

RELATIVE POINCARÉ LEMMA, CONTRACTIBILITY, QUASI-HOMOGENEITY AND VECTOR FIELDS TANGENT TO A SINGULAR VARIETY

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These are results of the joint work with S. Janeczko and M. Zhitomirskii.

Let N be the germ at 0 of a singular subset in \mathbb{R}^n . We study the interplay between Poincaré Lemma, contractibility, quasi-homogeneity of N and the algebra of vector fields tangent to N .

We work with germs in either the analytic or the C^∞ category. By Poincaré lemma property we mean the following property of N : any closed differential $(p+1)$ -form vanishing at any point of N is a differential of a p -form which also vanishes at any point of N .

The proof of the classical (global) Poincaré lemma uses contraction to a point, see for example [Bo-Tu]. This method also can be applied to singular varieties $N \subset \mathbb{R}^n$. The main corollary of results in [Re-3] is as follows : if \mathbb{R}^n is analytically contractible to 0 along N then N has the Poincaré lemma property. The analytic contraction of \mathbb{R}^n to 0 along N is an analytic family of maps $F_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that F_1 is the identity map, $F_0(\mathbb{R}^n) = 0$, and $F_t(N) \subset N$ for all t . We show that the Poincaré lemma property holds under weaker assumptions: it is enough to require that \mathbb{R}^n is analytically contractible to N along N (i.e. $F_0(\mathbb{R}^n) \subset N$ instead of $F_0(\mathbb{R}^n) = 0$). Also, the analytic contraction can be replaced by the piece-wise analytic contraction with respect to the parameter t . This result remains true in the C^∞ category.

The Poincaré lemma property of N can be expressed as the triviality of de Rham cohomology groups of N . Such cohomology groups were constructed in [Gr-Ke], [Gr]. See also [Re-3], [Bl-He]. We present a reduction theorem which helps to study the cohomology groups and to distinguish the cases when they are trivial.

Checking if there exists a smooth or analytic contraction to 0 along N is problematic. The simplest case where this is so is the case that N is quasi-homogeneous. This means that in some local coordinate system N contains, along with any point (x_1, \dots, x_n) , the curve $t^{\lambda_1}x_1, \dots, t^{\lambda_n}x_n$, where $\lambda_1, \dots, \lambda_n$ are positive numbers, called weights. Is it the only case of smooth (analytic) contractibility? This question was studied in many works, with attempt to give a positive answer for a wide class of singular varieties N . In [Re-1] it was proved that the analytic contractibility and quasi-homogeneity is the same property if N is singular plain curve, with an algebraically isolated singularity. Moreover, in [Re-1] it was proved that these properties are equivalent to the Poincaré lemma property. Later this result was generalized in [Sa], where the same was proved in the case that N is a singular hypersurface with an algebraically isolated singularity.

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We show that the classical quasi-homogeneity is not a necessary condition for contractibility (and consequently for the Poincaré lemma property). We give a definition of quasi-homogeneity of N with respect to a smooth submanifold $S \subset \mathbb{R}^n$. It can be understood as the classical quasi-homogeneity with some of the weights allowed to be 0. The classical quasi-homogeneity is the quasi-homogeneity with respect to $S = \{0\}$. We prove that if N is quasi-homogeneous with respect to S and S is contained in N then \mathbb{R}^n is contractible to N along N (and then by our results N has the Poincaré lemma property.) We give an example of an analytic singular set N which is quasi-homogeneous with respect to a certain smooth submanifold S in some coordinate system, but not quasi-homogeneous, i.e. not quasi-homogeneous with respect to $S = \{0\}$, in *any* coordinate system. We define quasi-homogeneity with respect to a chain of smooth submanifolds $S_1 \subset S_2 \subset \dots \subset S_r$ and show that the quasi-homogeneity of N with respect to the chain implies piece-wise smooth contractibility of \mathbb{R}^n to S_1 along N . If $S_1 \subset N$ then this implies contractibility to N and consequently the Poincaré lemma property. In general case, when S_1 is not contained in N our reduction theorem reduces the cohomology groups of $N \subset \mathbb{R}^n$ to the cohomology groups of $(N \cap S_1) \subset S_1$.

The quasi-homogeneity or its generalizations (quasi-homogeneity with respect to a smooth submanifold or a chain of smooth submanifolds) remain the main tool to check the (piece-wise) smooth or analytic contractibility and the Poincaré lemma property. According to A. Givental', positive quasi-homogeneity should be regarded as an analytic analog of contractibility, see [Gi].

