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# Heath-Jarrow-Morton-Musiela Equation of Bond Market

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# HEATH-JARROW-MORTON-MUSIELA EQUATION of BOND MARKET

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**Abstract**: The paper is devoted to a stochastic partial differential equation for the forward curve of the bond market, in the Musiela parameterization and the Heath-Jarrow-Morton framework. Special attention is paid to the existence and positivity of the solutions.

## 1 Introduction

The paper is devoted to stochastic partial differential equation

$$\mathrm{d}r(t)(\xi) = \left(\frac{\partial}{\xi} r(t)(\xi) + F(t, r(t))(\xi)\right)\mathrm{d}t + \sum_j G_j(t, r(t))(\xi)\mathrm{d}Z_j(t),\ \xi \geq 0$$

which appears in the theory of bond market, see Section 3. In the most studied case $Z_j$ are independent Brownian motions. Existence, asymptotics of solutions as well as stochastic invariance have been discussed, in particular, in Musiela [19], Bjork [2], Vargiolu [27], Filipovic [9], ,Tehranchi [25] . The case where $Z_j$ are Lévy processes is much less investigated and is the object of the present paper. To the best of our knowledge only papers by Filipovic and Tappe [10], and Rusinek [22], [23], were devoted to those more general equations.

In the first two sections we derive the equation from some financial assumptions. In Section 3 we investigate existence of the solutions. Our results are similar to those of Filipovic and Tappe but we work in different function spaces. Also, in the case of finite-dimensional noise, our conditions are more explicit. Then, in Section 4 we investigate positivity of solutions, a property important in applications. We show in particular, that if the noise is one-dimensional with jumps bounded from below then, under rather mild conditions on volatility, the equation preserves positivity. The final section treats equations with linear volatility. It has been observed by Morton in his PhD thesis that if the noise is Gaussian then solutions necessarily explode. We provide a large class of Lévy disturbances for which solutions do not explode. We show, in addition, that in the Gaussian case, when the linearity is random, then the non-exploding solutions exist provided the random coefficient behaves in a special way.

# 2 HJM Condition

Denote by $P(t, \theta)$, $0 \leq t \leq \theta$, the market price, at time $t$, of a bond paying 1 at time $\theta$, and by $(R(t),\, t \geq 0)$ the short rate process offered by a bank. Functions $f(t, \theta)$, $0 \leq t \leq \theta$, defined by the relation

$$P(t, \theta) = \mathrm{e}^{-\int_t^\theta f(t,\sigma)\,\mathrm{d}\sigma}, \qquad t \leq \theta,$$

are called *forward rate functions*. It is reasonable to assume that $f(t, t) = R(t)$, $t \geq 0$.

In Heath, Jarrow and Morton [12] it was assumed that

$$\mathrm{d}f(t, \theta) = \alpha(t, \theta)\,\mathrm{d}t + \langle \sigma(t, \theta), \mathrm{d}W(t) \rangle, \tag{1}$$

where $W$ is a $d$-dimensional Wiener process with covariance $Q$. According to the observed data, the (random) function $f(t, \theta)$ should be regular in $\theta$ for fixed $t$ and chaotic in $t$ for fixed $\theta$. The latter property is implied by the presence of $W$ in the representation, and the former is implied by regular dependence of $\alpha(t, \theta)$ and $\sigma(t, \theta)$ on $\theta$ for fixed $t$.

For practical implementation of the bond market models it is useful to replace the Wiener process $W$ by a Lévy process $Z$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and taking values in a possibly infinite-dimensional Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$. Thus we assume that the dynamics of the forward rate functions is given by the equation

$$\mathrm{d}f(t, \theta) = \alpha(t, \theta)\,\mathrm{d}t + \langle \sigma(t, \theta), \mathrm{d}Z(t) \rangle_U, \quad t \leq \theta. \tag{2}$$

For each $\theta \geq 0$, $\alpha(t,\theta)$, $\sigma(t,\theta)$ are predictable processes. One extends $\alpha$, $\sigma$ and $P$ putting

$$\alpha(t,\theta) := 0, \quad \sigma(t,\theta) := 0 \quad \text{for } t \geq \theta \tag{3}$$

and

$$P(t,\theta) := \mathrm{e}^{\int_\theta^t R(s)\,\mathrm{d}s} \qquad \text{for } t \geq \theta. \tag{4}$$

Let us note that if $t \leq \theta$ then

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)\,\mathrm{d}s + \int_0^t \langle \sigma(s,\theta), \mathrm{d}Z(s)\rangle_U$$

and if $\theta \leq t$ then

$$R(\theta) = f(\theta,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)\,\mathrm{d}s + \int_0^\theta \langle \sigma(s,\theta), \mathrm{d}Z(s)\rangle_U. \tag{5}$$

Let us recall (see e.g. Peszat and Zabczyk [20]), that any Lévy process $Z$ on a Hilbert space $U$ admits the representation

$$\begin{aligned} Z(t) = at + W(t) \quad &+ \quad \int_0^t \int_{\{|y|_U \leq 1\}} y\big(\pi(\mathrm{d}s,\mathrm{d}y) - \mathrm{d}s\,\nu(\mathrm{d}y)\big) \\ &+ \quad \int_0^t \int_{\{|y|_U > 1\}} y\,\pi(\mathrm{d}s,\mathrm{d}y), \end{aligned}$$

where $\pi$ is the Poisson random measure corresponding to $Z$ and $\nu$ is the jump intensity measure of $Z$. Moreover,

$$\int_U |y|_U^2 \wedge 1 \nu(\mathrm{d}y) < \infty,$$

and the exponential moments $\mathbb{E}\,\mathrm{e}^{-\langle x, Z(t)\rangle_U}$, $x \in U$, are finite, if and only if,

$$\int_{\{|y|_U > 1\}} \mathrm{e}^{-\langle x,\, y\rangle_U} \nu(\mathrm{d}y) < +\infty.$$

Finally

$$\mathbb{E}\,\mathrm{e}^{-\langle x, Z(t)\rangle_U} = \mathrm{e}^{tJ(x)}, \qquad x \in U,$$

where

$$\begin{aligned} J(x) &:= -\langle a, x\rangle_U + \tfrac{1}{2}\langle Qx, x\rangle_U + J_0(x), \\ J_0(x) &:= \int_U \left[ \mathrm{e}^{-\langle x,y\rangle_U} - 1 + \langle x, y\rangle_U \chi_{\{|y|_U \le 1\}} \right] \nu(\mathrm{d}y). \end{aligned} \tag{6}$$

Let $b$ be the Laplace transform of the measure $\nu$ restricted to the complement of the ball $\{y\colon |y|_U \le 1\}$, that is,

$$b(x) := \int_{\{|y|_U > 1\}} \mathrm{e}^{-\langle x, y\rangle_U} \nu(\mathrm{d}y),$$

and let $B$ be the set of those $x \in U$ for which the Laplace transform is finite. Thus $B = \{x \in U\colon b(x) < \infty\}$. We intend now to prove a theorem from Jakubowski and Zabczyk [14], which states "if and only if" conditions under which the discounted price processes are local martingales with respect to the probability $\mathbb{P}$. We will regard the coefficients $\alpha$ and $\sigma$ in (2) as, respectively, $H = L^2([0, T])$ and $L(U, H)$-valued, predictable processes given by

$$\alpha(t)(\theta) = \alpha(t, \theta), \quad \theta \in [0, T], \quad \sigma(t)x(\theta) = \langle \sigma(t, \theta), x\rangle_U, \ x \in U, \ \theta \in [0, T].$$

For our purposes it is convenient to introduce the following condition on the jump intensity measure $\nu$:

$$\forall\, r > 0 \text{ the function } b \text{ is bounded on } \{x \in B\colon |x|_U \le r\}. \tag{7}$$

Let

$$\hat{P}(t, \theta) := \mathrm{e}^{-\int_0^t R(s)\mathrm{d}s} P(t, \theta), \qquad t \ge 0 \tag{8}$$

be the *discounted price* of the bond.In the theorem below $J\colon U \mapsto \mathbb{R}$ is given by (6).

