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Conley index in Hilbert spaces and the Leray-Schauder degree

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Abstract. Let H be a real, infinite-dimensional and separable Hilbert space. With an isolated invariant set $\mathrm{inv}(N)$ of a flow ϕ^t generated by an $\mathscr{LS}$-vector field $f\colon H \supseteq \Omega \to H$, $f(x) = Lx + K(x)$, where $L\colon H \to H$ is strongly indefinite linear operator and $K\colon H \supseteq \Omega \to H$ is completely continuous, one can associate a homotopy invariant $h_{\mathscr{LS}}(\mathrm{inv}(N), \phi^t)$ called the $\mathscr{LS}$-Conley index. In fact, this is a homotopy type of a finite CW-complex. We define Betti numbers and hence Euler characteristic of such index and prove the formula relating this numbers to the Leray–Schauder degree $\deg_{\mathscr{LS}}(\hat{f}, N, 0)$, where $\hat{f} : H \supseteq \Omega \to H$ is defined as $\hat{f}(x) = x + L^{-1}K(x)$.

Introduction

The aim of this paper is to present certain generalization of the Poincaré–Hopf index theorem. This generalization goes in the direction of infinite dimensional nonlinear analysis and occurs when we are working with infinite dimensional Conley–type invariant for flows. Let H be a real, infinite dimensional Hilbert space. With a locally Lipschitz vector field $f\colon H \supseteq \Omega \to H$, which is completely continuous perturbation of an isomorphism $L\colon H \to H$, $f(x) = Lx + K(x)$ we can associate a flow $\phi_f^t\colon \Omega \to \Omega$ (at least locally) satisfying

$$\frac{d}{dt}\phi_f^t = -f \circ \phi_f^t, \qquad \phi_f^0 = \mathrm{id}.$$

Under certain assumptions we prove the formula

$$\deg_{\mathscr{LS}}(\hat{f}, \mathrm{int}(N), 0) = \chi(h_{\mathscr{LS}}(\mathrm{inv}(N), \phi_f^t)).$$

The left-hand side of above equality stands for the standard Leray–Schauder degree with respect to a bounded set $\mathrm{int}(N)$ and 0. A map $\hat{f}$ is defined by $\hat{f}(x) = x+L^{-1}K(x)$. On the other hand we have Euler characteristic. Here $h_{\mathscr{LS}}(\mathrm{inv}(N), \phi_f^t)$ is the Conley index of an isolated invariant set $\mathrm{inv}(N)$ of the flow ϕ_f^t on infinite dimensional Hilbert spaces.

An extension of the classical Conley's theory (for flows on locally compact metric spaces), we are going to work with, was introduced by K. Gęba, M. Izydorek and A. Pruszko in **[6]**. They considered a so-called $\mathscr{LS}$-vector fields i.e. completely continuous perturbation of an isomorphism $L\colon H \to H$ and have defined Conley index for flows induced by such maps. One of the most important fact is that this index admits situations, where L is strongly indefinite, i.e. both stable and unstable eigenspaces of L are infinite dimensional. This property makes this theory applicable to many variational problems occurring in Hamiltonian dynamics.

Further develop of this homotopy invariant has been done by Izydorek in **[7]**. He defined a cohomological Conley index in Hilbert spaces in order to obtain existance results in various strongly indefinite problems

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(variational problems, where gradient of action functional is $\mathscr{LS}$-vector field with strongly indefinite linear part). We briefly sketch out this definition. The definition of cohomology of Conley–type invariant was the reason to define Betti numbers and next Euler characteristic of Conley index in infinite dimensional case. Let E be an object representing Conley index, and denote by $H^*(E)$ the cohomology with coefficients in some fixed ring. We define Euler characteristic of Conley index E in the most natural way by

$$\chi(E) := \sum_{q\in\mathbb{Z}} (-1)^q \operatorname{rank} H^q(E).$$

The rank of $H^q(E)$ is called q^{th} Betti number of E.

The proof of our theorem is based on finite dimensional case of such Poincaré–Hopf relation. It has been first proved by C. McCord [**10**] in terms of local indices of zeros of a vector field. Earlier N. Dancer proved this kind of relation for considerably smaller class of isolated invariant sets, precisely for degenerate critical points (See [**4**]). We present simple proof of this fact given by M. Razvan and M. Fotouhi [**11**] based on Morse inequalities and Reineck continuation theorem. Similar result was obtained by W. Kryszewski and A. Szulkin in [**8**] in terms of critical groups of a smooth functional $\Phi\colon H \to \mathbb{R}$.

In order to gain insight into classical homotopy index theory we refer the reader to Conley's book "Isolated Invariant Sets and the Morse Theory" [**2**] or Salamon's paper [**13**].

1. Classical Conley theory

1.1. Finite dimensional case. First of all, we collect basic facts from Conley index theory for flows on a locally compact metric space X. Recall, that a continuous map $\phi\colon \mathbb{R}\times X \to X$ is called a *flow* on X if the following properties are satisfied:

(Fl.1) $\phi(0,x) = x$;
(Fl.2) $\phi(s,\phi(t,x)) = \phi(s+t,x)$ for all $s,t\in\mathbb{R}$.

We will sometimes write $\phi^t(x)$ instead of $\phi(t,x)$. This notation allows us to write above properties of flow in the shorter form, i.e. $\phi^0 = \mathrm{id}_X$ and $\phi^{t+s} = \phi^t\circ\phi^s$. The main objects of this theory are isolated invariant sets and associated to them isolating neighborhoods. Let ϕ^t be a flow on X. A subset S of X is called an *invariant set*, if $S = \bigcup_{t\in\mathbb{R}} \phi^t(S)$. So the invariant sets are precisely the sums of orbits $\bigcup_{x\in A\subset X} \{\phi^t(x);\, t\in\mathbb{R}\}$. For $N\subset X$ we define the maximal invariant set contained in N as

$$\operatorname{inv}(N) := \left\{x\in N;\, \phi^t(x)\in N,\, t\in\mathbb{R}\right\}.$$

If N is compact and $\operatorname{inv}(N)\subset\operatorname{int}(N)$, then N is called an *isolating neighborhood* and $S = \operatorname{inv}(N)$ is an *isolated invariant set*.

Let N be a compact subset of X. We say that $L\subset N$ is *positively invariant relative to* N, if $t>0$, $x\in L$ and $\phi^s(x)\in N$ for $s\in[0,t]$ then $\phi^s(x)\in L$ for $s\in[0,t]$.

Definition 1.1 (Index pair). A compact pair (N,L) is called an *index pair* for S, if:

(IP.1) $N\setminus L$ is a neighborhood of S and $S = \operatorname{inv}(\operatorname{cl}(N\setminus L))$;
(IP.2) L positively invariant relative to N;
(IP.3) if $x\in N$ and there exists $t>0$, such that $\phi^t(x)\notin N$, then there exists $s\in[0,t]$, such that $\phi^s(x)\in L$.

