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Abstract

We present tensor-product divergence-free and curl-free wavelets, and define associated projectors. These projectors permit the definition of an iterative algorithm to compute the Helmholtz decomposition in wavelet domain of any vector field. This decomposition is localized in space, in contrast to the Helmholtz decomposition calculated by Fourier transform. Then we prove the convergence of the algorithm in any dimension for the particular case of Shannon wavelets. We also present a modification of the algorithm which allows to apply it in an adaptive context. Finally, numerical tests show the validity of this approach for any choice of wavelets.

Introduction

In many physical problems, like the simulation of incompressible fluids (Stokes problem, Navier-Stokes equations [2, 17]), or in electromagnetism (Maxwell's equations [16]), the solution has to fulfill a divergence-free condition. The implementation of relevant numerical schemes often requires orthogonal projection on the set of divergence-free vector valued functions.

The Helmholtz decomposition [10, 2] consists in decomposing a vector field $\mathbf{u} \in (L^2(\mathbb{R}^n))^n$, into the sum of its divergence-free component $\mathbf{u}_{\rm div}$ and its curl-free component $\mathbf{u}_{\rm curl}$. More precisely, there exist a stream-function ψ and a potential-function p such that:

$$\mathbf{u} = \mathbf{u}_{\rm div} + \mathbf{u}_{\rm curl} \tag{0.1}$$

with

$$\mathbf{u}_{\rm div} = \mathbf{curl}\ \psi\ , \quad (\mathbf{div}\ \mathbf{u}_{\rm div} = 0) \quad \text{and} \quad \mathbf{u}_{\rm curl} = \nabla p\ , \quad (\mathbf{curl}\ \mathbf{u}_{\rm curl} = 0)\ .$$

Moreover, the functions $\mathbf{curl}\ \psi$ and ∇p are orthogonal in $(L^2(\mathbb{R}^n))^n$. The stream-function ψ and the potential-function p are unique, up to an additive constant.

This decomposition arises from the orthogonal direct sum of the two spaces $\mathbf{H}_{\rm div\,0}(\mathbb{R}^n)$, the space of divergence-free vector functions, and $\mathbf{H}_{\rm curl\,0}(\mathbb{R}^n)$, the space of curl-free vector functions. In short:

$$(L^2(\mathbb{R}^n))^n = \mathbf{H}_{\rm div\,0}(\mathbb{R}^n) \oplus^\perp \mathbf{H}_{\rm curl\,0}(\mathbb{R}^n)\ .$$

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This decomposition is straightforward in $(L^2(\mathbb{R}^n))^n$ thanks to the Leray projector (the orthogonal projector from $(L^2(\mathbb{R}^n))^n$ onto $\mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n)$) which can be explicitly described in Fourier domain. The Helmholtz decomposition (0.1) also exists for more general open sets Ω [10, 2].

The objective of the present paper is to propose an orthogonal Helmholtz decomposition in wavelet domain of any vector field. Since wavelet bases are localized both in physical and Fourier spaces [11], the advantages of such a decomposition will be, contrarily to the one based on the Fourier transform, first to be local in physical space, second to be available on the whole domain $\mathbb{R}^n$, as well as on bounded domains. Moreover, an accurate wavelet Helmholtz decomposition will be provided by a small number of degrees of freedom, thanks to nonlinear approximation properties of wavelet bases [4]. This last property will be of great interest, for instance in Direct Numerical Simulations (DNS) of turbulence [9].
In this context divergence-free wavelet bases have been originally designed by Lemarié [12], and investigated along with curl-free wavelets for the decomposition of $\mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n)$ and $\mathbf{H}_{\mathrm{curl}\,0}(\mathbb{R}^n)$ by Urban in [20]. Since divergence-free and curl-free wavelets are biorthogonal (and not orthogonal, see [13]) bases, the associated projectors do not provide directly the Helmholtz decomposition of a vector field. Therefore we have originally proposed an iterative algorithm in [7], of which we have proved the convergence in dimensions 2 and 3, using Shannon wavelets [8]. In order to achieve the wavelet Helmholtz decomposition in any dimension, we propose in this article a new formulation for the divergence-free and curl-free wavelets. This re-formulation will lead to an expression of the Leray projector in the wavelet domain, analogous of its expression in the Fourier one: all these ingredients will allow proving the convergence of the algorithm for Shannon wavelets, and to verify it experimentally for a large class of wavelets.

The article is organized as follows: in Section 1 we construct biorthogonal wavelet bases of $\mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n)$ and $\mathbf{H}_{\mathrm{curl}\,0}(\mathbb{R}^n)$, with their associated projectors. As the projectors associated to divergence-free wavelets are oblique, we define in Section 2.1 an iterative algorithm providing the wavelet Helmholtz decomposition of any vector field in dimension n. We prove the convergence of this algorithm in Section 2.2 in the case of Shannon wavelets. In Section 3, we modify the expression of the wavelet bases, to make their use possible in an adaptive scheme. Section 4 is devoted to numerical tests, in order to observe the convergence of the wavelet Helmholtz decomposition on 2D and 3D vector fields.

1 Divergence-free and curl-free wavelets

Divergence-free wavelets were defined by P.G. Lemarié in 1992 [12]. K. Urban used them in the numerical resolution of the Stokes problem [19], and extended the principle of their construction to derive curl-free wavelets [20]. These constructions are both based on the existence of *biorthogonal wavelet bases* (see [3, 14]) linked by differentiation. In particular, we will use the following result from [12]:

Proposition 1.1 *Let (V_j^1) be a multiresolution analysis (MRA) of $L^2(\mathbb{R})$, with associated wavelet ψ_1 and scaling function φ_1. Then there exists a MRA (V_j^0), with associated wavelet*

ψ_0 and scaling function φ_0, satisfying:

$$\psi_1'(x) \;=\; 4\;\psi_0(x) \qquad \text{and} \qquad \varphi_1'(x) \;=\; \varphi_0(x) \;-\; \varphi_0(x-1) \tag{1.1}$$

By this theorem, we have at our disposal two Riesz bases of $L^2(\mathbb{R})$:

$$\Big(\psi_{1,j,k}(x) = 2^{j/2}\psi_1(2^j x - k)\Big)_{j,k\in\mathbb{Z}}$$

and $(\psi_{0,j,k})_{j,k\in\mathbb{Z}}$, related by derivation. Hence a function decomposed into the first basis $\psi_{1,j,k}$ with coefficients $(d_{j,k})$, has for derivative the function with coefficients $(2^{j+2}d_{j,k})$ into the second basis $\psi_{0,j,k}$. Conversely an indefinite integral of the function of coefficients $(d_{j,k})$ into the second basis $\psi_{0,j,k}$, has for coefficients $(2^{-j-2}d_{j,k})$ in the first one.
Contrarily to the wavelets proposed in Lemarié's and Urban's works, we will prefer to use divergence-free and curl-free *anisotropic* wavelet bases, which means tensor-products of one dimensional wavelet bases [5, 7]. We will recall in the following the main ingredients for the definition of these wavelets and their associated projectors.

1.1 Anisotropic divergence-free wavelets

1.1.1 The two-dimensional case

Anisotropic divergence-free wavelets in two dimensions are constructed by taking the curl of wavelets in the tensor-product AMR $(V_j^1 \otimes V_j^1)$:

$$\Psi^{\text{div}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \frac{1}{4}\mathbf{curl}\left(\psi_1(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\right) = \left|\begin{array}{l} 2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \\ -2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\end{array}\right.$$

Here, $\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2$ is the scale parameter, and $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ the position parameter. For $\mathbf{j},\mathbf{k} \in \mathbb{Z}^2$, the set $\{\Psi^{\text{div}}_{\mathbf{j},\mathbf{k}}\}$ forms a Riesz basis of $\mathbf{H}_{\text{div}\,0}(\mathbb{R}^2)$. In order to complete this family to a basis of $(L^2(\mathbb{R}^2))^2$, we introduce the complementary functions:

$$\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \left|\begin{array}{l} 2^{j_1}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \\ 2^{j_2}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\end{array}\right.$$

This choice ensures that, for fixed $\mathbf{j}$ and $\mathbf{k}$, the complementary function $\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}$ is orthogonal to the divergence-free wavelet $\Psi^{\text{div}}_{\mathbf{j},\mathbf{k}}$. The exponent $\mathfrak{N}$ (and further $\mathcal{N}$) stands for "normal". Note that imposing this constraint of orthogonality yields a unique solution for the complementary function, contrarily to the several possible choices for the "natural" supplementary spaces introduced by K. Urban in [21].

To decompose any vector field $\mathbf{u}$ into this new basis, we begin with the standard tensor-product wavelet decomposition of $\mathbf{u}$ in the MRA $(V^1_{j_1} \otimes V^0_{j_2}) \times (V^0_{j_1} \otimes V^1_{j_2})$:

$$\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\left(d_{1,\mathbf{j},\mathbf{k}}\;\Psi^1_{\mathbf{j},\mathbf{k}} + d_{2,\mathbf{j},\mathbf{k}}\;\Psi^2_{\mathbf{j},\mathbf{k}}\right)$$

where, for $\mathbf{j},\mathbf{k} \in \mathbb{Z}^2$:

$$\Psi^1_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \left|\begin{array}{l} \psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \\ 0\end{array}\right.$$

$$\Psi^2_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \left|\begin{array}{l} 0 \\ \psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\end{array}\right.$$

are the tensor-product wavelets for each component (in L^∞ normalization).
We can now express $\mathbf{u}$ into the divergence-free wavelet basis and its complementary wavelet basis:

$$\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\left(d_{\mathrm{div}\,\mathbf{j},\mathbf{k}}\ \Psi^{\mathrm{div}}_{\mathbf{j},\mathbf{k}} + d_{\mathfrak{N}\mathbf{j},\mathbf{k}}\ \Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}\right) \tag{1.2}$$

which gives directly the coefficients $d_{\mathrm{div}\,\mathbf{j},\mathbf{k}}$ and $d_{\mathfrak{N}\mathbf{j},\mathbf{k}}$:

$$\begin{bmatrix} d_{\mathrm{div}\,\mathbf{j},\mathbf{k}} \\ d_{\mathfrak{N}\mathbf{j},\mathbf{k}} \end{bmatrix} = \begin{bmatrix} \frac{2^{j_2}}{2^{2j_1}+2^{2j_2}} & -\frac{2^{j_1}}{2^{2j_1}+2^{2j_2}} \\ \frac{2^{j_1}}{2^{2j_1}+2^{2j_2}} & \frac{2^{j_2}}{2^{2j_1}+2^{2j_2}} \end{bmatrix} \begin{bmatrix} d_{1,\mathbf{j},\mathbf{k}} \\ d_{2,\mathbf{j},\mathbf{k}} \end{bmatrix} \tag{1.3}$$

Remark 1.1 *Since the choice of the complementary wavelets $\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}$ is not unique, it influences the values of the coefficients $d_{\mathrm{div}\,\mathbf{j},\mathbf{k}}$ and $d_{\mathfrak{N}\mathbf{j},\mathbf{k}}$. We will see in Section 2.2 that this choice also influences the convergence of the wavelet Helmholtz decomposition. Of course, if $\mathbf{u}$ is divergence-free we retrieve $d_{\mathfrak{N}\mathbf{j},\mathbf{k}} = 0$.*

1.1.2 The three-dimensional case

As in dimension two, the 3D anisotropic divergence-free wavelets are constructed by taking the curl of standard tensor-product wavelets:

$$\Psi^{\mathrm{div}\,1}_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) = \tfrac{1}{4}\mathbf{curl}\left|\begin{array}{l} 0 \\ 0 \\ \psi_1\psi_1\psi_0 \end{array}\right. = \left|\begin{array}{l} 2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ -2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ 0 \end{array}\right.$$

$$\Psi^{\mathrm{div}\,2}_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) = \tfrac{1}{4}\mathbf{curl}\left|\begin{array}{l} \psi_0\psi_1\psi_1 \\ 0 \\ 0 \end{array}\right. = \left|\begin{array}{l} 0 \\ 2^{j_3}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ -2^{j_2}\psi_0(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \end{array}\right.$$

$$\Psi^{\mathrm{div}\,3}_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) = \tfrac{1}{4}\mathbf{curl}\left|\begin{array}{l} 0 \\ \psi_1\psi_0\psi_1 \\ 0 \end{array}\right. = \left|\begin{array}{l} -2^{j_3}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ 0 \\ 2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \end{array}\right.$$

