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Abstract

The PDO on $\mathbb{R}^{1+n}$ of the form $P = \partial_0 M(\partial_1, \dots, \partial_n) - L(\partial_1, \dots, \partial_n)$ is considered, M and L being square $m \times m$ matrices whose entries are scalar PDOs on $\mathbb{R}^n$ with constant coefficients. It is proved that

(i) the real parts of the λ-roots of $\det(\lambda M(i\xi) - L(i\xi))$ are bounded from above when ξ ranges over $\mathbb{R}^n$

if and only if

(ii) P has a fundamental solution with support in $H_+ = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : x_0 \geq 0\}$ having some special properties expressed in terms of the L. Schwartz space $\mathcal{O}'_C$ of rapidly decreasing distributions.

Moreover, it is proved that the fundamental solution with support in H_+ having these special properties is unique.

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1 Introduction and the main result

1.1 Rapidly decreasing distributions

By Theorem IX in Sec. VII.5 of L. Schwartz's book [13], for every distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ the following two conditions are equivalent:

(1.1) $T * \varphi \in \mathcal{S}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

(1.2) for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that $T = \sum_{|\alpha| \le m_k} \partial^\alpha F_{k,\alpha}$ where, for every multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ of length $|\alpha| = \alpha_1 + \dots + \alpha_n \le m_k$, $F_{k,\alpha}$ is a continuous function on $\mathbb{R}^n$ such that $\sup_{x \in \mathbb{R}^n} (1 + |x|)^k |F_{k,\alpha}(x)| < \infty$.

In the above $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ where $\partial_1, \dots, \partial_n$ are partial derivatives of the first order not multiplied by any factor. Each of the conditions (1.1), (1.2) is satisfied if and only if the distribution T is *rapidly decreasing*, where the definition of rapid decrease, due to L. Schwartz, refers to the notion of boundedness of a distribution. The space of rapidly decreasing distributions on $\mathbb{R}^n$ is denoted by $\mathcal{O}'_C(\mathbb{R}^n)$. From (1.2) it follows that

(1.3) whenever $T \in \mathcal{O}'_C(\mathbb{R}^n)$ and $\varphi \in C_b^\infty(\mathbb{R}^n)$, then $\varphi T \in \mathcal{O}'_C(\mathbb{R}^n)$.

It is visible from (1.2) that $\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, so that the Fourier transform $\mathcal{F}T$ makes sense for every $T \in \mathcal{O}'_C(\mathbb{R}^n)$. By Theorem XV in Sec. VII.8 of [13],

$$\mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n), \tag{1.4}$$

where $\mathcal{O}_M(\mathbb{R}^n)$ denotes the space of *infinitely differentiable slowly increasing functions* on $\mathbb{R}^n$. Recall that $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ if and only if $\phi \in C^\infty(\mathbb{R}^n)$ and for every $\alpha \in \mathbb{N}_0^n$ there is $m_\alpha \in \mathbb{N}_0$ such that

$$\sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m_\alpha} |\partial^\alpha \phi(\xi)| < \infty.$$

1.2 The main result

Denote by $M_{m \times m}$ the set of $m \times m$ matrices with complex entries. Let $M(\partial_1, \dots, \partial_n)$ and $L(\partial_1, \dots, \partial_n)$ be PDOs on $\mathbb{R}^n$ with constant coefficients belonging to $M_{m \times m}$. In other words, $M(\partial_1, \dots, \partial_n)$ and $L(\partial_1, \dots, \partial_n)$ are $m \times m$ matrices whose entries are scalar PDOs on $\mathbb{R}^n$ with constant complex coefficients. If we replace each ∂_ν by $i\xi_\nu$ where i is the imaginary unit and $\xi_\nu \in \mathbb{R}$, then $M(\partial_1, \dots, \partial_n)$ and $L(\partial_1, \dots, \partial_n)$ will change into matrices

$M(i\xi_1, \dots, i\xi_n)$ and $L(i\xi_1, \dots, i\xi_n)$ whose entries are complex polynomials of n real variables $\xi_1, \dots, \xi_n$.

Our object of interest will be the differential operator

$$P = \partial_0 M(\partial_1, \dots, \partial_n) - L(\partial_1, \dots, \partial_n)$$

on $\mathbb{R}^{1+n} = \{(x_0, x_1, \dots, x_n) : x_\nu \in \mathbb{R} \text{ for } \nu = 0, \dots, n\}$, and the associated $m \times m$ matrix $\lambda M(i\xi_1, \dots, i\xi_n) - L(i\xi_1, \dots, \xi_n)$ whose entries are polynomials of $1+n$ variables, $\lambda \in \mathbb{C}$ and $\xi_1, \dots, \xi_n \in \mathbb{R}$. Every $M_{m\times m}$-valued distribution N on $\mathbb{R}^{1+n}$ such that

$$PN = \delta \otimes \mathbb{1}_{m\times m}$$

is called a *fundamental solution* for the operator P.

Let $\mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$ be the space of $m \times m$ matrices whose entries are scalar distributions on $\mathbb{R}^{1+n}$ belonging to $\mathcal{O}'_C(\mathbb{R}^{1+n})$. For every fixed $\lambda \in \mathbb{C}$ let $e_{-\lambda}$ be the function on $\mathbb{R}^{1+n}$ given by $e_{-\lambda}(x_0, x_1, \dots, x_n) = \exp(-\lambda x_0)$ for $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{1+n}$. For $\vartheta \in \mathcal{D}(\mathbb{R})$, denote by ϑ_0 the function on $\mathbb{R}^{1+n}$ defined by $\vartheta_0(x_0, x_1, \dots, x_n) = \vartheta(x_0)$.

