Thermodynamic formalism for weak coarse expanding dynamical systems

Feliks Przytycki
Institute of Mathematics of the Polish Academy of Sciences

Joint with Tushar Das, Giulio Tiozzo, Mariusz Urbański, Anna Zdunik

Silkroad Mathematics Center, Shanghai, December 2020, online

1 – Introduction

I will follow our arXiv preprint, 2019-2020. For a background we refer to [HP] P. Haissinsky, K. Pilgrim: *Coarse expanding conformal dynamics*, Asterisque (2009). Remarkable is also [BM] M. Bonk, D. Meyer, *Expanding Thurston Maps*, AMS 2017.

Definition (Finite branched covering)

- $f : W_1 \to W_0$ continuous between locally compact Hausdorff topological spaces
- Local homeomorphism except finite set of branching points where it has bounded topological degree d bigger than 1. Another name *critical points*

– $T_{3\frac{1}{2}}$. Metrizable if having a countable basis.
– f is open.
– In surfaces in a neighbourhood of a branching point in adequate polar charts $f(r, \theta) = (r, d\theta)$.
If f is holomorphic in complex dimension 1, then $f(z) = z^d$ in holomorphic charts. However here z can be fixed repelling, belonging to X, whereas for holomorphic f it must be attracting.
– We assume W_0 is strongly path-connected, i.e path connected after removal of any countable set.

Definition (weakly coarse expanding system, WCX)

Let $f : W_0 \to W_1$ be as above with compact $\mathrm{cl}\, W_1 \subset W_0$, satisfying the
– [Expansion] axiom: There exists a finite family $\mathcal{U}\mathcal{U}_0 = \mathcal{U}_0$ of connected open subsets of W_0, intersecting the compact repellor $X := \bigcap_{n=0}^{\infty} f^{-n}(W_0)$, whose union covers it, and such that for every $\mathcal{Y}$, a finite open cover of X, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ the family $\mathcal{U}_n$ consisting of pullbacks of elements of $\mathcal{U}_0$ is subordinated to $\mathcal{Y}$.
– [Irreducibility] axiom: For every V open in X $f^n(V) = X$ (leo)
– [Non-triviality] axiom: The repellor X is not a single point.

1) We do not assume Haissinsky-Pilgrim, Bonk-Meyer [Degree] axiom, saying that the degrees of f^n are locally uniformly bounded. This property is called *Semi-hyperbolicity* for rational maps. Such are Thurston maps, where the postcritical set is finite.
2) [Expansion] axiom is (formally) stronger than just generating our topology at X. The latter follows if for every $x \in X$ and open $V \ni x$ we consider $\mathcal{Y}$ consisting of V and a neighbouhood in W_0 of its complement not containing x.
3) Merely $\mathcal{U}_N \prec \mathcal{U}_0$ so $\mathcal{U}_{kN+N} \prec \mathcal{U}_{kN}$, yields uniform backward shrinking.
4)Notice that if $p \in X$ is f-fixed then for open $V \ni p$ small enough $\bigcap_n \mathrm{Comp}_x f^{-n}(V) = \{p\}$, i.e. p is repelling.
5) Examples: Lattes, baricentric

Theorem

Suppose $f : W_1 \to W_0$ is a finite branched cover and axiom [Expansion] *holds. Then there exist a metric ρ on X compatible with the topology and constants $C > 0$, $\theta < 1$ such that for all $n \geq 0$*

$$\sup_{U \in \mathcal{U}_n} \operatorname{diam}_\rho(U) \leq C\theta^n.$$

– This holds e.g. for $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ rational function, $X = J(f)$, Riemann metric, f TCE.
– The same assertion follows from the [metric Expansion] axiom, that there exists ρ on X such that

$$\lim_{n \to \infty} \sup\{\operatorname{diam}_\rho(U) \ : \ U \in \mathcal{U}_n\} = 0.$$

– Proofs follow from Frink's metrization lemma. Haisssinsky & Pilgrim provide different proves via an adequate graph $\Gamma = \Gamma(\mathcal{U}_0)$ and Gromov's boundary. Metrics they construct are visual metrics. They are backward exponentially contracting as above and additionally satisfy $f(B(x,r)) = B(f(x), \lambda r)$ for some $\lambda > 1$ and all $x \in X$ and $r > 0$ small enough.

