equals zero”

Loop on X --→ Hor. lift --→ Reorientation

Metric on X measuring energy expenditure

▶ What is the most efficient way that the cat could change its shape so as to land on its feet?

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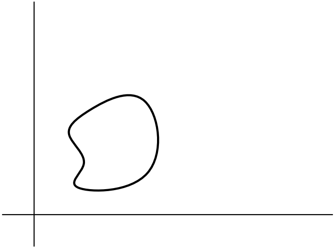
Locomotion of a microorganism in a fluid



$Q \subset$ mappings from S^1 into $\mathbb{R}^2$
$G = \mathrm{SE}(2)$
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$v \in TQ$: vector field on the membrane $\dashrightarrow$ response of the fluid

Define a principal connection on Q and a metric on X

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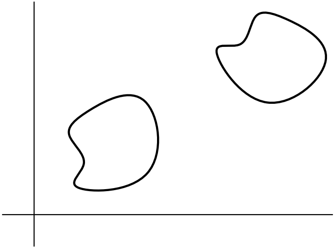

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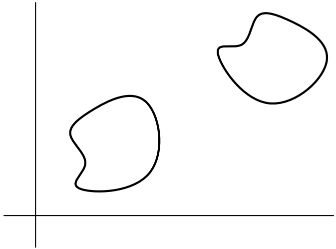

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Take a bi-invariant metric β on G (if there is any!)
If k is the metric on the base, define $L\colon TQ \to \mathbb{R}$ by

$$L(q,\dot{q}) = \frac{1}{2}\Big(k\left(T\pi(q,\dot{q}), T\pi(q,\dot{q})\right) + \beta\left(A(q,\dot{q}), A(q,\dot{q})\right)\Big)$$

(related to the metric $k \oplus \beta$ on Q)

- Then the (normal) solutions of the isoparallel problem are precisely the projections of the geodesics [Montgomery, 1990]

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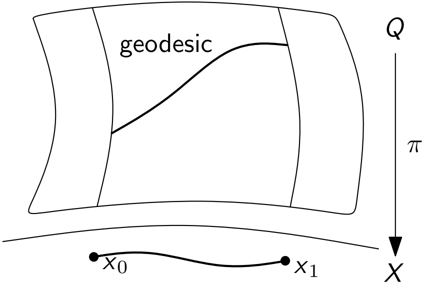

Lagrangian system ⇢ project (and lift)

But... some groups do not admit a bi-invariant metric (e.g. SE(2))

- Replace Lagrangian system by *generalized nonholonomic system* [Cendra and Ferraro, 2006]

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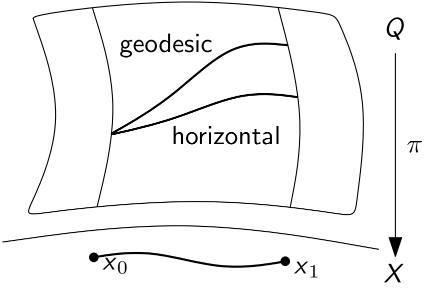

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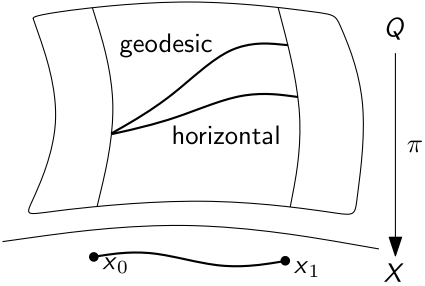

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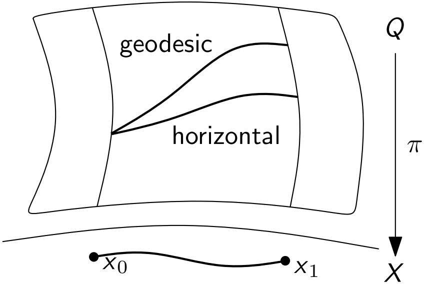

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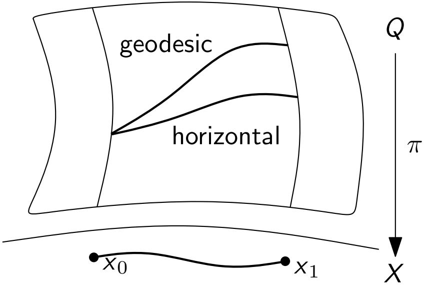

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Some definitions

$M =$ symmetric bilinear forms on $\mathfrak{g}$.

