IP: Algorithm of consistency with covariant derivations.

Abstract

Covariant derivations ∇, D^V and D^H, associated to the linear connection defined by a SODE [7], are used to develop an algorithm of consistency for Helmholtz conditions, the system of PDE corresponding to the inverse problem of the calculus of variations.

Carlos López *

Departamento de Matemáticas,
Universidad de Alcalá de Henares (Madrid). Spain

July 5, 2007

1 Covariant formulation of IP

In [4, 5, 7], a linear connection associated to a system of second order differential equations in normal form (SODE) is studied. The corresponding covariant derivation splits into the *dynamical* ∇, vertical D^V and horizontal D^H derivations. Within this framework, Helmholtz conditions for the inverse problem of the calculus of variations take a simple and economic form, through relations among the covariant derivatives of the unknown tensor of multipliers. This approach is exploited in [8] to study the classification *a la* Douglas [2] of variational SODE systems. The aim of the talk is to follow this covariant approach, applying the algorithm of consistency to the PDAE system on the unknown multipliers, that is, prolongations and projections of the equations in order to find new integrability conditions.. A comparison with alternative treatments, e.g. [1, 3], can be fruitfull. See [9] for a general description of jet bundles.

* **e**-*mail*: carlos.lopez@uah.es

1.1 Notation and basic expressions

$\pi : E \rightarrow \mathbb{R}$ is the fibre bundle of space–time over time. For the inverse problem the relevant operations are restricted to the space of π-vertical vector fields along $\pi_1^o : J^1\pi \rightarrow E$. The covariant derivations are determined by its particular action over functions and over a basis of vector fields $\partial_i = \partial_{x^i}$ (we present also its action over a dual basis of one forms d^i, identified with the contact forms $\theta^i = dx^i - v^i dt$ for the restriction to the fibres):

$$\begin{array}{rclrclrcl} \nabla F &=& \Gamma(F) & D_i^V F &=& \partial_{v^i} F & D_i^H F &=& H_i(F) \\ \nabla \partial_i &=& \Gamma_i^j \partial_j & D_i^V \partial_j &=& 0 & D_i^H \partial_j &=& \Gamma_{ij}^k \partial_k \\ \nabla d^i &=& -\Gamma_j^i d^j & D_i^V d^j &=& -\delta_i^j dt (\equiv 0) & D_i^H d^j &=& -\Gamma_{ik}^j d^k \end{array} \tag{1}$$

where $\Gamma = \partial_t + v^i \partial_{x^i} + f^i(t,x,v)\partial_{v^i}$ is the SODE system, $\Gamma_j^i = -\frac{1}{2}\frac{\partial f^i}{\partial v^j}$ the coefficients of the connection, $\Gamma_{jk}^i = \partial_{v^k}\Gamma_j^i$, and $H_i = \partial_{x^i} - \Gamma_i^j \partial_{v^j}$ the horizontal vectors.

Helmholtz conditions for the inverse problem represent a system of linear PDAEs (algebraic and differential equations) on the unknown multipliers $g_{ij}(t,x,v)$, the components of a symmetric $(0,2)$ tensor field $g = g_{ij} d^i \otimes d^j$ along π_1^o, corresponding to the Hessian of the Lagrangian [6]:

$$\Gamma(g_{ij}) = g_{ik}\Gamma_j^k + g_{jk}\Gamma_i^k \qquad \frac{\partial g_{ij}}{\partial v^k} = \frac{\partial g_{ik}}{\partial v^j} \qquad g_{ik}\Phi_j^k = g_{kj}\Phi_i^k \quad , \tag{2}$$

with $\Phi_i^j = -\partial_{x^i} f^j - \Gamma_k^j \Gamma_i^k - \Gamma(\Gamma_i^j)$ the components of the Jacobi endomorphism $\Phi = \Phi_i^j d^i \otimes \partial_j$, plus an inequation, the regularity condition $\det g \neq 0$. Its covariant formulation is

$$\nabla g = 0 \quad (D^V g)(X,Y,Z) = (D^V g)(X,Z,Y) \quad g(\Phi(X),Y) - g(X,\Phi(Y)) = 0 \tag{3}$$

with $(D^V g)(X,Y,Z) \equiv (D_Z^V g)(X,Y)$. Notice that, along the paper, the additional vector argument introduced by the D^V and D^H derivations appears in last possition, as e.g.

$$\begin{array}{rcl} (D^V\Phi)(X,Y) &=& (D_Y^V \Phi)(X) \\ (D^H D^V g)(X,Y,Z,W) &=& (D_W^H[D^V g])(X,Y,Z) = (D_W^H[D_Z^V g])(X,Y) - (D_{D_W^H Z}^V g)(X,Y) \end{array}$$