**Theorem 1** *Assume that predictable processes $\alpha$ and $\sigma$ have bounded trajectories and that* (7) *is satisfied.*

(*i*) *If the discounted price processes given by* (8) *are local martingales then for all $\theta \le T$,*

$$\int_t^\theta \sigma(t, v)\,\mathrm{d}v \in B, \quad \mathbb{P}\text{-}a.s. \text{ for almost all } t \in [0, \theta]. \tag{9}$$

(*ii*) *Assume* (9). *Then the discounted price processes* (8) *are local martingales if and only if*

$$\int_t^\theta \alpha(t,v)\,\mathrm{d}v = J\Big(\int_t^\theta \sigma(t,v)\,\mathrm{d}v\Big), \qquad \forall\,\theta \le T,\ \mathbb{P}\text{-a.s. for almost all } t\in[0,\theta]. \tag{10}$$

**Remark 1** We call (10) the *HJM Condition.* Let $D$ be the derivative operator acting on functions defined on $U$. Note that the theorem says that under very mild assumptions the discounted price processes are local martingales if and only if (10) holds, or equivalently if and only if

$$\alpha(t,\theta) \;=\; \frac{\mathrm{d}}{\mathrm{d}\theta} J\left(\int_t^\theta \sigma(t,v)\mathrm{d}v\right) = \left\langle DJ\left(\int_t^\theta \sigma(t,v)\mathrm{d}v\right), \sigma(t,\theta)\right\rangle_U .$$

Thus the dynamics of the forward rate functions is given by

$$\mathrm{d}f(t,\theta) = \left\langle DJ\left(\int_t^\theta \sigma(t,v)\mathrm{d}v\right), \sigma(t,\theta)\right\rangle_U \mathrm{d}t + \langle\sigma(t,\theta),\mathrm{d}Z(t)\rangle_U .$$

Note that the drift term is completely determined by the diffusion term.

**Remark 2** In the particular case of $Z$ being a Wiener process with covariance $Q$ one arrives at the *Classical HJM Condition*

$$\int_t^\theta \alpha(t,v)\,\mathrm{d}v = \frac{1}{2}\Big\langle Q\int_t^\theta \sigma(t,v)\,\mathrm{d}v, \int_t^\theta \sigma(t,v)\,\mathrm{d}v\Big\rangle_U$$

introduced in Heath, Jarrow and Morton [12]. Clearly the condition above holds if and only if

$$\alpha(t,\theta) = \Big\langle Q\sigma(t,\theta), \int_t^\theta \sigma(t,v)\,\mathrm{d}v\Big\rangle_U$$

for every $\theta \le T$, $\mathbb{P}$-a.s. for almost all $t\in[0,\theta]$.

**Remark 3** Formulae similar to (10) have been obtained earlier in Björk et al. [4], Björk, Kabanov and Runggaldier [5], and Eberlein and Raible [8].

# 3 HJMM equation

An important link between HJM modeling and stochastic partial differential equations is provided by the so-called *Musiela parametrization*. Assume that

$$\mathrm{d}f(t,\theta) = \alpha(t,\theta)\,\mathrm{d}t + \langle \sigma(t,\theta), \mathrm{d}Z(t)\rangle_U$$

and for $t \geq 0$, $\xi \geq 0$ and $u \in U$ define

$$r(t)(\xi) := f(t, t+\xi), \quad a(t)(\xi) := \alpha(t, t+\xi), \quad (b(t)u)(\xi) := \langle \sigma(t, t+\xi), u\rangle_U.$$

We will call $r$ the *forward curve*. Next, let $S(t)\varphi(\xi) = \varphi(\xi + t)$ be the shift semigroup. Then

$$\begin{aligned} r(t)(\xi) &= f(t, t+\xi) \\ &= f(0, t+\xi) + \int_0^t \alpha(s, t+\xi)\mathrm{d}s + \int_0^t \langle \sigma(s, t+\xi), \mathrm{d}Z(s)\rangle_U \\ &= r(0)(t+\xi) + \int_0^t a(s)(t-s+\xi)\mathrm{d}s + \int_0^t b(s)(t-s+\xi)\mathrm{d}Z(s) \\ &= S(t)r(0)(\xi) + \int_0^t S(t-s)a(s)(\xi)\mathrm{d}s + \int_0^t S(t-s)b(s)(\xi)\mathrm{d}Z(s). \end{aligned}$$

Thus

$$r(t) = S(t)r(0) + \int_0^t S(t-s)a(s)\mathrm{d}s + \int_0^t S(t-s)b(s)\mathrm{d}Z(s)$$

is a mild solution to the equation

$$\mathrm{d}r(t) = \left(\frac{\partial}{\partial \xi} r(t) + a(t)\right)\mathrm{d}t + b(t)\mathrm{d}Z(t),$$

where $\frac{\partial}{\partial \xi}$ denotes the generator of $(S(t),\, t \geq 0)$. Identifying the $L(U, \mathbb{R})$-valued processes $b(\cdot)(\xi)$, $\xi \geq 0$, with the $U$-valued process (denoted also by $b(\cdot)(\xi)$, $\xi \geq 0$) we note that if the HJM Condition is satisfied, then

$$\begin{aligned} \mathrm{d}r(t)(\xi) &= \left(\frac{\partial}{\partial \xi} r(t)(\xi) + \left\langle b(t)(\xi), DJ\left(\int_0^\xi b(t)(\eta)\mathrm{d}\eta\right)\right\rangle_U\right)\mathrm{d}t \\ &\quad + b(t)(\xi)\mathrm{d}Z(t) \\ &= \frac{\partial}{\partial \xi}\left(r(t)(\xi) + J\left(\int_0^\xi b(t)(\eta)\mathrm{d}\eta\right)\right)\mathrm{d}t + b(t)(\xi)\mathrm{d}Z(t). \end{aligned} \tag{11}$$

Let the *volatility* $b$ depend on the forward curve $r$, say $b(t)(\xi) = G(t, r(t))(\xi)$, and let

$$\begin{aligned} F(t,r)(\xi) :=& \left\langle G(t,r(t))(\xi), DJ\left(\int_0^\xi G(t,r(t))(\eta)\mathrm{d}\eta\right)\right\rangle_U \\ =& \frac{\partial}{\partial \xi} J\left(\int_0^\xi G(t,r(t))(\eta)\mathrm{d}\eta\right). \end{aligned} \tag{12}$$

Then the forward curve process becomes a solution of the so-called *Heath–Jarrow–Morton–Musiela equation*

$$\mathrm{d}r(t)(\xi) = \left(\frac{\partial}{\partial \xi} r(t)(\xi) + F(t, r(t))(\xi)\right)\mathrm{d}t + G(t, r(t))(\xi)\mathrm{d}Z(t). \tag{13}$$

# 4 Existence of solutions

In this section we deduce the existence of a solution to (13) from Theorem 9.7 of Peszat and Zabczyk [20]. We assume that the driving noise $Z$ is a finite-dimensional, say $\mathbb{R}^d$-valued, square integrable martingale. Thus $U = \mathbb{R}^d$, and $Z$ is a sum of a Wiener process and a compensated jump process, and therefore

$$J(z) = \frac{\langle Qz, z\rangle}{2} + \int_{\mathbb{R}^d} \left(\mathrm{e}^{-\langle z,y\rangle} - 1 + \langle z, y\rangle\right)\nu(\mathrm{d}y), \tag{14}$$

where $Q$ is a symmetric non-negative definite matrix and the jump measure $\nu$ satisfies $\int_{\mathbb{R}^d} |y|^2\nu(\mathrm{d}y) < \infty$. Here we denote by $\langle\cdot,\cdot\rangle$ the scalar product on $\mathbb{R}^d$ and by $|\cdot|$ the corresponding norm. We have the following elementary fact.