Indeed this family is linearly dependent ($2^{j_3}\Psi^{\mathrm{div}\,1}_{\mathbf{j},\mathbf{k}} + 2^{j_1}\Psi^{\mathrm{div}\,2}_{\mathbf{j},\mathbf{k}} + 2^{j_2}\Psi^{\mathrm{div}\,3}_{\mathbf{j},\mathbf{k}} = 0$), and we have to choose two functions among the three above, to form a basis of $\mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^3)$. In any case, we choose as the complementary wavelet a function which is orthogonal to the other three:

$$\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) = \left|\begin{array}{l} 2^{j_1}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ 2^{j_2}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ 2^{j_3}\psi_0(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \end{array}\right. \tag{1.4}$$

Again, to compute the decomposition of a vector field into this new basis, we introduce the tensor-product wavelets of the AMR $(V^1_{j_1}\otimes V^0_{j_2}\otimes V^0_{j_3})\times(V^0_{j_1}\otimes V^1_{j_2}\otimes V^0_{j_3})\times(V^0_{j_1}\otimes V^0_{j_2}\otimes V^1_{j_3})$, componentwise:

$$\Psi^1_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) = \left| \begin{array}{l} \psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ 0 \\ 0 \end{array} \right.$$

$$\Psi^2_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) = \left| \begin{array}{l} 0 \\ \psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \\ 0 \end{array} \right.$$

$$\Psi^3_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) = \left| \begin{array}{l} 0 \\ 0 \\ \psi_0(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \end{array} \right.$$

with $\mathbf{j}=(j_1,j_2,j_3)\in\mathbb{Z}^3$ and $\mathbf{k}=(k_1,k_2,k_3)\in\mathbb{Z}^3$.
Let

$$\mathbf{u} = \sum_{i=1}^{3}\sum_{\mathbf{j}\in\mathbb{Z}^3}\sum_{\mathbf{k}\in\mathbb{Z}^3} d_{i\,\mathbf{j},\mathbf{k}}\ \Psi^i_{\mathbf{j},\mathbf{k}}$$

be the decomposition of $\mathbf{u}$ into the above basis. To compute the coefficients of $\mathbf{u}$ in terms of the divergence-free vector wavelets and their complementary wavelets:

$$\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^3}\sum_{\mathbf{k}\in\mathbb{Z}^3}\left(d_{\mathrm{div}\,1,\mathbf{j},\mathbf{k}}\ \Psi^{\mathrm{div}\,1}_{\mathbf{j},\mathbf{k}} + d_{\mathrm{div}\,2,\mathbf{j},\mathbf{k}}\ \Psi^{\mathrm{div}\,2}_{\mathbf{j},\mathbf{k}} + d_{\mathrm{div}\,3,\mathbf{j},\mathbf{k}}\ \Psi^{\mathrm{div}\,3}_{\mathbf{j},\mathbf{k}} + d_{\mathfrak{N}\mathbf{j},\mathbf{k}}\ \Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}\right) \qquad (1.5)$$

we add the condition:

$$2^{j_3}\ d_{\mathrm{div}\,1,\mathbf{j},\mathbf{k}} + 2^{j_1}\ d_{\mathrm{div}\,2,\mathbf{j},\mathbf{k}} + 2^{j_2}\ d_{\mathrm{div}\,3,\mathbf{j},\mathbf{k}} = 0$$

to have as many equations as unknowns. This leads to the following system for the change of coordinates [5]:

$$\begin{bmatrix} d_{\mathrm{div}\,1,\mathbf{j},\mathbf{k}} \\ d_{\mathrm{div}\,2,\mathbf{j},\mathbf{k}} \\ d_{\mathrm{div}\,3,\mathbf{j},\mathbf{k}} \\ d_{\mathfrak{N}\mathbf{j},\mathbf{k}} \end{bmatrix} = \begin{bmatrix} \frac{2^{j_2}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} & -\frac{2^{j_1}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} & 0 \\ 0 & \frac{2^{j_3}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} & -\frac{2^{j_1}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} \\ -\frac{2^{j_3}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} & 0 & \frac{2^{j_1}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} \\ \frac{2^{j_1}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} & \frac{2^{j_2}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} & \frac{2^{j_3}}{2^{2j_1}+2^{2j_2}+2^{2j_3}} \end{bmatrix} \begin{bmatrix} d_{1,\mathbf{j},\mathbf{k}} \\ d_{2,\mathbf{j},\mathbf{k}} \\ d_{3,\mathbf{j},\mathbf{k}} \end{bmatrix} \qquad (1.6)$$

1.2 Generalization to the n-dimensional case

The first and natural idea for the general form of divergence-free wavelets in dimension n is to introduce:

$$
\Psi^{\mathrm{div}\,i}_{\mathbf{j},\mathbf{k}}(x_1,x_2,\dots,x_n) =
\begin{array}{ll}
 & \left| \begin{array}{l}
0\\
\vdots\\
0\\
2^{j_{i+1}}\psi_0(2^{j_1}x_1-k_1)\dots\psi_0(2^{j_{i-1}}x_{i-1}-k_{i-1})\psi_1(2^{j_i}x_i-k_i)\\
\qquad\qquad\psi_0(2^{j_{i+1}}x_{i+1}-k_{i+1})\dots\psi_0(2^{j_n}x_n-k_n)\\
\\
-2^{j_i}\psi_0(2^{j_1}x_1-k_1)\dots\psi_0(2^{j_i}x_i-k_i)\psi_1(2^{j_{i+1}}x_{i+1}-k_{i+1})\\
\qquad\qquad\psi_0(2^{j_{i+2}}x_{i+2}-k_{i+2})\dots\psi_0(2^{j_n}x_n-k_n)\\
0\\
\vdots\\
0
\end{array}\right.
\end{array}
\begin{array}{l}
\\ \\ \\ \leftarrow \text{line } i \\ \\ \\ \leftarrow \text{line } i+1 \\ \\ \\ \\ \\
\end{array}
\tag{1.7}
$$

for $1 \le i \le n$ (for $i = n$, the line $n+1$ is shifted to the first line and the index j_{n+1} is replaced by j_1). Taking $n-1$ vector wavelets out from these n wavelets allows to form a basis of $\mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n)$. Note that these functions are no longer derived from the curl operator since the curl operator takes a complicated form for $n \ge 4$.

The complementary functions are chosen to satisfy an orthogonality condition with the divergence-free wavelets of same index $(\mathbf{j},\mathbf{k})$:

$$
\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}(x_1,x_2,\dots,x_n) =
\left| \begin{array}{l}
2^{j_1}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\dots\psi_0(2^{j_n}x_n-k_n)\\
\vdots\\
2^{j_i}\psi_0(2^{j_1}x_1-k_1)\dots\psi_1(2^{j_i}x_i-k_i)\dots\psi_0(2^{j_n}x_n-k_n)\\
\vdots\\
2^{j_n}\psi_0(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\dots\psi_1(2^{j_n}x_n-k_n)
\end{array}\right.
\tag{1.8}
$$

Like in dimensions two and three, the anisotropic divergence-free wavelet transform will be related to the standard anisotropic wavelet transform. Each component u_i of a vector field $\mathbf{u}$ is decomposed in the tensor-product wavelet basis of the MRA $(V^0_{j_1}\otimes\dots\otimes V^1_{j_i}\otimes\dots\otimes V^0_{j_n})$:

$$
u_i(x_1,\dots,x_n) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^n} d_{i\,\mathbf{j},\mathbf{k}}\ \psi_0(2^{j_1}x_1-k_1)\dots\psi_1(2^{j_i}x_i-k_i)\dots\psi_0(2^{j_n}x_n-k_n)
$$

The divergence-free and complementary coefficients of $\mathbf{u}$ are given by:

$$
\mathbf{u} = \sum_{i=1}^{n}\sum_{\mathbf{j}\in\mathbb{Z}^3}\sum_{\mathbf{k}\in\mathbb{Z}^3} d_{\mathrm{div}\,i,\mathbf{j},\mathbf{k}}\ \Psi^{\mathrm{div\,i}}_{\mathbf{j},\mathbf{k}} + \sum_{\mathbf{j}\in\mathbb{Z}^3}\sum_{\mathbf{k}\in\mathbb{Z}^3} d_{\mathfrak{N}\mathbf{j},\mathbf{k}}\ \Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}
\tag{1.9}
$$

to which we add the relationship between divergence-free coefficients:

$$
\sum_{i=1}^{n} 2^{-j_i-j_{i+1}}\ d_{\mathrm{div}\,i,\mathbf{j},\mathbf{k}} = 0
$$

(with the convention $2^{-j_{n+1}} = 2^{-j_1}$) chosen such that the last row of the matrix in the system below is orthogonal to the others:

$$\begin{bmatrix} 2^{j_2} & 0 & 0 & \dots & \dots & 0 & -2^{j_n} & 2^{j_1} \\ -2^{j_1} & 2^{j_3} & 0 & \ddots & \ddots & \vdots & 0 & 2^{j_2} \\ 0 & -2^{j_2} & 2^{j_4} & \ddots & \ddots & \ddots & \vdots & 2^{j_3} \\ \vdots & \ddots & -2^{j_3} & 2^{j_5} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & -2^{j_{n-2}} & 2^{j_n} & 0 & 2^{j_{n-1}} \\ 0 & 0 & \dots & \dots & 0 & -2^{j_{n-1}} & 2^{j_1} & 2^{j_n} \\ 2^{-j_1-j_2} & 2^{-j_2-j_3} & 2^{-j_3-j_4} & \dots & 2^{-j_{n-2}-j_{n-1}} & 2^{-j_{n-1}-j_n} & 2^{-j_n-j_1} & 0 \end{bmatrix} \begin{bmatrix} d^{\mathrm{div}}_{1\,\mathbf{j},\mathbf{k}} \\ d^{\mathrm{div}}_{2\,\mathbf{j},\mathbf{k}} \\ \vdots \\ \vdots \\ \vdots \\ d^{\mathrm{div}}_{n-1\,\mathbf{j},\mathbf{k}} \\ d^{\mathrm{div}}_{n\,\mathbf{j},\mathbf{k}} \\ d^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}} \end{bmatrix} = \begin{bmatrix} d_{1\,\mathbf{j},\mathbf{k}} \\ d_{2\,\mathbf{j},\mathbf{k}} \\ \vdots \\ \vdots \\ \vdots \\ d_{n-1\,\mathbf{j},\mathbf{k}} \\ d_{n\,\mathbf{j},\mathbf{k}} \\ 0 \end{bmatrix} \tag{1.10}$$

Unfortunately, this matrix cannot be made orthogonal for $n \geq 4$ even with the choice of the last line orthogonal to the others. However for $n = 3$, the matrix of coordinate change is still orthogonal:

$$M = \begin{bmatrix} 2^{j_2} & 0 & -2^{j_3} & 2^{j_1} \\ -2^{j_1} & 2^{j_3} & 0 & 2^{j_2} \\ 0 & -2^{j_2} & 2^{j_1} & 2^{j_3} \\ 2^{j_3} & 2^{j_1} & 2^{j_2} & 0 \end{bmatrix}, \text{ with } M^{-1} = \frac{1}{2^{2j_1} + 2^{2j_2} + 2^{2j_3}} \begin{bmatrix} 2^{j_2} & 0 & -2^{j_3} & 2^{j_1} \\ -2^{j_1} & 2^{j_3} & 0 & 2^{j_2} \\ 0 & -2^{j_2} & 2^{j_1} & 2^{j_3} \\ 2^{j_3} & 2^{j_1} & 2^{j_2} & 0 \end{bmatrix} \tag{1.11}$$

This lack of orthogonality for $n \geq 4$ makes the inversion of the matrix more difficult. This is the reason why we will construct another divergence-free wavelet basis in which this matrix will be orthogonal. Moreover we will see in Section 3.1 that this new construction will be fruitful for a generalization of the method to isotropic and quasi-isotropic wavelets.