Theorem. *Let $P = \partial_0 M(\partial_1, \dots, \partial_n) - L(\partial_1, \dots, \partial_n)$ be the above differential operator on $\mathbb{R}^{1+n}$ with constant coefficients belonging to $M_{m\times m}$. Let*

$$\omega_0 = \sup\{\operatorname{Re}\lambda : \lambda \in \mathbb{C} \text{ and there is } (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \\ \text{such that } q(\lambda, i\xi_1, \dots, i\xi_n) = 0\}$$

where

$$q(\lambda, i\xi_1, \dots, i\xi_n) = \det(\lambda M(i\xi_1, \dots, i\xi_n) - L(i\xi_1, \dots, i\xi_n)).$$

Then the following two conditions are equivalent:

(i) $\omega_0 < \infty$,

(ii) *the differential operator P has a fundamental solution N with support in $H_+ = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : x_0 \geq 0\}$ such that $\vartheta_0 N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$ for every $\vartheta \in \mathcal{D}(\mathbb{R})$.*

Furthermore, if (i) *and* (ii) *are satisfied, then the fundamental solution N as in* (ii) *is unique and satisfies*

(iii) $\omega_0 = \inf\{\operatorname{Re}\lambda : \lambda \in \mathbb{C},\ e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})\}$, *and* $e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$ *whenever* $\operatorname{Re}\lambda > \omega_0$.

1.3 Remarks

Condition (i) can be called the Petrovskiĭ condition because it first appeared in I. G. Petrovskiĭ's paper [9]. Namely, in [9], in the footnote on p. 24, it was conjectured that, if $M(\partial_1, \dots, \partial_n) = \mathbb{1}_{m\times m}$, then (i) is equivalent to a certain formally weaker condition also concerning the λ-roots of $\det(\lambda \mathbb{1}_{m\times m} - L(i\xi_1, \dots, i\xi_n))$. The validity of this conjecture was proved by L. Gårding in [3]. I. G. Petrovskiĭ noticed the significance of smooth slowly increasing functions for the theory of evolutionary PDEs with constant coefficients. L. Schwartz explained in [11] how the results of Petrovskiĭ may be elucidated by placing them in the framework of rapidly decreasing distributions and smooth slowly increasing functions. (Condition (i) was not mentioned in [11]; notice that [11] was earlier than [3].)

L. Hörmander proved in [5] that if q is a polynomial of $1 + n$ variables with complex coefficients, then the following two conditions are equivalent:

(i)* there are constants $A \in]-\infty, \infty[$ and $r \in]0, \infty[$ such that

$$\inf\{\operatorname{Re} F(\zeta_1, \dots, \zeta_n) : (\zeta_1, \dots, \zeta_n) \in B_{i\xi_1,\dots,i\xi_n;r}\} \le A$$

for every $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and every function F holomorphic in the ball

$$B_{i\xi_1,\dots,i\xi_n;r} = \left\{(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n : \sum_{\nu=1}^{n} |\zeta_\nu - i\xi_\nu|^2 < r^2\right\}$$

such that $q(F(\zeta_1, \dots, \zeta_n), \zeta_1, \dots, \zeta_n) = 0$ in $B_{i\xi_1,\dots,i\xi_n;r}$,

(ii)* the differential operator $q(\partial_0, \partial_1, \dots, \partial_n)$ has a fundamental solution with support in H_+.

The equivalence (i)$^*\Leftrightarrow$(ii)* was reproved in Sec. 12.8 of [6]. The fundamental solution occurring in (ii)* need not be unique. It is non-unique if (i)* holds and the boundary of H_+ is characteristic for $q(\partial_0, \partial_1, \dots, \partial_n)$. Obviously (i) implies (i)*. Furthermore, as indicated in [5], the operator $\partial_0 - i(\partial_1 + 1)^2$ satisfies (i)* but does not satisfy (i). Therefore condition (i)* is essentially weaker than (i).

In [5], the largest power of λ in $q(\lambda, i\xi_1, \dots, i\xi_n)$ is multiplied by a polynomial of $\xi_1, \dots, \xi_n$ which, in contrast to the assumption (5) in Sec. 3.10 of [R], may vanish for some $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Similarly, in our Theorem it is allowed that for some $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ the matrix $M(i\xi_1, \dots, i\xi_n)$ is not invertible. It follows that, for some fixed $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$, the matricial differential operator $\partial_0 M(i\xi_1,\dots,i\xi_n)-L(i\xi_1,\dots,i\xi_n)$ may represent an implicit ordinary differential-algebraic system whose fundamental solution with support in $[0,\infty[$ is an $M_{m\times m}$-valued distribution on $\mathbb{R}$ not equal to a function. See [7], Sec. 6.4. Therefore it should not be expected that in our Theorem the fundamental solution is a function of the variable x_0 with values in $\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$. However, if (i) is satisfied and for every $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ the matrix $M(i\xi_1,\dots,i\xi_n)$ is invertible, then the restriction to H_+ of the fundamental solution N satisfying (ii) belongs to $C^\infty([0,\infty[;\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m}))$. See [8].

2 Existence of a fundamental solution satisfying (ii) and (iii)

2.1 Application of the Tarski–Seidenberg theorem

We are going to prove that if (i) holds, then the differential operator $\partial_0 M(\partial_1,\dots,\partial_n)-L(\partial_1,\dots,\partial_n)$ has a fundamental solution N satisfying the conditions (ii) and (iii). So, suppose that (i) holds and let

$$\mathcal{N}=\{(\sigma,\xi_0,\dots,\xi_n)\in\mathbb{R}^{2+n}:q(\sigma+i\xi_0,i\xi_1,\dots,i\xi_n)=0\}.$$

Then $\mathcal{N}\subset\{(\sigma,\xi_0,\dots,\xi_n)\in\mathbb{R}^{2+n}:\sigma\le\omega_0\}$, and hence, by Theorem A.3 from the Appendix to [14] or by Theorem 3.2 of [4]*), there are $c,\mu,\mu'\in\,]0,\infty[$ such that whenever $\sigma\in\,]\omega_0,\infty[$ and $(\xi_0,\dots,\xi_n)\in\mathbb{R}^{1+n}$, then

$$\begin{aligned}(2.1)\qquad |q(\sigma+i\xi_0,i\xi_1,\dots,i\xi_n)| &\ge c(\operatorname{dist}((\sigma,\xi_0,\dots,\xi_n);\mathcal{N}))^{\mu}\\ &\qquad\cdot(1+(\sigma^2+\xi_0^2+\dots+\xi_n^2)^{1/2})^{-\mu'}\\ &\ge c(\sigma-\omega_0)^{\mu}(1+|\sigma+i\xi_0|+(\xi_1^2+\dots+\xi_n^2)^{1/2})^{-\mu'}.\end{aligned}$$