The next two theorems are crucial in the definition of homotopy Conley index. The proofs can be found in Salamon's paper [**13**]

Theorem 1.2. *Every isolated invariant set S admits an index pair (N,L).*

If (N, L) is a pair of spaces, $L \subset N$, then we define the quotient space N/L obtained from N by collapsing L to a single point denoted by $[L]$, the base point of N/L.

Recall that $f\colon (X, x_0) \to (Y, y_0)$ is a *homotopy equivalence* if there exists $g\colon (Y, y_0) \to (X, x_0)$ such that $g \circ f$ is homotopic to $\mathrm{id}_{|X}$ rel. x_0 and $f \circ g$ is homotopic to $\mathrm{id}_{|Y}$ rel. y_0. If there is a homotopy equivalence $f\colon (X, x_0) \to (Y, y_0)$ we say that pairs (X, x_0) and (Y, y_0) are *homotopy equivalent* or they have the same *homotopy type*. The homotopy type of X is denoted by $[X, x_0]$.

THEOREM 1.3. *Let (N_0, L_0) and (N_1, L_1) be two index pairs for the isolated invariant set S. Then the pointed topological spaces N_0/L_0 and N_1/L_1 are homotopy equivalent.*

DEFINITION 1.4. If (N, L) is any index pair for the isolated invariant set S, then the homotopy type $h(S, \phi^t) = [N/L]$ is said to be the Conley (homotopy) index of S.

Theorem 1.3 says that $h(S, \phi^t)$ is independent of the choice of index pair. Let us illustrate the concept of Conley index by the following simple example.

Example. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $f\colon \mathrm{cl}\,\Omega \to \mathbb{R}$ be a smooth function such that $\nabla f^{-1}(0) \notin \partial\Omega$. Consider the positive gradient flow ϕ_f^t on Ω defined by

$$\frac{d}{dt}\phi_f^t = \nabla f \circ \phi_f^t, \qquad \phi_f^0 = \mathrm{id}.$$

The rest points of ϕ_f^t are the critical points of f. They are hyperbolic if f is a Morse function i.e. the Hessian of f is nonsingular at every $x \in \mathrm{Crit}(f)$. In this case the number

$$\mathrm{ind}_f(x) = \#\{\text{negative eigenvalues of the Hessian} \nabla^2 f(x)\}$$

i well defined. The Conley index of an isolated invariant set $S = \{x\}$, where $x \in \mathrm{Crit}(f)$ is the homotopy type of pointed k-sphere, where $k = n - \mathrm{ind}_f(x)$. We write it $h(\{x\}, \phi_f^t) = [S^k, *]$.

A *Morse decomposition* of an isolated invariant set S is a finite collection

$$\mathscr{M}(S) = \{M_i;\, 1 \le i \le l\}$$

of subsets $M_i \subset S$, which are disjoint, compact and invariant, and which can be ordered $(M_1, M_2, \ldots, M_l)$ so that for every $x \in S \setminus \bigcup_{1 \le j \le l} M_j$ there are indices $i < j$ such that

$$\omega(x) \subset M_i, \qquad \alpha(x) \subset M_j.$$

Notice that in the previous example the set $\mathrm{Crit}(f)$ of all critical points of f forms a Morse decomposition of $\mathrm{inv}(\Omega)$.

The formal power series

$$\mathscr{P}(t, A, B) = \sum_{q \in \mathbb{Z}} \mathrm{rank}\, H^q(A, B) \cdot t^q$$

is called the *Poincaré series of a pair* (A, B). One can prove, that for an isolated invariant set there is an index pair (N, L) for which the isomorphism $H^*(N, L) \cong H^*(N/L)$ holds. Such an index pair is called *regular*. We can therefore define the Poincaré polynomial for S as

$$\mathscr{P}(t, h(S, \phi^t)) := \mathscr{P}(t, N, L)$$

where (N, L) is any regular index pair for S. The next theorem gives us a useful tool in a Morse–theoretic methods. It is a generalization of a classical Morse inequalities.

THEOREM 1.5 (cf. [**3**],[**7**]). *Let S be an isolated invariant set with a Morse decomposition $\mathscr{M}(S) = \{M_i;\, 1 \le i \le l\}$. Then there is a polynomial $\mathscr{Q}$ with nonnegative coefficients such that*

$$\sum_{i=1}^{l} \mathscr{P}(t, h(M_i)) = \mathscr{P}(t, h(S)) + (1 + t)\mathscr{Q}(t).$$

1.2. Continuation to a gradient. Let $\phi\colon \mathbb{R}\times X\times[0,1]\to X$ be a continuous family of flows on X, i.e. $\phi^t_\lambda := \phi(t,\,\cdot\,,\lambda)\colon X\to X$ is a flow on X. Suppose that $N\subset X$ is compact and $S_i=\mathrm{inv}(N,\phi^t_i)$, $i=0,1$. We say that two isolated invariant sets S_0 and S_1 are *related by continuation* or S_0 *continues to* S_1 if N is an isolating neighborhood for all ϕ^t_λ for $\lambda\in[0,1]$. The notion of continuation is essential in Conley's theory because of the following statement.

Theorem 1.6 ([**2**]). *If S_0 and S_1 are related by continuation, then their Conley indices coincide.*

The formula we would like to prove is based on the Reineck continuation theorem.

Theorem 1.7 (Reineck [**12**]). *Let $F\colon\Omega\to\mathbb{R}^n$ be a smooth vector field with $\Omega\subset\mathbb{R}^n$ open. Let S be an isolated invariant set of the flow $\phi^t_F\colon\Omega\to\Omega$ generated by differential equation $\dot{x}(t)=-F(x(t))$ with isolating neighborhood N. Then S can be continued to an isolated invariant set in a positive gradient flow of ∇f, without changing F on $\Omega\setminus N$. Moreover, this can be done that the new flow is Morse–Smale.*

Remarks. The fact that such function f exists has been proved by Robbin and Salamon. They showed that for an isolated invariant set $S=\mathrm{inv}\,N$ there exists a smooth function $f\colon U\to\mathbb{R}$ defined on a neighborhood of N such that

(1) $f(x)=0$ iff $x\in S$ and
(2) $\frac{d}{dt}|_{t=0}f(\phi^t(x))<0$ for all $x\in\Omega\setminus S$.

The function which fulfil those properties is called the Lyaponov function. In general we can't expect that for an isolated invariant set the Lyaponov function has only nondegenerate critical points i.e. the rest points of gradient flow are hyperbolic. But this can be obtained via arbitrary small perturbation of ∇f. So without loss of generality we can assume that the gradient flow is Morse–Smale. Following by Reineck, we can explicit write the homotopy joining $-F$ and ∇f. Define $h\colon\Omega\times[0,1]\to\mathbb{R}^n$ as

$$h(x,\lambda)=\rho(x)[\lambda\nabla f(x)+(\lambda-1)F(x)]+(\rho(x)-1)F(x), \tag{1.1}$$

where $\rho\colon\Omega\to[0,1]$ is smooth function equal 1 on a compact neighborhood of S, say M ($\mathrm{cl}(M)\subset\mathrm{int}(N)$) and ρ is zero on $\Omega\setminus N$.