Commutation relations of the dynamical, vertical and horizontal derivations are

$$\begin{array}{rcl} \left[\nabla, D_*^V\right] &=& -D_*^H \\ \left[\nabla, D_*^H\right] &=& D_{\Phi(*)}^V + \mu[\Psi(*,.)] \\ \left[D_*^V, D_{**}^V\right] &=& 0 \\ \left[D_*^H, D_{**}^H\right] &=& D_{R(*,**)}^V - \mu[D^V R(*,**,.)] \\ \left[D_*^V, D_{**}^H\right] &=& \mu[\theta(*,**,.)] \quad , \end{array} \tag{4}$$

with $3R(X,Y) = (D^V_X\Phi)(Y) - (D^V_Y\Phi)(X)$, $\Psi(X,Y) = 2R(X,Y) - (D^V_X\Phi)(Y)$, θ a $(1,3)$ type symmetric tensor with local components $\Gamma^l_{ijk} = \partial_{v^k}\Gamma^l_{ij}$, and where the former expressions are understood acting over $(1,p)$ or $(0,p)$ tensor fields T according to the following rules:

(i) $*$ and $**$ represent vector arguments of the covariant derivatives (to be taken from the last arguments of the derived tensor), as for example

$$\begin{aligned}\left([\nabla, D^V_*]g\right)(X,Y,Z) &= [(\nabla\circ D^V)g - (D^V\circ\nabla)g](X,Y,Z) = \\ \nabla(D^V_Z g)(X,Y) &- (D^V_{\nabla(Z)}g)(X,Y) - (D^V_Z(\nabla g))(X,Y)\end{aligned}$$

or

$$\begin{aligned}&\left([D^V_*, D^H_{**}](g)\right)(X,Y,Z,W) = [D^V_W(D^H g)](X,Y,Z) - [D^H_Z(D^V g)](X,Y,W) = \\ &[D^V_W(D^H_Z g)](X,Y) - [D^H_{D^V_W Z} g](X,Y) - [D^H_Z(D^V_W g)](X,Y) + [D^V_{D^H_Z W} g](X,Y)\end{aligned}$$

etc.

(ii) For Φ, a $(1,1)$ tensor,

$$[D^V_{\Phi(*)}T](X,\ldots,Z) = [D^V_{\Phi(Z)}T](X,\ldots)$$

while for R, a $(1,2)$ tensor,

$$[D^V_{R(*,**)}T](X,\ldots,Y,Z) = [D^V_{R(Z,Y)}T](X,\ldots)$$

(iii) $\mu[A(.)]T$, for a $(1,1)$ tensor A, is $\mu[A(.)]T = a[A(.)]T - i[A(.)]T$, with

$$a[A(.)]T(X,Y,\ldots) = A(T(X,Y,\ldots))$$

if T is of $(1,p)$ type (vanishing for $(0,p)$ type T), and

$$i[A(.)]T(X_1,X_2,\ldots,X_p) = \sum_{i=1}^{p} T(X_1,\ldots,A(X_i),\ldots,X_p)$$

for T of $(1,p)$ or $(0,p)$ type. Consequently, $\mu[\Psi(*,.)]$ must be understood with Ψ (type $(1,2)$) already contracted with the last argument $(*)$ to give a $(1,1)$ type tensor, as in

$$(\mu[\Psi(*,.)]g)(X,Y,Z) = (\mu[\Psi(Z,.)]g)(X,Y) = -g(\Psi(Z,X),Y) - g(X,\Psi(Z,Y))$$

Similarly, for $\mu[D^V R(*,**,.)]$ we have, for example

$$\begin{aligned}\left(\mu[D^V R(*,**,.)]\Phi\right)(X,Y,Z) &= \left(\mu[D^V R(Z,Y,.)]\Phi\right)(X) = \\ D^V R(Z,Y,\Phi(X)) - \Phi(D^V R(Z,Y,X)) &= [D^V_{\Phi(X)}R](Z,Y) - \Phi([D^V_X R](Z,Y))\end{aligned}$$

etc.

Notice that all *structural* tensors appearing in the commutation relations, except for θ, are derived from the Jacobi endomorphism Φ. Bianchi identities for the connection determine some relations among the structural tensors and covariant derivatives, as e.g.

$$\sum_{cycl} D^V R(X,Y,Z) = 0$$

1.2 Framework for the PDE system

In Helmholtz conditions the independent variables are $\{t, x^i, v^i\}$, and the dependent variables (unknown functions) are the multipliers g_{ij}. The geometric framework is the vector bundle $\nu : F = S^2(V^*) \times_E J^1\pi \to J^1\pi = B$, with base manifold $J^1\pi$ the velocity–space–time manifold, and fibres the spaces $S^2(V_e^*)$ of symmetric $(0,2)$ tensors on $V_e\pi$, the π–vertical part of the tangent spaces T_eE. Points in F are symmetric $(0,2)$ tensors g over particular points $e \in E$, together with jets $j_t^1 s \in J_e^1\pi$, $s(t) = e$. We have algebraic equations, $g_{ik}\Phi_j^k - g_{kj}\Phi_i^k = 0$, and first order PDEs $(\nabla g)_{ij} = 0$ and $\partial_{v^k} g_{ij} = \partial_{v^j} g_{ik}$, determining a submanifold $R_1 \subset J^1\nu$, with $\nu_1^o(R_1) = F_1 \subset F$ the algebraic constraints submanifold. Local coordinates in $J^1\nu$ are

