Symmetries and Quantum Entanglement

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Motivations

- Quantum Information Theory
- Quantum Computing
- Quantum Teleportation
- ......
- structure of composite states in QM !!!

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Quantum entanglement — pure states

$$\mathcal{H}_{\text{total}} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Definition:

$\psi \in \mathcal{H}_{\text{total}}$ is separable iff

$$\psi = \psi_A \otimes \psi_B$$

with $\psi_A \in \mathcal{H}_A$ & $\psi_B \in \mathcal{H}_B$.

Definition:

$\psi \in \mathcal{H}_{\text{total}}$ is entangled (inseparable) iff it is NOT separable.

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Test

$$\psi \otimes \varphi \text{ ?}$$

$$\psi \otimes \chi + \varphi \otimes \omega + \psi \otimes \omega + \varphi \otimes \chi \text{ ?}$$

$$= \ (\psi + \varphi) \otimes (\chi + \omega) \text{ !!!}$$

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How to check for separability ???

Quantum entanglement — pure states

Schmidt decomposition

$$\psi \in \mathcal{H}_{\text{total}}$$

$$e_1, e_2, \ldots \in \mathcal{H}_A \quad \& \quad f_1, f_2, \ldots \in \mathcal{H}_B$$

$$\psi = \sum_\alpha \lambda_\alpha \, e_\alpha \otimes f_\alpha$$

$$\lambda_\alpha \geq 0 \, , \quad \sum_\alpha \lambda_\alpha^2 = 1$$

Theorem:

$\psi \in \mathcal{H}_{\text{total}}$ is separable iff ONLY ONE $\lambda_\alpha \neq 0$

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Mixed states

Density matrix in $\mathcal{H}$: $\rho \geq 0$, $\mathrm{Tr}\,\rho = 1$

A set of mixed states is CONVEX.

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Definition: (Werner 1989)

A bipartite state ρ is separable (classically correlated) iff

$$\rho \;=\; \sum_k p_k \, \rho_k^{(A)} \otimes \rho_k^{(B)}$$

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$\rho_k^{(A)}$ state in $\mathcal{H}_A$

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O_A & O_B observables in $\mathcal{H}_A$ & $\mathcal{H}_B$

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$$\langle O_A \otimes O_B \rangle_\rho = \sum_k p_k \, a_k b_k$$

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Separability problem

ρ — mixed state in $\mathcal{H}_{\text{total}} = \mathcal{H}_A \otimes \mathcal{H}_B$

Is ρ separable or entangled ?

To be separable is not a spectral property!!!

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Positive Partial Transpose (PPT)

$$\tau(X) = X^T$$

Definition:

A state ρ in $\mathcal{H}_A \otimes \mathcal{H}_B$ is PPT iff

$$(\mathbb{1} \otimes \tau)\rho \geq 0 .$$

Theorem: (Peres 1996)

Any separable state is PPT.

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Separability vs. PPT

SEPARABLE STATES $\subseteq$ PPT STATES

Theorem: (HHH 1996)

For

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \;,\; \mathbb{C}^2 \otimes \mathbb{C}^3 \;,\; \mathbb{C}^3 \otimes \mathbb{C}^2$$