1.3. Euler characteristic of $h(\mathrm{inv}(N),\phi^t)$. Recall that the Euler characteristic of a pair (E,E') is defined as the alternating sum of rank of the cohomology groups $H^q(E,E')$, i.e.

$$\chi(E,E')=\sum_{q\in\mathbb{Z}}(-1)^q\,\mathrm{rank}\,H^q(E,E'). \tag{1.2}$$

Notice that $\chi(E,E')=\mathscr{P}(-1,E,E')$. If both $H^q(E)$ and $H^q(E')$ are finitely generated (e.g. if E and E' are CW-complexes) the integer $\chi(E,E')$ is well defined. In particular if E' is a point in E (that is, E is a pointed space), then we have $\chi(E,*)=\sum_{q\in\mathbb{Z}}(-1)^q\,\mathrm{rank}\,\tilde{H}^q(E)$, where $\tilde{H}^q(E)$ stands for the reduced cohomology. Note that χ is independent of principal ideal domain used for define cohomology groups. Later on we will omit the point in $\chi(E,*)$ if it is clear from the context, that E is pointed space. The Euler characteristic is defined especially for the Conley index of an isolated invariant set for flows generated by equation $\dot{x}=-F(x)$.

The next proposition is due to Gęba (see Proposition 5.6 of [**5**]).

Proposition 1.8. *Let N be an isolating neighborhood for gradient Morse–Smale flow ϕ^t. Then $h(\mathrm{inv}(N),\phi^t)$ is a homotopy type of finite CW-complex.*

Conclusion 1.9. Let N be an isolating neighborhood for flow ϕ^t generated by $\dot{x}=-F(x)$. Then $h(\mathrm{inv}(N),\phi^t)$ is a homotopy type of finite CW-complex.

Proof. Since $\mathrm{inv}(N)$ is related by continuation to some isolated invariant set of gradient Morse–Smale flow, the result follows from Proposition 1.8. $\square$

In the Conley index theory we are working with pointed spaces. So we have to remember to take into account this distinguished point in calculation.

Example. Euler characteristic of the n-sphere. Since $H^q(S^n,*) = \mathbb{Z}$ if $q = n$ and is zero elsewhere we have $\chi(S^n,*) = (-1)^n$. In contrast $\chi(S^n) = 1 + (-1)^n$ when S^n is considered without base point.

1.4. Mapping degree. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. If $f\colon \mathrm{cl}\,\Omega \to \mathbb{R}^n$ is continuous map and does not vanish on the boundary $\partial\Omega$, then it is well known, that there is defined an integer $\deg(f,\Omega,0) \in \mathbb{Z}$ called the Brouwer topological degree. It is a powerfull tool in topology and analysis. For details we refer the reader to Lloyd's book [**9**]. Now, we are going to formulate only a few fundamental facts about the degree:

Nontriviality: If $0 \in \Omega$ then $\deg(I,\Omega,0) = 1$;

Existence: If $\deg(f,\Omega,0) \neq 0$ then $f^{-1}(0) \cap \Omega$ in nonempty;

Additivity: If Ω_1, Ω_2 are open, disjoint subsets of Ω and there is no zeros of f in the completion $\Omega \setminus (\Omega_1 \cup \Omega_2)$, then

$$\deg(f,\Omega,0) = \deg(f,\Omega_1,0) + \deg(f,\Omega_2,0);$$

Homotopy invariance: If $h\colon \mathrm{cl}\,\Omega \times [0,1] \to \mathbb{R}^n$ is continuous map such that $h(x,t) \neq 0$ for all $(x,t) \in \partial\Omega \times [0,1]$, then

$$\deg(h(\,\cdot\,,0),\Omega,0) = \deg(h(\,\cdot\,,1),\Omega,0)$$

There is a generic situation, when the degree is easy to calculate. If $\varphi\colon \mathrm{cl}\,\Omega \to \mathbb{R}$ is a Morse function of class C^1, such that $\deg(\nabla\varphi,\Omega,0)$ is defined, then

$$\deg(\nabla\varphi,\Omega,0) = \sum_{x \in \nabla\varphi^{-1}(0)\cap\Omega} (-1)^{\mathrm{ind}_\varphi(x)}.$$

Theorem 1.10 (cf. [**11**]). *Suppose that N is an isolating neighborhood for the flow ϕ_F^t generated by $\dot{x} = -F(x)$, where $F\colon \Omega \to \mathbb{R}^n$ is locally Lipschitz map. Then*

$$\chi(h(\mathrm{inv}(N),\phi_F^t)) = \deg(F,\mathrm{int}(N),0). \tag{1.3}$$

Notation. Now and subsequently we will sometimes write $\deg(F,N,0)$ instead of $\deg(F,\mathrm{int}(N),0)$.

Proof. By the Reineck continuation theorem we can deform $-F$ to ∇f on N using (1.1) to obtain the isolated invariant set of gradient flow, which consists only non-degenerate critical points of f and connecting orbits between them. Denote this set by $\mathrm{inv}_{\phi_f^t}(N)$. By the continuation property of Conley index we have $h(\mathrm{inv}(N),\phi_F^t) = h(\mathrm{inv}_{\phi_f^t}(N),\phi_f^t)$. The set of critical points $\{x_1,\ldots,x_m\}$ forms Morse decomposition of $\mathrm{inv}_{\phi_f^t}(N)$ and we can apply Morse inequalities. We know that $h(\{x_i\},\phi_f^t)$ has homotopy type of pointed k-sphere, where $k = n - \mathrm{ind}_f(x_i)$. So the Poincaré polynomial of $h(\{x_i\},\phi_f^t)$ is of the form

$$\mathscr{P}(t, h(\{x_i\},\phi_f^t)) = t^{n-\mathrm{ind}_f(x_i)}. \tag{1.4}$$

From the Morse inequalities we have

$$\begin{aligned}
\chi(h(\mathrm{inv}(N),\phi_F^t)) &= \chi(h(\mathrm{inv}_{\phi_f^t}(N),\phi_f^t)) \\
&= \mathscr{P}(-1, h(\mathrm{inv}_{\phi_f^t}(N),\phi_f^t)) = \sum_{i=1}^m \mathscr{P}(-1, h(\{x_i\},\phi_f^t)) \\
&= (-1)^n \sum_{i=1}^m (-1)^{\mathrm{ind}_f(x_i)}
\end{aligned} \tag{1.5}$$

For $1 \le i \le m$, let Ω_i be the neighborhood of x_i i N such that $\Omega_i \cap \Omega_j = \emptyset$. By the homotopy invariance of the Brouwer degree and additive property we can write

$$\deg(-F, N, 0) = \deg(\nabla f, N, 0) = \sum_{i=1}^{m} \deg(\nabla f, \Omega_i, 0). \tag{1.6}$$