**Lemma 1** *(i) If $z \in \mathbb{R}^d$ is such that $\int_{\{|y|\geq 1\}} |y|\mathrm{e}^{|z||y|}\nu(\mathrm{d}y) < \infty$, then $J$ is differentiable at $z$ and*

$$DJ(z) = Qz + \int_{\mathbb{R}^d} y\left(1 - \mathrm{e}^{-\langle z,y\rangle}\right)\nu(\mathrm{d}y).$$

*(ii) If $z \in \mathbb{R}^d$ is such that $\int_{\mathbb{R}^d} |y|^2\mathrm{e}^{|z||y|}\nu(\mathrm{d}y) < \infty$, then $J$ is twice differentiable at $z$ and*

$$D^2J(z) = Q + \int_{\mathbb{R}^d} y \otimes y\, \mathrm{e}^{-\langle z,y\rangle}\nu(\mathrm{d}y),$$

*where $y \otimes y[v] = \langle y, v\rangle y$, $v \in \mathbb{R}^d$.*

We assume that $G$ is of composition type, that is,

$$G(t,r(t))(\xi)[z] = \langle g(t,\xi,r(t)(\xi)),z\rangle, \qquad t,\xi \in [0,+\infty),\ z\in\mathbb{R}^d, \qquad (15)$$

where $g\colon [0,+\infty)\times[0,+\infty)\times\mathbb{R}\mapsto\mathbb{R}^d$. We identify $G(t,\psi)(\xi)$ with the vector $g(t,\xi,\psi(\xi))$ in $\mathbb{R}^d$.

Given $\gamma>0$ we consider the equation on the state space $\mathbf{H}_\gamma := H_\gamma \oplus \{\text{constant functions}\}$, where $H_\gamma := L^2\left([0,+\infty),\mathcal{B}([0,+\infty)),\mathrm{e}^{\gamma\xi}\mathrm{d}\xi\right)$. Note that $\mathbf{H}_\gamma$, equipped with the scalar product $\langle\psi+u,\varphi+v\rangle_{\mathbf{H}_\gamma} := \langle\psi,\varphi\rangle_{H_\gamma}+uv$, $\psi,\varphi\in H_\gamma$, $u,v\in\mathbb{R}$, is a real separable Hilbert space.

Let $S$ be the shift semigroup. Then for $\psi\in H_\gamma$,

$$|S(t)\psi)|^2_{H_\gamma} = \int_0^{+\infty}|\psi(\xi+t)|^2e^{\gamma\xi}\mathrm{d}\xi = \int_t^{+\infty}|\psi(\eta)|^2\mathrm{e}^{\gamma(\eta-t)}\mathrm{d}\eta \le \mathrm{e}^{-\gamma t}|\psi|^2_{H_\gamma}$$

and hence the following lemma holds.

**Lemma 2** *$S$ is a $C_0$-semigroup on $H_\gamma$ and $\mathbf{H}_\gamma$. Moreover,*

$$\|S(t)\|_{L(H_\gamma,H_\gamma)} \le \mathrm{e}^{-\frac{\gamma}{2}t}, \quad \|S(t)\|_{L(\mathbf{H}_\gamma,\mathbf{H}_\gamma)} = 1, \qquad t\ge 0.$$

By the Hölder inequality, for every $\gamma>0$, the space $H_\gamma$ is continuously embedded into $L^1 := L^1([0,+\infty),\mathcal{B}([0,+\infty)),\mathrm{d}\xi)$ and $|\psi|_{L^1}\le\gamma^{-1/2}|\psi|_{H_\gamma}$ for all $\psi\in H_\gamma$. We can formulate our first existence theorem.

**Theorem 2** *Let $Z$ be an $\mathbb{R}^d$-valued square integrable mean zero Lévy process with jump measure $\nu$, and let $G$ be given by* (15). *Assume that there are functions $\overline{g}\in H_\gamma$ and $\overline{h}\in H_\gamma\cap L^\infty$ such that*

*(i)* $\int_{\mathbb{R}} y^2\mathrm{e}^{|\overline{g}|_{L^1}|y|}\nu(\mathrm{d}y)<\infty$,

*(ii) for all $t,\xi\in[0,+\infty)$ and $u,v\in\mathbb{R}$,*

$$|g(t,\xi,u)|\le\overline{g}(\xi), \qquad |g(t,\xi,u)-g(t,\xi,v)|\le\overline{h}(\xi)|u-v|.$$

*Then for each $r_0\in\mathbf{H}_\gamma$, and for each $r_0\in H_\gamma$, there is a unique solution $r$ to* (13) *in $\mathbf{H}_\gamma$, respectively in $H_\gamma$, satisfying $r(0)=r_0$. Moreover, if the coefficient $g$ does not depend on $t$, then* (13) *defines (time homogeneous) Feller families on $\mathbf{H}_\gamma$ and on $H_\gamma$.*

Clearly (13) can be written as a time homogeneous equation of variables $X = (r,t)$ on the state space $\mathbf{H}_\gamma \times \mathbb{R}$. Therefore the theorem is a direct consequence of Lemmas 1 and 2, and the lemma below to formulate which it is convenient to introduce the following notation:

$$K_1(J,\overline{g}) := \sup_{z\colon |z|\le|\overline{g}|_{L^1}} |DJ(z)|\,, \quad K_2(J,\overline{g}) := \sup_{z\colon |z|\le|\overline{g}|_{L^1}} \left\|D^2J(z)\right\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\,.$$

Clearly, if assumption (i) of Theorem 2 is satisfied, then by Lemma 1, $J$ is twice differentiable at an arbitrary $z$ with $|z| \le |\overline{g}|_{L^1}$ and $K_i(J,\overline{g}) < \infty$, $i = 1,2$.

**Lemma 3** *Under the assumptions of Theorem 2, for every $t \ge 0$ one has $G(t,\cdot)\colon \mathbf{H}_\gamma \mapsto L_{(HS)}(\mathbb{R}^d, H_\gamma)$ and $F(t,\cdot)\colon \mathbf{H}_\gamma \mapsto H_\gamma$. Moreover, the following estimates hold:*

*(i) For all $t \ge 0$ and $\psi \in \mathbf{H}_\gamma$,*

$$|F(t,\psi)|^2_{H_\gamma} + \|G(t,\psi)\|^2_{L_{(HS)}(\mathbb{R}^d,H_\gamma)} \le \left(K_1^2(J,\overline{g}) + 1\right)|\overline{g}|^2_{H_\gamma}.$$

*(ii) For all $t \le 0$ and $\psi, \varphi \in H_\gamma$,*

$$\begin{aligned} \|G(t,\psi) - G(t,\varphi)\|_{L_{(HS)}(\mathbb{R}^d,H_\gamma)} &\le |\overline{h}|_{L^\infty}\,|\psi-\varphi|_{H_\gamma}\,, \\ |F(t,\psi) - F(t,\varphi)|_{H_\gamma} &\le K|\psi-\varphi|_{H_\gamma}, \end{aligned}$$

*where $K := |\overline{h}|_{L^\infty}\left(2K_2(J,\overline{g})|\overline{g}|^2_{H_\gamma} + 2K_1(J,\overline{g})\right)^{1/2}$.*

*(iii) For all $t \ge 0$ and $\psi, \varphi \in \mathbf{H}_\gamma$,*

$$|F(t,\psi) - F(t,\varphi)|^2_{H_\gamma} + \|G(t,\psi) - G(t,\varphi)\|^2_{L_{(HS)}(\mathbb{R}^d,H_\gamma)} \le \tilde{K}\;|\psi-\varphi|^2_{\mathbf{H}_\gamma},$$

*where $\tilde{K} := 2\left(|\overline{h}|^2_{L^\infty} + |\overline{h}|^2_{H_\gamma}\right)\left(1 + 2K_2(J,\overline{g})|\overline{g}|^2_{H_\gamma} + 2K_1(J,\overline{g})\right)$.*