SEPARABLE STATES $=$ PPT STATES

If $\quad \dim \mathcal{H}_A \cdot \dim \mathcal{H}_B \;>\; 6$

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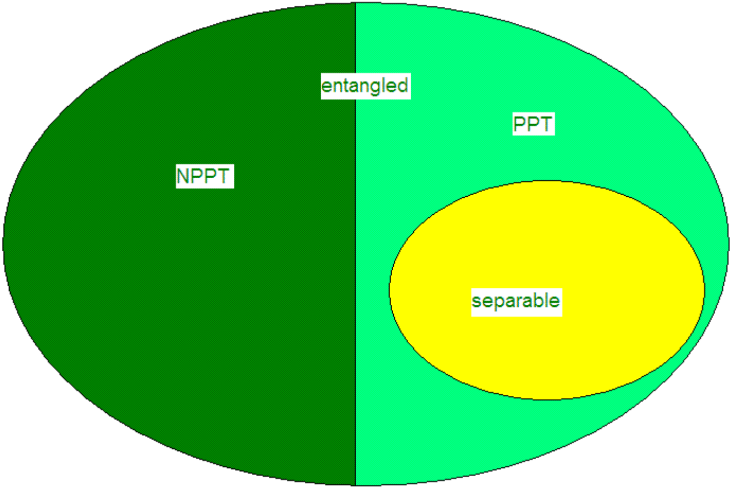
$3 \otimes 3$ PPT but entangled

$$\rho = \frac{1}{1+8a}\left[\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2}(1+a) & \cdot & \frac{1}{2}\sqrt{1-a^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \frac{1}{2}\sqrt{1-a^2} & \cdot & \frac{1}{2}(1+a) \end{array}\right] \begin{array}{c} |00\rangle \\ |01\rangle \\ |02\rangle \\ \hline |10\rangle \\ |11\rangle \\ |12\rangle \\ \hline |20\rangle \\ |21\rangle \\ |22\rangle \end{array}$$

$$\cdot \equiv 0 \,, \qquad a \in (0,1)$$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separable

Separability vs. positive maps

$$\mathcal{A} \ \& \ \mathcal{B} \ — \ C^*\text{–algebras}$$

$$\mathcal{A} \ni a \geq 0 \quad \Longleftrightarrow \quad a = xx^* \ .$$

Definition:

A linear map

$$\Lambda \, : \, \mathcal{A} \longrightarrow \mathcal{B}$$

is positive iff

$$a \geq 0 \quad \Longrightarrow \quad \Lambda(a) \geq 0 \ .$$

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Theorem: (HHH 1996)

A mixed state ρ in $\mathcal{H}_{\text{total}}$ is separable iff

$$(\mathbb{1} \otimes \Lambda)\rho \geq 0$$

for all positive maps

$$\Lambda \ : \ \mathcal{B}(\mathcal{H}_B) \longrightarrow \mathcal{B}(\mathcal{H}_A) \ .$$

Unfortunately, the structure of positive maps is poorly known!

17th Hilbert problem

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17th Hilbert problem

"Symmetry is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection."

Symmetric states

$$\mathcal{H}_{\text{total}} = \mathcal{H}_A \otimes \mathcal{H}_B$$

G — compact Lie group

$\mathfrak{D}_A$ & $\mathfrak{D}_B$ unitary IRREPs

$$\mathfrak{D}_A(g) : \mathcal{H}_A \longrightarrow \mathcal{H}_A$$

$$\mathfrak{D}_B(g) : \mathcal{H}_B \longrightarrow \mathcal{H}_B$$

for all $g \in G$

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A state ρ in $\mathcal{H}_{\text{total}}$ is $\mathfrak{D}_A \otimes \mathfrak{D}_B$–invariant iff

$$\left[\mathfrak{D}_A(g) \otimes \mathfrak{D}_B(g) \, , \, \rho \right] \; = \; 0$$

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$$\rho \;\in\; (\mathfrak{D}_A \otimes \mathfrak{D}_B)'$$

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Example — Werner state

$$\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^d$$

$$G \;=\; U(d)$$

$$\mathfrak{D}_A = \mathfrak{D}_B = \text{defining representation}$$

Werner state: $\longrightarrow$ $[U \otimes U \,,\, \rho] = 0$

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Werner state

$$\Big(U(d) \otimes U(d)\Big)' = \Big\{ \mathbb{I} \otimes \mathbb{I} \, , \, \mathbb{F} \Big\}$$

