FIVE MOST RESISTANT PROBLEMS IN DYNAMICS

ANATOLE KATOK

We will discuss five problems, or rather groups of problems, listed below, from various areas of dynamics which attracted great attention at various periods during the last half century, stimulated a variety of important developments but so far resisted attempts to penetrate core difficulties on the way to comprehensive solutions. Aside from giving detailed descriptions of the background and various versions of the formulations of the problems we will describe the most significant partial results and developments stimulated by the problems, offer (probably wild) guesses about the chances and possible timing of future breakthroughs and in some cases mention possible lines of attack. The names in brackets are of those mathematicians whose work led to realization of importance of the problem, sometimes by stating it (or its predecessor), sometimes by trying to solve it, in other cases by obtaining partial results.

1. [**Kolmogorov** (1950's), **Sinai** (early 1960's)] *Metric entropy for area preserving twist maps, or impossible problem of peaceful coexistence of KAM circles and stochastic behavior on a set of positive Lebesgue measure.*

The situation is quite paradoxical: expected answer is that the metric entropy is positive for a typical map and hence by Pesin theory there is an ergodic component of positive measure. But since such a component should avoid all KAM circles including the very problematic "last" ones it is unconceivable from the point of view of methods which exist or can be imagined how necessary estimates could be carried out.

2. [**Kushnirenko** (1965), **Anosov–A. K.** (1970), **Pesin** (1977)] *Smooth realization of ergodic measure preserving transformations with zero and finite entropy.*

Obstructions to smooth realization on any manifold are very likely to exist but their nature cannot be guessed at present. A variety of exciting questions concerning existence of non-standard smooth realizations look open to attack. There are prospects of finding obstructions to smooth realization of special kind (small dimension, particular homotopy classes).

3. [**H. Masur, A. K.**] (1980's) *Orbit growth, existence of periodic orbits and ergodicity of billiards in general (not rational) polygons.*

There is a huge gap between the known non-explicit sub-exponential estimate and the quadratic growth known for the rational case.

4. [**Furstenberg** (1967), **Rudolph** (1990)] *Invariant measures for hyperbolic actions of higher rank abelian groups including $\times 2$, $\times 3$ problem and Weyl chamber flow on $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$.*

Even expected answer is in doubt: are there weird non-algebraic measures of zero entropy?

5. [**Smale** (1967), **Franks** (1970)] *Topological classification of Anosov diffeomorphisms and differentiable classification of Anosov actions of $\mathbb{Z}^k$, $k \geq 2$.*

Again expected answer is in doubt: are there examples beyond infranilmanifolds?