If $(\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n$ and $\operatorname{Re}\lambda>\omega_0$, then

$$\begin{aligned}&(\lambda M(i\xi_1,\dots,i\xi_n)-L(i\xi_1,\dots,i\xi_n))^{-1}\\ &\qquad=(q(\lambda,i\xi_1,\dots,i\xi_n))^{-1}\operatorname{Adj}(\lambda M(i\xi_1,\dots,i\xi_n)-L(i\xi_1,\dots,i\xi_n))\end{aligned}$$

where the entries of the matrix $\operatorname{Adj}(\lambda M(i\xi_1,\dots,i\xi_n)-L(i\xi_1,\dots,i\xi_n))$ are polynomials on $\mathbb{C}\times\mathbb{R}^n$. Therefore, by (2.1), there are $C\in\,]0,\infty[$ and $k\in\,]0,\infty[$ such that

*) In accordance with the idea of L. Hörmander, these theorems are deduced from the Tarski–Seidenberg theorem about projections of semi-algebraic sets.

$$\text{(2.2)}\quad \|(\lambda M(i\xi_1,\ldots,i\xi_n)-L(i\xi_1,\ldots,i\xi_n))^{-1}\|_{M_{m\times m}} \\ \leq C(\operatorname{Re}\lambda-\omega_0)^{-\mu}(1+|\lambda|+|\xi|)^k$$

for every $(\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n$ such that $\operatorname{Re}\lambda>\omega_0$.

2.2 The slowly increasing functions $\widehat{N_\sigma}$ and the rapidly decreasing distributions N_σ

For every $\sigma\in\,]\omega_0,\infty[$ define the $M_{m\times m}$-valued function $\widehat{N_\sigma}$ on $\mathbb{R}^{1+n}$ by

$$\text{(2.3)}\quad \widehat{N_\sigma}(\xi_0,\ldots,\xi_n)=((\sigma+i\xi_0)M(i\xi_1,\ldots,i\xi_n)-L(i\xi_1,\ldots,i\xi_n))^{-1} \\ =(q(\sigma+i\xi_0,i\xi_1,\ldots,i\xi_n))^{-1}\operatorname{Adj}((\sigma+i\xi_0)M(i\xi_1,\ldots,i\xi_n)-L(i\xi_1,\ldots,i\xi_n))$$

for $(\xi_0,\ldots,\xi_n)\in\mathbb{R}^{1+n}$. Then, for every multiindex $\alpha\in\mathbb{N}^{1+n}$,

$$\partial^\alpha\widehat{N_\sigma}(\xi_0,\ldots,\xi_n)=(q(\sigma+i\xi_0,i\xi_1,\ldots,i\xi_n))^{-1-|\alpha|}P_\alpha(\sigma,\xi_0,\ldots,\xi_n)$$

where P_α is an $m\times m$ matrix with polynomial entries. Consequently, (2.1) implies that

$$\text{(2.4)}\qquad \widehat{N_\sigma}\in\mathcal{O}_M(\mathbb{R}^{1+n};M_{m\times m})\quad\text{for every }\sigma\in\,]\omega_0,\infty[.$$

Let

$$\text{(2.5)}\qquad N_\sigma=\mathcal{F}^{-1}\widehat{N_\sigma}$$

where $\mathcal{F}$ denotes the Fourier transformation on $\mathbb{R}^{1+n}$. From (1.4) and (2.4) it follows that

$$\text{(2.6)}\qquad N_\sigma\in\mathcal{O}'_C(\mathbb{R}^{1+n};M_{m\times m})\quad\text{for every }\sigma\in\,]\omega_0,\infty[.$$

Furthermore, from (2.3) it follows that

(2.7) if $\sigma\in\,]\omega_0,\infty[$ then N_σ is a fundamental solution for the differential operator $(\sigma+\partial_0)M(\partial_1,\ldots,\partial_n)-L(\partial_1,\ldots,\partial_n)$.

Let e_σ be the scalar function on $\mathbb{R}^{1+n}$ defined by $e_\sigma(x_0,\ldots,x_n)=\exp(\sigma x_0)$. Take $\sigma\in\,]\omega_0,\infty[$, and consider the distribution $e_\sigma N_\sigma\in\mathcal{D}'(\mathbb{R}^{1+n};M_{m\times m})$. By the Parseval equality, for every $\varphi\in\mathcal{D}(\mathbb{R}^{1+n})$ one has

$$\langle e_\sigma N_\sigma,\varphi\rangle=\langle N_\sigma,e_\sigma\varphi\rangle=(2\pi)^{-1-n}\langle\widehat{N_\sigma},\widehat{e_\sigma\varphi}^{\vee}\rangle \\ =(2\pi)^{-1-n}\underset{\mathbb{R}^{1+n}}{\int\cdots\int}(\widehat{e_\sigma\varphi}(-\xi_0,\ldots,-\xi_n)) \\ \cdot((\sigma+i\xi_0)M(i\xi_1,\ldots,i\xi_n)-L(i\xi_1,\ldots,i\xi_n))^{-1}\,d\xi_0\ldots d\xi_n.$$

For every $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ the Fourier integral

$$\widehat{\varphi}(\zeta_0, \dots, \zeta_n) = \underset{\mathbb{R}^{1+n}}{\int \cdots \int} e^{-i \sum_{\nu=0}^{n} x_\nu \zeta_\nu} \varphi(x_0, \dots, x_n)\, dx_0 \dots dx_n$$

makes sense for $(\zeta_0, \dots, \zeta_n) \in \mathbb{C}^{1+n}$ and defines the holomorphic extension of $\widehat{\varphi}$ from $\mathbb{R}^{1+n}$ onto $\mathbb{C}^{1+n}$. This holomorphic extension satisfies

$$\widehat{e_\sigma \varphi}(\zeta_0, \dots, \zeta_n) = \widehat{\varphi}(\zeta_0 + i\sigma, \zeta_1, \dots, \zeta_n).$$