$$\mathbb{F}(\psi \otimes \varphi) = \varphi \otimes \psi$$

$$\mathbb{F} = \sum_{i,j=1}^{d} e_{ij} \otimes e_{ji}$$

$$e_{ij} = |i\rangle\langle j|$$

Werner state

$$Q^0 = \frac{1}{2}(\mathbb{I}^{\otimes 2} + \mathbb{F}) , \qquad Q^1 = \frac{1}{2}(\mathbb{I}^{\otimes 2} - \mathbb{F})$$

$$Q^0 + Q^1 = \mathbb{I}^{\otimes 2}$$

General form

$$\rho = q_0 \widetilde{Q}^0 + q_1 \widetilde{Q}^1 ; \qquad \widetilde{Q}^\alpha = Q^\alpha / \mathrm{Tr}\, Q^\alpha ,$$

$$q_\alpha \geq 0 , \qquad q_0 + q_1 = 1$$

Theorem

$$\rho - \text{separable} \iff \rho - \text{PPT} \iff q_0 \geq \frac{1}{2}$$

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SU(2) Symmetry

two particles $\longleftrightarrow$ spin–$j_A \otimes$ spin–j_B $\quad (j_B \geq j_A)$

$$\mathcal{H}_A = \mathbb{C}^{d_A} \quad \& \quad \mathcal{H}_B = \mathbb{C}^{d_B}$$

$$d_A = 2j_A + 1 \quad \& \quad d_B = 2j_B + 1$$

$$G = SU(2) \longrightarrow \mathfrak{D}^{(j_A)} \otimes \mathfrak{D}^{(j_B)}$$

$$\mathfrak{D}^{(j_A)} \otimes \mathfrak{D}^{(j_B)} = \bigoplus_{J=j_B-j_A}^{j_A+j_B} \mathfrak{D}^{(J)} ,$$

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$$\mathcal{H}_{\text{total}} = \mathbb{C}^{2j_A+1} \otimes \mathbb{C}^{2j_B+1} \longrightarrow |j_A, m_A\rangle \otimes |j_B, m_B\rangle$$

$$\boxed{|j_A, m_A\rangle \otimes |j_B, m_B\rangle \longrightarrow |J, M\rangle}$$

$$J = j_B - j_A, \ldots, j_A + j_B$$

$$M = -J, \ldots, +J$$

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$$J = j_B - j_A, \ldots, j_A + j_B$$

$$M = -J, \ldots, +J$$

SU(2) symmetry

$$Q^J = \sum_M |JM\rangle\langle JM| \ , \quad J = j_B - j_A, \ldots, j_A + j_B$$

$$\left[\mathfrak{D}^{(j_A)} \otimes \mathfrak{D}^{(j_B)} , Q^J \right] = 0$$

General form

$$\rho = \sum_{J=j_B-j_A}^{j_A+j_B} q_J \, \widetilde{Q}^J \ ,$$

$$q_J \geq 0 \ , \quad \sum_J q_J = 1$$

$2 j_B$ dimensional simplex

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SU(2) symmetry vs. separability

- $j_A = 1/2$ and arbitrary $j_B \geq j_A$

$$d_A \otimes d_B \;=\; 2 \otimes k \;; \qquad k = 2, 3, \ldots$$

ρ is separable $\iff$ ρ is PPT

- $j_A = 1$ and arbitrary integer $j_B \geq j_A$

$$d_A \otimes d_B \;=\; 3 \otimes (2k+1) \;; \qquad k = 1, 2, \ldots$$

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One may investigate other groups $G \subset U(d)$

- orthogonal group
- symplectic group
- discrete groups
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One may generalize construction of symmetric states to multipartite case

$$\mathcal{H}_{\mathsf{total}} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$$

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Let us look for PPT states

It is very hard to characterize a set of separable states

SEPARABLE STATES $\subseteq$ PPT STATES

Let us try to characterize a set of PPT states

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How to construct PPT states?