Now it is easy to compute $\deg(\nabla f, \Omega_i, 0)$. Since f is Morse, the hessian $\nabla^2 f(x_i)$ is non-degenerate linear operator. The degree of ∇f with respect to Ω_i is just $(-1)^\mu$, where μ is the number of negative eigenvalues of $\nabla^2 f(x_i)$. So we have $\deg(\nabla f, \Omega_i, 0) = (-1)^{\mathrm{ind}_f(x_i)}$, and by (1.6)

$$\deg(F, N, 0) = (-1)^n \deg(-F, N, 0) = (-1)^n \sum_{i=1}^{m} (-1)^{\mathrm{ind}_f(x_i)} \tag{1.7}$$

Comparing (1.5) and (1.7) we obtain the formula (1.3). □

2. $\mathscr{LS}$-index

2.1. $\mathscr{LS}$-flows and the index. Let H be a real, separable Hilbert space and $L\colon H \to H$ be a linear bounded operator which satisfies following assumptions:

- $(L.1)$ L gives a splitting $H = \bigoplus_{n=0}^{\infty} H_n$ onto finite dimensional, mutually orthogonal L-invariant subspaces;
- $(L.2)$ $\dim H_0 < \infty$, where H_0 is subspace corresponding to the part of spectrum on imaginary axis, i.e. $\sigma_0(L) := \sigma(L_{|H_0}) = \sigma(L) \cap i\mathbb{R}$;
- $(L.3)$ $\sigma_0(L)$ is isolated in $\sigma(L)$.

We do not preclude the case $\dim H_\pm = \infty$, where H_- (resp. H_+) is invariant subspace corresponding to those part of spectrum of L which lies on the left (resp. right) half complex plane. Operators with above property are called *strongly indefinite.*

Let Λ be a compact metric space. A *family of flows indexed by* Λ is a continuous map $\phi\colon \mathbb{R} \times H \times \Lambda \to H$ such that $\phi_\lambda\colon \mathbb{R} \times H \to H$ defined by $\phi_\lambda(t,x) = \phi(t,x,\lambda)$ is a flow on H. As before we write $\phi^t(x,\lambda)$ instead of $\phi(t,x,\lambda)$. If $X \subset H$ and ϕ is a family of flows indexed by Λ then we define

$$\mathrm{inv}(X \times \Lambda) = \mathrm{inv}(X \times \Lambda, \phi) := \left\{(x,\lambda) \in X \times \Lambda;\ \phi^t(x,\lambda) \in X,\ t \in \mathbb{R}\right\}.$$

Definition 2.1. A family of flows $\phi^t\colon H \times \Lambda \to H$ is called a *family of $\mathscr{LS}$-flows* if

$$\phi^t(x,\lambda) = e^{tL}x + U(t,x,\lambda),$$

where $U\colon \mathbb{R} \times H \times \Lambda \to H$ is completely continuous.

Recall, that a map is completely continuous if it is continuous and maps bounded sets to relatively compact sets.

Definition 2.2. We say that a map $f\colon H \times \Lambda \to H$ is a *family of $\mathscr{LS}$-vector fields*, if f is of the form

$$f(x) = Lx + K(x,\lambda), \quad (x,\lambda) \in H \times \Lambda,$$

where $K\colon H \times \Lambda \to H$ is completely continuous and locally Lipschitz map.

If in the above definitions $\Lambda = \{\lambda_0\}$, we drop the parameter space out from notation, and we are talking about *$\mathscr{LS}$-flows* or *$\mathscr{LS}$-vector fields.*

Suppose that $f\colon H \to H$ is an $\mathscr{LS}$-vector field, $f(x) = Lx + K(x)$. We say that f is *subquadratic* if $|\langle K(x), x\rangle| \le a\,\|x\|^2 + b$ for some $a, b > 0$. One can prove that if f is subquadratic then f generates an $\mathscr{LS}$-flow (see **[7]** and references therein). That is for all $x \in H$, there exists a C^1-curve

$$\phi^{(\cdot)}(x)\colon \mathbb{R} \to H$$

satisfying

$$\frac{d}{dt}\phi^t(x) = -f \circ \phi^t(x), \qquad \phi^0(x) = x,$$

and is of the form $\phi^t(x) = e^{-tL}x + U(t,x)$, where $U\colon \mathbb{R} \times H \to H$ is completely continuous. Without loss of generality we will restrict our consideration to subquadratic $\mathscr{LS}$-vector fields.

An isolating neighborhood for a flow ϕ^t on infinite dimensional space is defined similarly to finite dimensional case. The difference lies in the fact that we cannot expect compactness of that set.

Definition 2.3. A bounded and closed set N is an isolating neighborhood for a flow ϕ^t if and only if $\operatorname{inv}(N) \subset \operatorname{int}(N)$.

The isolating neighborhoods are stable with respect to small perturbation of the flow. The sense of this concept is given by the following.

Proposition 2.4 (Gęba et al. [**6**]). *Let $\phi\colon \mathbb{R} \times H \times \Lambda \to H$ be a family of $\mathscr{LS}$-flows. For any bounded and closed $N \subset H$ the set*

$$\Lambda(N) = \{\lambda \in \Lambda;\ \operatorname{inv}(N, \phi_\lambda) \subset \operatorname{int}(N)\}$$

is open in Λ.

We are going to work in the category of compact metrizable spaces with a base point. The notion $f\colon (X, x_0) \to (Y, y_0)$ means that f is a continuous map preserving base points, i.e. $f(x_0) = y_0$. The product is defined in this category by $(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$. The *wedge* of two pointed spaces i.e. the space $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ is closed in $X \times Y$. Hence, the *smash product* $X \wedge Y = (X \times Y)/(X \vee Y)$ is also an object in that category. In addition, if $f\colon X \to Y$ and $g\colon X' \to Y'$ then there is defined $f \wedge g\colon X \wedge X' \to Y \wedge Y'$.

Consider the circle as the unit interval modulo its end points $S^1 = [0,1]/\{0,1\}$. The suspension functor is defined to be the smash product $SX := S^1 \wedge X$. For any $m \in \mathbb{N}$ we define $S^m X := S(S^{m-1}X)$.

$\mathscr{LS}$-index is defined as a sequence of pointed spaces with an extra information added. This leads us to a notion of spectra. Let $\nu\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be a fixed map and suppose that $(E_n)_{n=n(E)}^{\infty}$ is a sequence of spaces and $(\varepsilon_n\colon S^{\nu(n)}E_n \to E_{n+1})_{n=n(E)}^{\infty}$ is a sequence of maps.

Definition 2.5. We say that a pair $E = ((E_n)_{n=n(E)}^{\infty}, (\varepsilon_n)_{n=n(E)}^{\infty})$ is a *spectrum* if there exists $n_0 \geq n(E)$ such that $\varepsilon_n\colon S^{\nu(n)}E_n \to E_{n+1}$ is a homotopy equivalence for all $n \geq n_0$.

One can define the notion of maps of spectra, homotopy of spectra, their homotopy type etc. For us it is sufficient to know that a homotopy type $[E]$ of a spectrum E is uniquely determined by a homotopy type of a pointed space E_n for n sufficiently large. Moreover, in order to define the homotopy type $[E]$ one only needs a sequence $(E_n)_{n=n(E)}^{\infty}$ such that $S^{\nu(n)}E_n$ is homotopy equivalent to E_{n+1} for n sufficiently large.