$$\{t, x^i, v^i; g_{ij}; f_{ij}^o = \partial_t g_{ij}, f_{ij;k}^v = \partial_{v^k} g_{ij}, f_{ij;k}^h = \partial_{x^k} g_{ij}\} \tag{5}$$

A covariant description of points $j_v^1\sigma \in J^1\nu$, $\sigma \in \Gamma(\nu)$ a section, is obtained by associating to $j_v^1\sigma$ the three tensors $z^o = \nabla g$, $z^v = D^V g$ and $z^h = D^H g$, where the base point $v \in B$ is understood, and $g = \mathrm{Im}\sigma$. It corresponds to the use of ∇ instead of ∂_t (Douglas utilised $\Gamma = \frac{d}{dt}$), D_i^V instead of ∂_{v^i} (in fact, they are identical), and D_i^H instead of ∂_{x^i}. Similarly, for $J^1\nu_1$ ($\nu_1 : J^1\nu \to B$) we can use the representation of $j_v^1\sigma_1$, $\sigma_1 \in \Gamma(\nu_1)$, $\sigma_1(v') = (g(v'), w^o(v'), w^v(v'), w^h(v'))$ (where g and w^o are $(0,2)$, and w^v and w^h $(0,3)$ tensors), given by

$$\{v \in B; g \in S^2(V_v^*), w^o, w^v, w^h; z^o, z^v, z^h, u^{oo}, u^{ov}, u^{oh}, u^{vo}, u^{vv}, \ldots, u^{hh}\} \tag{6}$$

where $z^o = \nabla g$, $z^v = D^V g$, $z^h = D^H g$, $u^{oo} = \nabla w^o$, $\ldots$, $u^{vo} = \nabla w^v$, $u^{vv} = D^V w^v$, $\ldots$, $u^{hh} = D^H w^h$, and similarly for higher order jets (e.g., $u^{vhv} = D^V u^{vh}$). By the original symmetry of g, all tensors are symmetric in their first two indices. For generic σ_1, the w are not derivatives of g, and only if $j_v^1(\nu_1^o \circ \sigma_1) = \sigma_1(v)$ we are in $J^{1,1}\nu \subset J^1\nu_1$ where $w = z$. Moreover, if $j_v^1\sigma_1 = i(j_v^2(\nu_1^o \circ \sigma_1))$, $i : J^2\nu \to J^1\nu_1$, the u are the second derivatives of g.

The original Helmholtz conditions are given by (3), with

$$\Phi_{12}(g) = 0 \quad \Phi_{12}(g)(X,Y) \equiv g(\Phi(X),Y) - g(X,\Phi(Y))$$

the algebraic tensorial equation defining $F_1 \subset F$, and

$$z^o = 0 \quad z^v \text{ totally symmetric}$$

the first order partial differential equations, determining the fibres of R_1 over F_1. Prolongation of F_1 to $J^1\nu$ is performed by computing the covariant ∇, D^V and D^H derivatives of the algebraic equation; the prolonged equations are then combined with the original PDEs to cancel the (z^o, z^v, z^h) terms, obtaining new algebraic equations, linear integrability conditions of the system. Once the system is consistent at first order, i.e., no new algebraic equations are obtained, we proceed to higher order prolongations.

Prolongations of the PDAE R_1 (more precisely, the final $R_1^f \subset R_1$ obtained in the previous step) to $J^2\nu$ are performed in three steps

(i) jet prolongation to $J^1\nu_1$ of the PDAE submanifold R_1 by computing the covariant ∇, D^V and D^H derivatives of the equations

(ii) identification (trivial) of the w with the z (determining $J^{1,1}\nu \subset J^1\nu_1$, i.e., first holonomy conditions)

(iii) second holonomy conditions (identity of crossed partial derivatives, $J^2\nu \subset J^{1,1}\nu \subset J^1\nu_1$) through the commutation relations applied to g

$$\begin{aligned}
u^{ov} - u^{vo} &= [D^V_*, \nabla](g) = D^H(g) = z^h \\
(u^{ho} - u^{oh})(X,Y,Z) &= z^v(X,Y,\Phi(Z)) - i[\Psi(Z,.)]g(X,Y) \\
u^{hh}(X,Y,Z,W) - u^{hh}(X,Y,W,Z) &= z^v(X,Y,R(W,Z)) + i[D^V R(W,Z,.)]g(X,Y) \\
&\dots
\end{aligned}$$

Again, the prolonged equations and second holonomy conditions can generate by combination new first order (or even algebraic) integrability conditions. A similar procedure can be applied to higher order prolongations.