The method that we propose now is inspired by the expression of the Leray projector in Fourier domain. We recall that the n-dimensional Leray projector $\mathbb{P}$ in Fourier space takes the form:

$$\widehat{\mathbb{P}(\mathbf{u})} = \begin{bmatrix} \widehat{u_{\mathrm{div}\,1}} \\ \widehat{u_{\mathrm{div}\,2}} \\ \vdots \\ \vdots \\ \widehat{u_{\mathrm{div}\,n-1}} \\ \widehat{u_{\mathrm{div}\,n}} \end{bmatrix} = \begin{bmatrix} 1-\frac{\xi_1^2}{|\xi|^2} & -\frac{\xi_2\xi_1}{|\xi|^2} & \dots & \dots & -\frac{\xi_n\xi_1}{|\xi|^2} \\ -\frac{\xi_1\xi_2}{|\xi|^2} & 1-\frac{\xi_2^2}{|\xi|^2} & \ddots & \ddots & -\frac{\xi_n\xi_2}{|\xi|^2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -\frac{\xi_1\xi_n}{|\xi|^2} & -\frac{\xi_2\xi_n}{|\xi|^2} & \dots & -\frac{\xi_{n-1}\xi_n}{|\xi|^2} & 1-\frac{\xi_n^2}{|\xi|^2} \end{bmatrix} \begin{bmatrix} \widehat{u_1} \\ \widehat{u_2} \\ \vdots \\ \vdots \\ \widehat{u_{n-1}} \\ \widehat{u_n} \end{bmatrix} \tag{1.12}$$

where $\widehat{u_k}$ denotes the Fourier transform[1] of the k-th component u_k of $\mathbf{u}$, on $\mathbb{R}^n$.

By analogy, we introduce the following expressions for the divergence-free wavelets

[1] The Fourier transform of a function $f \in L^1(\mathbb{R})$ is noted $\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-ix\xi} dx$, we recall that $f \mapsto \frac{1}{\sqrt{2\pi}}\, \hat{f}$ defines an isometry on $L^2(\mathbb{R})$.

(with the same notation as before): for $1 \le i \le n$,

$$\Psi^{\mathrm{div}\,i}_{\mathbf{j},\mathbf{k}}(x_1,\dots,x_n) = \frac{1}{|\omega|^2}\left|\begin{array}{l} -\omega_i\omega_1\psi_1(\omega_1x_1-k_1)\psi_0(\omega_2x_2-k_2)\dots\psi_0(\omega_nx_n-k_n) \\ \vdots \\ -\omega_i\omega_{i-1}\psi_0(\omega_1x_1-k_1)\dots\psi_0(\omega_{i-2}x_{i-2}-k_{i-2})\psi_1(\omega_{i-1}x_{i-1}-k_{i-1}) \\ \hfill \psi_0(\omega_ix_i-k_i)\dots\psi_0(\omega_nx_n-k_n) \\ \\ \left(\sum_{\ell\neq i}\omega_\ell^2\right)\psi_0(\omega_1x_1-k_1)\dots\psi_0(\omega_{i-1}x_{i-1}-k_{i-1})\psi_1(\omega_ix_i-k_i) \\ \hfill \psi_0(\omega_{i+1}x_{i+1}-k_{i+1})\dots\psi_0(\omega_nx_n-k_n) \\ \\ -\omega_i\omega_{i+1}\psi_0(\omega_1x_1-k_1)\dots\psi_0(\omega_ix_i-k_i)\psi_1(\omega_{i+1}x_{i+1}-k_{i+1}) \\ \hfill \psi_0(\omega_{i+2}x_{i+2}-k_{i+2})\dots\psi_0(\omega_nx_n-k_n) \\ \vdots \\ -\omega_i\omega_n\psi_0(\omega_1x_1-k_1)\dots\psi_0(\omega_{n-1}x_{n-1}-k_{n-1})\psi_1(\omega_nx_n-k_n) \end{array}\right.$$

where $\omega_i = 2^{j_i}$ and $|\omega|^2 = \sum_{i=1}^n 2^{2j_i}$.
The complementary vector wavelet is chosen as before with renormalization:

$$\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}} \mapsto \frac{1}{|\omega|}\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}$$

For this choice of functions, the matrix expressing the standard coefficients $d_{i\,\mathbf{j},\mathbf{k}}$ in terms of the divergence-free coefficients $d_{\mathrm{div}\,i,\mathbf{j},\mathbf{k}}$ in the system (1.10), is now given by:

$$M = \begin{bmatrix} 1-\frac{\omega_1^2}{|\omega|^2} & -\frac{\omega_2\omega_1}{|\omega|^2} & \dots & \dots & -\frac{\omega_n\omega_1}{|\omega|^2} & \frac{\omega_1}{|\omega|} \\ -\frac{\omega_1\omega_2}{|\omega|^2} & 1-\frac{\omega_2^2}{|\omega|^2} & \ddots & \ddots & -\frac{\omega_n\omega_2}{|\omega|^2} & \frac{\omega_2}{|\omega|} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -\frac{\omega_1\omega_n}{|\omega|^2} & -\frac{\omega_2\omega_n}{|\omega|^2} & \dots & -\frac{\omega_{n-1}\omega_n}{|\omega|^2} & 1-\frac{\omega_n^2}{|\omega|^2} & \frac{\omega_n}{|\omega|} \\ \frac{\omega_1}{|\omega|} & \frac{\omega_2}{|\omega|} & \dots & \frac{\omega_{n-1}}{|\omega|} & \frac{\omega_n}{|\omega|} & 0 \end{bmatrix} \tag{1.13}$$

and this is a symetric orthogonal matrix ($M^{-1} = {}^t M = M$).

Note that these new divergence-free wavelets are linear combinations of (1.7). Then, the projections onto the divergence-free spaces generated by divergence-free wavelets of same index $(\mathbf{j},\mathbf{k})$ are the same in both cases. The interest of this new formulation for the basis functions lies in the easy inversion of the matrix M (1.13). It allows to deduce the divergence-free and complementary wavelet coefficients from the standard ones without solving a system.

1.3 Curl-free wavelets

The construction of curl-free wavelets **in two dimensions** is given by considering the gradient of the tensor-product wavelets of the MRA ($V^1_{j_1}\otimes V^1_{j_2}$):

$$\Psi^{\mathrm{curl}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \frac{1}{4}\nabla\left(\psi_1(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\right) = \left|\begin{array}{l} 2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2) \\ 2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \end{array}\right.$$

These vector wavelets form a basis of the space $\mathbf{H}_{\mathrm{curl}\,0}(\mathbb{R}^2)$. We complete it to a basis of $(L^2(\mathbb{R}^2))^2$ by adding the complementary functions:

$$\Psi^{\mathcal{N}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \left| \begin{array}{l} 2^{j_2}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2) \\ -2^{j_1}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \end{array} \right.$$

The decomposition of any vector field into this wavelet basis can be obtained from the standard decomposition into the canonical wavelets:

$$\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\left(d_{1,\mathbf{j},\mathbf{k}}\ \Psi^{\#}_{1,\mathbf{j},\mathbf{k}} + d_{2,\mathbf{j},\mathbf{k}}\ \Psi^{\#}_{2,\mathbf{j},\mathbf{k}}\right)$$

with

$$\Psi^{\#}_{1,\mathbf{j},\mathbf{k}}(x_1,x_2) = \left| \begin{array}{l} \psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2) \\ 0 \end{array} \right. \quad \Psi^{\#}_{2,\mathbf{j},\mathbf{k}}(x_1,x_2) = \left| \begin{array}{l} 0 \\ \psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \end{array} \right.$$

The new decomposition:

$$\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\left(d_{\mathrm{curl}\,\mathbf{j},\mathbf{k}}\ \Psi^{\mathrm{curl}}_{\mathbf{j},\mathbf{k}} + d_{\mathcal{N}\,\mathbf{j},\mathbf{k}}\Psi^{\mathcal{N}}_{\mathbf{j},\mathbf{k}}\right) \tag{1.14}$$

is thus given by:

$$\left[\begin{array}{c} d_{\mathcal{N}\,\mathbf{j},\mathbf{k}} \\ d_{\mathrm{curl}\,\mathbf{j},\mathbf{k}} \end{array}\right] = \left[\begin{array}{cc} \frac{2^{j_2}}{2^{2j_1}+2^{2j_2}} & -\frac{2^{j_1}}{2^{2j_1}+2^{2j_2}} \\ \frac{2^{j_1}}{2^{2j_1}+2^{2j_2}} & \frac{2^{j_2}}{2^{2j_1}+2^{2j_2}} \end{array}\right]\left[\begin{array}{c} d_{1,\mathbf{j},\mathbf{k}} \\ d_{2,\mathbf{j},\mathbf{k}} \end{array}\right] \tag{1.15}$$

Remark 1.2 *One can notice the similarity between the divergence-free and curl-free transforms, emphasized by the equality of matrices (1.3) and (1.15).*

In the same way, **three-dimensional** curl-free wavelets are constructed by taking the gradient of tensor-product wavelets of the MRA $(V^1_{j_1}\otimes V^1_{j_2}\otimes V^1_{j_3})$:

$$\begin{aligned} \Psi^{\mathrm{curl}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) &= \frac{1}{4}\nabla\left(\psi_1(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3)\right) \\ &= \left| \begin{array}{l} 2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \\ 2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \\ 2^{j_3}\psi_1(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \end{array} \right. \end{aligned}$$

These wavelets generate a basis of the space $\mathbf{H}_{\mathrm{curl}\,0}(\mathbb{R}^3)$ which will be completed into a basis of $(L^2(\mathbb{R}^3))^3$ with the following complementary functions (linearly dependent since $\sum_{i=1}^{3} 2^{j_{i-1[3]}}\Psi^{\mathcal{N}\,i}_{\mathbf{j},\mathbf{k}} = 0$):

$$\begin{aligned} \Psi^{\mathcal{N}\,1}_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) &= \left| \begin{array}{l} 2^{j_2}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \\ -2^{j_1}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \\ 0 \end{array} \right. \\ \Psi^{\mathcal{N}\,2}_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) &= \left| \begin{array}{l} 0 \\ 2^{j_3}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \\ -2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \end{array} \right. \\ \Psi^{\mathcal{N}\,3}_{\mathbf{j},\mathbf{k}}(x_1,x_2,x_3) &= \left| \begin{array}{l} -2^{j_3}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_1(2^{j_3}x_3-k_3) \\ 0 \\ 2^{j_1}\psi_1(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\psi_0(2^{j_3}x_3-k_3) \end{array} \right. \end{aligned} \tag{1.16}$$

The operation for obtaining the coefficients associated to $\Psi^{\mathcal{N}\,i}_{\mathbf{j},\mathbf{k}}$ and $\Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}}$ from the standard wavelet coefficients uses the same matrix (1.11) as for the divergence-free wavelet transform.

In dimension $n \geq 4$, we will call wavelets that are a gradient of a potential function curl-free. Therefore the n**-dimensional** curl-free vector wavelets are constructed by taking the gradient of the tensor-product wavelets of the MRA $(V^1_{j_1} \otimes \cdots \otimes V^1_{j_n})$:

$$\Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}}(x_1,\ldots,x_n) = \frac{1}{4}\nabla\left(\psi_1(2^{j_1}x_1-k_1)\ldots\psi_1(2^{j_n}x_n-k_n)\right)$$
$$= \left| \begin{array}{l} 2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\ldots\psi_1(2^{j_n}x_n-k_n) \\ \vdots \\ 2^{j_i}\psi_1(2^{j_1}x_1-k_1)\ldots\psi_0(2^{j_i}x_i-k_i)\ldots\psi_1(2^{j_n}x_n-k_n) \\ \vdots \\ 2^{j_n}\psi_1(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2)\ldots\psi_0(2^{j_n}x_n-k_n) \end{array} \right. \quad (1.17)$$

In this case, the complementary wavelets (which will correspond to imperfect divergence-free wavelets) are defined like the anisotropic divergence-free wavelets, simply by exchanging the 0's and the 1's in the wavelet indices .