Assume that $f\colon H \to H$ is an $\mathscr{LS}$-vector field, $f(x) = Lx + K(x)$. Let $\phi^t\colon H \to H$ be the $\mathscr{LS}$-flow generated by f and assume that $N \subset H$ is an isolating neighborhood for ϕ^t. Denote by $P_n\colon H \to H$ the orthogonal projection onto $H^n = \bigoplus_{i=1}^n H_i$. Define

$$f_n\colon H^n \to H^n, \qquad f_n(x) = Lx + P_nK(x).$$

Let $\phi_n^t\colon H \to H$ be a flow induced by f_n. The definition of $\mathscr{LS}$-Conley index is based on the following.

Lemma 2.6 (Gęba et al. [**6**]). *There exists $n_0 \in \mathbb{N}$ such that $N^n = N \cap H^n$ is an isolating neighborhood for a flow ϕ_n^t provided that $n \geq n_0$.*

By the above lemma the set $\operatorname{inv}(N^n, \phi_n^t)$ is an isolated and invariant (by definition) and thus admits an index pair (Y_n, Z_n) by Theorem 1.2. The Conley index of $\operatorname{inv}(N^n)$ is the homotopy type $[Y_n/Z_n]$. Fix a map $\nu\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by setting $\nu(n) := \dim H_{n+1}^{-}$. Using the continuation property of the Conley index one can prove that the pointed space Y_{n+1}/Z_{n+1} is in fact homotopy equivalent to $\nu(n)$-fold suspension of Y_n/Z_n, that is

$$[Y_{n+1}/Z_{n+1}] = [S^{\nu(n)}(Y_n/Z_n)]$$

for all $n \geq n_0$. In the light of earlier observation the sequence $(E_n)_{n=n_0}^{\infty} = (Y_n/Z_n)_{n=n_0}^{\infty}$ represents the spectrum, say E and uniquely determines its homotopy type $[E]$. This leads us to the definition.

Definition 2.7. Let ϕ^t be an $\mathscr{LS}$-flow generated by an $\mathscr{LS}$-vector field. If N is an isolating neighborhood for ϕ^t, then the homotopy type of spectrum

$$h_{\mathscr{LS}}(\mathrm{inv}(N), \phi^t) := [E]$$

is well defined and we call it the $\mathscr{LS}$-Conley index of $\mathrm{inv}(N)$ with respect to ϕ^t.

Let $\mathbf{0}$ represents the homotopy type of spectrum such that for all $n \geq 0$ E_n is just a point and ε_n maps this point into the point in E_{n+1}.

Proposition 2.8 (Gęba et al. [**6**]). *The $\mathscr{LS}$-Conley index has the following properties:*

Nontriviality: *Let $\phi^t\colon H \to H$ be an $\mathscr{LS}$-flow and $N \subset H$ be an isolating neighborhood for ϕ^t. If $h_{\mathscr{LS}}(\mathrm{inv}(N), \phi^t) \neq \mathbf{0}$, then $\mathrm{inv}(N, \phi^t) \neq \emptyset$;*

Continuation: *Let Λ be a compact, connected and locally contractible metric space. Assume that $\phi^t\colon H \times \Lambda \to H$ is a family of $\mathscr{LS}$-flows. Let N be an isolating neighborhood for a flow ϕ^t_λ for some $\lambda \in \Lambda$. Then there is a compact neighborhood $\mathscr{U}_\lambda \subset \Lambda$ such that*

$$h_{\mathscr{LS}}(\mathrm{inv}(N), \phi^t_\mu) = h_{\mathscr{LS}}(\mathrm{inv}(N), \phi^t_\nu)$$

for all $\mu, \nu \in \mathscr{U}_\lambda$.

2.2. Cohomological $\mathscr{LS}$-Conley index. The main reference for this section is [**7**]. Now and subsequently H denotes the Alexander–Spanier cohomology functor. Let $E = (E_n, \varepsilon_n)_{n=n(E)}^{\infty}$ be a spectrum. Define $\rho\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by setting $\rho(0) = 0$ and $\rho(n) = \sum_{i=0}^{n-1} \nu(i)$ for $n \geq 1$. For a fixed $q \in \mathbb{Z}$ consider a sequence of cohomology groups

$$H^{q+\rho(n)}(E_n), \quad n \geq n(E).$$

Define a sequence of homomorphisms by a composition $h_n = (S^*)^{-\nu(n)} \circ \varepsilon_n^{q+\rho(n+1)}$:

$$h_n\colon H^{q+\rho(n+1)}(E_{n+1}) \longrightarrow H^{q+\rho(n+1)}(S^{\nu(n)}E_n) \longrightarrow H^{q+\rho(n)}(E_n),$$

where S^* denotes the suspension isomorphism. Thus we see that $\{H^{q+\rho(n)}(E_n), h_n\}$ forms an inverse system and we are ready to make the following definition.

Definition 2.9. The q^{th} cohomology group of a spectrum E is $CH^q(E) := \varprojlim\{H^{q+\rho(n)}(E_n), h_n\}$.

Since E_{n+1} is homotopically equivalent to $S^{\nu(n)}E_n$ for $n \geq n_0$, we see that

$$h_n\colon H^{q+\rho(n+1)}(E_{n+1}) \to H^{q+\rho(n)}(E_n)$$

is an isomorphism for $n \geq n_0$ and the sequence of groups $H^{q+\rho(n)}(E_n)$ stabilizes. This simply observation implies that:

- $CH^q(E) \cong H^{q+\rho(n)}(E_n)$ for $n \geq n_0$;
- the graded group $CH^*(E)$ is finitely generated if $H^*(E_{n_0})$ is finitely generated;
- the spectrum E is of finite type if the space E_{n_0} is of finite type.

These groups may be nonzero for positive and also negative integers (see [**7**]).

Now we are able to define Betti numbers and Euler characteristic of an $\mathscr{LS}$-Conley index represented by the spectrum E in the obvious way.

Definition 2.10. Let E be a fixed spectrum. The q^{th} Betti number of E is defined as

$$\beta_q(E) := \operatorname{rank} CH^q(E),$$

and the Euler characteristic is given by

$$\chi(E) := \sum_{q\in\mathbb{Z}} (-1)^q \beta_q(E).$$

Remark 2.11. Since $CH^q(E) \cong H^{q+\rho(n)}(E_n)$ for $n \geq n_0$ we have

$$\begin{aligned}(-1)^{\rho(n_0)}\chi(E) &= (-1)^{\rho(n_0)} \sum_{q\in\mathbb{Z}} (-1)^q \beta_q(E)\\ &= \sum_{q\in\mathbb{Z}} (-1)^{q+\rho(n_0)} \beta_{q+\rho(n_0)}(E_{n_0}) = \chi(E_{n_0}).\end{aligned}$$