Frink's metrization lemma

Lemma (Aline Frink, BAMS 1937)

Let X be a topological space, and let $(\Omega_n)_{n\geq 0}$ be a sequence of open neighborhoods of the diagonal $\Delta \subseteq X \times X$, such that

(a) $\Omega_0 = X \times X$

(b) $\bigcap_{n=0}^{\infty} \Omega_n = \Delta$, *where Δ is the diagonal in $X \times X$.*

(c) *For any $n \geq 1$,*

$$\Omega_n \circ \Omega_n \circ \Omega_n \subseteq \Omega_{n-1}$$

where $\circ$ is the composition in the sense of relations: i.e., $R \circ S = \{(x, y) \in X \times X \ : \ \exists z \in X \text{ s.t. } (x, z) \in R \text{ and } (z, y) \in S\}$.

Then there exists a metric ρ on X, compatible with the topology, such that

$$\Omega_n \subseteq \{(x, y) \in X \times X \ : \ \rho(x, y) < 2^{-n}\} \subseteq \Omega_{n-1}$$

for any $n \geq 1$.

We apply it to $\Omega_n := \mathcal{U}^{(2)}_{Mn} = \{(x, y) \in X \times X : \exists U \in \mathcal{U}_{nM},\ x, y \in U\}$ for M large enough.

3 – Extension to graph Γ and visual metrics

Let $f : W_0 \to W_1, \mathcal{U}_n$ and X be as in the definition of weakly coarse expanding map, WCX.
– Following [HP], the vertices of generation n are elements of $\mathcal{U}_{n-1}$, where $n = 1, 2, \ldots$. The set of them is denoted $S(n)$. We add the base vertex of generation 0, denoted o. For $n \in \mathbb{N}$ and a vertex $W \in S(n)$ we set $|W| = n$.
Two vertices W_1, W_2 are joined by an edge (W_1, W_2) if

$$||W_1| - |W_2|| \leq 1 \quad \text{and} \quad W_1 \cap W_2 \cap X \neq \varnothing.$$

– We decree that each edge is isometric to the Euclidean interval. The resulting graph $\Gamma = \Gamma(f, \mathcal{U}_0)$ is a geodesic metric space, 1/2-quasi-starlike from the base point o, proper (bounded closed balls are compact). Due to [Expansion] axiom Γ is Gromov hyperbolic, [HP, Thm.3.3.1] .
– We can compactify it taking the above metric multiplied by the factor $e^{-\varepsilon n}$ on each edge of "generation" n. Then $\partial\Gamma = X$. f can be extended to $F : \mathrm{cl}\,\Gamma \to \mathrm{cl}\,\Gamma$ by $F((W_1, W_2)) = (f(W_1), f(W_2))$, "locally affine", expanding by e^{ε}. For $\mathcal{V}$ also satisfying [Expansion] $\Gamma(f, \mathcal{U})$ is quasi-isometric to $\Gamma(f, \mathcal{V})$. The identity on X is quasi-symmetric.