2 Orthogonal wavelet Helmholtz decomposition: convergence of an iterative algorithm

The objective now is to compute the Helmholtz decomposition of any vector field $\mathbf{u}$, using divergence-free and curl-free wavelets. More precisely, let $\mathbb{P}$ be the Leray projector and $\mathbb{Q}$ the orthogonal projector onto the curl-free vector functions; we want to rewrite equation (0.1) as:

$$\mathbf{u} = \mathbb{P}\,\mathbf{u} + \mathbb{Q}\,\mathbf{u} \quad , \qquad \mathbb{P}\,\mathbf{u} = \mathbf{u}_{\text{div}} \quad , \quad \mathbb{Q}\,\mathbf{u} = \mathbf{u}_{\text{curl}} \qquad (2.1)$$

such that $\mathbf{u}_{\text{div}}$ and $\mathbf{u}_{\text{curl}}$ will be expanded into the divergence-free and curl-free wavelet bases:

$$\mathbf{u}_{\text{div}} = \mathbb{P}\,\mathbf{u} = \sum_{\mathbf{j},\mathbf{k}} d_{\text{div}\,\mathbf{j},\mathbf{k}}\Psi^{\text{div}}_{\mathbf{j},\mathbf{k}} \qquad \text{and} \qquad \mathbf{u}_{\text{curl}} = \mathbb{Q}\,\mathbf{u} = \sum_{\mathbf{j},\mathbf{k}} d_{\text{curl}\,\mathbf{j},\mathbf{k}}\Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}} \qquad (2.2)$$

However, the divergence-free wavelet basis as well as the curl-free wavelet basis are not orthogonal bases, therefore their associated projectors are oblique and depend on the choice of the supplementary spaces $\text{H}_{\mathfrak{N}} = Span\{\Psi_{\mathfrak{N}}\}$ and $\text{H}_{\mathcal{N}} = Span\{\Psi_{\mathcal{N}}\}$ introduced in Section 1. Below we will introduce an iterative algorithm to provide such a decomposition.

2.1 Iterative computation of divergence-free and curl-free components of any vector field

The iterative algorithm will be based on the following two non-orthogonal decompositions:

$$(L^2(\mathbb{R}^n))^n = \mathbf{H}_{\text{div}\,0} \oplus \text{H}_{\mathfrak{N}} \quad \text{and} \quad (L^2(\mathbb{R}^n))^n = \text{H}_{\mathcal{N}} \oplus \mathbf{H}_{\text{curl}\,0} \qquad (2.3)$$

and we will let, for a vector field $\mathbf{u} \in \left(L^2(\mathbb{R}^n)\right)^n$,

$$\mathbf{u} = \text{P}_{\text{div}}\,\mathbf{u} + \text{Q}_{\mathfrak{N}}\,\mathbf{u} \qquad (2.4)$$

be its decomposition into the divergence-free wavelet space plus its complement in $\mathrm{H}_{\mathfrak{N}}$, and

$$\mathbf{u} = \mathrm{P}_{\mathcal{N}}\,\mathbf{u} + \mathrm{Q}_{\text{curl}}\,\mathbf{u} \tag{2.5}$$

be the decomposition into the curl-free wavelet space and its complement $\mathrm{H}_{\mathcal{N}}$.

Iterative algorithm :

The decomposition (2.4) allows to extract the divergence-free part of the field $\mathbf{u}$, whereas the decomposition (2.5) allows to extract its curl-free part, both in an approximate way. We propose to apply them successively until the residue become sufficiently close to 0. We expect the convergence of the residue to proceed as indicated in figure 1.

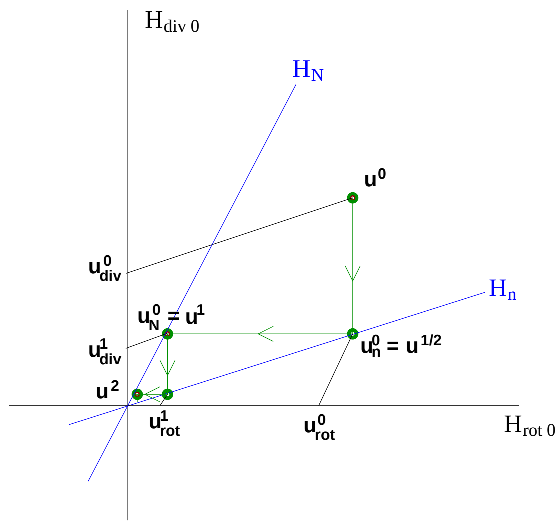

Figure 1: *Idealistic schematization of the convergence process of the algorithm with* $H_{\mathfrak{N}} = vect\{\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}\}$ *and* $H_{\mathcal{N}} = vect\{\Psi^{\mathcal{N}}_{\mathbf{j},\mathbf{k}}\}$.

Using the same notation as above, and starting from $\mathbf{u}^0 = \mathbf{u}$, the first step of the algorithm gives:

$$\begin{cases} \mathbf{u}^{1/2} = \mathbf{u}^0 - \mathrm{P}_{\text{div}}\mathbf{u}^0 = \mathrm{Q}_{\mathfrak{N}}\mathbf{u}^0 \\ \\ \mathbf{u}^1 = \mathbf{u}^{1/2} - \mathrm{Q}_{\text{curl}}\,\mathbf{u}^{1/2} = \mathrm{P}_{\mathcal{N}}\,\mathbf{u}^{1/2} \end{cases}$$

Then the next steps are defined by:

$$\begin{cases} \mathbf{u}^1 = \mathrm{P}_{\mathcal{N}}\,\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}^0 \\ \\ \mathbf{u}^{p+1} = \mathrm{P}_{\mathcal{N}}\,\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}^p \quad \forall p \geq 1 \end{cases} \tag{2.6}$$

The sequence $\mathbf{u}^p$ defined like this satisfies:

$$\mathbf{u}^p = \underbrace{\mathrm{P}_{\text{div}}\,\mathbf{u}^p}_{\mathbf{u}^p_{\text{div}}} + \underbrace{\mathrm{Q}_{\text{curl}}\,\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}^p}_{\mathbf{u}^p_{\text{curl}}} + \underbrace{\mathrm{P}_{\mathcal{N}}\,\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}^p}_{\mathbf{u}^{p+1}} \tag{2.7}$$

Asymptotically, if the sequence $(\mathbf{u}^p)_{p\in\mathbb{N}}$ converges to 0, then the decomposition (2.1) holds with:

$$\mathbf{u}_{\rm div} = \sum_{p=0}^{+\infty} \mathbf{u}^p_{\rm div} \quad \text{and} \quad \mathbf{u}_{\rm curl} = \sum_{p=0}^{+\infty} \mathbf{u}^p_{\rm curl}$$

Ideally, this algorithm will converge in the same way as the sequence $(\mathbf{u}^p)$ tends to 0 in Figure 1: in the next chapter, this convergence will be demonstrated in every dimension, for the particular case of Shannon wavelets which have infinite support but whose Fourier transforms are optimally localized. This convergence has also been tested and observed on various two-dimensional and three-dimensional fields (regular or irregular, random or arising from numerical simulations) with spline wavelets [5].
However, a brief analysis of figure 1 gives a clear idea of what it is crucial for convergence: the closeness of the spaces $\mathrm{H}_{\mathfrak{N}}$ spanned by the set $\{\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}\}$, and $\mathrm{H}_{\mathcal{N}}$ spanned by the set $\{\Psi^{\mathcal{N}}_{\mathbf{j},\mathbf{k}}\}$ to respectively $\mathrm{H}_{\mathrm{curl}\,0}$ and $\mathrm{H}_{\mathrm{div}\,0}$. Hence, the oblique wavelet projectors $\mathrm{P}_{\mathrm{div}}$ and $\mathrm{Q}_{\mathrm{curl}}$ have to be as close as possible to the original orthogonal projectors $\mathbb{P}$ and $\mathbb{Q}$.

2.2 Convergence of the algorithm

The wavelet algorithm of Section 2.1 was originally designed in dimension two and three [7]. We prove its convergence for the Shannon wavelets, and for all dimensions.

The Shannon wavelet ψ is compactly supported in Fourier space, thus it has infinite support in physical space:

$$\widehat{\psi}(\xi) = e^{-i\xi/2}\ \chi_{[-2\pi,-\pi]\cup[\pi,2\pi]}(\xi) \qquad , \qquad \psi(x) = \frac{\sin 2\pi(x-1/2)}{\pi(x-1/2)} - \frac{\sin \pi(x-1/2)}{\pi(x-1/2)}$$

where χ stands for the characteristic function, i.e. $\chi_E(x) = 1$ if $x \in E, \chi_E(x) = 0$ if $x \notin E$. The corresponding scaling function is:

$$\hat{\varphi}(\xi) = \chi_{[-\pi,\pi]}(\xi) \qquad , \qquad \varphi(x) = \frac{\sin \pi x}{\pi x}$$

Then

$$\forall j,k \in \mathbb{Z},\ \mathrm{supp}(\widehat{\psi}_{j,k}) = [-2^{j+1}\pi, -2^j\pi] \cup [2^j\pi, 2^{j+1}\pi]$$

Theorem 2.1 *Let* $\mathbf{u}$ *in* $\left(L^2(\mathbb{R}^n)\right)^n$*, and let the sequence* $(\mathbf{u}^p)_{p\geq 0}$ *be defined by (2.6):*

$$\mathbf{u}^0 = \mathbf{u} \qquad \text{and} \qquad \mathbf{u}^{p+1} = \mathrm{P}_{\mathcal{N}}\mathrm{Q}_{\mathfrak{N}}\ \mathbf{u}^p, \quad p \geq 0 \tag{2.8}$$

where $\mathrm{Q}_{\mathfrak{N}}$ *and* $\mathrm{P}_{\mathcal{N}}$ *are the complementary projectors associated respectively to divergence-free wavelets (2.4) and curl-free wavelets (2.5). We assume that the wavelet* ψ_1 *used for constructing the divergence-free and curl-free wavelets of Section 1.2 is the Shannon wavelet.*

Then the sequence $(\mathbf{u}^p)$ *satisfies, in* L^2 *norm:*

$$\|\mathbf{u}^p\| \leq \left(\frac{9}{16}\right)^p \|\mathbf{u}\|$$

and converges to zero in L^2*.*

Moreover, the Helmholtz decomposition (0.1) of $\mathbf{u}$ *is given by:*

$$\mathbf{u}_{\text{div}} = \sum_{p\in\mathbb{N}} \mathrm{P}_{\text{div}}\,\mathbf{u}^p, \qquad \mathbf{u}_{\text{curl}} = \sum_{p\in\mathbb{N}} \mathrm{Q}_{\text{curl}}\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}^p$$

Proof: Let ψ_1 and ψ_0 be two wavelets linked by derivation like in Proposition 1.1 of Section 1. Then for $j,k\in\mathbb{Z}$:

$$\psi_1\widehat{(2^j\cdot-k)} = 4\,\frac{2^j}{i\xi}\psi_0\widehat{(2^j\cdot-k)}, \qquad \text{which gives} \qquad \widehat{\psi_{1,j,k}} = \frac{4\omega}{i\xi}\widehat{\psi_{0,j,k}}$$

with $\omega = 2^j$.