Consequently, whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ and $\sigma \in \,]\omega_0, \infty[$, then

$$\begin{aligned} \text{(2.8)} \quad \langle e_\sigma N_\sigma, \varphi \rangle = (2\pi)^{-1-n} & \underset{\mathbb{R}^{1+n}}{\int \cdots \int} \widehat{\varphi}(-\xi_0 + i\sigma, -\xi_1, \dots, -\xi_n) \\ & \cdot ((\sigma + i\xi_0) M(i\xi_1, \dots, i\xi_n) - L(i\xi_1, \dots, i\xi_n))^{-1}\, d\xi_0 \dots d\xi_n. \end{aligned}$$

An integration by parts shows that whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ and $l \in \mathbb{N}$, then

$$\begin{aligned} \text{(2.9)} \quad (1 + |\xi_0 - i\sigma|^l + |\xi_1|^l + \dots + |\xi_n|^l) & |\widehat{\varphi}(-\xi_0 + i\sigma, -\xi_1, \dots, -\xi_n)| \\ & \le \Big(\|\varphi\|_{L^1(\mathbb{R}^{1+n})} + \sum_{\nu=0}^{n} \|\partial_\nu^l \varphi\|_{L^1(\mathbb{R}^{1+n})} \Big) \exp(H_\varphi(\sigma)) \end{aligned}$$

for every $\sigma, \xi_0, \dots, \xi_n \in \mathbb{R}$ where

$$\text{(2.10)} \qquad H_\varphi(\sigma) = \sup\{\sigma x_0 : (x_0, \dots, x_n) \in \operatorname{supp} \varphi\}.$$

Furthermore, by (2.2), there are $C, k \in \,]0, \infty[$ such that

$$\begin{aligned} \text{(2.11)} \quad & \|((\sigma + i\xi_0) M(i\xi_1, \dots, i\xi_n) - L(i\xi_1, \dots, i\xi_n))^{-1}\|_{M_{m\times m}} \\ & \qquad \le C(\sigma - \omega_0)^{-\mu}(1 + |\xi_0 - i\sigma| + |\xi_1| + \dots + |\xi_n|)^k \end{aligned}$$

for every $\sigma \in \,]\omega_0, \infty[$ and $\xi_0, \dots, \xi_n \in \mathbb{R}$. From (2.8)–(2.11) and the Cauchy integral theorem it follows that

(2.12) the distribution $e_\sigma N_\sigma \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$ does not depend on σ provided that $\sigma \in \,]\omega_0, \infty[$,

(2.13) $\lim_{\sigma\to\infty} \langle e_\sigma N_\sigma, \varphi \rangle = 0$ whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ and $\operatorname{supp} \varphi \subset \mathbb{R}^{1+n} \backslash H_+$.

2.3 The fundamental solution N

Thanks to (2.12) we may define the distribution $N \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$ by the equality

$$\text{(2.14)} \qquad N = e_\sigma N_\sigma \quad \text{for every } \sigma \in \,]\omega_0, \infty[.$$

From (2.13) it follows that

$$\operatorname{supp} N \subset H_+. \tag{2.15}$$

For every $\sigma \in \mathbb{R}$ let

$$S_\sigma = ((\sigma + \partial_0)M(\partial_1, \dots, \partial_n) - L(\partial_1, \dots, \partial_n))(\delta \otimes \mathbb{1}_{m\times m}). \tag{2.16}$$

Then $S_\sigma = \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$, $\operatorname{supp} S_\sigma = \{0\}$, and

$$S_\sigma = e_{-\sigma} S_0 \quad \text{for every } \sigma \in \mathbb{R}, \tag{2.17}$$

because whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$, then

$$\begin{aligned}
\langle e_{-\sigma} S_0, \varphi\rangle &= \langle S_0, e_{-\sigma}\varphi\rangle \\
&= [(-\partial_0 M(-\partial_1, \dots, -\partial_n) - L(-\partial_1, \dots, -\partial_n))(e_{-\sigma}\varphi)](0) \\
&= [((\sigma - \partial_0)M(-\partial_1, \dots, -\partial_n) - L(-\partial_1, \dots, -\partial_n))\varphi](0) \\
&= \langle S_\sigma, \varphi\rangle.
\end{aligned}$$

From (2.7), (2.14) and (2.17) it follows that whenever $\sigma \in \,]\omega_0, \infty[$, then

$$\begin{aligned}
PN &= (\partial_0 M(\partial_1, \dots, \partial_n) - L(\partial_1, \dots, \partial_n))N \\
&= S_0 * N = (e_\sigma S_\sigma) * (e_\sigma N_\sigma) = e_\sigma(\delta \otimes \mathbb{1}_{m\times m}) = \delta \otimes \mathbb{1}_{m\times m},
\end{aligned}$$

proving that

(2.18) N is a fundamental solution for the operator P.

Above we have used the fact that whenever $T, U \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$, $\sigma \in \mathbb{R}$, and one of T, U has compact support, then $e_\sigma(T * U) = (e_\sigma T) * (e_\sigma U)$. This is true under the additional assumption that $T, U \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n}; M_{m\times m})$, and this case implies the general assertion by regularization.

2.4 Properties of N

If $\vartheta \in \mathcal{D}(\mathbb{R})$ and $\sigma \in \,]\omega_0, \infty[$, then $\vartheta_0 e_\sigma$ is bounded on $\mathbb{R}^{1+n}$ together with all its partial derivatives, so that, by (1.3), $\vartheta_0 N = (\vartheta_0 e_\sigma) N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$ because $N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$. Therefore

(2.19) whenever $\vartheta \in \mathcal{D}(\mathbb{R})$, then $\vartheta_0 N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$.

By (2.15), (2.18) and (2.19), N has all the properties specified in (ii).

It remains to prove that N also satisfies (iii). To this end, take $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \in \,]\omega_0, \infty[$. Let $\sigma = \frac{1}{2}(\omega_0 + \operatorname{Re}\lambda)$. Then $e_{-\lambda}N = e_{\sigma-\lambda}N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$ because $N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$, $\operatorname{supp} N_\sigma \subset H_+$, and $e_{\sigma-\lambda}$ is bounded together with all its partial derivatives on the set $\{(x_0, \dots, x_n) \in \mathbb{R}^{1+n} : x_0 > -1\}$.