3. Relationship between $\mathscr{LS}$-index and degree

3.1. The Leray–Schauder degree with respect to L. Let U be an open and bounded subset of H. Denote by $\deg_{\mathscr{LS}}(f, U, 0)$ the Leray-Schauder degree, defined for completely continuous perturbation of identity. For more details about degree theory see **[9]**. Consider an $\mathscr{LS}$-vector field f in H, $f(x) = Lx + K(x)$, where L is strongly indefinite linear bounded and invertible operator and K is completely continuous map. Suppose that f does not vanish on ∂U. We will define degree for the class of such maps in the following manner:

$$\deg_L(f, U, 0) := \deg_{\mathscr{LS}}(I + L^{-1}K, U, 0).$$

Since the zero sets for both f and $I + L^{-1}K$ are the same and $L^{-1}K$ is completely continuous map, the above definition works. The $\deg_L$ inherits all the properties of the Leray–Schauder degree. In particular one has:

Nontriviality: If $0 \in U$ then $\deg_L(L, U, 0) = 1$;

Existence: If $\deg_L(f, U, 0) \neq 0$ then f has a zero inside U;

Additivity: If U_1, U_2 are open, disjoint subsets of U and there are no zeros of f in the completion $U \setminus (U_1 \cup U_2)$, then

$$\deg_L(f, U, 0) = \deg_L(f, U_1, 0) + \deg_L(f, U_2, 0);$$

Homotopy invariance: If $h\colon H \times [0, 1] \to H$ is an $\mathscr{LS}$-vector field for all $t \in [0, 1]$ such that $h(x, t) \neq 0$ for all $(x, t) \in \partial U \times [0, 1]$, then $\deg_L(h(\,\cdot\,, t), U, 0))$ is independent of $t \in [0, 1]$.

3.2. Main theorem.

Theorem 3.1. *Let H be a Hilbert space and $L\colon H \to H$ be a linear bounded operator satisfying assumptions $(L.1)$-$(L.4)$ of Paragraph 2.1. Moreover assume that:*

(1) *L is an isomorphism;*
(2) *$\Omega \subseteq H$ is open and bounded, and $f\colon \Omega \to H$ is of the form*

$$f(x) = Lx + K(x),$$

where $K\colon \Omega \to H$ is completely continuous map of class C^1;

(3) *ϕ^t is the local flow of equation $\dot{x} = -f(x)$;*
(4) *N is an isolating neighborhood for ϕ^t.*

Then the equality

$$\chi(h_{\mathscr{LS}}(\operatorname{inv}(N), \phi^t)) = \deg_L(f, \operatorname{int}(N), 0). \tag{3.1}$$

holds.

PROOF. Suppose that $h_{\mathscr{LS}}(\mathrm{inv}(N), \phi^t)$ is represented by spectrum $E = (E_n, \varepsilon_n)_{n \geq n(E)}$. According to finite dimensional formula (1.3)

$$(-1)^{\rho(n)} \chi(E_n) = (-1)^{\rho(n)} \deg(L + P_n K, N^n, 0). \tag{3.2}$$

The left–hand side of the above equation represents $\chi(E)$ for large n, so we have to show that (3.2) is independent of $n \geq n_0$. We have

$$\begin{aligned}(-1)^{\rho(n)} \chi(E_n) &= (-1)^{\rho(n)} \deg(L + P_n K, N^n, 0) \\ &= (-1)^{\rho(n)} \deg L_{|H^n} \cdot \deg(I + P_n L^{-1} K, N^n, 0) = \deg(I + P_n L^{-1} K, N^n, 0),\end{aligned} \tag{3.3}$$

since the degree of linear isomorphism with respect to 0 is $(-1)^\nu$, where ν is the number of negative eigenvalues of L. But in this case it is exactly $\dim H^n_- = \sum_{i=1}^n \dim H_i^- = \sum_{i=0}^{n-1} \nu(i) = \rho(n)$. Assume that n_0 is chosen such that

$$\deg_{\mathscr{LS}}(I + L^{-1}K, N, 0) = \deg(I + P_n L^{-1} K, N^n, 0) = \deg(I + P_{n+k} L^{-1} K, N^{n+k}, 0)$$

for all $n \geq n_0$ and $k \in \mathbb{N}$. Replacing in (3.3) n by $n+k$, $k \in \mathbb{N}$ we obtain

$$(-1)^{\rho(n+k)} \chi(E_{n+k}) = \deg(I + P_{n+k} L^{-1} K, N^{n+k}, 0). \tag{3.4}$$

On the other hand

$$\begin{aligned}(-1)^{\rho(n+k)} \chi(E_{n+k}) &= (-1)^{\rho(n+k)} \chi(S^{\nu(n)+\ldots+\nu(n+k-1)} E_n) \\ &= (-1)^{\rho(n+k)} \chi(S^{\nu(n)}) \cdot \ldots \cdot \chi(S^{\nu(n+k-1)}) \chi(E_n) \\ &= (-1)^{\rho(n)} \chi(E_n)\end{aligned}$$

and the proof follows from induction. □

3.3. **L is not an isomorphism.** Now consider weaker assumption about an operator $L\colon H \to H$. We would like to admit the case, when L is not invertible operator but is selfadjoint, i.e. $\langle Lx, y\rangle = \langle x, Ly\rangle$ for all $x, y \in H$. Let $P_0\colon H \to H$ denote the orthogonal projection onto H_0, the kernel of L. Define a map $\widehat{L}\colon H \to H$ by $\widehat{L}x := Lx + P_0 x$. Since the kernel of L is orthogonal to the image of L we see, that $\widehat{L}$ is an isomorphism. In particular, if L is invertible, then $\widehat{L} = L$. If f is vector filed of the form $Lx + K(x)$, where K is completely continuous, we can write it equivalently as

$$f(x) = \widehat{L}x + \widehat{K}(x),$$

where $\widehat{K}(x) = K(x) - P_0 x$. Note that $\widehat{K}$ is completely continuous as well, since $\dim H_0 < \infty$. As before for an open bounded subset $U \subset H$ and $\mathscr{LS}$-vector field $f = L + K$ such that $0 \notin f(\partial U)$ we set

$$\deg_L(f, U, 0) := \deg_{\mathscr{LS}}(I + \widehat{L}^{-1}\widehat{K}, U, 0).$$

PROPOSITION 3.2. *Let assumptions of Theorem 3.1 are satisfied. Suppose that $L\colon H \to H$ is selfadjoint (instead of isomorphism). Then the equality* (3.1) *is valid.*

PROOF. If L is selfadjoint then

$$\deg(L + P_n K, N^n, 0) = \deg(\widehat{L} + P_n \widehat{K}, N^n, 0),$$

since $P_n P_0 = P_0$ and L preserves the splitting of $H = \bigoplus_{n=1}^\infty H_n$. Next

$$\deg(\widehat{L} + P_n \widehat{K}, N^n, 0) = \deg \widehat{L}_{|H^n} \cdot \deg(I + P_n \widehat{L}^{-1} \widehat{K}, N^n, 0).$$

Observe that $\deg \widehat{L}_{|H^n} = (-1)^{\rho(n)}$. Indeed, the number of negative eigenvalues of L and $\widehat{L}$ coincide, because $\widehat{L}$ differs from L only on the kernel of L by identity. That is there are only the $\lambda = 1$ of multiplicity $\dim H_0$ added to spectrum of L in places of zeros. The $\deg(I + P_n \widehat{L}^{-1} \widehat{K}, N^n, 0)$ stabilizes for large n and represents $\deg_{\mathscr{LS}}(I + \widehat{L}^{-1}\widehat{K}, N, 0)$. In the light of the proof of preceding theorem it gives us the required result. □

In fact this theorem can be formulated for much bigger class of operators L. It is easy to see that L is admissible if $H = \ker L \oplus \operatorname{im} L$, where $\oplus$ means a direct sum (not orthogonal). This condition allows us to define the $\deg_L$ in the above way.