**Proof** Take $t \ge 0$ and $\psi \in \mathbf{H}_\gamma$. Then

$$\|G(t,\psi)\|^2_{L_{(HS)}(\mathbb{R}^d,H_\gamma)} = \int_0^\infty |g(t,\xi,\psi(\xi))|^2 \mathrm{e}^{\gamma\xi}\mathrm{d}\xi \le |\overline{g}|^2_{H_\gamma}.$$

Next, for $G(t,\psi)(\eta)$ treated as a vector in $\mathbb{R}^d$,

$$\left|\int_0^\xi G(t,\psi)(\eta)\mathrm{d}\eta\right| \le \int_0^\infty |\overline{g}(\eta)|\mathrm{d}\eta = |\overline{g}|_{L^1}. \tag{16}$$

Hence, by the first assumption of the theorem and Lemma 1, for every $\xi > 0$, $\int_0^\xi G(t,\psi)(\eta)\mathrm{d}\eta$ belongs to the domain of the derivative of $J$,

$$\left|DJ\left(\int_0^\xi G(t,\psi)(\eta)\mathrm{d}\eta\right)\right| \le K_1(J,\overline{g}) < \infty,$$

and (i) follows.

To see the Lipschitz continuity note that for $\psi,\varphi\in\mathbf{H}_\gamma$ and $t\ge 0$,

$$\|G(t,\psi)-G(t,\varphi)\|^2_{L_{(HS)}(\mathbb{R}^d,H_\gamma)} \le \int_0^\infty |\overline{h}(\xi)|^2|\psi(\xi)-\varphi(\xi)|^2\mathrm{e}^{\gamma\xi}\mathrm{d}\xi.$$

Thus $C(\psi,\varphi) := \|G(t,\psi)-G(t,\varphi)\|^2_{L_{(HS)}(\mathbb{R}^d,H_\gamma)}$ can be estimated by

$$\begin{cases} |\overline{h}|^2_{L^\infty}|\psi-\varphi|^2_{H_\gamma} & \text{if } \psi,\varphi\in H_\gamma,\\ \overline{h}|^2_{H_\gamma}|\psi-\varphi|^2 & \text{if } \psi,\varphi\in\mathbb{R},\\ 2\left(|\overline{h}|^2_{H_\gamma}+|\overline{h}|^2_{L^\infty}\right)\left(|\psi|^2_{H_\gamma}+|\varphi|^2\right) & \text{if } \psi\in H_\gamma,\ \varphi\in\mathbb{R}. \end{cases}$$

Since for $\psi\in H_\gamma$ and $\varphi\in\mathbb{R}$, $|\psi-\varphi|^2_{\mathbf{H}_\gamma} = |\psi|^2_{H_\gamma}+|\varphi|^2$, we have

$$C(\psi,\varphi) \le \begin{cases} |\overline{h}|^2_{L^\infty}|\psi-\varphi|^2_{H_\gamma}, & \text{if } \psi,\varphi\in H_\gamma,\\ 2\left(|\overline{h}|^2_{L^\infty}+|\overline{h}|^2_{H_\gamma}\right)|\psi-\varphi|^2_{\mathbf{H}_\gamma}, & \text{if } \psi,\varphi\in\mathbf{H}_\gamma. \end{cases}$$

We show the Lipschitz continuity of $F$. Let $\psi,\varphi\in\mathbf{H}_\gamma$ and $t\ge 0$. Clearly

$$|F(t,\psi)-F(t,\varphi)|^2_{H_\gamma} \le 2(I_1+I_2),$$

where $I_1$ is equal to

$$\int_0^\infty |G(t,\psi)(\xi)|^2\left|DJ\left(\int_0^\xi G(t,\psi)(\eta)\mathrm{d}\eta\right)-DJ\left(\int_0^\xi G(t,\varphi)(\eta)\mathrm{d}\eta\right)\right|^2\mathrm{e}^{\gamma\xi}\mathrm{d}\xi$$

and

$$I_2 := \int_0^\infty |G(t,\psi)(\xi)-G(t,\varphi)(\xi)|^2\left|DJ\left(\int_0^\xi G(t,\varphi)(\eta)\mathrm{d}\eta\right)\right|^2\mathrm{e}^{\gamma\xi}\mathrm{d}\xi.$$

By (16),

$$
\begin{aligned}
I_1 &\le K_2(J,\overline{g}) \int_0^\infty |\overline{g}(\xi)|^2 \left( \int_0^\xi |G(t,\psi)(\eta) - G(t,\varphi)(\eta)| \mathrm{d}\eta \right)^2 \mathrm{e}^{\gamma\xi} \mathrm{d}\xi \\
&\le K_2(J,\overline{g})\, |\overline{g}|^2_{H_\gamma} \int_0^\infty |G(t,\psi)(\eta) - G(t,\varphi)(\eta)|^2 \mathrm{d}\eta \\
&\le K_2(J,\overline{g})\, |\overline{g}|^2_{H_\gamma} \|G(t,\psi) - G(t,\varphi)\|^2_{L_{(HS)}(\mathbb{R}^d, H_\gamma)}
\end{aligned}
$$

and

$$
I_2 \le K_1(J,\overline{g})\, \|G(t,\psi) - G(t,\varphi)\|^2_{L_{(HS)}(\mathbb{R}^d, H_\gamma)}.
$$

□

**Remark 4** Note that if $r_0 - c \in H_\gamma$ then the solution $r(t)$, $t \ge 0$ is a càdlàg process in $\mathbf{H}_\gamma$ and $r(t) - c \in H_\gamma$ $t \ge 0$. Indeed, the stochastic term $\int_0^t S(t-s)G(s,r(s))\mathrm{d}Z(s)$ can be written as $\int_0^t S(t-s)\mathrm{d}M(s)$, where $M(t) = \int_0^t G(s,r(s))\mathrm{d}Z(s)$ is a square integrable martingale in $H_\gamma$. Since, by Lemma 2, $S$ is a semigroup of contractions on $H_\gamma$ we infer by the Kotelenez theorem; see Theorem 9.3 in Peszat and Zabczyk [20]. Next, if $Z$ is a Wiener process, then by the factorization one obtains the continuity of $r$ in $\mathbf{H}_\gamma$; see Theorem 11.6 of Peszat and Zabczyk [20].

## 4.1 Existence in a special case

We restrict our attention to the special case of (13), in which $Z$ is two-dimensional and $G$ is, as in Section 4, of composition type. In fact we assume that $Z = (W, L)$ where $W$ is a standard real-valued Wiener process, and $L$ is an independent square integrable real-valued Lévy martingale with Laplace exponent

$$
\log \mathbb{E}\, \mathrm{e}^{-zL(1)} = \int_{\mathbb{R}} \left(\mathrm{e}^{-zy} - 1 + zy\right) \nu(\mathrm{d}y), \qquad \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) < \infty. \tag{17}
$$

Note that in the notation from the previous subsection, $d = 2$ and

$$
J(z_1, z_2) = \frac{z_1^2}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^{-z_2 y} - 1 + z_2 y\right) \nu(\mathrm{d}y). \tag{18}
$$

Therefore we are concerned with the following equation:

$$\begin{aligned}\mathrm{d}r(t)(\xi) = &\left(\frac{\partial}{\partial\xi}r(t)(\xi) + F(r)(\xi)\right)\mathrm{d}t + g_1(t,\xi,r(t)(\xi))\mathrm{d}W(t)\\ &+ g_2(t,\xi,r(t)(\xi))\mathrm{d}L(t),\end{aligned} \tag{19}$$

where $g_i\colon [0,+\infty)\times[0,+\infty)\times\mathbb{R}\mapsto\mathbb{R}$, $i=1,2$, and

$$\begin{aligned}F(t,\psi)(\xi) = &\, g_1(t,\xi,\psi(\xi))\int_0^\xi g_1(t,\eta,\psi(\eta))\mathrm{d}\eta\\ &+ g_2(t,\xi,\psi(\xi))\int_{\mathbb{R}} y\left(1-\mathrm{e}^{-y\int_0^\xi g_2(t,\eta,\psi(\eta))\mathrm{d}\eta}\right)\nu(\mathrm{d}y).\end{aligned}$$

Note that $G(t,\psi)[z_1,z_2](\xi) = g_1(t,\xi,\psi(\xi))z_1 + g_2(t,\xi,\psi(\xi))z_2$.