G-action: For $g \in G$, $\beta \in M$ and $\eta_1, \eta_2 \in \mathfrak{g}$, define

$$(g\beta)(\eta_1, \eta_2) = \beta\left(\mathrm{Ad}_{g^{-1}}\,\eta_1, \mathrm{Ad}_{g^{-1}}\,\eta_2\right).$$

Infinitesimal generator: If $\xi \in \mathfrak{g}$ and $\beta \in M$ then

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The GNHS

Generalized nonholonomic system:

- It is a system on $Q \times M$
- $L\colon\ T(Q \times M) \to \mathbb{R}$,

$$L(q,\beta,\dot{q},\dot{\beta}) = \frac{1}{2}k\left(T\pi\left(q,\dot{q}\right), T\pi\left(q,\dot{q}\right)\right) + \frac{1}{2}\beta\left(A\left(q,\dot{q}\right), A\left(q,\dot{q}\right)\right)$$

- Distributions in $T(Q \times M)$
  - Kinematic $C_K\colon\ \dot{\beta} = A(q,\dot{q})\beta$
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Notation

Associated bundles:

$$\widetilde{M} = (Q \times M)/G$$

$$\widetilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$$

These are bundles over $X = Q/G$ with standard fiber M (resp. $\mathfrak{g}$).

Notation:

$$\bar{\beta} = [q, \beta]_G \in \widetilde{M}$$

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$$T(Q \times M) \equiv TQ \times TM \equiv TQ \oplus (Q \times M) \oplus (Q \times M)$$

$$\big(T(Q \times M)\big)/G \equiv TX \oplus \tilde{\mathfrak{g}} \oplus 2\widetilde{M}$$

The reduced Lagrangian $\ell\colon TX \oplus \tilde{\mathfrak{g}} \oplus 2\widetilde{M} \to \mathbb{R}$ is

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Reduced equations for the GNHS

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- Find a curve from q_0 to q_1
- Extremizing $\int_{t_0}^{t_1} \sqrt{k\big(T\pi(\dot{q}), T\pi(\dot{q})\big)}\, dt$
- Constrained to $A(q, \dot{q}) = 0$
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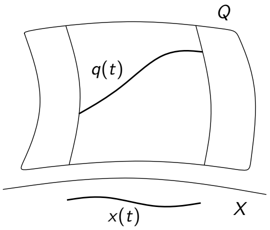

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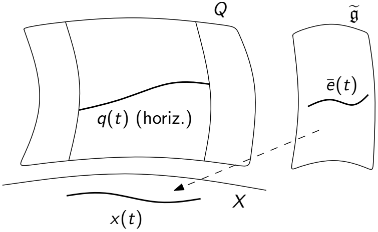

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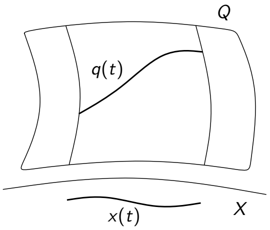

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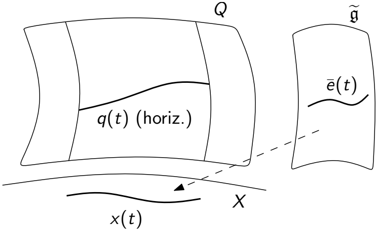

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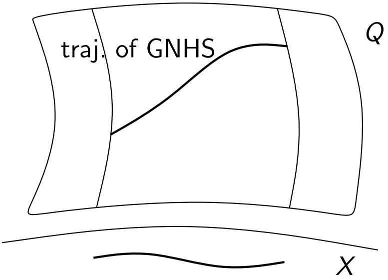

Trajectory of GNHS $\dashrightarrow$ project and lift

Geodesic $\dashrightarrow$ project and lift

With a bi-invariant metric, $C_K = TQ \oplus 0$ and $C_V = TQ \oplus 0$, and the GNHS becomes Montgomery's Lagrangian system

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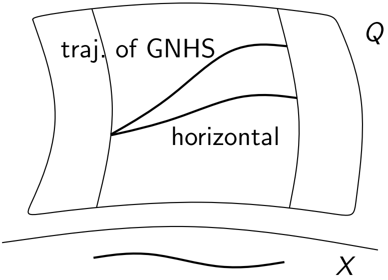

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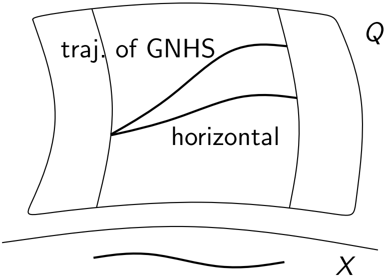

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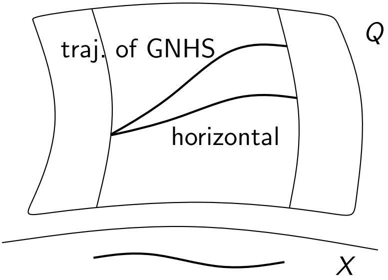

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  - Reduction
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