A state ρ in $\mathcal{H}_A \otimes \mathcal{H}_B$ is PPT iff

$$(\mathbb{1} \otimes \tau)\rho \geq 0 \ .$$

PPT is easy to check!

But, it is not easy to construct ρ which is PPT!!!

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Symmetry vs. PPT

$$\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^d$$

$$G \;=\; U(1) \times \ldots \times U(1) \;\cong\; T(d) \;\subset\; U(d)$$

$$[\,U \otimes \overline{U}\,,\,\rho\,] \;=\; 0\,, \quad \text{for} \;\; U \in \mathbb{T}(d)$$

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Abelian symmetry

General form

$$\rho \;=\; \sum_{i,j=0}^{d-1} a_{ij}\, e_{ij} \otimes e_{ij} \;+\; \sum_{i\neq j=0}^{d-1} b_{ij}\, e_{ii} \otimes e_{jj}$$

ρ is positive iff

- $[a_{ij}] \geq 0$
- $b_{ij} \geq 0$ for all $i \neq j$

$$\mathrm{Tr}\,\rho = 1 \quad \Longleftrightarrow \quad \mathrm{Tr}\,[a_{ij}] = 1$$

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$$\rho \;=\; \sum_{i,j=0}^{2} a_{ij}\, e_{ij} \otimes e_{ij} \;+\; \sum_{i \neq j=0}^{2} b_{ij}\, e_{ii} \otimes e_{jj}$$

$$\rho \;=\; \left[\begin{array}{ccc|ccc|ccc}
a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\
\cdot & b_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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General result

Theorem: D.C. & A. Kossakowski, PRA (2006)

$$\rho \;=\; \sum_{i,j=0}^{d-1} a_{ij}\, e_{ij} \otimes e_{ij} + \sum_{i \neq j=0}^{d-1} b_{ij}\, e_{ii} \otimes e_{jj}$$

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Many examples of PPT states belong to this class!

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New class in $d \otimes d$

Now we construct a wide class of states in $\mathcal{H}_{\mathrm{total}} = \mathbb{C}^d \otimes \mathbb{C}^d$ which is based on symmetric decompositions of $\mathcal{H}_{\mathrm{total}}$.

It generalizes the previous class invariant under $\mathbb{T}(d) \subset U(d)$.

Let us consider $d = 3$.

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Circulant decomposition of $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\mathbb{C}^3 \quad \longleftrightarrow \quad e_0\,,\, e_1\,,\, e_2$$

$$S \;:\; \mathbb{C}^3 \longrightarrow \mathbb{C}^3$$

$$S\, e_i \;=\; e_{i+1} \qquad \text{mod } 3$$

$$\Sigma_0 \;=\; \text{span}\,\{e_0 \otimes e_0\,,\, e_1 \otimes e_1\,,\, e_2 \otimes e_2\} \;\in\; \mathbb{C}^3 \otimes \mathbb{C}^3$$

$$\Sigma_1 \;=\; (\mathbb{1} \otimes S)\Sigma_0$$

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Circulant decomposition of $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\Sigma_0 = \text{span}\{e_0 \otimes e_0 \,,\, e_1 \otimes e_1 \,,\, e_2 \otimes e_2\}$$

$$\Sigma_1 = \text{span}\{e_0 \otimes e_1 \,,\, e_1 \otimes e_2 \,,\, e_2 \otimes e_0\}$$

$$\Sigma_2 = \text{span}\{e_0 \otimes e_2 \,,\, e_1 \otimes e_0 \,,\, e_2 \otimes e_1\}$$

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = \Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2$$

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$$\Sigma_2 = \text{span}\{e_0 \otimes e_2\,, e_1 \otimes e_0\,, e_2 \otimes e_1\}$$

$$\boxed{\mathbb{C}^3 \otimes \mathbb{C}^3 = \Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2}$$