The formal theory of integrability of PDE systems guarantees that the algorithm of consistency is finite. By redefining in the standard way the final PDE system as first order (identification of lower order partial derivatives with new algebraic dependent variables, and the highest order derivatives as first order) the system becomes a formally integrable first order PDAE (and locally integrable in the analytic framework). The problem of determining the stop of the algorithm is solved through the concept of involution, and the use of some test (Cartan test of involution, or homological condition for the Spencer sequence). We will not study here these aspects in detail.

2 Covariant algorithm of consistency

The algorithm of consistency for a given system of partial differential equations consists of recursive prolongation and projection of the PDE submanifold and its jet prolongations, in order to generate all possible integrability conditions. The algorithm stops when the system becomes involutive, which is characterised by a homological condition; in the old theory of Janet–Riquier, the system is involutive if it is one step consistent (no integrability conditions by one prolongation) and the *alternants* do not promote parametric derivatives into principal (no passivity conditions). In the particular case of Helmholtz conditions, there is always trivial solution $g = 0$, but in order to fulfil the regularity condition $\det g \neq 0$ a non trivial solution must be found. Therefore, a scheme of classification can be given as follows

(i) apply the consistency algorithm, generating new algebraic and partial differential equations

(ii) each case is determined by the stop conditions (e.g., Cartan test) of the algorithm; that is, if some conditions are fulfilled the algorithm stops, otherwise, we must continue with the prolongation–projection process

(iii) for each case, we find a final $F_f \subset F$ algebraic manifold; three possibilities appear

  (a) $F_f = 0$; trivial vanishing solution, not variational

  (b) $0 \neq F_f \subset K$, with K the subset of singular tensors $\det(g) = 0$; not variational because all not trivial solutions (Hessians) are singular

  (c) $0 \neq F_f$ and $F_f - K \neq \emptyset$, variational

(iv) number and type (number of variables) of arbitrary functions in the general solution is determined by standard theory of involution, e.g. through the Cartan characters of the final involutive system

This scheme does not match the one of Douglas for $N = 2$ ($N = \dim E_t$, $E_t = \pi^{-1}(t)$), because he used particular coordinate systems in cases II (one algebraic condition) and III (two) according to the position of the plane or line with regard to K, a cone in the three dimensional spaces $F_v = \nu^{-1}(v)$. For $N = 3$, K is a five dimensional third degree algebraic variety in the six dimensional F_v, and more cumbersome for higher N. Our proposal is to avoid the non linear regularity condition (inequation) up to the end, once the more systematic linear study has finished. As we see, a fully covariant algorithm of consistency can be applied, and the study becomes algebraic and linear at each level by computing ranks of the tensorial equations involving the g, z and u.

From [2] it is clear that even for $N = 2$ the complete classification is quite long (cases IIa1, IIa2, ...) defining the whole graph of cases and subcases. Douglas did not compute exhaustively all conditions determining the selection of subcases; sometimes, he gave just the indication of linear expressions without explicit presentation of the coefficients. In particular, it can happen that the set of first order PDE defines an explicit system, with all partial derivatives determined. Then, it is enough to check for the closeness of the corresponding one form; either it is closed or new algebraic conditions appear. An exhaustive classification for $N = 3$ (or higher) could use this type of shortcuts too, in order to maintain the size of the study manageable.

2.1 Some steps for N generic

For generic N, it seems very difficult to enter into details of the involution requirement, unless we particularise to simple cases (e.g. Case I). We next generate some linear algebraic integrability conditions through the algorithm of consistency, without applying a stopping test.

Let us first generate a new tensorial PDE by prolongation of $z^o = 0$ and z^vt.s., in order to *symmetrise* Helmholtz conditions. From $D^V z^o = u^{ov} = 0$, $\nabla z^v = u^{vo}$t.s. and the commutation $u^{vo} - u^{ov} = -z^h$ we find z^ht.s.. There is some (partial) symmetry among vertical and horizontal variables coming from this property, already indicated in [2].

Another remarkable property found by Douglas (page 106, *"It is remarkable that precisely the same equation (12.4) is also the integrability condition of ..."*; page 110, *" ... is also obtained as the integrability condition of ..."*) is grounded on the symmetry of θ, making some D^V–D^H alternants identical by the commutation relations

$$[D^V_*, D^H_{**}] = \mu[\theta(*, **, .)] = [D^V_{**}, D^H_*]$$

We now prolong the algebraic tensorial equation, rewritten for convenience as

$$0 = \sum_{s\in P_2} (-1)^{|s|} g(\Phi(X_{s1}), X_{s2})$$

with P_2 the group of permutations of $\{1, 2\}$, $s \in P_2$ a permutation (either $s = (s1 = 1, s2 = 2)$ or $s = (s1 = 2, s2 = 1)$) and $|s|$ its signature, $+$ and $-$ respectively.