Now, to show (iii), it remains to prove that

$$\text{(2.20)} \qquad \text{if } \lambda \in \mathbb{C} \text{ and } e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m}), \text{ then } \operatorname{Re}\lambda \geq \omega_0.$$

So, suppose that $\lambda \in \mathbb{C}$ and $e_{-\lambda}N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$. Take any $\sigma \in [\operatorname{Re}\lambda, \infty)$. Since $e_{\lambda-\sigma}$ is bounded on $\{(x_0, \dots, x_n) \in \mathbb{R}^{1+n} : x_0 > -1\}$ together with all its partial derivatives, from (1.3) it follows that $e_{-\sigma}N = e_{\lambda-\sigma}(e_{-\lambda}N) \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$. Furthermore

$$S_\sigma * (e_{-\sigma}N) = (e_{-\sigma}S_0) * (e_{-\sigma}N) = e_{-\sigma}(S_0 * N) = e_{-\sigma}(\delta \otimes \mathbb{1}_{m\times m}) = \delta \otimes \mathbb{1}_{m\times m}.$$

Let $\phi = \mathcal{F}(e_{-\sigma}N)$. Then $\phi \in \mathcal{O}_M(\mathbb{R}^{1+n}; M_{m\times m})$ and

$$\begin{aligned}((\sigma + i\xi_0)M(i\xi_1, \dots, i\xi_n) - L(i\xi_1, \dots, i\xi_n) \cdot \phi(\xi_0, \dots, \xi_n)&\\ = [\mathcal{F}(S_\sigma * (e_{-\sigma}N))](\xi_0, \dots, \xi_n) &= \mathbb{1}_{m\times m}\end{aligned}$$

for every $(\xi_0, \dots, \xi_n) \in \mathbb{R}^{1+n}$. This implies that for every $(\xi_0, \dots, \xi_n) \in \mathbb{R}^{1+n}$ the matrix $((\sigma + i\xi_0)M(i\xi_1, \dots, i\xi_n) - L(i\xi_1, \dots, i\xi_n))$ is invertible, so that $q(\sigma + i\xi_0, i\xi_1, \dots, i\xi_n) \neq 0$. Since the last is true for every $\sigma \in [\operatorname{Re}\lambda, \infty)$, it follows that $\operatorname{Re}\lambda \geq \omega_0$, proving (2.20).

3 Uniqueness of the fundamental solution satisfying (ii)

3.1 Some general associativity relations for convolution

In what follows, convolution $*$ will occur only in the following circumstances:

(a) $\varphi * \psi$ where $\varphi, \psi \in \mathcal{D}(\mathbb{R}^{1+n})$,

(b) $T * \varphi$ where $T \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$ and $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$,

(c) $S * T$ where $S, T \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$ and one of the distributions S, T has compact support.

In (b) the convolution is compatible with multiplication of matrices by scalars. In (c) the convolution is compatible with multiplication of matrices by matrices. We shall use the identities

$$S * (T * \varphi) = (S * T) * \varphi = (S * \varphi) * T \tag{3.1}$$

whenever $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$, $S, T \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$, and one of S, T has compact support. See [12], the formulas (II, 7;2), p. 152, and (II, 7;9), p. 168. However the advanced theory of [12] is not necessary to prove (3.1) in our case of $M_{m\times m}$-valued distributions. From (3.1) it is easy to deduce that

$$(S * T) * (\varphi * \psi) = (S * \varphi) * (T * \psi) \tag{3.2}$$

whenever $\phi, \psi \in \mathcal{D}(\mathbb{R}^{1+n})$, $S, T \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$, and one of S, T has compact support, where it is important that $S * \varphi$ and $T * \psi$ are functions.

3.2 N_σ as a left-side fundamental solution

Let $S_\sigma \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$ be defined by (2.16). Then $\operatorname{supp} S_\sigma = \{0\}$.

Lemma 3.1. *If* (i) *holds,* $\sigma \in\,]\omega_0, \infty[$ *and* $N_\sigma \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$ *is defined by* (2.5)*, then*

$$N_\sigma * S_\sigma = \delta \otimes \mathbb{1}_{m\times m}. \tag{3.3}$$

Proof. By (1.4) the equality (3.3) is equivalent to the equality

$$\widehat{N_\sigma}(\xi_0, \ldots, \xi_n) \cdot \widehat{S_\sigma}(\xi_0, \ldots, \xi_n) = \mathbb{1}_{m\times m} \tag*{(3.3)^}$$

for every $(\xi_0, \ldots, \xi_n) \in \mathbb{R}^{1+n}$, where ^ denotes the Fourier transformation on $\mathbb{R}^{1+n}$ and

$$\widehat{S_\sigma}(\xi_0, \ldots, \xi_n) = (i\xi_0 + \sigma) M(i\xi_1, \ldots, i\xi_n) - L(i\xi_1, \ldots, i\xi_n).$$

The meaning of $(3.3)^\wedge$ is exactly this: whenever $(\xi_0, \ldots, \xi_n) \in \mathbb{R}^{1+n}$, then $\widehat{N_\sigma}(\xi_0, \ldots, \xi_n)$ is the matrix inverse to $\widehat{S_\sigma}(\xi_0, \ldots, \xi_n)$. The last is equivalent to the definition (2.3) of $\widehat{N_\sigma}$.

3.3 Reduction to an associativity relation for convolution

Suppose now that (i) holds, and let N be a fundamental solution satisfying (ii). Fix $\sigma \in\,]\omega_0, \infty[$, and define N_σ by (2.5), and S_σ by (2.16).