Circulant state — construction

$a\,,\,b\,,\,c \quad 3\times 3$ matrices

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} , \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} , \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix} .$$

$$\boxed{a \geq 0\,, \quad b \geq 0\,, \quad c \geq 0}$$

$$\Sigma_0 \;=\; \mathrm{span}\left\{e_0 \otimes e_0\,,\, e_1 \otimes e_1\,,\, e_2 \otimes e_2\right\}$$

$$\rho_0 \;=\; \left(\begin{array}{ccc|ccc|ccc}
a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22}
\end{array}\right) \;.$$

$$\Sigma_1 \; = \; \mathsf{span}\left\{e_0 \otimes e_1 \,, e_1 \otimes e_2 \,, e_2 \otimes e_0\right\}$$

$$\rho_1 \; = \; \left(\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\
\hline
\cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right) .$$

$$\Sigma_2 \;=\; \text{span}\left\{e_0 \otimes e_2\,,\, e_1 \otimes e_0\,,\, e_2 \otimes e_1\right\}$$

$$\rho_2 \;=\; \left(\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\
\hline
\cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right) .$$

Circulant state

$$\rho \;=\; \rho_0 + \rho_1 + \rho_2$$

$$\rho \;=\; \left(\begin{array}{ccc|ccc|ccc}
a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\
\cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\
\cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\
\hline
\cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\
a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\
\cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\
\hline
\cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\
\cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\
a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22}
\end{array}\right) .$$

Circulant state

$$\rho \;=\; \rho_0 + \rho_1 + \rho_2$$

$$\rho \;=\; \left(\begin{array}{ccc|ccc|ccc}
a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\
\cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\
\cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\
\hline
\cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\
a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\
\cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\
\hline
\cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\
\cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\
a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22}
\end{array}\right) .$$

$$\rho_{\text{old}} = \left[\begin{array}{ccc|ccc|ccc} a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\ \cdot & b_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & b_{10} & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & b_{12} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_{20} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_{21} & \cdot \\ a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22} \end{array}\right]$$

$$\rho = \left(\begin{array}{ccc|ccc|ccc} a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\ \cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\ \cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\ \hline \cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\ a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\ \cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\ \hline \cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\ \cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\ a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22} \end{array}\right).$$

$$\rho_{\text{old}} = \left[\begin{array}{ccc|ccc|ccc}
a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\
\cdot & b_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & b_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & b_{10} & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & b_{12} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_{20} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_{21} & \cdot \\
a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22}
\end{array}\right]$$

$$\rho = \left(\begin{array}{ccc|ccc|ccc}
a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\
\cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\
\cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\
\hline
\cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\
a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\
\cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\
\hline
\cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\
\cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\
a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22}
\end{array}\right).$$

Partial transposition $\rho^\tau = (\mathbb{1} \otimes \tau)\rho$ — a MIRACLE!

$$\rho^\tau = \left(\begin{array}{ccc|ccc|ccc}
\widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \widetilde{a}_{02} & \cdot \\
\cdot & \widetilde{b}_{00} & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{02} \\
\cdot & \cdot & \widetilde{c}_{00} & \cdot & \widetilde{c}_{01} & \cdot & \widetilde{c}_{02} & \cdot & \cdot \\
\hline
\cdot & \widetilde{b}_{10} & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{12} \\
\cdot & \cdot & \widetilde{c}_{10} & \cdot & \widetilde{c}_{11} & \cdot & \widetilde{c}_{12} & \cdot & \cdot \\
\widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \widetilde{a}_{12} & \cdot \\
\hline
\cdot & \cdot & \widetilde{c}_{20} & \cdot & \widetilde{c}_{21} & \cdot & \widetilde{c}_{22} & \cdot & \cdot \\
\widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \widetilde{a}_{22} & \cdot \\
\cdot & \widetilde{b}_{20} & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{22}
\end{array}\right)$$

$$\rho^\tau = \widetilde{\rho}_0 + \widetilde{\rho}_1 + \widetilde{\rho}_2$$

Partial transposition $\rho^\tau = (\mathbb{1} \otimes \tau)\rho$ — a MIRACLE!