Theorem (DPTUZ1: Tushar Das, FP, Giulio Tiozzo, Mariusz Urbański, Anna Zdunik)

Let $f : W_1 \to W_0$ be a weakly coarse expanding dynamical system without periodic critical points, let X be its repellor, ρ_X an exponentially contracting metric on X compatible with the topology. Then, for every $\varphi : (X, \rho_X) \to \mathbb{R}$ a Hölder continuous function (potential):
(1) there exists a unique equilibrium state μ_φ. Let $\psi : (X, \rho_X) \to \mathbb{R}$ be a Hölder continuous function (observable), and denote $S_n\psi(x) := \sum_{k=0}^{n-1} \psi(f^k(x))$. Then there exists the finite limit

$$\sigma^2 := \lim_{n\to\infty} \frac{1}{n} \int_X \left(S_n\psi(x) - n\int \psi \, d\mu_\varphi\right)^2 d\mu_\varphi \geq 0$$

such that the following statistical laws hold for the Hölder observable ψ:
(2) Central Limit Theorem, CLT, for the sequence $\psi \circ f^k$,
(3) Law of Iterated Logarithm LIL,
(4) Exponential Decay of Correlations, EDC:
There exist constants $\alpha > 0$ and $C \geq 0$ such that for any μ_φ–integrable function $\chi : X \to \mathbb{R}$, for any β-Hölder function $\psi : X \to \mathbb{R}$, and $n \geq 0$,

$$\left| \int_X \psi \cdot (\chi \circ f^n) \, d\mu_\varphi - \int_X \psi \, d\mu_\varphi \cdot \int_X \chi \, d\mu_\varphi \right| \leq Ce^{-n\alpha} \|\underline{\chi}\|_1 \cdot \|\underline{\psi}\|_\beta,$$

where $\underline{\chi} := \chi - \int_X \chi \, d\mu_\varphi$, and $\| \cdot \|_\beta$ is the β-Hölder norm.

Theorem (DPTUZ1 continuation)

(5) Large Deviations – level 1.
For every $t \in \mathbb{R}$, we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\varphi\left(\left\{x \in X : \mathrm{sgn}(t) S_n\psi(x) \geq \mathrm{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi}\right\}\right)$$
$$= -t \int_X \psi \, d\mu_{\varphi+t\psi} + P_{top}(\varphi + t\psi) - P_{top}(\varphi).$$

(6) Moreover, $\sigma = 0$ if and only if there exists a continuous $u : X \to \mathbb{R}$ such that $\psi - \int_X \psi \, d\mu_\varphi = u \circ f - u$. (cohomology equation)

(7) Finally, $\mu_{\varphi_1} = \mu_{\varphi_2}$ if and only if there exist $K \in \mathbb{R}$ and a continuous $u : X \to \mathbb{R}$ such that $\varphi_1 - \varphi_2 = u \circ f - u + K$.

In (6) and (7) the function u is Hölder continuous with respect to a visual metric.

We shall reduce (most of) this theorem to a standard one for $\varsigma : \Sigma^d \to \Sigma^d$ the shift on the one-sided symbolic space of $d = \deg f$ symbols and $\phi : \Sigma^d \to \mathbb{R}$ Hölder.

Topological pressure and equilibrium states – definitions

Let $f : X \to X$ be a continuous map of a compact metric space. A probability measure μ on X is *f-invariant* if $f_\star\mu = \mu$, and we let $M(f)$ be the set of f-invariant probability measures on X. We denote as $h_\mu(f)$ the metric entropy of f with respect to μ.
Now, consider a continuous function $\varphi : X \to \mathbb{R}$, which we call a *potential*. The *topological pressure* of f with potential φ is defined as

$$P_{top}(f, \varphi) := \sup_{\mu \in M(f)} \left\{ h_\mu(f) + \int_X \varphi \, d\mu \right\}.$$

Note that $P_{top}(\varphi)$ may also be defined topologically or metrically (via separated or spanning sets, [PU]). An f-invariant probability measure μ on X is an *equilibrium state* for φ if it realizes the supremum.

As in holomorphic dynamics, a special role is played by branching (*critical*) points, where the map is not locally injective. A major source of difficulty in the study of weakly coarse expanding systems is the presence of repelling periodic critical points in the repellor (not possible if they attract since they are outside Julia set then).
The systems we consider do not satisfy the [Degree] condition, hence they are not coarse expanding in the sense of [HP].