For one ∇ prolongation, we find

$$0 = \sum_{s\in P_2} (-1)^{|s|} \left((\nabla g)(\Phi(X_{s1}), X_{s2}) + g((\nabla\Phi)(X_{s1}), X_{s2})\right)$$

and, taking into account $z^o = 0$, recursively

$$0 = \sum_{s\in P_2} (-1)^{|s|} g((\nabla^k\Phi)(X_{s1}), X_{s2})$$

where $k = 0, 1, \ldots$, $\nabla^k = \nabla \circ \nabla \cdots \text{k times} \cdots \nabla$, well known integrability conditions [6].

D^V prolongation of each former equation

$$0 = \sum_{s \in P_2} (-1)^{|s|} \Big((D^V g)((\nabla^k \Phi)(X_{s1}), X_{s2}, Z) + g((D^V \nabla^k \Phi)(X_{s1}, Z), X_{s2}) \Big)$$

and the z^vt.s. condition gives

$$0 = \sum_{s \in P_3} (-1)^{|s|} g((D^V \nabla^k \Phi)(X_{s1}, X_{s2}), X_{s3})$$

toghether with (∇ prolongation again)

$$0 = \sum_{s \in P_3} (-1)^{|s|} g((\nabla^{k_2} D^V \nabla^{k_1} \Phi)(X_{s1}, X_{s2}), X_{s3})$$

These families contain the equations $\sum_{\text{cycl}} g(\nabla^k R(X, Y), Z) = 0$ [6, 8], and some additional ones. We can rewrite these equations tranfering the D^V operation to the begining by succesive commutation with ∇, defining $D^V = C_o$, $[C_o, \nabla] = D^H = C_1$, $[C_k, \nabla] = C_{k+1}$ (e.g., $C_2 = -D^V_{\Phi(*)} - \mu[\Psi(*, .)]$. We obtain the equivalent family

$$0 = \sum_{s \in P_3} (-1)^{|s|} g((\nabla^{k_2} C_{k_1} \Phi)(X_{s1}, X_{s2}), X_{s3})$$

Some of these expressions are linearly dependent of the former ones, e.g., for $k_2 = 0$ and $k_1 = 1$,

$$0 = \sum_{s \in P_3} (-1)^{|s|} g((D^H \Phi)(X_{s1}, X_{s2}), X_{s3}) = \sum_{cycl} g(\nabla R(X, Y), Z)$$

according to the Bianchi identity $D^H_X \Phi(Y) - D^H_Y \Phi(X) = \nabla R(X, Y)$, identical to the case $k_2 = 1$, $k_1 = 0$. However, for $k_2 = 0$ and $k_1 = 2$, we find after simplifications

$$0 = \sum_{s \in P_3} (-1)^{|s|} g((D^V_\Phi \Phi)(X_{s1}, X_{s2}), X_{s3})$$

a true independent equation, which seems to have been passed over in [8].

Again, recursive D^V and ∇ prolongations and (exclusively) the equation z^vt.s. generates

$$0 = \sum_{s \in P_l} (-1)^{|s|} g((\nabla^{k_{l-1}} D^V \nabla^{k_{l-2}} D^V \cdots D^V \nabla^{k_1} \Phi)(X_{s1}, X_{s2}, \ldots, X_{s[l-1]}), X_{sl})$$

or, equivalently

$$0 = \sum_{s \in P_l} (-1)^{|s|} g((\nabla^{k_{l-1}} C_{k_{l-2}} C_{k_{l-3}} \cdots C_{k_1} \Phi)(X_{s1}, X_{s2}, \dots, X_{s[l-1]}), X_{sl})$$

Finally, D^H prolongations and the symmetry of z^h gives way to similar algebraic integrability conditions, with the obvious substitution of derivations

$$0 = \sum_{s \in P_l} (-1)^{|s|} g((\nabla^{k_{l-1}} D^H \nabla^{k_{l-2}} D^H \cdots D^H \nabla^{k_1} \Phi)(X_{s1}, X_{s2}, \dots, X_{s[l-1]}), X_{sl})$$

but taking into account the commutation relation $D^H = [D^V, \nabla]$ they happen to be linearly dependent of these already generated. The same discussion is valid for higher order prolongations, the D^H ones being obtained already by an alternant of D^V and ∇. Notice that we have used linear combinations of one particular D^V–prolongation and the symmetry of z^v to generate new relatively simple algebaric equations, but we should consider the whole rank of the prolonged equations; linear combinations of the z^vt.s. and two or more D^V–prolongations could also generate new conditions.

2.2 $N = 2$

By fixing N, a more detailed study of the algorithm can be developed; the involution test can be applied when appropriate (δ or quasi–regular, [3]) local coordinates are chosen, and the whole rank of the family of equations can be computed.