$$\rho^\tau = \left(\begin{array}{ccc|ccc|ccc}
\widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \widetilde{a}_{02} & \cdot \\
\cdot & \widetilde{b}_{00} & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{02} \\
\cdot & \cdot & \widetilde{c}_{00} & \cdot & \widetilde{c}_{01} & \cdot & \widetilde{c}_{02} & \cdot & \cdot \\
\hline
\cdot & \widetilde{b}_{10} & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{12} \\
\cdot & \cdot & \widetilde{c}_{10} & \cdot & \widetilde{c}_{11} & \cdot & \widetilde{c}_{12} & \cdot & \cdot \\
\widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \widetilde{a}_{12} & \cdot \\
\hline
\cdot & \cdot & \widetilde{c}_{20} & \cdot & \widetilde{c}_{21} & \cdot & \widetilde{c}_{22} & \cdot & \cdot \\
\widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \widetilde{a}_{22} & \cdot \\
\cdot & \widetilde{b}_{20} & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{22}
\end{array}\right)$$

$$\rho^\tau = \widetilde{\rho}_0 + \widetilde{\rho}_1 + \widetilde{\rho}_2$$

Circulant decomposition of $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\widetilde{\Sigma}_0 \;=\; \text{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_2\,, e_2 \otimes e_1\}$$

$$\widetilde{\Sigma}_1 \;=\; \text{span}\,\{e_0 \otimes e_1\,, e_1 \otimes e_0\,, e_2 \otimes e_2\}$$

$$\widetilde{\Sigma}_2 \;=\; \text{span}\,\{e_0 \otimes e_2\,, e_1 \otimes e_1\,, e_2 \otimes e_0\}$$

$$\widetilde{\Sigma}_1 \;=\; (\mathbb{1} \otimes S)\widetilde{\Sigma}_0\;, \qquad \widetilde{\Sigma}_2 \;=\; (\mathbb{1} \otimes S^2)\widetilde{\Sigma}_0$$

$$\boxed{\mathbb{C}^3 \otimes \mathbb{C}^3 \;=\; \widetilde{\Sigma}_0 \oplus \widetilde{\Sigma}_1 \oplus \widetilde{\Sigma}_2}$$

Circulant decomposition of $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\widetilde{\Sigma}_0 = \text{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_2\,, e_2 \otimes e_1\}$$

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$$\widetilde{\Sigma}_2 = \text{span}\,\{e_0 \otimes e_2\,, e_1 \otimes e_1\,, e_2 \otimes e_0\}$$

$$\widetilde{\Sigma}_1 = (\mathbb{1} \otimes S)\widetilde{\Sigma}_0\,, \qquad \widetilde{\Sigma}_2 = (\mathbb{1} \otimes S^2)\widetilde{\Sigma}_0$$

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = \widetilde{\Sigma}_0 \oplus \widetilde{\Sigma}_1 \oplus \widetilde{\Sigma}_2$$

Circulant decomposition of $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\widetilde{\Sigma}_0 = \text{span}\{e_0 \otimes e_0 \,,\, e_1 \otimes e_2 \,,\, e_2 \otimes e_1\}$$

$$\widetilde{\Sigma}_1 = \text{span}\{e_0 \otimes e_1 \,,\, e_1 \otimes e_0 \,,\, e_2 \otimes e_2\}$$

$$\widetilde{\Sigma}_2 = \text{span}\{e_0 \otimes e_2 \,,\, e_1 \otimes e_1 \,,\, e_2 \otimes e_0\}$$

$$\widetilde{\Sigma}_1 = (\mathbb{1} \otimes S)\widetilde{\Sigma}_0 \,, \qquad \widetilde{\Sigma}_2 = (\mathbb{1} \otimes S^2)\widetilde{\Sigma}_0$$