For each $\mathbf{j}\in\mathbb{Z}^n$, we consider the level $\mathbf{j}$ of the wavelet decomposition of a vector field $\mathbf{u}$:

$$\mathbf{u_j} = \left|\begin{array}{l} u_{\mathbf{j}\,1} = \sum_{\mathbf{k}\in\mathbb{Z}^n} d_{1,\mathbf{j},\mathbf{k}}\ \psi_{1,j_1,k_1}(x_1)\psi_{0,j_2,k_2}(x_2)\dots\psi_{0,j_n,k_n}(x_n) \\ \quad\vdots \\ u_{\mathbf{j}\,i} = \sum_{\mathbf{k}\in\mathbb{Z}^n} d_{i,\mathbf{j},\mathbf{k}}\ \psi_{0,j_1,k_1}(x_1)\dots\psi_{1,j_i,k_i}(x_i)\dots\psi_{0,j_n,k_n}(x_n) \\ \quad\vdots \\ u_{\mathbf{j}\,n} = \sum_{\mathbf{k}\in\mathbb{Z}^n} d_{1,\mathbf{j},\mathbf{k}}\ \psi_{0,j_1,k_1}(x_1)\dots\psi_{0,j_{n-1},k_{n-1}}(x_{n-1})\psi_{1,j_n,k_n}(x_n) \end{array}\right.$$

Applying the Fourier transform yields for $1\le i\le n$, with $\omega_i = 2^{j_i}$:

$$\widehat{u_{\mathbf{j}\,i}} = \sum_{\mathbf{k}\in\mathbb{Z}^n} \frac{4\omega_i}{i\xi_i}\, d_{i,\mathbf{j},\mathbf{k}}\ \widehat{\psi_{0,j_1,k_1}}(\xi_1)\dots\widehat{\psi_{0,j_i,k_i}}(\xi_i)\dots\widehat{\psi_{0,j_n,k_n}}(\xi_n)$$

Then we obtain the complementary part (the residue after the oblique projection on the divergence-free wavelet space) by applying the orthogonal matrix (1.13) to the wavelet coefficients, and considering the last component:

$$\begin{aligned}\widehat{\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u_j}} &= \sum_{\mathbf{k}\in\mathbb{Z}^n}\left(\sum_{i=1}^n \frac{\omega_i}{|\omega|}d_{i,\mathbf{j},\mathbf{k}}\right)\frac{1}{|\omega|}\widehat{\Psi^{\mathfrak{N}}_{\mathbf{j},\mathbf{k}}} \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^n}\left(\sum_{i=1}^n \frac{\omega_i}{|\omega|}d_{i,\mathbf{j},\mathbf{k}}\right)\left|\begin{array}{l} \frac{\omega_1}{|\omega|}\widehat{\psi_{1,j_1,k_1}}(\xi_1)\widehat{\psi_{0,j_2,k_2}}(\xi_2)\dots\widehat{\psi_{0,j_n,k_n}}(\xi_n) \\ \quad\vdots \\ \frac{\omega_\ell}{|\omega|}\widehat{\psi_{0,j_1,k_1}}(\xi_1)\dots\widehat{\psi_{1,j_\ell,k_\ell}}(\xi_\ell)\dots\widehat{\psi_{0,j_n,k_n}}(\xi_n) \\ \quad\vdots \\ \frac{\omega_n}{|\omega|}\widehat{\psi_{0,j_1,k_1}}(\xi_1)\dots\widehat{\psi_{0,j_{n-1},k_{n-1}}}(\xi_{n-1})\widehat{\psi_{1,j_n,k_n}}(\xi_n) \end{array}\right.\end{aligned}$$

$\widehat{\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u_j}}$ may be express in terms of $\widehat{\mathbf{u}}_{\mathbf{j}}$:

$$\begin{aligned}\left(\widehat{\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u_j}}\right)_\ell &= \frac{\omega_\ell}{|\omega|^2}\left(\frac{4\omega_\ell}{i\xi_\ell}\right)\sum_{i=1}^n \omega_i \sum_{\mathbf{k}\in\mathbb{Z}^n} d_{i,\mathbf{j},\mathbf{k}}\ \widehat{\psi_{0,j_1,k_1}}(\xi_1)\dots\widehat{\psi_{0,j_n,k_n}}(\xi_n) \\ &= \frac{\omega_\ell^2}{|\omega|^2\xi_\ell}\sum_{i=1}^n \xi_i\widehat{u_{\mathbf{j}\,i}}\end{aligned} \tag{2.9}$$

and we can write: $\widehat{\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u_j}} = A_{\mathfrak{N}}\widehat{\mathbf{u_j}}$, where

$$A_{\mathfrak{N}} = \begin{bmatrix} \frac{\omega_1^2}{|\omega|^2\xi_1} \\ \vdots \\ \frac{\omega_n^2}{|\omega|^2\xi_n} \end{bmatrix} \times [\xi_1 \dots \xi_n]$$

Similarly, we can express the Fourier transform of $\mathrm{P}_{\mathcal{N}}\,\mathbf{u_j}$ as $A_{\mathcal{N}}\widehat{\mathbf{u_j}}$ with:

$$A_{\mathcal{N}} = Id - \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times \left[\frac{\omega_1^2}{|\omega|^2\xi_1} \dots \frac{\omega_n^2}{|\omega|^2\xi_n}\right] \tag{2.10}$$

Since the wavelet basis we use for the projection $\mathrm{P}_{\mathcal{N}}$ differs from the one we use for the projection $\mathrm{Q}_{\mathfrak{N}}$, the level $\mathbf{j}$ of the wavelet decomposition $\mathbf{u_j}$ corresponds to two different projections of $\mathbf{u}$ when considering either $\mathrm{P}_{\mathcal{N}}$, or $\mathrm{Q}_{\mathfrak{N}}$. As a result, usually we can't write:

$$\widehat{\mathrm{P}_{\mathcal{N}}\mathrm{Q}_{\mathfrak{N}}\mathbf{u_j}} = A_{\mathcal{N}}A_{\mathfrak{N}}\widehat{\mathbf{u_j}} \tag{2.11}$$

For simplicity, we consider the special case where the component $\mathbf{u_j}$ is the same for the two decompositions. Then equality (2.11) holds. For Shannon wavelets, this condition is satisfied.

We assume now that the function $\widehat{\psi}_1$ is a Shannon wavelet. Then, the wavelet levels $\widehat{\mathbf{u_j}}$ of the vector function $\mathbf{u}$ have almost disjoint compact supports. Hence $\mathbf{u}_{\mathbf{j}}^p$ the level $\mathbf{j}$ of the wavelet decomposition is stable under the different projections. Expression 2.11 is valid on $\widehat{\mathbf{u}}(\xi)$ instead of $\mathbf{u_j}$ under the condition that $\xi \in \prod_{i=1}^n \pm(2^{j_i}\pi, 2^{j_i+1}\pi)$.

Each iteration $\mathbf{u}^{p+1} = \mathrm{P}_{\mathcal{N}}\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}^p$ of the algorithm can be written in Fourier space:

$$\widehat{\mathbf{u}_{\mathbf{j}}^{p+1}} = A_{\mathcal{N}}A_{\mathfrak{N}}\widehat{\mathbf{u}_{\mathbf{j}}^{p}}$$

where the matrix

$$A_{\mathcal{N}}A_{\mathfrak{N}} = \left(Id - \frac{1}{|\omega|^2}\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times \left[\frac{\omega_1^2}{\xi_1} \dots \frac{\omega_n^2}{\xi_n}\right]\right) \times \frac{1}{|\omega|^2}\begin{bmatrix} \frac{\omega_1^2}{\xi_1} \\ \vdots \\ \frac{\omega_n^2}{\xi_n} \end{bmatrix} \times [\xi_1 \dots \xi_n]$$

is of rank one. It has thus only one non-zero eigenvalue, noted $\lambda(\xi)$, which can be computed by:

$$\lambda(\xi) = \mathrm{trace}(A_{\mathcal{N}}A_{\mathfrak{N}}) = 1 - \left(\sum_{i=1}^n \xi_i^2\right)\left(\sum_{i=1}^n \frac{\omega_i^4}{|\omega|^4\xi_i^2}\right)$$

Introducing $\zeta_i = \frac{\xi_i}{\omega_i}$, and $\alpha_i = \frac{\omega_i}{|\omega|}$, we obtain:

$$\lambda(\xi) = 1 - F(\zeta, \alpha) = 1 - \left(\sum_{i=1}^n \alpha_i^2\zeta_i^2\right)\left(\sum_{i=1}^n \alpha_i^2\zeta_i^{-2}\right)$$

Since we are using Shannon wavelets, $\text{supp}(\widehat{\psi}_{0,j,k}) = \text{supp}(\widehat{\psi}_{1,j,k}) = \pm[2^j\pi, 2^{j+1}\pi]$. Therefore if $\xi_i \in supp(\widehat{\psi}_{0,j,k})$ for $1 \leq i \leq n$, then $|\zeta_i| \in [\pi, 2\pi]$. We have also the constraint $\sum_{i=1}^n \alpha_i^2 = 1$, and the maximization of F is no more than a Kantorovitch inequality, which yields for a fixed ζ:

$$\max_{\alpha_i, \sum \alpha_i^2 = 1} F(\zeta, \alpha) = \frac{1}{4}\left(\frac{\min|\zeta_i|}{\max|\zeta_i|} + \frac{\max|\zeta_i|}{\min|\zeta_i|}\right)^2 \tag{2.12}$$

As $F(\zeta, \alpha) \geq 1$, we have

$$|\lambda(\xi)| \leq \max_{\zeta} \max_{\alpha_i, \sum \alpha_i^2 = 1} F(\zeta, \alpha) - 1 = \frac{1}{4}\left(\frac{1}{2} + \frac{2}{1}\right)^2 - 1 = \frac{9}{16}$$

Hence,

$$\forall \xi \in \prod_{i=1}^n \pm[2^{j_i}\pi, 2^{j_i+1}\pi], \qquad |\widehat{\mathbf{u}_{\mathbf{j}}^{p+1}}(\xi)| \leq \frac{9}{16}|\widehat{\mathbf{u}_{\mathbf{j}}^{p}}(\xi)|$$

As for Shannon wavelets, $\|\mathbf{u}\|_{L^2}^2 = \sum_{\mathbf{j} \in \mathbb{Z}^n} \|\mathbf{u_j}\|_{L^2}^2$, by adding the different scale decomposition, we obtain:

$$\|\mathbf{u}^{p+1}\|_{L^2} \leq \frac{9}{16} \|\mathbf{u}^p\|_{L^2}$$

which leads to the result.

Finally the divergence-free components and the curl-free components arising from (2.7) are added to form the divergence-free part and the curl-free part of $\mathbf{u}$ in the wavelet domain:

$$\mathbf{u}_{\text{div}} = \sum_{p \in \mathbb{N}} \mathrm{P}_{\text{div}}\, \mathbf{u}^p, \qquad \mathbf{u}_{\text{curl}} = \sum_{p \in \mathbb{N}} \mathrm{Q}_{\text{curl}} \mathrm{Q}_{\mathfrak{N}}\, \mathbf{u}^p$$

Remark 2.1 *In the algorithm (2.6), if we replace* $\mathbf{u}^{p+1} = \mathbf{u}^p - \mathbf{u}_{\text{div}}^p - \mathbf{u}_{\text{curl}}^p$ *by* $\mathbf{u}^{p+1} = \mathbf{u}^p - \mathbf{u}_{\text{div}}^p - b\mathbf{u}_{\text{curl}}^p$ *with some* $b > 0$*, i.e. if we replace* $A_{\mathcal{N}}$ *in (2.10) by:*

$$A_{\mathcal{N}} = Id - b \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times \left[\frac{\omega_1^2}{|\omega|^2 \xi_1} \cdots \frac{\omega_n^2}{|\omega|^2 \xi_n}\right]$$

then the eigenvalue $\lambda(\xi)$ *is given by*

$$\lambda(\xi) = 1 - b\ F(\zeta, \alpha_i)$$

and verifies, using Shannon wavelets:

$$1 - \frac{25}{16}\, b \leq \lambda(\xi) \leq 1 - b$$

The optimal choice for b *is thus* $b = \frac{32}{41}$*, and then* $|\lambda(\xi)| \leq \frac{9}{41} = 1 - b = \frac{25}{16}b - 1$.

On the other hand, in numerical experiments of section 4, we will use spline wavelets, and for these wavelets, the optimal b *is* 1.24*. For* $b = 1.24$*, the convergence rate is equal to* 0.41 *instead of* 0.56 *for* $b = 1$*. This means that spline wavelets are not as optimal as Shannon wavelets and damage the term* $F(\zeta, \alpha)$ *in two ways:* $F(\zeta, \alpha)$ *behaves, first, as if it was multiplicated by* 0.48*, and secondly, as if* $\max|\zeta_i| = 2.7 \min|\zeta_i|$ *(the spline wavelets don't verify this compact support condition, but the algorithm behaves as if they did so).*

3 Generalization of the isotropic divergence-free wavelets construction

An important drawback of anisotropic wavelets, is that they are not well suited for adaptive schemes. Indeed, if one refines the grid somewhere, then the anisotropic wavelets lengthen across the whole domain instead of remaining localized (the refinement grid has to be cartesian).