Our second result addresses this issue in the case the underlying space is an open subset of the 2-sphere (which is the case considered in [BM] and [Zhiqiang Li, *Ergodic Theory of Expanding Thurston Maps*].
Namely, our results also hold in the presence of periodic critical points as follows.

Theorem (DPTUZ2)

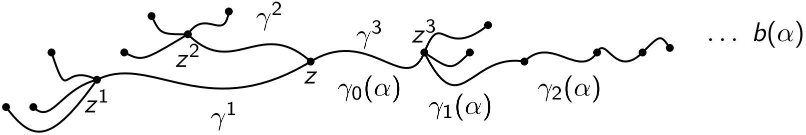
If $f : W_1 \to W_0$ is a weakly coarse expanding system and $W_0 \subseteq S^2$ is an open subset of the 2-sphere, with the Euclidean topology, then all claims (1)-(2)-(3)-(4)-(5)-(6)-(7) of Theorem DPTUZ1 hold even if there are periodic critical points.

Geometric coding tree

Consider an important example of *distance expanding* map, the shift to the left, on the symbolic space $\varsigma : \Sigma^d \to \Sigma^d$, where Σ^d is the space of all sequences $(\alpha_n)_{n=0,1,...}$ with $\alpha_n \in \{1, ..., d\}$, and ς is the left shift $\varsigma((\alpha_n)) = (\alpha_{n+1})$, with the metric $\rho_\Sigma((\alpha_n), (\alpha'_n)) := 2^{-\inf\{k \geq 0 \,:\, \alpha_k \neq \alpha'_k\}}$.

We shall use it for 'coding' our weak coarse expanding maps. (It itself is not weak coarse expanding since it does not contain non-trivial continuous paths.)

Given $z \in X$ and curves $\gamma^j : [0,1] \to W_0$, $\; j = 1, \ldots, d$, joining z to $z^j \in f^{-1}(z)$, we define a graph $\mathscr{T}$ consisting of the set of vertices $f^{-n}(z) \in X$ and edges $f^{-n}(\gamma^j) \subset W_0$, $n = 0, 1, ...$ and $j = 1, ..., d$, such that denoting the edges in $f^{-n}(\gamma^j)$ by $\gamma_n(\alpha)$ for all $\alpha \in \Sigma^d$ the following conditions hold



$\gamma_0(\alpha) := \gamma^{\alpha_0}$, $f \circ \gamma_n(\alpha) = \gamma_{n-1}(\varsigma(\alpha))$, $\gamma_n(\alpha)(0) = \gamma_{n-1}(\alpha)(1)$.

The vertices are defined as the ends of $\gamma_n(\alpha)$, denoted $z_n(\alpha)$, $z_{n-1}(\alpha)$.

Coding

For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called an *infinite geometric branch* and denoted by $b(\alpha)$.

Lemma

For each $\alpha \in \Sigma^d$ the sequence $z_n(\alpha)$ converges exponentially to a point $\pi(\alpha) \in X$. The mapping $\pi : \Sigma^d \to X$ is Hölder continuous and onto.

In the proof one uses the fact [HP] that there exists N such that if $U, U' \in \mathcal{U}_n$ intersect, maybe outside X, then they belong to a common $\hat{U} \in \mathcal{U}_{n-N}$. (We used it already checking conditions of Frink's lemma.)
Hence for a chain of consecutively intersecting $U(j) \in \mathcal{U}_n$ for $j = 1, ..., k$ given k, $\rho_X(U(1), U(k))$ is (exponentially) small for n large.
Hint: to prove the fact use Hausdorff and local compactess properties of the topology on W_1.

No entropy drop

Let $f : W_1 \to W_0$ be a finite branched cover with repellor X, let ρ be a metric on X, and let $\varphi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous potential. Suppose as in Theorem DPTUZ1, that there are no periodic critical (branching) points.