$$\boxed{\mathbb{C}^3 \otimes \mathbb{C}^3 = \widetilde{\Sigma}_0 \oplus \widetilde{\Sigma}_1 \oplus \widetilde{\Sigma}_2}$$

$$\widetilde{\Sigma}_0 \;=\; \text{span}\,\{e_0 \otimes e_0\,,\, e_1 \otimes e_2\,,\, e_2 \otimes e_1\}$$

$$\widetilde{\rho}_0 \;=\; \left(\begin{array}{ccc|ccc|ccc}
\widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \widetilde{a}_{02} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \widetilde{a}_{12} & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \widetilde{a}_{22} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$\widetilde{\Sigma}_1 \;=\; \mathrm{span}\,\{e_0 \otimes e_1\,,\, e_1 \otimes e_0\,,\, e_2 \otimes e_2\}$$

$$\widetilde{\rho}_1 \;=\; \left(\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \widetilde{b}_{00} & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{02} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \widetilde{b}_{10} & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \widetilde{b}_{20} & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{22}
\end{array}\right)$$

$$\widetilde{\Sigma}_2 \;=\; \mathsf{span}\,\{e_0 \otimes e_2\,,\, e_1 \otimes e_1\,,\, e_2 \otimes e_0\}$$

$$\widetilde{\rho}_2 \;=\; \left(\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \widetilde{c}_{00} & \cdot & \widetilde{c}_{01} & \cdot & \widetilde{c}_{02} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \widetilde{c}_{10} & \cdot & \widetilde{c}_{11} & \cdot & \widetilde{c}_{12} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \widetilde{c}_{20} & \cdot & \widetilde{c}_{21} & \cdot & \widetilde{c}_{22} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$a\,,\,b\,,\,c \quad\longrightarrow\quad \widetilde{a}\,,\,\widetilde{b}\,,\,\widetilde{c}$$

$$\begin{pmatrix} a_{00} & c_{01} & b_{02} \\ c_{10} & b_{11} & a_{12} \\ b_{20} & a_{21} & c_{22} \end{pmatrix}, \begin{pmatrix} b_{00} & a_{01} & c_{02} \\ a_{10} & c_{11} & b_{12} \\ c_{20} & b_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} c_{00} & b_{01} & a_{02} \\ b_{10} & a_{11} & c_{12} \\ a_{20} & c_{21} & b_{22} \end{pmatrix}.$$

Theorem

Circulant state ρ is PPT iff

$$\boxed{\widetilde{a} \geq 0\,, \quad \widetilde{b} \geq 0\,, \quad \widetilde{c} \geq 0}$$

$$a\,,\,b\,,\,c \quad\longrightarrow\quad \widetilde{a}\,,\,\widetilde{b}\,,\,\widetilde{c}$$

$$\begin{pmatrix} a_{00} & c_{01} & b_{02} \\ c_{10} & b_{11} & a_{12} \\ b_{20} & a_{21} & c_{22} \end{pmatrix}, \quad \begin{pmatrix} b_{00} & a_{01} & c_{02} \\ a_{10} & c_{11} & b_{12} \\ c_{20} & b_{21} & a_{22} \end{pmatrix}, \quad \begin{pmatrix} c_{00} & b_{01} & a_{02} \\ b_{10} & a_{11} & c_{12} \\ a_{20} & c_{21} & b_{22} \end{pmatrix}.$$

Theorem

Circulant state ρ is PPT iff

$$\boxed{\widetilde{a} \geq 0\,, \quad \widetilde{b} \geq 0\,, \quad \widetilde{c} \geq 0}$$