For the simplest case $N = 2$ we can recover the main scheme of Douglas classification, with the differences already pointed out if we do not consider the regularity condition up to the end. The fibres of $\nu : F \to B$ are three dimensional vector spaces equivalent to the set $S^2(\mathbb{R}^2)$ of symmetric 2×2 matrices. All linear equations obtained in the previous paragraph with three or more entries are trivial (totally skew symmetric, $\Omega^k(\mathbb{R}^2) \equiv 0$ for $k > 2$).

We have then a primary classification criterion given by the rank of the family

$$\{\Phi_{12}(g), (\nabla\Phi)_{12}(g), (\nabla^2\Phi)_{12}(g)\}$$

being 0 (case I), 1 (case II), 2 (case III) or 3 (case IV). Let us denote by $F_2 \subset F$ the initial algebraic submanifold defined by vanishing of the former family, $\dim F_2 = 3, 2, 1, 0$.

Case IV is trivially not variational, with just the vanishing solution $g = 0$. Case I is variational because no algebraic conditions appear when the algorithm stops (see for example [8, 3] for a proof with arbitrary N). In Douglas's paper, he generates by alternants the z^ht.s. condition, and an additional passivity condition (there were two that *turn out to be the same* by the D^V–D^H alternants symmetry) of second order, after which the system becomes passive and orthonormal.

Case III corresponds to a line; in [2] Douglass defines cases IIIa and IIIb according to the line being contained in K or not. One case is not variational because the algebraic set of solutions has just singular matrices $F_2 \subset K$, while the other case generates an explicit first order PDE, and we must just check for the closedeness condition, i.e., computing the alternants of the first order PDE we generate new (linearly dependent) first order, and they project to give possible new algebraic equations. If we insist in avoiding the regularity condition, the algorithm works as follows.

From $g(\Phi(X), Y) - g(X, \Phi(Y)) = 0$ and $g(\nabla\Phi(X), Y) - g(X, \nabla\Phi(Y)) = 0$ (two independent equations; the non trivial components are those with $X \neq Y$) we get by D^V and D^H prolongations,

$$z^v(\Phi(X), Y, Z) - z^v(X, \Phi(Y), Z) + g(D^V\Phi(X, Z), Y) - g(X, D^V\Phi(Y, Z)) = 0$$

and similarly for $\nabla\Phi$, and for z^h. Toghether with the symmetry condition of z^v and z^h (I consider z^ht.s. already incorporated into the Helmholtz original equations), and $z^o = 0$ we find 15 equations. More explicitely, each one of the algebraic equations determines two D^V prolongations (D^V_1 and D^V_2) and also two D^H prolongations, making $2 \times (2 + 2)$ total prolongations, z^v and z^h totally symmetric are also $2 + 2$ equations, and finally $z^o = 0$ gives another 3.

Either the whole system is of maximal rank, and we have an explicit first order PDE system, or not. Let us express the equations in components, taking advantage of the symmetry to use restricted variables ($z^v_{11;2} = z^v_{12;1} \equiv z^v_{112}$, $z^v_{12;2} = z^v_{22;1} \equiv z^v_{122}$):

1) for the algebraic equations, once selected local coordinates $\{x^1, x^2\}$

$$g(\Phi(X_2), X_1) - g(X_2, \Phi(X_1) = g(\Phi^1_2 X_1 + \Phi^2_2 X_2, X_1) - g(X_2, \Phi^1_1 X_1 + \Phi^2_1 X_2) = 0$$

and similarly with $\nabla\Phi$, we obtain

$$A.g_{11} + B.g_{12} + C.g_{22} = 0 \quad A_1.g_{11} + B_1.g_{12} + C_1.g_{22} = 0$$

with $A = \Phi^1_2$, $B = \Phi^2_2 - \Phi^1_1$, $C = -\Phi^2_1$, and $A_1 = (\nabla\Phi)^1_2$, $B_1 = (\nabla\Phi)^2_2 - (\nabla\Phi)^1_1$, $C_1 = -(\nabla\Phi)^2_1$.