We say x is an ϵ-*singular point* if it lies within distance at most ϵ of a branching point. Clearly for each non-periodic (branching) point the times n so that $f^n(x)$ is ϵ close to it, are rare for each $x \in X$. Hence

Lemma

Suppose that no branching point is periodic. For any $x \in X$, denote as $S_{n,x}$ the number of cylinders in Σ^d of depth n which intersect $\pi^{-1}(x)$. Then

$$\lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} S_{n,x} = 0.$$

Corollary

Let μ be a ς-invariant measure on the symbolic space Σ^d, and let $\nu = \pi_\star \mu$. Then $\quad h_\mu(\sigma) = h_\nu(f)$.

Proof of Theorem DPTUZ1

For every Hölder $\phi : X \to \mathbb{R}$ for $\hat{\phi} = \phi \circ \pi$, we find an equilibrium $\mu_{\hat{\phi}}$. Then $\pi_*(\mu_{\hat{\phi}})$ is an equilibrium for (f, ϕ) on X since π_* is *onto* and for all μ and $\nu = \pi_*(\mu)$ as in Corollary above

$$h_\mu(\varsigma) + \int \hat{\phi}\, d\mu = h_\nu(f) + \int \phi\, d\nu.$$

The latter yields the equality of the pressures $P(f, \phi) = P(\varsigma, \hat{\phi})$. Uniqueness follows from the uniqueness of $\mu_{\hat{\phi}}$. Theorem DPTUZ1, the laws (2)-(5), follow from these laws on Σ^d and Hölder continuity of π.

Notice that the proof works for all ϕ *topologically Hölder* on X, that is such that

$$\mathrm{var}_{\mathcal{U}_n}\phi := \sup_{U \in \mathcal{U}_n} \sup_{x,y \in U \cap X} |\phi(x) - \phi(y)|$$

converge to 0 exponentially fast.
This may happen to be weaker than Hölder if $\mathrm{mesh}_{\rho_X}(\mathcal{U}_n)$ converge to 0 faster than exponentially.

To prove (6) and (7) use a visual metric ρ_ε, for which $f(B(\xi, re^{-\epsilon})) = B(f(\xi), r)$ for all $\xi \in X$ and $r > 0$ small enough.

Then, following [Bowen], for $\eta = \phi_1 - \phi_2 - \int(\phi_1 - \phi_2)\, d\mu$ we choose $x \in X$ so that $O(x) = \{f^n(x)\}_{n \geq 0}$ is dense in X, and prove it is Hölder on $O(x)$ hence extends to a Hölder continuous function on X.

$\sigma^2 = 0$ implies there exists $C > 0$ such that $|S_n\eta(x)| < C$ for all $x \in X$ and $n \in \mathbb{N}$. Hence $S_n\eta(x) = 0$ for every x periodic, $f^n(x) = x$.

Finally one uses periodic shadowing, by considering $f^k(x), f^m(x)$ for $k < m$, $\rho_\varepsilon(f^k(x), f^m(x)) < \delta$, and pulling back $f^k(x)$ along $f^{m-1}(x), ..., f^{m-k}$. Keeping repeating it one gets an exponential convergence to a periodic limit $y, ..., f^{m-k}(y) = y$, with $S_{m-k}\eta(y) = 0$. Hence $|\eta(f^k(x)) - \eta(f^m(x))| \leq \mathrm{Const}\, \delta^\alpha$.

FIGURE

Periodic branching points – blowing up

Lemma (local model)

Let $f : W_1 \to W_0 \subseteq S^2$ be a weakly coarse expanding map, and let $p \in X$ be a fixed critical point. Then for any $\lambda > 1$ there exist $d \in \mathbb{Z} \setminus \{0\}$, a neighborhood U of p and a homeomorphism $h : \overline{\mathbb{D}} \to \overline{U}$ such that $f \circ h = h \circ g$ where $g : \overline{\mathbb{D}}_{\lambda^{-1}} \to \overline{\mathbb{D}}$ is defined as $g(re^{i\theta}) := \lambda re^{id\theta}$ for any $r \leq 1$, $\theta \in \mathbb{R}$.