The uniqueness of N is a consequence of the equality $e_{-\sigma}N = N_\sigma$. Since $(\partial_0 M(\partial_1, \ldots, \partial_n) - L(\partial_1, \ldots, \partial_n))N = \delta \otimes \mathbb{1}_{m\times m}$, it follows that

$$\begin{aligned} S_\sigma * (e_{-\sigma}N) &= ((\partial_0 + \sigma)M(\partial_1, \ldots, \partial_n) - L(\partial_1, \ldots, \partial_n))(e_{-\sigma}N) \\ &= M(\partial_1, \ldots, \partial_n)[\sigma e_{-\sigma}N + (\partial_0 e_{-\sigma})N + e_{-\sigma}\partial_0 N] \\ &\quad + e_{-\sigma}L(\partial_1, \ldots, \partial_n)N \\ &= e_{-\sigma}(\partial_0 M(\partial_1, \ldots, \partial_n) - L(\partial_1, \ldots, \partial_n))N \\ &= e_{-\sigma}(\delta \otimes \mathbb{1}_{m\times m}) = \delta \otimes \mathbb{1}_{m\times m}. \end{aligned}$$

Hence

$$N_\sigma = N_\sigma * (S_\sigma * (e_{-\sigma}N)).$$

On the other hand, by Lemma 3.1,

$$e_{-\sigma}N = (N_\sigma * S_\sigma) * (e_{-\sigma}N).$$

Therefore the proof of the uniqueness of N reduces to the proof of the equality

$$(N_\sigma * S_\sigma) * (e_{-\sigma}N) = N_\sigma * (S_\sigma * (e_{-\sigma}N)). \tag{3.4}$$

The associavity in (3.4) is not obvious because from among the three factors $N_\sigma, S_\sigma, e_{-\sigma}N$ only S_σ has compact support.

3.4 Proof of (3.4) by the method of C. Chevalley

We shall prove (3.4) following the argument of C. Chevalley from the proof of Theorem 2.2 on pp. 120–121 of [2]. This argument is based on (3.2) and the Fubini theorem. Notice that our $*$ will always denote the "classical" convolution (where one factor has a compact support), and not the "generalized" convolution of C. Chevalley. We shall use the following lemma.

Lemma 3.2. *Suppose that $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$, $N \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$, and $\vartheta_0 N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$ whenever $\vartheta \in \mathcal{D}(\mathbb{R})$. Then $N * \varphi \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n; M_{m\times m}))$.*

Proof. It is sufficient to prove that $(N * \varphi)|_{[-a,a]\times\mathbb{R}^n} \in C^\infty([-a, a]; \mathcal{S}(\mathbb{R}^n; M_{m\times m}))$ for every $a \in]0, \infty[$. So, take $a, b \in]0, \infty[$ such that $\operatorname{supp} \varphi \subset [-b, b] \times \mathbb{R}^n$, and $\vartheta \in \mathcal{D}(\mathbb{R})$ such that $\vartheta = 1$ on $[-a - b, a + b]$. Then

$$(N * \varphi)|_{[-a,a]\times\mathbb{R}^n} = ((\vartheta_0 N) * \varphi)|_{[-a,a]\times\mathbb{R}^n}.$$

Since $\vartheta_0 N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$, by (1.1) one has $(\vartheta_0 N) * \varphi \in \mathcal{S}(\mathbb{R}^{1+n}; M_{m\times m})$, and a fortiori $((\vartheta_0 N) * \varphi)|_{[-a,a]\times\mathbb{R}^n} \in C^\infty([-a, a]; \mathcal{S}(\mathbb{R}^n; M_{m\times m}))$.

Since the set $\{\varphi_1 * \varphi_2 * \varphi_3 : \varphi_i \in \mathcal{D}(\mathbb{R}^{1+n}) \text{ for } i = 1,2,3\}$ is dense in $\mathcal{D}(\mathbb{R}^{1+n})$, (3.4) will follow once it is proved that

$$[(N_\sigma * S_\sigma) * (e_{-\sigma}N)] * [\varphi_1 * \varphi_2 * \varphi_3] = [N_\sigma * (S_\sigma * (e_{-\sigma}N)) * [\varphi_1 * \varphi_2 * \varphi_3] \tag{3.5}$$

for every $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{D}(\mathbb{R}^{1+n})$. To prove (3.5), define the $M_{m\times m}$-valued functions on $\mathbb{R}^{1+n}$ by

$$f = N_\sigma * \varphi_1, \quad g = S_\sigma * \varphi_2, \quad h = (e_{-\sigma}N) * \varphi_3.$$

Then, by (1.1), Lemma 3.2 and the equality $h = (e_{-\sigma}N) * (e_{-\sigma}e_\sigma\varphi_3) = e_{-\sigma}(N * (e_\sigma\varphi_3))$,

$$\begin{gathered} f \in \mathcal{S}(\mathbb{R}^{1+n}; M_{m\times m}), \quad g \in \mathcal{D}(\mathbb{R}^{1+n}; M_{m\times m}), \\ h \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n; M_{m\times m})). \end{gathered} \tag{3.6}$$

Furthermore, since $\operatorname{supp} N_\sigma \subset H_+$, $\operatorname{supp} S_\sigma = \{0\}$, and $\operatorname{supp} e_{-\sigma}N \subset H_+$, it follows that there is $b \in \mathbb{R}$ such that

$$\operatorname{supp} f, \operatorname{supp} g, \operatorname{supp} h \subset [b, \infty[\times \mathbb{R}^n. \tag{3.7}$$

From (3.6) and (3.7) it follows that for every $(x_0, \ldots, x_n) \in \mathbb{R}^{1+n}$ the multiple integral

$$\int \cdots \int_{\mathbb{R}^{1+n}\times\mathbb{R}^{1+n}} f(v_0, \ldots, v_n) g(u_0 - v_0, \ldots, u_n - v_n) h(x_0 - u_0, \ldots, x_n - u_n) \, dv_0 \ldots dv_n du_0 \ldots du_n$$

is absolutely convergent. Consequently, $(f * g) * h = f * (g * h)$ by the Fubini theorem. This implies (3.5) because

$$[(N_\sigma * S_\sigma) * (e_{-\sigma}N)] * [\varphi_1 * \varphi_2 * \varphi_3] = (f * g) * h$$

and

$$[N_\sigma * (S_\sigma * (e_{-\sigma}N))] * [\varphi_1 * \varphi_2 * \varphi_3] = f * (g * h),$$

as may be deduced from (3.2).