4. Alternative approach

4.1. Finite-dimensional approximation. In this section the equality (3.1) will be obtained via direct calculation, in the case when $L = (-I, I)\colon H_- \oplus H_+ \to H_- \oplus H_+$ and S is an isolated zero of a given vector field.

We say, that an operator sequence $\{P_n\}_{n=1}^{\infty}$, $P_n\colon H \to H$ is *strongly convergent* to the identity operator $I\colon H \to H$, if $\lim\limits_{n\to\infty} P_n x = x$ for all $x \in H$.

LEMMA 4.1. *If $K\colon H \to H$ is compact operator and $P_n\colon H \to H$, $n = 1, 2, \dots$ is a sequence of orthogonal projections onto H^n strongly convergent to the identity, then*

(a) $\lim\limits_{n\to\infty} \|P_n K - K\| = 0$;
(b) $\lim\limits_{n\to\infty} \|P_n K P_n - K\| = 0$;
(c) $\lim\limits_{n\to\infty} \|Q_n K\| = 0$, *where $Q_n\colon H \to H$ is an orthogonal projection onto H_n.*

PROOF. Statement (a) is a well known fact from the Riesz–Schauder theory. Since

$$\|P_n K P_n - K\| \leq \|P_n K P_n - P_n K\| + \|P_n K - K\|$$

and since $P_n K$ is compact, in order to prove (b) it is enough to show that for any compact A we have $\lim\limits_{n} \|AP_n - A\| = 0$. If A is compact, then the adjoint operator A^* is compact as well and we may write $\|AP_n - A\| = \|(AP_n - A)^*\| = \|P_n A^* - A^*\| \longrightarrow 0$. Finally, we have an estimation $0 \leq \|Q_n K\| \leq \|\left(\sum_{i=n}^{\infty} Q_i\right) K\| = \|(I - P_{n-1})K\| < \varepsilon$ provided $n \geq n_0$. This proves (c). $\square$

DEFINITION 4.2. We say that $A \in \mathfrak{B}(H)$ is *hyperbolic*, if

$$\operatorname{dist}(\sigma(A), i\mathbb{R}) := \inf_{\lambda\in\sigma(A),\, x\in i\mathbb{R}} d(x, \lambda) > 0.$$

The set of all hyperbolic operators will be denoted by $\mathfrak{B}_{\mathrm{hip}}(H)$.

Here $d(\,\cdot\,, \cdot\,)$ stands for the distance function on $\mathbb{C}$.

Recall, that the multivalued map $\mathfrak{B}(H) \ni A \mapsto \sigma(A) \subset \mathbb{C}$ is *upper semi continuous*, that is for all $A \in \mathfrak{B}(H)$ and $\epsilon > 0$, there exists $\delta > 0$, such that inequality $\|A - B\| < \delta$ implies $\sup\limits_{\lambda\in\sigma(B)} \operatorname{dist}(\lambda, \sigma(A)) < \epsilon$.

LEMMA 4.3. *$\mathfrak{B}_{\mathrm{hip}}(H)$ is an open subset of $\mathfrak{B}(H)$.*

PROOF. Set $\rho := \operatorname{dist}(\sigma(A), i\mathbb{R})$. There exists $\delta > 0$ such that for all B in δ-neighborhood of A

$$\sup_{\lambda\in\sigma(B)} \operatorname{dist}(\lambda, \sigma(A)) < \rho/2.$$

Thus, the triangle inequality gives us the following estimation

$$\begin{aligned}\operatorname{dist}(\sigma(B), i\mathbb{R}) &= \inf_{\mu\in\sigma(B),\, x\in i\mathbb{R}} d(\mu, x) \geq \inf_{\mu\in\sigma(B),\, \lambda\in\sigma(A),\, x\in i\mathbb{R}} (d(x,\lambda) - d(\lambda,\mu)) \\ &\geq \inf_{\lambda\in\sigma(A),\, x\in i\mathbb{R}} d(x,\lambda) - \sup_{\mu\in\sigma(B)} \big(\inf_{\lambda\in\sigma(A)} d(\lambda,\mu)\big) > \rho - \frac{\rho}{2} = \frac{\rho}{2} > 0,\end{aligned}$$

which completes the proof. $\square$

4.2. Conley index and the $\mathscr{LS}$-degree.

THEOREM 4.4. *Let H be a real Hilbert space and let L satisfy all the assumptions (L.1)–(L.4) of 2.1. Moreover, assume that:*

1. *L is of the form $(-I, I)\colon H_- \oplus H_+ \to H_- \oplus H_+$, where both $H_\pm$ are of infinite dimension;*
2. *$\Omega \subseteq H$ is a neighborhood of the origin in H and $f\colon \Omega \to H$ is an $\mathscr{LS}$-vector field, with $K\colon \Omega \to H$ being continuously differentiable;*
3. *$f(0) = 0$ and $Df(0) \in \mathfrak{B}_{\mathrm{hip}}(H)$.*

Let ϕ^t be a flow generated by equation $\dot{x} = -f(x)$. Then there exists $\rho > 0$, such that

$$\chi(h_{\mathscr{LS}}(\{0\}, \phi^t)) = \deg_L(f, B(0,\rho), 0). \tag{4.1}$$

REMARK 4.5. Assumption (3) guarantees that $S = \{0\}$ is an isolated invariant set and $x_0 = 0$ is isolated in the set $f^{-1}(0)$ (cf. Remark 1.11 of [**1**]).

In order to compute the index on the left-hand side of (4.1) consider a sequence of finite dimensional approximations $f_n\colon H^n \to H^n$, $f_n(x) = Lx + P_nK(x)$. Since the derivative $Df(0) = L + DK(0)$ is a hyperbolic operator, then by Lemmas 4.1 and 4.3 there exists $n_0 \in \mathbb{N}$ such that $Df_n(0) = L + P_nDK(0)$ is hyperbolic, provided $n \geq n_0$. Let us note that $DK(0)$ is a compact linear operator.