It relies on

Lemma (On backward shrinking Jordan domain)

For every B a neighbourhood of p in W_0, there exists a Jordan curve γ in $B \setminus \{p\}$ such that a component of $f^{-1}(\gamma)$ is a Jordan curve disjoint from γ and separates γ from p.

Proof. Start with a Jordan domain $p \in U \subset B$ and take $N > 0$ such that the closure of $U_N := \mathrm{Comp}_p\, f^{-N}(U)$ is contained in U, existing by [Expansion] axiom. Consider $V := \mathrm{Comp}_p \left(\bigcap_{n=0}^{N-1} U_n \right)$, where Comp_p means the component containing p. Its slightly corrected boundary can be our γ.

Construction of the blowup.

Let $\mathcal{C}$ denote the (finite) set of all periodic critical points in X, and let $\mathcal{E} = \bigcup_{n \geq 0} f^{-n}(\mathcal{C})$. We can define a space $\widetilde{S}$ which is given by blowing up every point of $\mathcal{E}$ to a circle.

Namely, for each $q \in \mathcal{E}$ let S_q be a copy of S^1. The space $\widetilde{S}$ is defined as a *set* as

$$\widetilde{S} := (S^2 \setminus \mathcal{E}) \sqcup \bigsqcup_{q \in \mathcal{E}} S_q.$$

The topology on $\widetilde{S}$ will be defined shortly. Note there is a natural projection map $\pi : \widetilde{S} \to S^2$ which sends each S_q to q. Let $\widetilde{W_0} := \pi^{-1}(W_0)$, $\widetilde{W_1} := \pi^{-1}(W_1)$.

Definition of g. Let us now extend f to a map $g : \widetilde{W_1} \to \widetilde{W_0}$. In order to do so, let us identify all S_q for $q \in \mathcal{E}$ with $\mathbb{R}/\mathbb{Z}$, and define $g : S_q \to S_{f(q)}$ as $g(\theta) := d\theta \mod 1$, where d is the local (signed) degree of f at q. Finally, let us define $g := f$ on $\widetilde{W_1} \setminus \bigcup_{q \in \mathcal{E}} S_q$.

Topology

Let us choose one element from any periodic orbit in X containing critical points, and let us call $\mathcal{E}^*$ the union of such points. Moreover, for each $q \in \mathcal{E}$ let $m = m(q)$ be the minimal $m \geq 1$ such that $f^m(q) \in \mathcal{E}^*$. In particular for $p \in \mathcal{E}^*$ this is the minimal period.
Let now $p \in \mathcal{E}^*$. Then by Lemma on the local model, there exists a neighborhood U_p of p such that the map $f^m : U_p \to f^m(U_p)$ is topologically conjugate to $(r, \theta) \mapsto (\lambda r, d\theta \mod 1)$ for some $\lambda > 1$.
We now fix some small values $r_0, \epsilon > 0$ and define for any $\theta_0 \in [0, 2\pi)$ the set

$$V_{p,\theta_0} := \pi^{-1}(h(\{0 < r < r_0, \theta_0 - \epsilon < \theta < \theta_0 + \epsilon\})) \cup \{\theta \in S_p \ : \ \theta_0 - \epsilon < \theta < \theta_0 + \epsilon\}$$

For $q \in \mathcal{E} \setminus \mathcal{E}^*$ we define V_{q,θ_0} by taking pullbacks from $\mathcal{E}^*$.
We now define the topology on $\widetilde{W}_0$ to be the topology generated by

$$\{\pi^{-1}(U) \ : \ U \text{ open in } W_0\} \cup \{g^{-n}(V_{p,\theta_0}) \ : \ n \geq 0, p \in \mathcal{C}, \theta_0 \in [0, 2\pi)\}.$$

The continuity of π is immediate. Also **Hausdorff property** of the topology on $\widetilde{W}_0$, **compactness** of the closure of $\widetilde{W}_1$ and **local connectivity** of $\widetilde{W}_0$ are clear.