How to check if N is quasi-homogeneous? Assume that the set of non-singular points of N is dense and that the ideal $I(N)$ of functions vanishing on N is p -generated ($p < \infty$) and can be identified with N (this is always so for analytic varieties). Then the simplest way to prove that N is quasi-homogeneous is to prove that the ideal $I(N)$ is quasi-homogeneous, i.e. there exist (a) a local coordinate system x and (b) tuple of generators H_1, \dots, H_p such that in the coordinate system x each of the generators H_1, \dots, H_p is quasi-homogeneous with the same weights. How to check that (a) and (b) exist or to prove that they do not exist if one works with arbitrary generators and arbitrary local coordinate system? One should not expect an algorithm, but it is important to give an answer in terms of some canonical object.

It is clear that the quasi-homogeneity of N (or the ideal $I(N)$) is related to the following property of the algebra of all smooth or analytic vector fields V tangent to N (or to the ideal $I(N)$ which means that $V(f) \in I(N)$ for any $f \in I(N)$). If N is quasi-homogeneous then one of these vector fields must have positive eigenvalues. In fact, it follows from the definition of quasi-homogeneity of N that in suitable coordinates the Euler vector field $E = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \dots + \lambda_n x_n \frac{\partial}{\partial x_n}$, where $\lambda_1, \dots, \lambda_n$ are the weights, is tangent to N . If we change the coordinate system then E will be transformed to a vector field of another form, but the new vector field has the same eigenvalues $\lambda_1, \dots, \lambda_n$. Therefore the quasi-homogeneity of N implies the existence of a smooth (analytic) vector field V which is tangent to N and has positive eigenvalues at the singular point 0. Is this also a sufficient property for quasi-homogeneity?

We give the positive answer: *N is quasi-homogeneous if and only if there exists a smooth (analytic) vector field V , $V(0) = 0$, which is tangent to N and has positive eigenvalues at the singular point 0 .* We also generalize the above theorem from the classical quasi-homogeneity (i.e. quasi-homogeneity with respect to $\{0\}$) to the quasi-homogeneity with respect to a smooth submanifold $S \subset \mathbb{R}^n$. A necessary and sufficient condition for such quasi-homogeneity is the existence of a vector field V which vanishes at any point of S and has at any point of S the same positive eigenvalues corresponding to directions transversal to S .

We also reformulate these results in terms of ideal $I(N)$ with an additional statement on the degree of quasi-homogeneity of the ideal. The tangency of a vector field V to the ideal $I(N)$ implies that $V(H) = R(\cdot)H$, where $H = (H_1, \dots, H_p)^t$ is any tuple of generators of $I(N)$ and $R(\cdot)$ is a matrix-function. It is easy to see that the eigenvalues d_1, \dots, d_p of the matrix $R(0)$ do not depend on the choice of generators, i.e. they are the invariants of a vector field V tangent to $I(N)$. We prove that if V has positive eigenvalues then there exist a coordinate system and a tuple of generators $\hat{H}_1, \dots, \hat{H}_p$ such that in this coordinate system \hat{H}_i is quasi-homogeneous of degree d_i .

We generalize these results from the classical quasi-homogeneity (i.e. quasi-homogeneity with respect to $\{0\}$) to the quasi-homogeneity with respect to a smooth submanifold $S \subset \mathbb{R}^n$.

All results hold in either the analytic or the C^∞ category. In the analytic category the particular case $p = 1$ they can be compared with the distinguished Saito theorem [Sa] stating that function-germ H with algebraically isolated singularity is quasi-homogeneous if and only if it belongs to the ideal generated by its partial derivatives. Recently the Saito theorem was generalized in [Vo] (see also references in [Vo]) for complete intersection singularities. The relation between the results in [Vo] and Theorem 4.3 is yet to be understood.

We compare the Poincaré lemma property used in this paper with a different version of this property studied in [Fe],[Gi]: the property of an analytic set N that any closed $(p + 1)$ -form with vanishing pullback to the regular part of N is a differential of a p -form satisfying the same condition. The corresponding de Rham complexes are a priori different. The certain types of contractibility are sufficient conditions for exactness of the both complexes. Nevertheless, we do not know if the cohomology groups of the two complexes are isomorphic for any analytic varieties.

We also define the algebraic restriction of a differential form β to a subset N of \mathbb{R}^n as an equivalence class of the following relation:

p -forms β_1 and β_2 have the same algebraic restriction to N if $\beta_1 - \beta_2 = \alpha - d\gamma$, where α is a p -form which vanishes at any point of N and β is a $(p - 1)$ -form which vanishes at any point of N (see also [Zh1]).

Let N be a subset of \mathbb{R}^{2k} and let ω_0 and ω_1 be symplectic forms on \mathbb{R}^{2k} .

Assume that N has the Poincaré lemma property. If ω_0 and ω_1 have the same algebraic restriction to N then there exists a diffeomorphism Φ of \mathbb{R}^{2k} such that $\Phi^*\omega_1 = \omega_0$ and $\Phi|_N = Id_N$.

Using this result we get many local classification results in symplectic geometry. For example classifications of curves of the type A_{2k} obtained by V. I. Arnol'd in [Ar].

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