At this moment, it is preferable not to develop the total spectra of scale parameters $\mathbf{j} \in \mathbb{Z}^n$, but to restrict ourselves to anisotropic wavelets with a parameter $\mathbf{j}$ verifying $\max(\mathbf{j}) - \min(\mathbf{j}) \le m$ with m equals to one or two, without affecting the convergence of the wavelet Helmholtz algorithm to much. For this purpose, we need the scale function φ which, associated with ψ, forms an MRA.

By doing this, we show that a slight modification of the isotropic wavelets (case $m = 0$) originally designed by P.G. Lemarié [12] and used by K. Urban [20] allows one to discard the arbitrary choice of the index i for which $\varepsilon_i = 1$ as a pivotal element to form the divergence-free wavelets. The price to pay for this modification is that the divergence-free wavelets are redundant – and even zero in some cases. There are $(2^n-1)n$ of them instead of $(2^n-1)(n-1)$ to form a basis in dimension n. However, we profit from this redundancy to add a linear relation on the wavelet coefficients. This construction permits to give a clearer understanding of the divergence-free wavelet transform in the isotropic case.

3.1 Mixed isotropic/anisotropic divergence-free and curl-free wavelets

In the following, we will define generalized quasi-isotropic divergence-free wavelets by considering two kinds of divergence-free wavelets, with limitations on the scale parameter $\mathbf{j}$. Let $m \ge 0$ be given, we will only consider scale indices $\mathbf{j} = (j_1, j_2, \dots, j_n)$ such that $\max(j_1, j_2, \dots, j_n) - \min(j_1, j_2, \dots, j_n) \le m$. Then the family will be formed with:

- usual anisotropic divergence-free wavelets $\Psi^{\mathrm{div}\, i}_{\mathbf{j},\mathbf{k}}$ with $1 \le i \le n$ as in Section 1.2,
- modified isotropic divergence-free vector wavelets $\Psi^{\mathrm{div}\, i}_{\varepsilon\, \mathbf{j},\mathbf{k}}$, with $\varepsilon \in \{0,1\}^n \backslash \{(0,\dots,0)\}$, $1 \le i \le n$ and with components of the type $\eta^{(\varepsilon_1)}_{j_1,k_1} \dots \eta^{(\varepsilon_n)}_{j_n,k_n}$, with $\eta^{(1)} = \psi$, $\eta^{(0)} = \varphi$, and $j_\ell = \max(j_1, j_2, \dots, j_n) - m$ if $\varepsilon_\ell = 0$.

To construct these last functions, we will introduce linear combinations of componentwise wavelets that are a mix between isotropic and anisotropic wavelets:

$$\Psi^i_{\varepsilon\, \mathbf{j},\mathbf{k}}(x_1, \dots, x_n) = \left| \begin{array}{l} 0 \\ \vdots \\ \eta_0^{(\varepsilon_1)}(2^{j_1} x_1 - k_1) \dots \eta_1^{(\varepsilon_i)}(2^{j_i} x_i - k_i) \dots \eta_0^{(\varepsilon_n)}(2^{j_n} x_n - k_n) \\ \vdots \\ 0 \end{array} \right.$$

where, taking into account the derivation relations of proposition 1.1, for $1 \le i \le n$,

$$\eta_\ell^{(\varepsilon_i)} = \left\{ \begin{array}{ll} \psi_\ell & \text{if} \quad \varepsilon_i = 1 \\ \varphi_\ell & \text{if} \quad \varepsilon_i = 0 \end{array} \right. , \quad \ell = 0, 1, \qquad \left(\eta_1^{(\varepsilon_i)}\right)'(x) = 4^{\varepsilon_i} \left(\eta_0^{(\varepsilon_i)}(x) - (1 - \varepsilon_i)\eta_0^{(\varepsilon_i)}(x-1)\right)$$

The set $\{\Psi^i_{\varepsilon\,\mathbf{j},\mathbf{k}} : 1 \le i \le n, \mathbf{j},\mathbf{k} \in \mathbb{Z}^n, \varepsilon \in \{0,1\}^n \setminus \{(0,\dots,0)\}$, $\max(j_1,j_2,\dots,j_n) - \min(j_1,j_2,\dots,j_n) \le m$, $\varepsilon_\ell = 0 \implies j_\ell = \max(j_1,j_2,\dots,j_n) - m\}$ forms a Riesz basis of $\big(L^2(\mathbb{R}^n)\big)^n$.
The n-dimensional quasi-isotropic divergence-free wavelets are defined by: for $1 \le i \le n$,

$$\Psi^{\mathrm{div}\,i}_{\varepsilon\,\mathbf{j},\mathbf{k}}(x_1,\dots,x_n)$$

$$= \left| \begin{array}{l} -2^{j_i+j_1}\varepsilon_1\eta_1^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\Big(\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i)-(1-\varepsilon_i)\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i-1)\Big)\dots\eta_0^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \\ \vdots \\ 4^{1-\varepsilon_i}\left(\sum_{\ell\ne i,\,\varepsilon_\ell=1}2^{2j_\ell}\right)\eta_0^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\eta_1^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\eta_0^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \\ \vdots \\ -2^{j_i+j_n}\varepsilon_n\eta_0^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\Big(\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i)-(1-\varepsilon_i)\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i-1)\Big)\dots\eta_1^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \end{array} \right.$$

and the complementary wavelet by:

$$\Psi^{\mathfrak{N}}_{\varepsilon\,\mathbf{j},\mathbf{k}}(x_1,\dots,x_n) = \left| \begin{array}{l} 2^{j_1}\varepsilon_1\eta_1^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\eta_0^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \\ \vdots \\ 2^{j_n}\varepsilon_n\eta_0^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\eta_1^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \end{array} \right.$$

Remark that the case $m = 0$ corresponds to isotropic divergence-free wavelets as proposed by P.G. Lemarié and K. Urban [12, 20] where we have made linear combinations among the possible cases. Note also that $\Psi^{\mathfrak{N}}_{\varepsilon\,\mathbf{j},\mathbf{k}}$ is no more orthogonal to $\Psi^{\mathrm{div}\,i}_{\varepsilon\,\mathbf{j},\mathbf{k}}$, except if $\varepsilon_i = 1$.

If we denote by $(d_{i,\varepsilon\,\mathbf{j},\mathbf{k}})$ the wavelet coefficients of the standard wavelet decomposition verifying the limitations on the scale parameter $\mathbf{j}$ as indicated previously, then we obtain the divergence-free wavelet and complementary wavelet coefficients by solving the following system for fixed $\mathbf{j},\mathbf{k},\varepsilon$:

$$M_{\mathrm{div}} \begin{bmatrix} d_{\mathrm{div}\,1,\varepsilon\,\mathbf{j},\mathbf{k}} \\ \vdots \\ d_{\mathrm{div}\,n,\varepsilon\,\mathbf{j},\mathbf{k}} \\ d_{\mathfrak{N}\varepsilon\,\mathbf{j},\mathbf{k}} \end{bmatrix} + \sum_{i,\,\varepsilon_i=0} M^{(i)}_{\mathrm{div}} \begin{bmatrix} d_{\mathrm{div}\,1,\varepsilon\,\mathbf{j},\mathbf{k}-e_i} \\ \vdots \\ d_{\mathrm{div}\,n,\varepsilon\,\mathbf{j},\mathbf{k}-e_i} \\ d_{\mathfrak{N}\varepsilon\,\mathbf{j},\mathbf{k}-e_i} \end{bmatrix} = \begin{bmatrix} d_{1,\varepsilon\,\mathbf{j},\mathbf{k}} \\ \vdots \\ d_{n,\varepsilon\,\mathbf{j},\mathbf{k}} \\ 0 \end{bmatrix} \tag{3.1}$$

with $(e_i)_{1\le i\le n}$ the canonical basis of $\mathbb{R}^n$. With the notations $\omega_i = 2^{j_i}$ and $|\varepsilon\omega|^2 = \sum_{i,\,\varepsilon_i=1}\omega_i^2$, after normalization of the wavelets:

$$\Psi^{\mathrm{div}\,i}_{\varepsilon\,\mathbf{j},\mathbf{k}} \mapsto \frac{1}{|\varepsilon\omega|^2}\Psi^{\mathrm{div}\,i}_{\varepsilon\,\mathbf{j},\mathbf{k}}, \qquad \Psi^{\mathfrak{N}}_{\varepsilon\,\mathbf{j},\mathbf{k}} \mapsto \frac{1}{|\varepsilon\omega|}\Psi^{\mathfrak{N}}_{\varepsilon\,\mathbf{j},\mathbf{k}},$$

the expressions of M_{div} and $M_{\text{div}}^{(i)}$ are chosen as follows (the last line indicating the desired linear relationship between the divergence-free wavelet coefficients is arbitrary):

$$M_{\text{div}} = \begin{bmatrix} 4^{1-\varepsilon_1}(1-\varepsilon_1\frac{\omega_1^2}{|\varepsilon\omega|^2}) & -\varepsilon_1\frac{\omega_2\omega_1}{|\varepsilon\omega|^2} & \dots & -\varepsilon_1\frac{\omega_n\omega_1}{|\varepsilon\omega|^2} & \varepsilon_1\frac{\omega_1}{|\varepsilon\omega|} \\ -\varepsilon_2\frac{\omega_1\omega_2}{|\varepsilon\omega|^2} & 4^{1-\varepsilon_2}(1-\varepsilon_2\frac{\omega_2^2}{|\varepsilon\omega|^2}) & \ddots & -\varepsilon_2\frac{\omega_n\omega_2}{|\varepsilon\omega|^2} & \varepsilon_2\frac{\omega_2}{|\varepsilon\omega|} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon_n\frac{\omega_1\omega_n}{|\varepsilon\omega|^2} & -\varepsilon_n\frac{\omega_2\omega_n}{|\varepsilon\omega|^2} & \dots & 4^{1-\varepsilon_n}(1-\varepsilon_n\frac{\omega_n^2}{|\varepsilon\omega|^2}) & \varepsilon_n\frac{\omega_n}{|\varepsilon\omega|} \\ \varepsilon_1\frac{\omega_1}{|\varepsilon\omega|} & \varepsilon_2\frac{\omega_2}{|\varepsilon\omega|} & \dots & \varepsilon_n\frac{\omega_n}{|\varepsilon\omega|} & 0 \end{bmatrix} \quad (3.2)$$

and

$$M_{\text{div}}^{(i)} = \begin{bmatrix} 0 & \dots 0 & \varepsilon_1\frac{\omega_i\omega_1}{|\varepsilon\omega|^2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots 0 & \varepsilon_n\frac{\omega_i\omega_n}{|\varepsilon\omega|^2} & 0 & \dots & 0 \\ 0 & \dots 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (3.3)$$

where only column number i of $M_{\text{div}}^{(i)}$ is different from zero.