The repellor Y and the [Irreducibility] axiom.

Denote as Y the repellor for $g : \widetilde{W_1} \to \widetilde{W_0}$. Clearly $Y = \pi^{-1}(X)$. So, for $p \in \mathcal{E}$, $S_p \subset Y$. We prove that g is leo on Y.
For this, note that S_p, for say p f-fixed, is contained in the closure of the lift of $X \setminus \{p\}$, and even is contained in the closure of $Y \setminus \pi^{-1}(\mathcal{E})$, using the chart h in the local model and backward invariance of X.
For any V being a neighbourhood in Y of $y \in Y \setminus \pi^{-1}\mathcal{E}$ there is n such that $g^n(V) = Y$ as lifting leo of f on X. Hence leo on the closure Y holds. The irreducibility is proved.

Note that if X is not a neighbourhood of p in W_0, e.g. there is a curve in $W_0 \setminus X$ converging to p then we can blow up p so that S_p is not in Y. For this we use a different model and can get g on S_p of degree d but having an attracting periodic orbit.

verifying the [Expansion] axiom

Let $\widetilde{\mathcal{U}}_0$ be the lift of $\mathcal{U}_0$ to a neighborhood of Y after blowing up construction.
For each periodic branching p add to $\widetilde{\mathcal{U}}_0$ two neighborhoods of arcs in the circle $S_p = \mathbb{R}/2\pi\mathbb{Z}$, V_0 and V_1 with $0 \leq r < r_\star$ and

$$-\pi/4 < \theta < \pi + \pi/4 \quad \text{and} \quad \pi - \pi/4 < \theta < 2\pi + \pi/4$$

respectively, in the polar coordinates of the model. We replace by them the lift of the set $U_0(p) \in \mathcal{U}_0$.
Denote this cover by $\widetilde{\mathcal{W}}_0$, and for each n by $\widetilde{\mathcal{W}}_n$ the cover given by connected components of the sets $g^{-n}(U)$ for any $U \in \widetilde{\mathcal{W}}_0$.

– Clearly $\bigcup_n \widetilde{\mathcal{W}}_n$ provides a (countable) basis for the topology at all points of Y. Hence there exists a cover of Y by the sets $U(x,n) \in \widetilde{\mathcal{W}}_n$, each being a subset of an element V of $\mathcal{V}$ for $n \geq N(V,x)$.

– By compactness and the Hausdorff property, we can find, for each $V \in \mathcal{V}$, a compact set $V^* \subset V \cap Y$ so that the union of the V^* covers Y. Now for an arbitrary $V \in \mathcal{V}$ one proves that over all $x \in V^*$, the values $N(V,x)$ are uniformly bounded.

– **Case 1.** Suppose that $\pi(V^*) \cap \pi(V^c) = \varnothing$. Since π is continuous, these sets are compact, so their Euclidean distance in S^2 is positive, say $\delta_V > 0$. So if $U \in \widetilde{\mathcal{U}}_n$ intersects both V^* and V^c, the Euclidean diameter of $\pi(U)$ is at least δ_V, so n is bounded from above by some N_V, independent of $x \in V^*$. In other words $U(x, n)$ cannot intersect both V^* and V^c for $n > N_V$, hence is in V.

– **Case 2.** If $\pi(V^*)$ and $\pi(V^c)$ intersect, then there exists $q \in \mathcal{E}$ such that both V^* and V^c intersect S_q. In such a case we have the:

Claim. Among all q's so that V^* and V^c intersect S_q, the set of possible $m(q)$ is bounded.