2) for the PDE

$$z^v(\Phi(X_2), X_1, X_1) - z^v(X_2, \Phi(X_1), X_1) + g(D^V\Phi(X_2, X_1), X_1) - g(X_2, D^V\Phi(X_1, X_1)) = 0$$

we obtain

$$A.z^v_{111} + B.z^v_{112} + C.z^v_{122} + A^{v1}.g_{11} + B^{v1}.g_{12} + C^{v1}.g_{22} = 0$$

where $A^{v1} = (D^V_1\Phi)^1_2$, etc.

$$
\begin{aligned}
A.z^v_{111} + B.z^v_{112} + C.z^v_{122} + A^{v1}.g_{11} + B^{v1}.g_{12} + C^{v1}.g_{22} &= 0\\
A.z^v_{112} + B.z^v_{122} + C.z^v_{222} + A^{v2}.g_{11} + B^{v2}.g_{12} + C^{v2}.g_{22} &= 0\\
A_1.z^v_{111} + B_1.z^v_{112} + C_1.z^v_{122} + A^{v1}_1.g_{11} + B^{v1}_1.g_{12} + C^{v1}_1.g_{22} &= 0\\
A_1.z^v_{112} + B_1.z^v_{122} + C_1.z^v_{222} + A^{v2}_1.g_{11} + B^{v2}_1.g_{12} + C^{v2}_1.g_{22} &= 0\\
A.z^h_{111} + B.z^h_{112} + C.z^h_{122} + A^{h1}.g_{11} + B^{h1}.g_{12} + C^{h1}.g_{22} &= 0\\
A.z^h_{112} + B.z^h_{122} + C.z^h_{222} + A^{h2}.g_{11} + B^{h2}.g_{12} + C^{h2}.g_{22} &= 0\\
A_1.z^h_{111} + B_1.z^h_{112} + C_1.z^h_{122} + A^{h1}_1.g_{11} + B^{h1}_1.g_{12} + C^{h1}_1.g_{22} &= 0\\
A_1.z^h_{112} + B_1.z^h_{122} + C_1.z^h_{222} + A^{h2}_1.g_{11} + B^{h2}_1.g_{12} + C^{h2}_1.g_{22} &= 0
\end{aligned}
$$

and

$$z^o_{11} = 0 \quad z^o_{12} = 0 \quad z^o_{22} = 0$$

Notice that the symmetry has reduced the variables (and equations) to 11.

Taking into account the z^v–z^h symmetry of the equations, we must analise just

$$\begin{pmatrix} A & B & C & 0 \\ 0 & A & B & C \\ A_1 & B_1 & C_1 & 0 \\ 0 & A_1 & B_1 & C_1 \end{pmatrix} \begin{pmatrix} z^v_{111} \\ z^v_{112} \\ z^v_{122} \\ z^v_{222} \end{pmatrix} = - \begin{pmatrix} A^{v1} & B^{v1} & C^{v1} \\ A^{v2} & B^{v2} & C^{v2} \\ A^{v1}_1 & B^{v1}_1 & C^{v1}_1 \\ A^{v2}_1 & B^{v2}_1 & C^{v2}_1 \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix}$$

Defining

$$\Delta_1 = BC_1 - CB_1 \quad \Delta_2 = CA_1 - AC_1 \quad \Delta_3 = AB_1 - BA_1$$

we can express the determinant of the principal matrix, the symbol, as $\Delta_1.\Delta_3 - (\Delta_2)^2$. But it happens that the line defined by the original algebraic equations

$$A.g_{11} + B.g_{12} + C.g_{22} = 0 \quad A_1.g_{11} + B_1.g_{12} + C_1.g_{22} = 0$$

is

$$(g_{11}, g_{12}, g_{22}) = \lambda(\Delta_1, \Delta_2, \Delta_3)$$

so that $\Delta_1.\Delta_3 - (\Delta_2)^2 = 0$ corresponds to it being inside $K = \{g_{11}g_{22} - g^2_{12} = 0\}$. Then, either we are in K or the first order system is explicit. It is straightforward to prolong the explicit system to second order and project it through the commutation relations, computation of alternants, to look for additional algebraic equations.

Notice that the prolongation to second order and following projection by combination with the holonomy conditions can be performed without a previous discussion of the rank. By doing so, we should be ignoring the regularity condition and the distinction of cases IIIa and IIIb; we can then compute new algebraic conditions up to the stop of the algorithm. After that, in cases with F_f non trivial we can check for regularity. This approach is less economical than Douglas's one because for the singular case he stopped the analysis without additional prolongations, but it is probably more easily generalisable to fulfil a whole study of the linear problem before analising the regularity condition. Case II follows a similar algorithm, but it is obviously much longer.

2.3 Begining $N = 3$

In the former discussion we can find some ideas to attack higher dimensional cases. One is to use restricted variables making use of the symmetry conditions, both in $J^1\nu$ and in higher order jet bundles. For example, for $N = 3$ we can retrict the computations to the variables $\{g_{11}, g_{12}, g_{13}, g_{22}, g_{23}, g_{33}\}$ in F_v;

$$\{z^o_{11}, z^o_{12}, z^o_{13}, z^o_{22}, z^o_{23}, z^o_{33}\}$$

all of them vanishing,

$$\{z^v_{111}, z^v_{112}, z^v_{113}, z^v_{122}, z^v_{123}, z^v_{133}, z^v_{222}, z^v_{223}, z^v_{233}, z^v_{333}\}$$