**Theorem 3** *Assume that $\nu$ is supported in $[-m,+\infty)$ for some $m\geq 0$, and that $g_2(t,\xi,u)\geq 0$ for all $t,\xi\geq 0$ and $u\in\mathbb{R}$. Moreover, we assume that there are functions $\overline{g}\in\mathbf{H}_\gamma$ and $\overline{h}\in H_\gamma\cap L^\infty$ such that for all $t,\xi\in[0,+\infty)$ and $u,v\in\mathbb{R}$,*

$$|g_i(t,\xi,u)|\leq\overline{g}(\xi),\qquad |g_i(t,\xi,u)-g_i(t,\xi,v)|\leq\overline{h}(\xi)\,|u-v|,\qquad i=1,2.$$

*Then for each $r_0\in\mathbf{H}_\gamma$ and each $r_0\in H_\gamma$ there is a unique solution $r$ to* (19) *in $\mathbf{H}_\gamma$, respectively in $H_\gamma$, satisfying $r(0)=r_0$. Moreover, if the coefficients $g_i$ do not depend on $t$, then* (19) *defines (time homogeneous) Feller families on $\mathbf{H}_\gamma$ and on $H_\gamma$.*

**Proof** The proof follows the ideas of the proof of Theorem 2. Only the fact that we do not assume that $\int_{\mathbb{R}} y^2\mathrm{e}^{|\overline{g}|_{L^1}|y|}\nu(\mathrm{d}y)<\infty$ should be explained. This assumption is compensated by the fact the $g_2$ is non-negative and $\nu$ has support in $[-m,+\infty)$. In fact we have the following version of Lemma 3. Its proof is left to the reader. □

To formulate the result we need the following analogues of $K_1(J,\overline{g})$ and $K_2(J,\overline{g})$. Let

$$\tilde{J}(z) = \int_{-m}^{+\infty}\left(\mathrm{e}^{-zy}-1+zy\right)\nu(\mathrm{d}y)$$

and let

$$\tilde{K}_1(\tilde{J},\overline{g}) := \sup_{0\leq z\leq|\overline{g}|_{L^1}}\left|\tilde{J}'(z)\right|,\quad K_2(\tilde{J},\overline{g}) := \sup_{0\leq z\leq|\overline{g}|_{L^1}}\left|\tilde{J}''(z)\right|.$$

Note that $\tilde{K}_i(\tilde{J},\overline{g}_2)<\infty$, $i=1,2$.

**Lemma 4** *Under the assumptions of Theorem* 3*, for every* $t \geq 0$*, one has* $G(t,\cdot)\colon \mathbf{H}_\gamma \mapsto L_{(HS)}(\mathbb{R}^2, H_\gamma)$ *and* $F(t,\cdot)\colon \mathbf{H}_\gamma \mapsto H_\gamma$*. Moreover,*

(*i*) *For all* $t \geq 0$ *and* $\psi \in \mathbf{H}_\gamma$*,*

$$|F(t,\psi)|^2_{H_\gamma} + \|G(t,\psi)\|^2_{L_{(HS)}(\mathbb{R}^2,H_\gamma)} \leq 2|\overline{g}|^2_{H_\gamma}\left(1 + |\overline{g}|^2_{L^1} + \tilde{K}_1^2(\tilde{J},\overline{g})\right).$$

(*ii*) *For all* $t \leq 0$ *and* $\psi, \varphi \in H_\gamma$*,*

$$\begin{aligned} \|G(t,\psi) - G(t,\varphi)\|_{L_{(HS)}(\mathbb{R}^2,H_\gamma)} &\leq 2|\overline{h}|_{L^\infty}\,|\psi-\varphi|_{H_\gamma}\,, \\ |F(t,\psi) - F(t,\varphi)|_{H_\gamma} &\leq K|\psi-\varphi|_{H_\gamma}, \end{aligned}$$

*where* $K := |\overline{h}|_{L^\infty}\left(1 + |\overline{g}|_{H_\gamma} + 2\tilde{K}_2(\tilde{J},\overline{g})|\overline{g}|^2_{H_\gamma} + 2\tilde{K}_1(\tilde{J},\overline{g})\right)^{1/2}.$

(*iii*) *For all* $t \geq 0$ *and* $\psi, \varphi \in \mathbf{H}_\gamma$*,*

$$|F(t,\psi) - F(t,\varphi)|^2_{H_\gamma} + \|G(t,\psi) - G(t,\varphi)\|^2_{L_{(HS)}(\mathbb{R}^2,H_\gamma)} \leq \tilde{K}\ |\psi-\varphi|^2_{\mathbf{H}_\gamma},$$

*where*

$$\tilde{K} := 8\left(|\overline{h}|^2_{L^\infty} + |\overline{h}|^2_{H_\gamma}\right)\left(1 + |\overline{g}|^2_{H_\gamma} + 2\tilde{K}_2(\tilde{J},\overline{g})|\overline{g}|^2_{H_\gamma} + 2\tilde{K}_1(\tilde{J},\overline{g})\right).$$

# 5 Positivity

We call a function $\psi\colon [0,+\infty) \mapsto \mathbb{R}$ *non-negative* if $\psi(\xi) \geq 0$ for almost all $\xi \geq 0$. Clearly, in all models, the forward curve functions should take non-negative values. We present here sufficient conditions on the coefficient $G$ and the noise $Z$ under which (13) preserves positivity, that is, for every non-negative initial value $r(0)$ the functions $r(t)$, $t \geq 0$, are-non negative.

We restrict our attention to the special case of (13) considered in Section 4.1. To simplify the exposition we consider only the time independent coefficients $g_i$, $i = 1, 2$.

**Theorem 4** *Assume that* $g_1(\xi, 0) = 0$ *for* $\xi \in [0,\infty)$ *and that one of the following conditions holds:*

(*i*) $g = (g_1, g_2)$ *satisfy the assumptions of Theorem* 2*,* $\nu$ *is supported on* $[-m, M]$ *for some* $m, M > 0$ *and* $|g_2(\xi,u)| \leq u\,M^{-1} \wedge m^{-1}$ *for all* $\xi \geq 0$ *and* $u \geq 0$*,* *(ii)* *$\nu$ is supported on $[-m, +\infty)$ for some $m$, $g_1$ and $g_2$ satisfy the assumptions of Theorem 3, and $0 \le g_2(\xi, u) \le u\, m^{-1}$ for all $\xi \ge 0$ and $u \ge 0$.*

*Then (16) preserves positivity, that is, for every non-negative $r(0) \in \mathbf{H}_\gamma$, functions $r(t)$, $t \ge 0$, are non-negative.*

By Theorems 2 and 3, for any $r(0) \in \mathbf{H}_\gamma$ there is a unique solution $(r(t), t \ge 0)$ starting from $r(0)$. Moreover, see Remark 4, $r$ is a càdlàg process in $\mathbf{H}_\gamma$. In the proof we will use the following theorem from Milian [18] dealing with the preserving positivity by the equation

$$\mathrm{d}X = (AX + F(t, X))\,\mathrm{d}t + B(t, X)\mathrm{d}W \tag{20}$$

driven by a Wiener process $W$ taking values in a Hilbert space $U$. In its formulation the state space $H = L^2(\mathcal{O}, \mathcal{B}(\mathcal{O}), \rho(\xi)\mathrm{d}\xi)$, where $\mathcal{O}$ is an open domain in $\mathbb{R}^d$ and $\rho$ is a non-negative weight, $A$ generates a $C_0$-semigroup $S$ on $H$ and $F\colon [0, +\infty) \times H \mapsto H$, $B\colon [0, +\infty) \times H \mapsto L_{(HS)}(U, H)$.