4 Proof of (ii)⇒(i)

4.1 The distributions $\vartheta_0 N(\varphi \otimes \cdot)$

Take N satisfying (ii). Fix a, b such that $0 < a < b < \infty$, and $\vartheta \in \mathcal{D}(\mathbb{R})$ such that $\vartheta = 1$ on $[-b, b]$. For every $\varphi \in \mathcal{D}(\mathbb{R})$ consider the mapping

$$T(\varphi) : \mathcal{D}(\mathbb{R}^n) \ni \phi \mapsto \langle \vartheta_0 N, \varphi \otimes \phi \rangle \in M_{m\times m}.$$

Obviously, the correspondence $\varphi \mapsto T(\varphi)$ is linear. Since $\vartheta_0 N \in \mathcal{O}'_C(\mathbb{R}^{1+n}; M_{m\times m})$, from (1.2) it follows that for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that

$$\vartheta_0 N = \sum_{p+|\alpha|\le m_k} \partial_0^p \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} F_{k;p,\alpha} \tag{4.1}$$

where every $F_{k;p,\alpha}$ is a continuous $M_{m\times m}$-valued function on $\mathbb{R}^{1+n} = \{(t,x) \in \mathbb{R}\times\mathbb{R}^n\}$ for which

$$\sup_{(t,x)\in\mathbb{R}^{1+n}} (1+|t|+|x|)^k |F_{k;p,\alpha}(t,x)| < \infty.$$

Consequently, whenever $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$T(\varphi) = \sum_{|\alpha|\le m_k} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f_{k;\alpha;\varphi} \tag{4.2}$$

where

$$f_{k;\alpha;\varphi}(x) = \sum_{p\le m_k-|\alpha|} \int_{\mathbb{R}} ((-\partial_0)^p \varphi(t)) F_{k;p;\alpha}(t,x)\, dt.$$

It follows that, whenever $|\alpha| \le m_k$, $\varphi \in \mathcal{D}(\mathbb{R})$, and $x \in \mathbb{R}^n$, then

$$\begin{aligned} |f_{k;\alpha;\varphi}(x)| &\le C_k \sum_{p\le m_k-|\alpha|} \int_{\operatorname{supp}\vartheta} |\partial_0^p \varphi(t)| (1+|t|+|x|)^{-k}\, dt \\ &\le D_k (1+|x|)^{-k} \sup\{|\partial_0^p\varphi(t)| : p = 0,\dots,m_k,\ t\in\mathbb{R}\}, \end{aligned} \tag{4.3}$$

where $C_k, D_k \in\,]0,\infty[$ depend only on k. In particular this shows that

$$T(\varphi) \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m}) \quad \text{for every } \varphi \in \mathcal{D}(\mathbb{R}).^{*)} \tag{4.4}$$

Since N is the fundamental solution for $\partial_0 M(\partial_1,\dots,\partial_n) - L(\partial_1,\dots,\partial_n)$ with support in H_+, and $\vartheta = 1$ on $[-b,b]$, it follows that

(4.5) $T(\varphi) = 0$ whenever $\operatorname{supp}\varphi \subset\,]-\infty, 0[$,

(4.6) $M(\partial_1,\dots,\partial_n)(T(-\partial_0\varphi)) - L(\partial_1,\dots,\partial_n)(T(\varphi)) = \varphi(0)(\delta\otimes \mathbb{1}_{m\times m})$ for every $\varphi \in C^\infty_{[-b,b]}(\mathbb{R})$ where δ is the Dirac distribution on $\mathbb{R}^n$.

In the subsequent lemmas it will be tacitly assumed that (ii) holds and N, a, b, ϑ, T are fixed. For every $\varphi \in \mathcal{D}(\mathbb{R})$ denote by $\widehat{T}(\varphi)$ the image of $T(\varphi)$ under the Fourier transformation on $\mathbb{R}^n$. Then $\widehat{T}(\varphi) \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$, by (4.4) and (1.4).

*) After introducing the topology in $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$, it is possible to prove that the mapping $\mathcal{D}(\mathbb{R}) \ni \varphi \mapsto T(\varphi) \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is a vector-valued distribution. However this is insignificant for the present proof.

Lemma 4.1. *There are* $p_0, m_0 \in \mathbb{N}_0$ *and* $C \in]0,\infty[$ *such that*

$$\|\widehat{T}(\varphi)(\xi)\|_{M_{m\times m}} \le C(1+|\xi|)^{m_0} \sup\{|\partial_0^p \varphi(t)| : p = 0,\dots,p_0,\ a \le t \le b\}$$

for every $\xi \in \mathbb{R}^n$ *and* $\varphi \in C^\infty_{[a,b]}(\mathbb{R})$.

Proof. If in (4.1) we take $k > n$, then, by (4.2) and (4.3),

$$T(\varphi) = \sum_{|\alpha|\le m_k} \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f_{k;\alpha;\varphi} \qquad \text{for every } \varphi \in C^\infty_{[a,b]}(\mathbb{R})$$

where

$$\|f_{k;\alpha;\varphi}\|_{L^1(\mathbb{R}^n;M_{m\times m})} \le D \sup\{|\partial_0^p\varphi(t)| : p = 0,\dots,m_k,\ a\le t\le b\}$$

for every α with $|\alpha| \le m_k$ and every $\varphi \in C^\infty_{[a,b]}(\mathbb{R})$, with $D \in]0,\infty[$ depending only on k. Consequently, whenever $\varphi \in C^\infty_{[a,b]}(\mathbb{R})$, then

$$\|\widehat{T}(\varphi)(\xi)\|_{M_{m\times m}} \le (1+|\xi|)^{m_k} \|g_\varphi(\xi)\|_{M_{m\times m}} \qquad \text{for every } \xi \in \mathbb{R}^n$$

for some $g_\varphi \in C_b(\mathbb{R}^n; M_{m\times m})$ with

$$\sup_{\xi\in\mathbb{R}^n} \|g_\varphi(\xi)\|_{M_{m\times m}} \le C \sup\{|\partial_0^p\varphi(t)| : p = 0,\dots,m_k,\ a \le t \le b\}$$

for some $C \in]0,\infty[$ depending only on k.

4.2 An inequality of Chazarain type

Lemma 4.2. *There are* $a \in \mathbb{R}$ *and* $b \in]0,\infty[$ *such that whenever* $(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n$ *and*

$$\operatorname{Re}\lambda > a + b\log(1+|\lambda|+|\xi|)\ ^{*)},$$

then the matrix $\lambda M(i\xi) - L(i\xi)$ *is invertible, so that* $q(\lambda, i\xi) \ne 0$.