One can notice that for $\varepsilon_i = 0$,

$$d_{\text{div}\, i,\varepsilon\,\mathbf{j},\mathbf{k}} = \frac{1}{4} d_{i,\varepsilon\,\mathbf{j},\mathbf{k}}$$

Then system (3.1) is equivalent to:

$$M_{\text{div}} \begin{bmatrix} \varepsilon_1 d_{\text{div}\,1,\varepsilon\,\mathbf{j},\mathbf{k}} \\ \vdots \\ \varepsilon_n d_{\text{div}\,n,\varepsilon\,\mathbf{j},\mathbf{k}} \\ d_{\mathfrak{N}\varepsilon\,\mathbf{j},\mathbf{k}} \end{bmatrix} = \begin{bmatrix} d_{1,\varepsilon\,\mathbf{j},\mathbf{k}} \\ \vdots \\ d_{n,\varepsilon\,\mathbf{j},\mathbf{k}} \\ 0 \end{bmatrix} - \frac{1}{4} M_{\text{div}} \begin{bmatrix} (1-\varepsilon_1) d_{1,\varepsilon\,\mathbf{j},\mathbf{k}} \\ \vdots \\ (1-\varepsilon_n) d_{n,\varepsilon\,\mathbf{j},\mathbf{k}} \\ 0 \end{bmatrix} - \frac{1}{4} \sum_{i,\,\varepsilon_i=0} M_{\text{div}}^{(i)} \begin{bmatrix} d_{1,\varepsilon\,\mathbf{j},\mathbf{k}-e_i} \\ \vdots \\ d_{n,\varepsilon\,\mathbf{j},\mathbf{k}-e_i} \\ 0 \end{bmatrix}$$

To solve system (3.1), we multiply the above system by the matrix:

$$\begin{bmatrix} (1-\frac{\varepsilon_1\omega_1^2}{|\varepsilon\omega|^2}) & -\varepsilon_2\varepsilon_1\frac{\omega_2\omega_1}{|\varepsilon\omega|^2} & \dots & -\varepsilon_n\varepsilon_1\frac{\omega_n\omega_1}{|\varepsilon\omega|^2} & \varepsilon_1\frac{\omega_1}{|\varepsilon\omega|} \\ -\varepsilon_1\varepsilon_2\frac{\omega_1\omega_2}{|\varepsilon\omega|^2} & (1-\frac{\varepsilon_2\omega_2^2}{|\varepsilon\omega|^2}) & \ddots & -\varepsilon_n\varepsilon_2\frac{\omega_n\omega_2}{|\varepsilon\omega|^2} & \varepsilon_2\frac{\omega_2}{|\varepsilon\omega|} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon_1\varepsilon_n\frac{\omega_1\omega_n}{|\varepsilon\omega|^2} & -\varepsilon_2\varepsilon_n\frac{\omega_2\omega_n}{|\varepsilon\omega|^2} & \dots & (1-\frac{\varepsilon_n\omega_n^2}{|\varepsilon\omega|^2}) & \varepsilon_n\frac{\omega_n}{|\varepsilon\omega|} \\ \varepsilon_1\frac{\omega_1}{|\varepsilon\omega|} & \varepsilon_2\frac{\omega_2}{|\varepsilon\omega|} & \dots & \varepsilon_n\frac{\omega_n}{|\varepsilon\omega|} & 0 \end{bmatrix} \quad (3.4)$$

and we find that for $\varepsilon_i = 1$,

$$d_{\text{div}\, i,\varepsilon\,\mathbf{j},\mathbf{k}} = d_{i,\varepsilon\,\mathbf{j},\mathbf{k}} - \frac{\omega_i}{|\varepsilon\omega|^2} \sum_{\ell=1}^{n} \varepsilon_\ell \omega_\ell d_{\ell,\varepsilon\,\mathbf{j},\mathbf{k}}$$

and

$$d_{\mathfrak{N}\varepsilon\,\mathbf{j},\mathbf{k}} = \sum_{i,\,\varepsilon_i=1} \frac{\omega_i}{|\varepsilon\omega|} d_{i,\varepsilon\,\mathbf{j},\mathbf{k}} + \sum_{i,\,\varepsilon_i=0} \frac{\omega_i}{4|\varepsilon\omega|} (d_{i,\varepsilon\,\mathbf{j},\mathbf{k}} - d_{i,\varepsilon\,\mathbf{j},\mathbf{k}-e_i})$$

We proceed similarly as in Section 2.2 to establish the expression (2.9), then the Fourier transform of $\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}_{\varepsilon\,\mathbf{j}}$ satisfies:

$$\widehat{\mathrm{Q}_{\mathfrak{N}}\,\mathbf{u}_{\varepsilon\,\mathbf{j}}} = \frac{1}{|\varepsilon\omega|^2}\begin{bmatrix} \varepsilon_1\frac{\omega_1^2}{\xi_1} \\ \vdots \\ \varepsilon_n\frac{\omega_n^2}{\xi_n}\end{bmatrix} \times [\xi_1 \dots \xi_n]\ \widehat{\mathbf{u}_{\varepsilon\,\mathbf{j}}}$$

We proceed similarly for the curl-free wavelet transform.
The n-dimensional quasi-isotropic curl-free wavelets are defined by:

$$\Psi^{\mathrm{curl}}_{\varepsilon\,\mathbf{j},\mathbf{k}}(x_1,\dots,x_n)$$
$$= \left| \begin{array}{l} 2^{j_1}\cdot 4^{\varepsilon_1-1}\left(\eta_0^{(\varepsilon_1)}(2^{j_1}x_1-k_1)-(1-\varepsilon_1)\eta_0^{(\varepsilon_1)}(2^{j_1}x_1-k_1-1)\right)\dots\eta_1^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\eta_1^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \\ \vdots \\ 2^{j_i}\cdot 4^{\varepsilon_i-1}\eta_1^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\left(\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i)-(1-\varepsilon_i)\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i-1)\right)\dots\eta_1^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \\ \vdots \\ 2^{j_n}\cdot 4^{\varepsilon_n-1}\eta_1^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\eta_1^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\left(\eta_0^{(\varepsilon_n)}(2^{j_n}x_n-k_n)-(1-\varepsilon_n)\eta_0^{(\varepsilon_n)}(2^{j_n}x_n-k_n-1)\right) \end{array}\right.$$

and the complementary wavelets are defined for $1\le i\le n$ by:

$$\Psi^{\mathcal{N}\,i}_{\varepsilon\,\mathbf{j},\mathbf{k}}(x_1,\dots,x_n)$$
$$= \left| \begin{array}{l} -2^{j_i+j_1}\varepsilon_i\varepsilon_1\eta_0^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\eta_1^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\eta_1^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \\ \vdots \\ \left(\sum_{\ell\neq i,\,\varepsilon_\ell=1}2^{2j_\ell}\right)\eta_1^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\eta_0^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\eta_1^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \\ \vdots \\ -2^{j_i+j_n}\varepsilon_i\varepsilon_n\eta_1^{(\varepsilon_1)}(2^{j_1}x_1-k_1)\dots\eta_1^{(\varepsilon_i)}(2^{j_i}x_i-k_i)\dots\eta_0^{(\varepsilon_n)}(2^{j_n}x_n-k_n) \end{array}\right.$$

After renormalization of the wavelets ($\Psi^{\mathrm{curl}}_{\varepsilon\,\mathbf{j},\mathbf{k}}$ is divided by $|\varepsilon\omega|$ and $\Psi^{\mathcal{N}\,i}_{\varepsilon\,\mathbf{j},\mathbf{k}}$ by $|\varepsilon\omega|^2$), the wavelet coefficients verify, for fixed $\mathbf{j},\mathbf{k},\varepsilon$:

$$M_{\mathrm{curl}}\begin{bmatrix} d_{\mathcal{N}\,1,\varepsilon\,\mathbf{j},\mathbf{k}} \\ \vdots \\ d_{\mathcal{N}\,n,\varepsilon\,\mathbf{j},\mathbf{k}} \\ d_{\mathrm{curl}\,\varepsilon\,\mathbf{j},\mathbf{k}}\end{bmatrix} + \sum_{i,\,\varepsilon_i=0} M^{(i)}_{\mathrm{curl}}\begin{bmatrix} d_{\mathcal{N}\,1,\varepsilon\,\mathbf{j},\mathbf{k}-e_i} \\ \vdots \\ d_{\mathcal{N}\,n,\varepsilon\,\mathbf{j},\mathbf{k}-e_i} \\ d_{\mathrm{curl}\,\varepsilon\,\mathbf{j},\mathbf{k}-e_i}\end{bmatrix} = \begin{bmatrix} d_{1,\varepsilon\,\mathbf{j},\mathbf{k}} \\ \vdots \\ d_{n,\varepsilon\,\mathbf{j},\mathbf{k}} \\ 0\end{bmatrix} \tag{3.5}$$

where the matrices M_{curl} and $M^{(i)}_{\mathrm{curl}}$ for $1\le i\le n$ are given by:

$$M_{\mathrm{curl}} = \begin{bmatrix} (1-\varepsilon_1\frac{\omega_1^2}{|\varepsilon\omega|^2}) & -\varepsilon_2\varepsilon_1\frac{\omega_2\omega_1}{|\varepsilon\omega|^2} & \dots & -\varepsilon_n\varepsilon_1\frac{\omega_n\omega_1}{|\varepsilon\omega|^2} & 4^{\varepsilon_1-1}\frac{\omega_1}{|\varepsilon\omega|} \\ -\varepsilon_1\varepsilon_2\frac{\omega_1\omega_2}{|\varepsilon\omega|^2} & (1-\varepsilon_2\frac{\omega_2^2}{|\varepsilon\omega|^2}) & \ddots & -\varepsilon_n\varepsilon_2\frac{\omega_n\omega_2}{|\varepsilon\omega|^2} & 4^{\varepsilon_2-1}\frac{\omega_2}{|\varepsilon\omega|} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon_1\varepsilon_n\frac{\omega_1\omega_n}{|\varepsilon\omega|^2} & -\varepsilon_2\varepsilon_n\frac{\omega_2\omega_n}{|\varepsilon\omega|^2} & \dots & (1-\varepsilon_n\frac{\omega_n^2}{|\varepsilon\omega|^2}) & 4^{\varepsilon_n-1}\frac{\omega_n}{|\varepsilon\omega|} \\ \varepsilon_1\frac{\omega_1}{|\varepsilon\omega|} & \varepsilon_2\frac{\omega_2}{|\varepsilon\omega|} & \dots & \varepsilon_n\frac{\omega_n}{|\varepsilon\omega|} & 0 \end{bmatrix} \tag{3.6}$$

and

$$M^{(i)}_{\text{curl}} = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & -\frac{\omega_i}{4|\varepsilon\omega|} \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \tag{3.7}$$

where only the line number i of $M^{(i)}_{\text{curl}}$ is non zero.
To solve the system of equations (3.5), we multiply it by matrix (3.4) and we find:

$$d_{\text{curl}\,\varepsilon\,\mathbf{j},\mathbf{k}} = \sum_{i=1}^{n} \varepsilon_i \frac{\omega_i}{|\varepsilon\omega|} d_{i,\varepsilon\,\mathbf{j},\mathbf{k}}$$

which means that the Fourier transform of $\mathrm{P}_{\mathcal{N}}\,\mathbf{u}_{\varepsilon\,\mathbf{j}}$ satisfies:

$$\widehat{\mathrm{P}_{\mathcal{N}}\,\mathbf{u}_{\varepsilon\,\mathbf{j}}} = \left(Id - \frac{1}{|\varepsilon\omega|^2} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times \left[\varepsilon_1 \frac{\omega_1^2}{\xi_1} \dots \varepsilon_n \frac{\omega_n^2}{\xi_n} \right] \right) \widehat{\mathbf{u}_{\varepsilon\,\mathbf{j}}}$$

The expressions of the wavelet coefficients $d_{\mathcal{N}\,i,\varepsilon\,\mathbf{j},\mathbf{k}}$ are, for $\varepsilon_i = 1$:

$$d_{\mathcal{N}\,i,\varepsilon\,\mathbf{j},\mathbf{k}} = d_{i,\varepsilon\,\mathbf{j},\mathbf{k}} - \frac{\omega_i}{|\varepsilon\omega|^2} \sum_{\ell=1}^{n} \varepsilon_\ell \omega_\ell d_{\ell,\varepsilon\,\mathbf{j},\mathbf{k}}$$

and for $\varepsilon_i = 0$:

$$d_{\mathcal{N}\,i,\varepsilon\,\mathbf{j},\mathbf{k}} = d_{i,\varepsilon\,\mathbf{j},\mathbf{k}} + \frac{\omega_i}{4|\varepsilon\omega|^2} \sum_{\ell=1}^{n} \varepsilon_\ell \omega_\ell d_{\ell,\varepsilon\,\mathbf{j},\mathbf{k}-e_i}$$

3.2 Convergence of the iterative Helmholtz decomposition in the mixed isotropic/anisotropic case

Theorem 3.1 *In dimension n, the Helmholtz wavelet algorithm (2.6) defined in Section 2.1 converges using Shannon wavelets, if*

$$m > \frac{1}{2\ln 2} \ln \frac{25n^2}{7}$$

in the construction of the mixed isotropic/anisotropic divergence-free and curl-free wavelets (cf Section 3.1).