**Theorem 5** ***(Milian)*** *Assume that:*

*(i)* *The semigroup $S$ preserves positivity.*

*(ii)* *There is a constant $C$ such that for all $t, s > 0$ and $x, y \in H$,*

$$\begin{aligned}&|(F(t,x) - F(s,y)| + \|B(t,x) - B(s,y)\|_{L_{(HS)}(U,H)}\\&\quad \le C\left(|t-s| + |x-y|_H\right).\end{aligned}$$

*(iii)* *For every $t \ge 0$ and for all non-negative continuous $x, f \in H$ satisfying $\langle f, x\rangle_H = 0$ one has $\langle F(t,x), f\rangle_H \ge 0$ and $\langle B(t,x)v, f\rangle_H = 0$ for every $v \in U$.*

*Then (20) preserves positivity.*

**Remark 5** The original Milian theorem is a little more general. In particular it covers also the case of $W$ being a cylindrical Wiener process. Also it shows that (iii) is necessary for preserving positivity. The assumption of Lipschitz continuity in $t$ can be easily replaced by uniform continuity.

The problem of preserving positivity and the so-called comparison principle have been studied by several authors; see e.g. Aubin and Da Prato [1], Goncharuk and Kotelenez [11], Jachimiak [13], Kotelenez [17].

**Proof of Theorem 4** First of all not that the semigroup $S$ preserves positivity. Let $D(x)(\xi) = g_2(\xi, x(\xi))$ and let us approximate $L$ by a sequence $\{L_n\}$ of processes satisfying $|\Delta L_n(t)| \geq 1/n$, $t \geq 0$, $n \in \mathbb{N}$. We assume that $L_n$ converges $\mathbb{P}$-a.s. to $L$ uniformly on each compact time interval. The existence of such a sequence follows from the Lévy–Khinchin decomposition. Let $r_n$ be the solution to the problem

$$\mathrm{d}r = \left(\frac{\partial}{\partial \xi} r + F(r)\right) \mathrm{d}t + B(r)\mathrm{d}W(t) + D(r)\mathrm{d}L_n, \quad r_n(0) = r(0). \tag{21}$$

Since $r_n$ converges to $r$ it is enough to show that (21) preserves positivity. To do this note that $L_n$ has only isolated jumps. Between the jumps positivity is preserved by Theorem 5, as the driving process is Wiener. Assume that the solution is positive till the jump at time $\tau$. Then

$$r_n(\tau)(\xi) = r_n(\tau-)(\xi) + g_2(\xi, r_n(\tau-)(\xi)) \left(L_n(\tau) - L_n(\tau-)\right).$$

Hence, if $(a)$ holds, then

$$\begin{aligned} &r_n(\tau-)(\xi) + g_2(\xi, r_n(\tau-)(\xi)) \left(L_n(\tau) - L_n(\tau-)\right) \\ &\qquad \geq r_n(\tau-)(\xi) - (m \vee M) \left|g_2(\xi, r_n(\tau-)(\xi))\right| \geq 0 \end{aligned}$$

and, if $(b)$ holds, then

$$\begin{aligned} &r_n(\tau-)(\xi) + g_2(\xi, r_n(\tau-)(\xi)) \left(L_n(\tau) - L_n(\tau-)\right) \\ &\qquad \geq r_n(\tau-)(\xi) - m g_2(\xi, r_n(\tau-)(\xi)) \geq 0, \end{aligned}$$

and the result follows. $\square$

**Example 1** Let $Z = (W, L)$ be as in Theorem 4, and let the jump measure $\nu$ of $L$ be supported on $[-m, +\infty)$ for a certain $m > 0$. Let $g_i(\xi, u) = h_i(\xi) v_i(u)$, $i = 1, 2$. Assume that:

- $(i)$ $v_i$, $i = 1, 2$ are bounded and Lipschitz continuous, $h_i$, $i = 1, 2$ are bounded and belong to $H_\gamma$,
- $(ii)$ $v_1(0) = 0$, $h_2$ and $v_2$ are non-negative, $0 \leq v_2(u) \leq u/(m|h_2|_{L^\infty})$ for $u \geq 0$.

Then the assumptions of Theorem 4 are satisfy and (19) defines Feller family preserving positivity.

An important example of a jump measure supported on $[0, +\infty)$ is given below.

**Example 2** Given $\alpha > 0$ let $\nu(\mathrm{d}\xi) = \chi_{\{\xi>0\}}\xi^{-1-\beta}\mathrm{e}^{-\alpha\xi}$. Then, there is a constant $c = c(\alpha, \beta)$ such that for $z > 0$,

$$J'(z) = \begin{cases} c\left[\alpha^{-1+\beta} - (z+\alpha)^{-1+\beta}\right] & \text{if} \quad 0 < \beta < 1; \\ c\left[(z+\alpha)^{-1+\beta} - \alpha^{-1+\beta}\right] & \text{if} \quad 1 < \beta < 2. \end{cases}$$

# 6 Linear volatility

As in the section concerning special case we assume that $Z = (W, L)$, where $W$ is a standard Wiener process in $\mathbb{R}$, and $L$ is a real-valued Lévy martingale with Laplace transform (17). Moreover we assume that the volatility $G$ is a linear function of $r$, that is,

$$G(t,\psi)[z_1, z_2](\xi) = g_1(t)\psi(\xi)z_1 + g_2(t)\psi(\xi)z_2, \qquad z_1, z_2 \in \mathbb{R},\ \xi, t \geq 0.$$

Above $g_1$ and $g_2$ are predictable random processes independent of $\xi$. Then (19) becomes

$$\begin{aligned} \mathrm{d}r(t)(\xi) = {} & \left(\frac{\partial}{\partial\xi}r(t)(\xi) + F(t,r)(\xi)\right)\mathrm{d}t + g_1(t)r(t)(\xi)\mathrm{d}W(t) \\ & + g_2(t)r(t)(\xi)\mathrm{d}L(t), \end{aligned} \tag{22}$$

where

$$\begin{aligned} F(t,\psi)(\xi) = {} & g_1^2(t)\psi(\xi)\int_0^\xi \psi(\eta)\mathrm{d}\eta \\ & + g_2(t)\psi(\xi)\int_{\mathbb{R}} y\left(1 - \exp\left\{-yg_2(t)\int_0^\xi \psi(\eta)\mathrm{d}\eta\right\}\right)\nu(\mathrm{d}y). \end{aligned}$$

We will always assume that $r_0$ is a non-negative function. Let $u(t)(\xi) = \int_0^\xi r(t)(\eta)\mathrm{d}\eta$ be a primitive of $r(t)$. Then

$$\begin{aligned} \mathrm{d}u(t)(\xi) = {} & \left(\frac{\partial}{\partial\xi}u(t)(\xi) + \frac{(g_1(t)u(t)(\xi))^2}{2} + \tilde{J}\left(g_2(t)u(t)\right)\right)\mathrm{d}t \\ & + g_1(t)u(t)\mathrm{d}W(t) + g_2(t)u(t)(\xi)\mathrm{d}L(t), \end{aligned} \tag{23}$$

where

$$\tilde{J}(z) := \int_{\mathbb{R}} \left(\mathrm{e}^{-zy} - 1 + zu\right) \nu(\mathrm{d}y). \tag{24}$$

We assume that the jump measure $\nu$ of $L$ satisfies $\int_{\mathbb{R}} |y|^2 \nu(\mathrm{d}y) < \infty$.

## 6.1 Pure jump case

Assume that $g_1 \equiv 0$. To simplify we assume that $g_2 \equiv 1$. Then

$$\mathrm{d}u(t)(\xi) = \left(\frac{\partial}{\partial \xi} u(t)(\xi) + \tilde{J}\left(u(t)(\xi)\right)\right) \mathrm{d}t + u(t)(\xi)\mathrm{d}L(t). \tag{25}$$

To ensure the positivity we assume that $\nu$ is supported on $[-1, +\infty)$.