Proof. From (4.6) it follows that

$$M(i\xi)\widehat{T}(-\partial_0\varphi)(\xi) - L(i\xi)\widehat{T}(\varphi)(\xi) = \varphi(0)\mathbb{1}_{m\times m} \tag{4.7}$$

for every $\varphi \in C^\infty_{[-b,b]}(\mathbb{R})$ and $\xi \in \mathbb{R}^n$. Take $\varphi_0 \in C^\infty_{[-b,b]}(\mathbb{R})$ such that $\varphi_0 = 1$ on $[-a,a]$. Following J. Chazarain [1], p. 394, consider functions of the form

*) This inequality and its proof are similar to the inequality (1.2) on p. 394 of [1] and the argument on p. 395 of [1]. There is however an important difference. In [1] the inequality (1.2) does not involve ξ and determines the “logarithmic region” $\Lambda \subset \mathbb{C}$ such that for every $\lambda \in \Lambda$ an abstract operator $Q(\lambda) = \lambda^n A_m + \dots + \lambda A_1 + A_0$ is invertible. In our case the inequality involves ξ but the operator $Q(\lambda)$ is replaced by the *polynomial* $q(\lambda, i\xi)$, and Lemma 4.2 is not the final step of the argument.

$\varphi = e_{-\lambda}\varphi_0$ where λ ranges over $\mathbb{C}$. By (4.5) and (4.7), for every $(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n$ one has $\mathbb{1}_{[a,b]}\partial_0\varphi_0 \in C^\infty_{[a,b]}(\mathbb{R})$ and

$$(\lambda M(i\xi) - L(i\xi))\widehat{T}(e_{-\lambda}\varphi_0)(\xi) = \mathbb{1}_{m\times m} + M(i\xi)\widehat{T}(e_{-\lambda}\mathbb{1}_{[a,b]}\partial_0\varphi_0)(\xi).$$

This implies that if

$$\|M(i\xi)\widehat{T}(e_{-\lambda}\mathbb{1}_{[a,b]}\partial_0\varphi_0)(\xi)\|_{M_{m\times m}} < 1$$

for some $(\lambda, \xi) \in \mathbb{C}\times\mathbb{R}^n$, then the $m \times m$ matrix $\lambda M(i\xi) - L(i\xi)$ is invertible. Therefore, by Lemma 4.1, it is invertible for every $(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n$ such that

$$\begin{aligned}(4.8)\qquad \sup\{|\partial_0^p(e^{-\lambda t}\partial_0\varphi_0(t))| : p = 0, \dots, p_0,\ a \le t \le b\}&\\ \cdot \|M(i\xi)\|_{M_{m\times m}}(1 + |\xi|)^{m_0} < C^{-1}&.\end{aligned}$$

Since $M(i\xi)$ is a matrix with polynomial entries, there are $K \in]0, \infty[$ and $k \in \mathbb{N}_0$ such that

$$(4.9)\qquad \|M(i\xi)\|_{M_{m\times m}} \le K(1 + |\xi|)^k \quad \text{for every } \xi \in \mathbb{R}^n.$$

Since $\sup\{|\partial_0^p\varphi_0(t)| : p = 1, \dots, p_0+1,\ a \le t \le b\}$ is finite and $\sup_{a\le t\le b} |e^{-\lambda t}| = e^{-a \operatorname{Re}\lambda}$ whenever $\operatorname{Re}\lambda \ge 0$, there is $L \in]0, \infty[$ such that

$$\begin{aligned}(4.10)\qquad \sup\{|\partial_0^p(e^{-\lambda t}\partial_0\varphi_0(t))| : p = 0, \dots, p_0,\ a \le t \le b\}&\\ \le Le^{-a \operatorname{Re}\lambda}(1 + |\lambda|)^{p_0} \quad \text{whenever } \operatorname{Re}\lambda \ge 0&.\end{aligned}$$

By (4.8)–(4.10), the matrix $\lambda M(i\xi) - L(i\xi)$ is invertible if $\operatorname{Re}\lambda \ge 0$ and

$$e^{-a \operatorname{Re}\lambda}(1 + |\lambda|)^{p_0}(1 + |\xi|)^{k+m_0} < (CKL)^{-1},$$

and therefore if

$$\operatorname{Re}\lambda > a^{-1}\log(CKL + 1) + a^{-1}(k + m_0 + p_0)\log(1 + |\lambda| + |\xi|).$$

4.3 The Chazarain type inequality implies (i)

The implication (ii)⇒(i) is an immediate consequence of Lemma 4.2 and the following

Lemma 4.3. *Let p be a polynomial of $1 + n$ variables with complex coefficients. Suppose that there are $a \in \mathbb{R}$ and $b \in]0, \infty[$ such that*

$$\begin{aligned}(4.11)\qquad \operatorname{Re}\lambda \le a + b\log(1 + |\lambda| + |\xi|)&\\ \textit{whenever } (\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n \textit{ and } p(\lambda, \xi) = 0&.\end{aligned}$$

Then

$$\sup\{\operatorname{Re}\lambda : (\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n,\ p(\lambda, \xi) = 0\} < \infty.$$

The proof follows the scheme due to L. Gårding and L. Hörmander. Let

$$\sigma(r) = \sup\{\operatorname{Re}\lambda : \lambda \in \mathbb{C} \text{ and there is } \xi \in \mathbb{R}^n \text{ such that } |\lambda^2| + |\xi^2| \le \tfrac{1}{2}r^2 \text{ and } p(\lambda,\xi) = 0\}.$$

Then, by (4.11),

$$\sigma(r) \le a + b\log(1+r) \quad \text{for every } r \in [0,\infty[. \tag{4.12}$$

Following an idea of L. Hörmander (presented in the Appendix to [6]), the Tarski–Seidenberg theorem is used to show that there is a polynomial $V(z,w)$ (not vanishing identically) of two variables such that $V(r,\sigma(r)) = 0$ for every $r \in [0,\infty[$. Then, as in L. Gårding's proof of the Lemma on p. 11 of [3], the Puiseux expansions of the w-roots of $V(z,w)$ for large $|z|$ show that (4.12) is possible only if $\sup\{\sigma(r) : r \in [0,\infty[\} < \infty$.

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