Σ_α vs. $\widetilde{\Sigma}_\alpha$

$$\Sigma_0 = \text{span}\,\{e_0 \otimes e_0\,,\, e_1 \otimes e_1\,,\, e_2 \otimes e_2\}$$

$$\boxed{\Sigma_\alpha = (\mathbb{1} \otimes S^\alpha)\Sigma_0}$$

$$\widetilde{\Sigma}_0 = \text{span}\,\{e_0 \otimes e_0\,,\, e_1 \otimes e_2\,,\, e_2 \otimes e_1\}$$

$$\boxed{\widetilde{\Sigma}_\alpha = (\mathbb{1} \otimes S^\alpha)\widetilde{\Sigma}_0}$$

Σ_α vs. $\widetilde{\Sigma}_\alpha$

$$\Sigma_0 = \text{span}\,\{e_0 \otimes e_0\,,\, e_1 \otimes e_1\,,\, e_2 \otimes e_2\}$$

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Σ_α vs. $\widetilde{\Sigma}_\alpha$

$$\Sigma_0 \;=\; \text{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_1\,, e_2 \otimes e_2\}$$

$$e_0 \otimes e_{\pi(0)}\,, e_1 \otimes e_{\pi(1)}\,, e_2 \otimes e_{\pi(2)}$$

$$\boxed{\pi(0) = 0\,,\ \pi(1) = 1\,,\ \pi(2) = 2}$$

$$\widetilde{\Sigma}_0 \;=\; \text{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_2\,, e_2 \otimes e_1\}$$

$$e_0 \otimes e_{\widetilde{\pi}(0)}\,, e_1 \otimes e_{\widetilde{\pi}(1)}\,, e_2 \otimes e_{\widetilde{\pi}(2)}$$

$$\boxed{\widetilde{\pi}(0) = 0\,,\ \widetilde{\pi}(1) = 2\,,\ \widetilde{\pi}(2) = 1}$$

Σ_α vs. $\widetilde{\Sigma}_\alpha$

$$\Sigma_0 \;=\; \text{span}\,\{e_0 \otimes e_0 \,,\, e_1 \otimes e_1 \,,\, e_2 \otimes e_2\}$$

$$e_0 \otimes e_{\pi(0)} \,,\, e_1 \otimes e_{\pi(1)} \,,\, e_2 \otimes e_{\pi(2)}$$

$$\boxed{\pi(0) = 0 \,,\, \pi(1) = 1 \,,\, \pi(2) = 2}$$

$$\widetilde{\Sigma}_0 \;=\; \text{span}\,\{e_0 \otimes e_0 \,,\, e_1 \otimes e_2 \,,\, e_2 \otimes e_1\}$$

$$e_0 \otimes e_{\widetilde{\pi}(0)} \,,\, e_1 \otimes e_{\widetilde{\pi}(1)} \,,\, e_2 \otimes e_{\widetilde{\pi}(2)}$$

$$\boxed{\widetilde{\pi}(0) = 0 \,,\, \widetilde{\pi}(1) = 2 \,,\, \widetilde{\pi}(2) = 1}$$

π vs. $\widetilde{\pi}$

$$\pi(0) = 0\,,\ \pi(1) = 1\,,\ \pi(2) = 2$$

$$\widetilde{\pi}(0) = 0\,,\ \widetilde{\pi}(1) = 2\,,\ \widetilde{\pi}(2) = 1$$

π & $\widetilde{\pi}$ are "complementary"

$$\pi(i) + \widetilde{\pi}(i) = 3 \qquad i = 1, 2$$

$$\Pi = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \qquad \widetilde{\Pi} = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}$$

π vs. $\widetilde{\pi}$

$$\boxed{\pi(0) = 0\,,\ \pi(1) = 1\,,\ \pi(2) = 2}$$

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Examples

- Werner state
- Isotropic state
- DiVincenzo *et al.* BE state
- Størmer state
- Ha state
- ... and many others

Known PPT states which are not circulant

- Horodecki $3 \otimes 3$ state
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Open problems

- separability
- "circulant" entanglement witnesses
- "circulant" positive maps
- generalization to $d \otimes d \otimes \ldots \otimes d$
- do there exist other characteristic decompositions?