and the corresponding z^h in $J^1_g\nu$; from $D^V z^v = u^{vv}$ symmetric in the first three indices, and the commutation relation $[D^V_*, D^V_{**}] = 0$ we get u^{vv} totally symmetric, while $D^H z^h = u^{hh}$ $(1,2,3)$–symmetric and the commutation rule allows to restrict the variables to u^{hh}_{1111}, u^{hh}_{1112}, $\ldots$, because, for example $u^{hh}_{112;1} = u^{hh}_{111;2} + \cdots$, etc. Moreover, we can select a particular ordering on the upper indices, so that for example $u^{vo} = u^{ov} - z^h$ is written in terms of u^{ov} and lower order terms, etc. (notice that $u^{oo} = u^{ov} = u^{oh} = 0$). In this way, we can use $\{u^{oo} = u^{ov} = u^{oh} = 0, u^{vv}, u^{vh}, u^{hh}\}$ in $J^2\nu$ with symmetrised subindices, diminishing the dimensions of the systems of equations.

Another idea is to begin with a primary–secondary classification $F_{in} \subset F$ based on the family obtained in the N–generic discussion, which for $N = 3$ is restricted to equations with two or three entries ($\Omega^k(\mathbb{R}^3) \equiv 0$ for $k > 3$):

$$\{(\nabla^k\Phi)_{12}(g) = 0, k \leq 6\}$$

each tensorial equation having at most three independent components, and

$$0 = \sum_{s\in P_3} (-1)^{|s|} g((\nabla^{k_2} D^V \nabla^{k_1}\Phi)(X_{s1}, X_{s2}), X_{s3})$$

with one independent component each.

We find in this way seven primary cases, $\dim F_{in} = 6, \ldots, 0$, with different secondary cases according to the origin of the restrictions. For example, $\dim F_{in} = 4$ can come from just $\Phi_{12}(g) = 0$ with two nontrivial components, or from $\Phi_{12}(g) = 0$, $(\nabla\Phi)_{12}(g) = 0$ with one nontrivial component each, or from $\Phi_{12}(g) = 0$ (one), $\sum_{s\in P_3}(-1)^{|s|}g(D^V\Phi)(X_{s1}, X_{s2}), X_{s3}) = 0$ (one).

Third, we can exploit the commutation $[D^V, \nabla] = D^H$ to avoid the D^H–prolongations in the analisys because they will be equivalent to alternants of ∇ and D^V. Similarly, some D^V–D^H alternants are redundant by the symmetry of θ. Fourth, whenever we get an explicit system we can leave unfinished the computation by making reference to the closedness condition; also, some cases can be *sent* to former ones when the involved variables are the same, even if the coefficients differ. Fifth, I think the notation $A^{v1} = (D_1^V\Phi)_2^1$, etc. can be also helpful to shorten the expressions, and the prolongations become quite straightforward.

Finally, while K for $N = 2$ K contains vector subspaces of at most dimension one, for $N = 3$ they are of dimension two, so that if $\dim F_f \geq 3$ the regularity condition is authomatically satisfied.

Even through, it seems an unbeatable *tour de force* to get an exahustive classification unless some symbolic computation programm is used. Perhaps, restriction of the SODE systems considered, as in [3] for the isotropic case, can be maneaged within this framework of the covariant algorithm, e.g., SODE with $\theta = 0$, Φ diagonal, etc.

References

[1] J. E. Aldridge, G. E. Prince, W. Sarlet, G. Thompson; "An EDS approach to the inverse problem in the calculus of variations " *J. Math. Phys.***47**, 103508 (2006)

[2] J. Douglas; "Solution of the inverse problem of the calculus of variations", *Trans. Am. Math. Soc.***50** (1941), 71–128.

[3] J. Grifone and Z. Muzsnay; Variational principles for second-order differential equations. (World Scientific, 2000)

[4] E. Martínez, J.F. Cariñena and W. Sarlet; " Derivations of differential forms along the tangent bundle projection", *Diff. Geometry and its Appl.***2** (1992), 17–43.

[5] E. Martínez, J.F. Cariñena and W. Sarlet; " Derivations of differential forms along the tangent bundle projection. Part II", *Diff. Geometry and its Appl.***3** (1993), 1–29.

[6] W. Sarlet; " The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics", *J. Phys. A: Math. Gen.***15** (1982), 1503–1517.

[7] W. Sarlet, A. Vandecasteele, F. Cantrijn and E. Martínez; " Derivations of forms along a map: the framework for time-dependent second-order equations", *Diff. Geometry and its Appl.***5** (1995), 171–203.

[8] W. Sarlet, M. Crampin and E. Martínez; " The integrability conditions in the inverse problem of the calculus of variations for second order ordinary differential equations", *Acta Appl. Math.***54** (1998), 233–273.

[9] D. J. Saunders; The geometry of Jet bundles. (London Math. Soc. Lect. Note Series **142**, Cambridge Univ. Press, 1989)