The set $\mathrm{cl}(B(0,\rho)) \cap H^n$ is an isolating neighborhood for the invariant set $\{0\} \in H^n$ for $n \geq n_1$ (comp. Lemma 2.6). Assume that n_0 is chosen such that $n_0 \geq n_1$. W have a splitting $H^{n_0} = \widehat{H}_-^{n_0} \oplus \widehat{H}_+^{n_0}$ where $\widehat{H}_-^{n_0}$ (resp. $\widehat{H}_+^{n_0}$) stands for unstable (resp. stable) subspace of the linear equation $\dot{x} = -Df_{n_0}(0)x$. In the hyperbolic case, the Conley index is exactly the homotopy type of pointed sphere: $h(\{0\}, \phi_{n_0}^t) = [S^{\dim \widehat{H}_-^{n_0}}, *]$.

Denote by E_{n_0} the space that is homotopy equivalent to $(S^{\dim \widehat{H}_-^{n_0}}, *)$. In order to establish the relation between E_{n_0} and E_{n_0+1}, we have to compute the index of the flow generated by $f_{n_0+1}\colon H^{n_0+1} \to H^{n_0+1}$. Note that the derivative $Df_{n_0+1}(0) = L + P_{n_0+1}DK(0)$ preserve the splitting $H^{n_0+1} = H^{n_0} \oplus H_{n_0+1}$. It is easy to see if we write it as

$$L_{|H^{n_0}} + P_{n_0}DK(0) + L_{|H_{n_0+1}} + Q_{n_0+1}DK(0)\colon H^{n_0} \oplus H_{n_0+1} \to H^{n_0} \oplus H_{n_0+1}.$$

In this situation we have the formula $h(\{0\}, \phi_{n_0+1}^t) = h(\{0\}, \phi_{n_0}^t) \wedge h(\{0\}, \eta)$, where $h(\{0\}, \eta)$ is an index of $\{0\}$ with respect to flow generated by $\dot{x} = -L_{|H_{n_0+1}}x - Q_{n_0+1}DK(0)x$.

Since $\|Q_nDK(0)\| \to 0$, the maps $L_{|H_{n_0+1}}$ and $L_{|H_{n_0+1}} + Q_{n_0+1}DK(0)$ are homotopic for sufficiently large n_0 and the index $h(\{0\}, \eta)$ is determined by dimension of the unstable subspace of linear equation

$$\dot{x} = -L_{|H_{n_0+1}}x.$$

Set $H_{n_0+1} = H_{n_0+1}^- \oplus H_{n_0+1}^+$, where $H_{n_0+1}^-$ (resp. $H_{n_0+1}^+$) is the unstable (resp. stable) subspace of L and define $\nu\colon \mathbb{N}^* \to \mathbb{N}^*$ by $\nu(n) = \dim H_{n+1}^-$. We have

$$h(\{0\}, \phi_{n_0+1}^t) = [S^{\dim \widehat{H}_-^{n_0}}, *] \wedge [S^{\nu(n_0)}, *] = [S^{\nu(n_0)} S^{\dim \widehat{H}_-^{n_0}}, *]$$

CONCLUSION 4.6. E_{n+1} is the $\nu(n)$-fold suspension of E_n, provided that n is sufficiently large.

Define $\rho\colon \mathbb{N}^* \to \mathbb{N}^*$ by $\rho(0) = 0$ and $\rho(n) = \sum_{i=0}^{n-1} \nu(i)$. According to definition of cohomological Conley index we have an isomorphism

$$CH^q(h_{\mathscr{LS}}(\{0\}, \phi^t)) \cong H^{q+\rho(n)}(h(\{0\}, \phi_n^t)), \quad n \geq n_0.$$

It follows that $CH^q(h_{\mathscr{LS}}(\{0\}, \phi^t)) \cong H^{q+\rho(n)}(S^{\dim \widehat{H}_-^n}, *) \cong \mathbb{Z}$ for $q = \dim \widehat{H}_-^n - \rho(n)$ and hence

$$\chi(h_{\mathscr{LS}}(\{0\}, \phi^t)) = (-1)^{\dim \widehat{H}_-^n - \rho(n)}, \quad n \geq n_0.$$

In particular we have $\chi(h_{\mathscr{LS}}(\{0\}, \phi^t)) = (-1)^{\dim \widehat{H}_-^{n_0} - \rho(n_0)}$.

By the stability of an $\mathscr{LS}$-degree we have

$$\deg_L(f, B(0,\rho), 0) = \deg_{\mathscr{LS}}(I + L^{-1}K, B(0,\rho), 0) = \deg(I + L^{-1}P_nK, B^n(0,\rho), 0)$$

for $n \geq n_0$. From the fact that $\deg(L_{|H^n}, B^n, 0) = (-1)^{\rho(n)}$ and $\deg(L + P_nK, B^n, 0) = (-1)^{\dim \widehat{H}^n_-}$ we conclude that

$$\deg(I + P_nL^{-1}K, B^n, 0) = \deg(L + P_nK, B^n, 0) \cdot [\deg(L_{|H^n}, B^n, 0)]^{-1} = (-1)^{\dim \widehat{H}^n_- - \rho(n)}$$

for $n \geq n_0$ and the proof of (4.1) is completed.

References

[1] A. Abbondandolo, P. Majer, *Lectures on the Morse Complex for Infinite–Dimensional Manifolds*, Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology **27**, Springer 2004, NATO Science Series, pp. 1–74

[2] C. Conley, *Isolated Invariant Sets and the Morse Theory*, CBMS, Vol. 38, AMS, Providence, 1978

[3] C. Conley, E. Zehnder *Morse-type Index Theory for Flows and Periodic Solutions for Hamiltonian Equations*, Comm. on Pure and Appl. Math., Vol. **37**, 1984, pp. 207–253

[4] E.N. Dancer, *Degenerate Critical Points, Homotopy Indices and Morse Inequalities*, Journ. für Math.

[5] K. Gęba, *Degree for Gradient Equivariant Maps and Equivariant Conley Index*, Topological Nonlinear Analysis, Progress in Nonlinear Diff. Eq. and Their Appl. **27**, Birkhäuser,

[6] K. Gęba, M. Izydorek, A. Pruszko *The Conley Index in Hilbert Spaces and its Applications*, Studia Math., **134**, No.**3**, 1999, pp. 217-233

[7] M. Izydorek, *A Cohomological Conley Index in Hilbert Spaces and Applications to Strongly Indefinite Problems*, Journ. of Diff. Eq., **170**, 2001, pp. 22-55

[8] W. Kryszewski, A. Szulkin *An Infinite Dimensional Morse Theory with Applications*, Trans. of the Amer. Math. Soc., Vol. **349**, No. **8**, 1997, pp. 3181-3234

[9] N.G. LLoyd, *Degree theory*, Cambridge University Press, 1978

[10] C.K. McCord, *On the Hopf Index and the Conley Index* , Trans. of the Amer. Math. Soc., Vol. **313**, No. **2**, 1989, pp. 853-860

[11] M.R. Razvan, M. Fotouhi, *On the Poincaré Index of Isolated Invariant Set*, arXiv Math

[12] J.F. Reineck, *Continuation to gradient flows*, Duke Math. Journal, Vol. **64**, No. **2**, 1991.

[13] D. Salamon, *Connected simple systems and the Conley index of isolated invariant sets*, Trans. of the Amer. Math. Soc., Vol. **291**, 1985, pp. 1–41

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