Proof: Assume that we are using Shannon wavelets, each level of the wavelet decomposition (indexed by $\mathbf{j} \in \mathbb{Z}^n$ with $\max(\mathbf{j}) - \min(\mathbf{j}) \le m$ and by $\varepsilon \in \{0,1\}^n \setminus \{(0,\dots,0)\}$) evolves independently during the wavelet Helmholtz decomposition algorithm (2.6). Hence:

$$\widehat{\mathbf{u}^{p+1}_{\varepsilon\,\mathbf{j}}} = A\widehat{\mathbf{u}^{p}_{\varepsilon\,\mathbf{j}}}$$

with

$$A=\left(Id-\frac{1}{|\varepsilon\omega|^2}\begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix}\times\left[\varepsilon_1\frac{\omega_1^2}{\xi_1}\dots\varepsilon_n\frac{\omega_n^2}{\xi_n}\right]\right)\times\frac{1}{|\varepsilon\omega|^2}\begin{bmatrix}\varepsilon_1\frac{\omega_1^2}{\xi_1}\\ \vdots\\ \varepsilon_n\frac{\omega_n^2}{\xi_n}\end{bmatrix}\times[\xi_1\dots\xi_n]$$

This matrix is of rank one and has a single non-zero eigenvalue λ, equal to the trace of A:

$$\lambda=1-\left(\sum_{i=1}^{n}\frac{\xi_i^2}{|\varepsilon\omega|^4}\right)\left(\sum_{i=1}^{n}\frac{\varepsilon_i\omega_i^4}{\xi_i^2}\right)$$

i.e., with $\zeta_i=\frac{\xi_i}{\omega_i}$,

$$\lambda=1-\left(\sum_{i=1}^{n}\frac{\omega_i^2}{|\varepsilon\omega|^2}\zeta_i^2\right)\left(\sum_{i=1}^{n}\frac{\omega_i^2}{|\varepsilon\omega|^2}\varepsilon_i\zeta_i^{-2}\right)$$

which can be rewritten, if we distinguish the case $\varepsilon_i=1$ and $\varepsilon_i=0$ in the first sum:

$$\lambda=1-\left(\sum_{i=1}^{n}\frac{\varepsilon_i\omega_i^2}{|\varepsilon\omega|^2}\zeta_i^2\right)\left(\sum_{i=1}^{n}\frac{\varepsilon_i\omega_i^2}{|\varepsilon\omega|^2}\zeta_i^{-2}\right)-\left(\sum_{i=1}^{n}(1-\varepsilon_i)\frac{\omega_i^2}{|\varepsilon\omega|^2}\zeta_i^2\right)\left(\sum_{i=1}^{n}\frac{\omega_i^2}{|\varepsilon\omega|^2}\varepsilon_i\zeta_i^{-2}\right)$$

For the first term, we use the Kantorovich inequality (2.12). We denote by μ the second term:

$$\mu=\left(\sum_{i=1}^{n}(1-\varepsilon_i)\frac{\omega_i^2}{|\varepsilon\omega|^2}\zeta_i^2\right)\left(\sum_{i=1}^{n}\frac{\omega_i^2}{|\varepsilon\omega|^2}\varepsilon_i\zeta_i^{-2}\right)\geq 0,$$

then

$$-\frac{9}{16}-\mu\leq\lambda\leq-\mu \tag{3.8}$$

As for i s.t. $\varepsilon_i=0$, $\omega_i\leq 2^{-m}|\varepsilon\omega|$ and for i s.t. $\varepsilon_i=1$, $\omega_i\leq|\varepsilon\omega|$, μ is bounded by

$$\mu\leq 2^{-2m}\sum_{i=1}^{n}\zeta_i^2\ \sum_{i=1}^{n}\zeta_i^{-2}=2^{-2m}n^2\sum_{i=1}^{n}\frac{1}{n}\zeta_i^2\ \sum_{i=1}^{n}\frac{1}{n}\zeta_i^{-2}$$

Again, the Kantorovitch inequality yields

$$\mu\leq\frac{25}{16}2^{-2m}n^2$$

Since according to (3.8), a sufficient condition for the convergence is $\mu<\frac{7}{16}$, this condition will be satisfied provided that

$$m>\frac{1}{2\ln 2}\ln\frac{25n^2}{7}$$

Consequently the number m of additional wavelet transforms we have to apply after an isotropic wavelet transform, is not excessive and allows to use the wavelet Helmholtz decomposition in an adaptive scheme.

Remark 3.1 *Again, as it was noted in remark 2.1, we may introduce a parameter $b > 0$ in the algorithm (2.6): $\mathbf{u}^{p+1} = \mathbf{u}^p - \mathbf{u}^p_{\text{div}} - b\mathbf{u}^p_{\text{curl}}$. Then, according to the previous study, the following bounds holds for the eigenvalue $\lambda(\xi)$, using Shannon wavelets:*

$$1 - \frac{25}{16} b \left(1 + n^2 2^{-2m}\right) \leq \lambda(\xi) \leq 1 - b$$

The algorithm converges for b sufficiently small. In practice, with spline wavelets, whatever the value of b, the algorithm (2.6) diverges for isotropic wavelets ($m = 0$) and presents the same profile as in figure 2.

4 Numerical experiments

The wavelet Helmholtz algorithm was applied to non divergence-free periodic vector fields on the cube $[0, 1]^n$ ($n = 2, 3$), in order to observe the convergence rate of the iterative algorithm. We used spline wavelets for the experiments: the basis functions (φ_0, ψ_0) are splines of order two (i.e. piecewise polynomials of degree one), and the basis functions (φ_1, ψ_1) are splines of order three (i.e. piecewise polynomials of degree two).

The reference [7] provides technical explanations for the implementation of the method. We have already tested in [7] the convergence of the iterative algorithm, using spline wavelets of different orders, successfully applied to a large class of two and three-dimensional fields. The observed convergence rates were about 0.5 (see also figure 2).

We will now compare the convergence rates obtained with several families of mixed isotropic/anisotropic divergence-free and curl-free wavelets. We will apply the wavelet Helmholtz algorithm to the 2D vector field

$$\mathbf{u} = \left| \begin{array}{l} \sin(2\pi x)\cos(2\pi y) - 2\sin(2\pi x)\cos(2\pi y) \\ -\cos(2\pi x)\sin(2\pi y) - 2\cos(2\pi x)\sin(2\pi y) \end{array} \right.$$

discretized on 256^2 grid points, and to the 3D vector field:

$$\mathbf{u} = \left| \begin{array}{l} -\sin(2\pi x)\cos(2\pi y)\cos(2\pi z) + \sin(2\pi x)\cos(2\pi y) - \sin(2\pi x)\cos(2\pi z) \\ -\cos(2\pi x)\sin(2\pi y)\cos(2\pi z) + \sin(2\pi y)\cos(2\pi z) - \cos(2\pi x)\sin(2\pi y) \\ -\cos(2\pi x)\cos(2\pi y)\sin(2\pi z) + \cos(2\pi x)\sin(2\pi z) - \cos(2\pi y)\sin(2\pi z) \end{array} \right.$$

discretized on 32^2 grid points. Figure 2 shows the evolution of the residue $\|\mathbf{u}^p\|_{L^2}$ in terms of the number of iterations p. In figure 2 we have used four different wavelet bases:

- two-dimensional isotropic (i.e. $m = 0$) functions, and this choice leads to the divergence of the algorithm.
- two-dimensional anisotropic (i.e. $m = +\infty$) functions,
- two-dimensional quasi-isotropic (with $m = 1$) functions,
- and three-dimensional anisotropic functions.

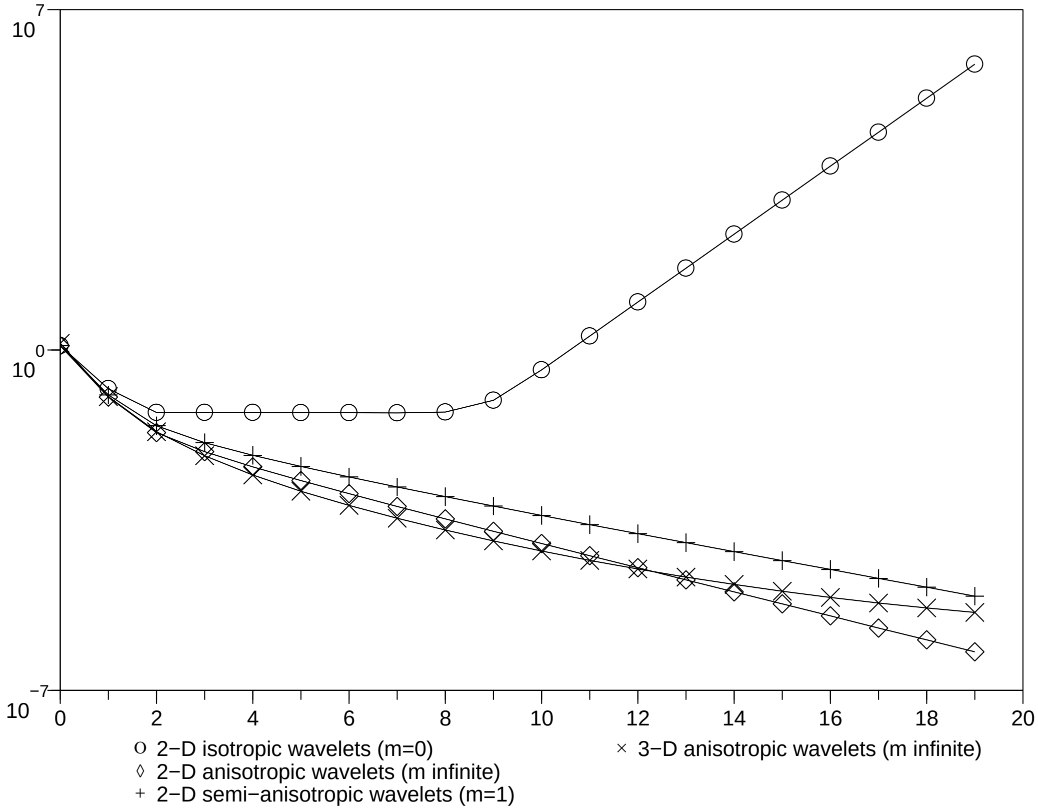

Figure 2: Convergence profiles of the Helmholtz algorithm with spline wavelets

These expriments clearly show that isotropic functions are not well suited to be used in the wavelet Helmholtz algorithm. In all other cases (anisotropic and quasi-isotropic), the algorithm converges. At the end of the execution, the accuracy depends on the spline order of the used wavelets. The change of behaviour for the three-dimensional case in the last steps is related to the spline approximation: the forcing of convergence by one half-shifting of the velocity components was used in dimension two (cf [7]) but not in dimension three.

Conclusion

In this article, we have constructed anisotropic divergence-free and curl-free wavelets in dimension n, by generalization of the constructions in 2D and 3D. To obtain small orthogonal systems for the computation of related coefficients, we have modified the previous constructions of divergence-free wavelets (and thus curl-free wavelets) by analogy with the Leray projector written in Fourier domain. These new formulations have allowed us to define an iterative algorithm for the wavelet Helmholtz decomposition of any vector field, and we have proved its convergence in the particular case of Shannon wavelets. Moreover we have proved its convergence for quasi-isotropic wavelets. We have observed in numerical experiments that the convergence rate of the method doesn't depend on the space dimension.

The interest of such wavelet Helmholtz decomposition is that it is localized in space contrarily to a decomposition computed by Fourier transform. Moreover this algorithm work in wavelet adaptive schemes, by using quasi-isotropic wavelets. This makes the method very attractive for large dimensional problems and it opens new prospects, for example for the direct simulation of turbulence using wavelet bases [5, 7]. Moreover, this decomposition may be generalized to bounded and non periodic domains, using wavelets on the interval in the construction of divergence-free and curl-free functions [20].

However, some questions remain open: the convergence of the iterative algorithm using general wavelets, or how to speed up the convergence rate. For this last point, a solution using wavelet packets has been investigated in [5].

These constructions also address the issue of numerical algorithms based on divergence-free and curl-free wavelets for solving differential problems. Generalizations of such constructions to other linear differential problems prove useful and provide original wavelet solvers (see [6]).

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