Proof Otherwise, let $(q_n) \subseteq X$ be a sequence with $m(q_n) \to \infty$, and choose a convergent subsequence $q_n \to q^*$ using the metric ρ_1 on X. Let $x_n \in S_{q_n} \cap V^*$ and $y_n \in S_{q_n} \cap V^c$.
Now a simple analyses shows that both the cases $q^* \notin \mathcal{E}$ or $q^* \in \mathcal{E}$ are impossible, which ends the proof.

Conclusion: Proof of DPTUZ2

For Hölder, hence topologically Hölder $\varphi : X \to \mathbb{R}$ consider $\hat{\varphi} := \varphi \circ \pi \circ \Pi$, where $\pi : Y \to X$ is the blowdown map and $\Pi : \Sigma^d \to Y$ is the coding map. Notice that the composition is Hölder. Then consider the unique equilibrium $\mu_{\hat{\varphi}}$ on Σ^d and define $\mu_\phi := \Pi \circ \pi)_*(\mu_{\hat{\varphi}})$.

Then, as in the case without periodic critical points, for any Hölder continuous observable ψ statistical laws for the sequence $(\psi \circ f^n)_{n\in\mathbb{N}}$ follow from the adequate statistical laws for the shift map with respect to the sequence $(\psi \circ \pi \circ \Pi \circ \varsigma^n)_{n\in\mathbb{N}}$.
Caution: we do not know/use that π is Hölder.

One should however additionally explain in the periodic critical case that this μ_ϕ is indeed a unique equilibrium on X, since an entropy drop can happen.
The measure $\mu = \mu_{\hat{\varphi}}$ is ergodic with respect to the shift ς, and positive on non-empty open sets (as a consequence of the Gibbs property [Bowen]. Notice that $\Pi_*(\mu)$ is 0 on S_p and its g^n-pre-images. Otherwise, by ergodicity it would be supported on S_p (if it charged other preimages of S_p it would charge preimages under iterates, therefore being infinite). So μ would be 0 on the open nonempty set $\Sigma \setminus \Pi^{-1}(S_p)$, a contradiction.

So $\pi \circ \Pi$ preserves the entropy of μ since Π does "no entropy drop" and π does it too as it is a measurable isomorphism $\pi : Y \setminus \bigsqcup_{p \in \mathcal{E}} S_p \to X \setminus \mathcal{E}$. Moreover, for $\mu_\varphi = (\pi \circ \Pi)_*(\mu)$ we have $\int_\Sigma \varphi \circ \pi \circ \Pi \, d\mu = \int_X \varphi \, d\mu_\varphi$ by definition. Hence

$$P_{top}(f,\varphi) \geq h_{\mu_\varphi}(f) + \int_X \varphi \, d\mu_\varphi = h_\mu(\sigma) + \int_\Sigma \varphi \circ \pi \circ \Pi \, d\mu = P_{top}(\varsigma, \varphi \circ \pi \circ \Pi).$$

So $\boxed{P_{top}(f,\varphi) = P_{top}(\varsigma, \varphi \circ \pi \circ \Pi)}$. In particular, we have proved that μ_X is an equilibrium state. Uniqueness follows easily.

Let us finish with

Proposition

The lift $g : \widetilde{W_1} \to \widetilde{W_0}$ resulting from blowing up periodic repelling branching points of $f : W_1 \to W_0 \subset S^2$, can be continuously embedded into the sphere S^2, where Y becomes a repellor for the extended system. More precisely, there exist a continuous embedding $\iota : \widetilde{W_0} \to S^2$, a continuous map $g' : S^2 \to S^2$ with $g' \circ \iota = \iota \circ g$ and an open set W_0' which contains $\iota(\widetilde{W_0})$ so that $\iota(Y) = \bigcap_{n \geq 0} (g')^{-n}(W_0')$.

The components of $S^2 \setminus \iota(S_p)$ not intersecting $\iota(\widetilde{W_0})$, become basins of attracting periodic orbits. If $X = S^2$ then $\iota(Y)$ is a Sierpiński carpet.

THANK YOU FOR YOUR ATTENTION