In order to solve (25) and consequently the HJMM equation for $r$ we solve the following auxiliary problem:

$$\mathrm{d}v(t) = \tilde{J}(v(t))\mathrm{d}t + v(t)\mathrm{d}L(t), \qquad v(0) = \xi. \tag{26}$$

Clearly $\tilde{J} \colon [0, +\infty) \mapsto \mathbb{R}$ is Lipschitz continuous if its derivative is bounded. This is guaranteed by the assumption

$$\nu \text{ has support in } [0, +\infty) \text{ and } \int_0^\infty y\nu(\mathrm{d}y) < \infty. \tag{27}$$

**Proposition 1** *Assume* (27). *Then:*

*(i) For each non-negative $\xi \geq 0$ there is a unique non-negative solution $(v(t)(\xi),\, t \geq 0)$ to* (26). *Moreover, for each $t$, $v(t)(\xi)$ depends continuously on $\xi$, $\mathbb{P}$-a.s.*

*(ii) There exists a version of $v$ differentiable in $\xi$ and*

$$\frac{\partial}{\partial \xi} v(t)(\xi) = \mathrm{e}^{\int_0^t \tilde{J}'(v(s)(\xi))\mathrm{d}s} A(t) \tag{28}$$

*where $A$ is a positive càdlàg process.*

**Proof** By Protter [21], Theorem 37, p. 308, the unique solution to (26) depends continuously on $\xi$. Since $\tilde{J}''(z) = \int_{[0,\infty)} y^2 \mathrm{e}^{-zy} \nu(\mathrm{d}y)$, $z \geq 0$, $\tilde{J}'$ is locally Lipschitz, by Protter [21], Theorem 39, p. 312, the solution $v$ is differentiable in $\xi$ and $v_\xi(t)(\xi) = \frac{\partial}{\partial \xi} v(t)(\xi)$ satisfies

$$\mathrm{d}v_\xi(t)(\xi) = \tilde{J}'\left(v(t-)(\xi)\right) v_\xi(t-)(\xi)\mathrm{d}t + v_\xi(t-)(\xi)\mathrm{d}L(t) = v_\xi(t-)(\xi)\mathrm{d}X(t),$$

where

$$X(t) = L(t) + \int_0^t \tilde{J}'\left(v(s-)(\xi)\right) \mathrm{d}s.$$

By Doléan's formula (see e.g. Protter [21]),

$$v_\xi(t)(\xi) = \mathrm{e}^{X(t)} \prod_{s \le t} \left(1 + \Delta X(s)\right) \mathrm{e}^{-\Delta X(s)}.$$

Since $\Delta X(s) = \Delta L(s)$ and $\int_0^t \tilde{J}'\left(v_\xi(s-)(\xi)\right) \mathrm{d}s = \int_0^t \tilde{J}'\left(v_\xi(s)(\xi)\right) \mathrm{d}s$,

$$v_\xi(t)(\xi) = \mathrm{e}^{\int_0^t \tilde{J}'(v(s)(\xi))\mathrm{d}s} \mathrm{e}^{L(t)} \prod_{s \le t} \left(1 + \Delta L(s)\right) \mathrm{e}^{-\Delta L(s)}.$$

□

**Proposition 2** *Assume* (27). *Let* $v(t)(\xi)$, $t \ge 0$, $\xi \ge 0$, *be the solution to* (26). *Then*

$$u(t)(\xi) = v(t)(t + u_0(\xi)), \qquad t \ge 0, \ \xi \ge 0,$$

*is the unique solution to* (25).

**Proof** We claim that

$$\begin{aligned}\mathrm{d}v(t)(t + u_0(\xi)) = &\left(\frac{\partial}{\partial \xi} v(t)(t + u_0(\xi)) + \tilde{J}\left(v(t)(t + u_0(\xi))\right)\right) \mathrm{d}t \\ &+ v(t)(t + u_0(\xi))\mathrm{d}L(t).\end{aligned}$$

Let $\psi\colon [0, +\infty) \times [0, +\infty) \mapsto \mathbb{R}$ be a differentiable function of both variables. Given $\xi > 0$ consider the process $(v(t)(\psi(t, \xi)),\ t \ge 0)$. Then for any partition $0 = t_0 < t_1 < \ldots < t_N = t$,

$$v(t)\left(\psi(t, \xi)\right) - v(0)\left(\psi(0, \xi)\right) = I_1 + I_2,$$

where

$$I_1 := \sum_n \left(v\left(t_{n+1}\right)\left(\psi(t_{n+1},\xi)\right) - v\left(t_{n+1}\right)\left(\psi(t_n,\xi)\right)\right),$$

$$I_2 := \sum_n \left(v\left(t_{n+1}\right)\left(\psi(t_n,\xi)\right) - v\left(t_n\right)\left(\psi(t_n,\xi)\right)\right).$$

But

$$\begin{aligned} I_1 &= \sum_n \left[v_\xi\left(t_{n+1}\right)\left(\psi(t_{n+1},\xi)\right.\right. \\ &\qquad \left.\left. +\tilde{\varepsilon}\left(\psi(t_n,\xi)-\psi(t_{n+1},\xi)\right)\left(\psi(t_{n+1},\xi)-\psi(t_n,\xi)\right)\right)\right] \\ &= \sum_n v_\xi\left(t_{n+1}\right)\left(\psi(t_{n+1},\xi)+\tilde{\varepsilon}\left(\psi(t_n,\xi)-\psi(t_{n+1},\xi)\right)\right) \\ &\qquad \times \psi'\left(t_n+\tilde{\eta}(t_{n+1}-t_n),\xi\right)\left(t_{n+1}-t_n\right), \end{aligned}$$

where $\tilde{\varepsilon}$ and $\tilde{\eta}$ are such that $|\tilde{\varepsilon}| \leq 1$ and $|\tilde{\eta}| \leq 1$. Taking into account continuous dependence of $v_\xi$ on the second variable (see Proposition 1) we obtain $I_1 \to \int_0^t v_\xi(s)(\psi(s,\xi))\mathrm{d}s$ as $n \uparrow \infty$. Since $v$ satisfies (26),

$$I_2 = \sum_n \left(\int_{t_n}^{t_{n+1}} J\left(v(s-)(\psi(t_{n+1},\xi)\right)\mathrm{d}s + \int_{t_n}^{t_{n+1}} v(s-)\left(\psi(t_n,\xi)\right)\mathrm{d}L(s)\right)$$

and therefore

$$I_2 \to \int_0^t J\left(v(s-)(\psi(s,\xi)\right)\mathrm{d}s + \int_0^t v(s-)(\psi(s,\xi))\mathrm{d}L(s).$$

□ As a corollary we obtain the following basic result.

**Theorem 6** *For any nonnegative $r_0$, the unique solution $r$ to* (25) *is given by*

$$r(t)(\xi) = \frac{\partial}{\partial \xi}u(t)(\xi) = \frac{\partial}{\partial \xi}v(t)(t+u_0(\xi)) = v_\xi(t)(t+u_0(\xi))r_0(\xi),$$

*where $v_\xi$ has representation* (28).

## 6.2 Gaussian case

This section is concerned with the Gaussian case. Namely, we assume that the jump case in (22) vanishes, and consequently $L = W$ is a standard Brownian motion. To shorten the notation we write $g_1(t) = h(t)$. We assume that

$h$ is a predictable process satisfying $\mathbb{E}\int_0^T h^2(t)\mathrm{d}t < \infty$, $\forall T > 0$. Therefore (22) becomes

$$\begin{cases} \mathrm{d}r(t)(\xi) = \left(\dfrac{\partial}{\partial\xi}r(t)(\xi) + h^2(t)r(t)(\xi)\displaystyle\int_0^\xi r(t)(\eta)\mathrm{d}\eta\right)\mathrm{d}t \\ \qquad\qquad + h(t)r(t)(\xi)\mathrm{d}W(t), \\ r(0)(\xi) = r_0(\xi). \end{cases} \tag{29}$$

We assume that $r_0$ is a non-negative function. Set

$$M_h(t) := \exp\left\{-\frac{1}{2}\int_0^t h^2(s)\mathrm{d}s + \int_0^t h(s)\mathrm{d}W(s)\right\}.$$

The theorem below shows that Eq. (29) can be solved explicitly. However the solution may blow up in finite time.

**Theorem 7** *Then unique solution to* (29) *is given by*

$$r(t)(\xi) = \frac{\partial}{\partial\xi}\left[\left(\int_0^{t+\xi} r_0(\eta)\mathrm{d}\eta\right)^{-1} - \frac{1}{2}\int_0^t h^2(s)M_h(s)\mathrm{d}s\right]^{-1} M_h(t). \tag{30}$$

**Proof** Let us denote by $u(t)$ the following primitive of $r(t)$, $u(t)(\xi) := \int_0^\xi r(t)(\eta)\mathrm{d}\eta$. Then $r(t)(\xi) = \frac{\partial u}{\partial\xi}(t)(\xi)$ and consequently we have the following equation on $u$:

$$\begin{aligned} \mathrm{d}\frac{\partial u}{\partial\xi} &= \left(\frac{\partial^2 u}{\partial\xi^2} + h^2\frac{\partial u}{\partial\xi}u\right)\mathrm{d}t + h\frac{\partial u}{\partial\xi}\mathrm{d}W \\ &= \frac{\partial}{\partial\xi}\left\{\left(\frac{\partial u}{\partial\xi} + h^2\frac{u^2}{2}\right)\mathrm{d}t + hu\mathrm{d}W\right\}. \end{aligned}$$

Hence,

$$\begin{cases} \mathrm{d}u(t)(\xi) = \left(\dfrac{\partial u}{\partial\xi}(t)(\xi) + h^2(t)\dfrac{u^2(t)(\xi)}{2}\right)\mathrm{d}t + h(t)u(t)(\xi)\mathrm{d}W(t), \\ u(0)(\xi) = \displaystyle\int_0^\xi r_0(\eta)\mathrm{d}\eta. \end{cases}$$

In order to solve this equation we use the ideas from the previous subsection. Namely, $u(t)(\xi) = v(t)(t + u(0)(\xi))$, where $v(t)(\xi)$ is the unique solution to the stochastic Bernoulli problem

$$\mathrm{d}v(t)(\xi) = h^2(t)\frac{v^2(t)(\xi)}{2}\mathrm{d}t + h(t)v(t)(\xi)\mathrm{d}W(t), \quad v(0)(\xi) = \xi.$$

Let us fix $\xi$. We solve the equation using the substitution $z(s) = 1/v(s)(\xi)$. By Itô's formula,

$$\begin{aligned}
\mathrm{d}z(s) &= \left(-\frac{1}{v^2(s)}h^2(s)\frac{v^2(s)}{2} + \frac{1}{2}2\frac{1}{v^3(s)}h(s)v^2(s)\right)\mathrm{d}s \\
&\quad - \frac{1}{v^2(s)}h(s)v(s)\mathrm{d}W(s) \\
&= h^2(s)\left(-\frac{1}{2} + z(s)\right)\mathrm{d}s - h(s)z(s)\mathrm{d}W(s).
\end{aligned}$$

To solve the equation we use the variation of constants formula. Thus we look for a solution in the form $z(s) = c(s)z_0(s)$, where $z_0$ is a solution to the homogeneous equation

$$\mathrm{d}z_0(s) = h^2(s)z_0(s)\mathrm{d}s - h(s)z_0(s)\mathrm{d}W(s).$$

Then

$$\mathrm{d}z = \mathrm{d}(cz_0) = c'z_0\mathrm{d}s + c\mathrm{d}z_0 = c'z_0\mathrm{d}s + h^2z\mathrm{d}s - hz\mathrm{d}W.$$

This leads to the condition $c'(s) = -\frac{h^2(s)}{2}\frac{1}{z_0(s)}$. Clearly, for $z_0$ we can take

$$z_0(s) = \exp\left\{\frac{1}{2}\int_0^s h^2(\eta)\mathrm{d}\eta - \int_0^s h(\eta)\mathrm{d}W(\eta)\right\} = \frac{1}{M_h(s)}.$$

Taking into account the initial value condition $z(0) = 1/\xi$ we obtain

$$z(s) = \left(\xi^{-1} - \frac{1}{2}\int_0^s h^2(\eta)M_h(\eta)\mathrm{d}\eta\right)\frac{1}{M_h(s)},$$

which gives the desired formula. □

**Corollary 1** *As a direct consequence of* (30)*, the set $\mathcal{D}$ of all $(t,\xi)$ for which the solution blows up is given by*

$$\mathcal{D} = \left\{(t,\xi)\colon \frac{1}{2}\int_0^t h^2(s)M_h(s)\mathrm{d}s = \left(\int_0^{t+\xi} r_0(\eta)\mathrm{d}\eta\right)^{-1}\right\}.$$

*Since $r_0 \geq 0$, the processes $t \mapsto \left(\int_0^{t+\xi} r_0(\eta)\mathrm{d}\eta\right)^{-1}$, where $\xi \geq 0$ is fixed, and $\xi \mapsto \left(\int_0^{t+\xi} r_0(\eta)\mathrm{d}\eta\right)^{-1}$, where $t \geq 0$ is fixed, have decreasing trajectories. Clearly, the process $2^{-1}\int_0^t h^2(s)M_h(s)\mathrm{d}s$, $t \geq 0$, is increasing and starts from 0. Therefore, if $\int_0^\infty r_0(\eta)\mathrm{d}\eta = +\infty$ and $\mathbb{P}\left(\int_0^\infty h^2(s)\mathrm{d}s > 0\right) = 1$, then with probability 1, for any $\xi$ there is a time $t$ such that $(t,\xi) \in \mathcal{D}$.*

Let $\phi\colon \mathbb{R} \mapsto \mathbb{R}$ be a bounded non-negative Lipschitz function satisfying $\phi(1) \neq 0$, $\phi(0) = 0$ and

$$m(\phi) := \sup_{x\neq 0} \frac{|\phi(x)|^2}{|x|} < \infty. \tag{31}$$

Let $X$ be a solution to the equation

$$\mathrm{d}X = \phi(X)\mathrm{d}W, \qquad X(0) = 1.$$

Note, that $X$ is not identically 0 as $\phi(1) \neq 0$. Clearly, $X = M_h$ with non-negative process

$$h(t) = \frac{\phi(X(t))}{X(t)}, \qquad t \geq 0.$$

From (31), $\mathbb{P}$-a.s. $h^2(s)M_h(s) \leq \frac{|\phi(X(s))|^2}{X(s)} \leq m(\phi)$, $t \geq 0$. Hence,

$$\left|\frac{1}{2}\int_0^t h^2(s)M_h(s)\mathrm{d}s\right| \leq \frac{tm(\phi)}{2}$$

and we have the following consequence of Theorem 7.

**Theorem 8** *Let $T > 0$. Then for each non-negative initial value $r_0 \in L^1 := L^1([0,+\infty),\mathcal{B}([0,+\infty),\mathrm{d}\xi)$ satisfying $|\psi|_{L^1} < \frac{2}{Tm(\phi)}$, the process $r$ given by* (30) *has trajectories $\mathbb{P}$-a.s. in $C([0,T);L^1)$ and is a unique strong solution to* (29) *on the open interval $[0,T)$. Additionally, if $r_0 \in C_b^n([0,+\infty))$, then $r \in C\left([0,T);C_b^n([0,+\infty)\right)$, $\mathbb{P}$-a.s.*

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