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A note on completeness of bond market in infinite dimension

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A note on completeness of bond market in infinite dimension *

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Abstract

The completeness of bond market model with infinite number of sources of randomness on a finite time interval is studied. It is proved that under natural conditions the market is not complete.

Key words: bond market, completeness, infinite dimensional model
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JEL Classification Numbers: G10,G11

1 Introduction

We investigate the completeness of continuous time zero-coupon bond market driven by an infinite number of independent Wiener processes. By $P(t, T)$ we denote the value at time t of a bond paying 1 at time $T \leq \bar{S}$, where $\bar{S} < \infty$ is the time horizon of the model. For any $t \in [0, \bar{S}]$, $P(t, T)$ is a function of T on the interval $[0, \bar{S}]$ called the bond price curve. We are interested in the problem of replicating contingent claims which depend on the information on the market up to time S, where $S \leq \bar{S}$. So, we consider the situation when a trader can change his portfolio on the time interval $[0, S]$ using bonds which can expire also later than S, namely bonds with maturities up to time $\bar{S}$. Notice that for $S = \bar{S}$ our model covers situations which occur in practice. If trader wants to hedge a payoff which

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depends on the bond prices at time S, he always uses bonds with maturities not exceeding S. Such strategies will be called *natural*.

We work with the model where the bond prices are driven by infinite dimensional Wiener process. The bond price curves are allowed to be in the space $L^2(0,\bar{S})$ or in $H^1(0,\bar{S})$. We prove that the bond market is not complete for any choice of the bond price curve space, i.e. there always exists a *bounded* contingent claim, measurable with respect to the σ- field generated by the Wiener process up to time S, which can not be replicated. Our results are in a strong contrast with Theorem 4.1 in [1] which states that the bond market is complete (see Remark 1).

The problem of replicating with the use of natural strategies was stated and partially solved in [4]. It was shown with the help of Malliavin calculus that each contingent claim which is of special form can be replicated. The lack of completeness in the case when $S<\infty$ and $\bar{S}=\infty$ was shown in [8] however, for non-standard definition of the set of contingent claims, namely for $D_0 := \bigcap_{p>1} L^p(\Omega)$. The problem of replicating in the class of all bounded random variables was still open.

2 The model

We consider a bond market with finite horizon $\bar{S}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\bar{S}]}, \boldsymbol{Q})$, where filtration is generated by a cylindrical Wiener process W in l^2 i.e. $W_t=(W_t^1, W_t^2, ...)$ is a sequence of independent standard Wiener processes. Let the bond market price $P(t,T)$ be of the form

$$P(t,T)=e^{-\int_t^T f(t,u)du},$$

where $f(t,T)$ is a forward rate at time t for a time interval $[t,T]$. Since $f(t,T), P(t,T)$ are functions of T for a fixed t, we will treat them as elements of some function space and sometimes write $f(t), P(t)$ instead. Having the forward rates we can define the short rate r_t at time t by $r_t=f(t,t)$.

Let the dynamic of the forward rate curve be given by the equation:

$$df(t,T)=\alpha(t,T)dt+\sum_{i=1}^{\infty}\sigma^i(t,T)dW^i(t) \qquad t\in[0,\bar{S}], \tag{1}$$

where we put

$$\alpha(t,T)=\sigma^i(t,T)=0, \tag{2}$$

for $t\geq T$ and $i=1,2,\ldots$, or using shorter notation:

$$df(t)=\alpha(t)dt+<\sigma(t),dW(t)>_{l^2,l^2},$$

where the drift coefficient $\alpha(t)$ is a random variable taking values in the function space $H:=L^2[0,\bar{S}]$ for all $t\in[0,\bar{S}]$ and is assumed to be Bochner integrable, i.e.

$$\int_0^{\bar{S}}\|\alpha(t)\|_H dt=\int_0^{\bar{S}}\left(\int_0^{\bar{S}}\alpha^2(t,T)dT\right)^{\frac{1}{2}}dt<\infty \qquad \boldsymbol{Q}-a.s. \tag{3}$$

and $\sigma(t) : l^2 \longrightarrow H$ is a random variable taking values in the space of Hilbert-Schmidt operators, i.e.

$$\|\sigma(t)\|^2_{L_{HS}(l^2,H)} = \sum_{i=1}^{\infty} \|\sigma^i(t)\|^2_H = \sum_{i=1}^{\infty} \left(\int_0^{\bar{S}} \sigma^i(t,T)^2 dT \right) < \infty \qquad \boldsymbol{Q} - a.s.$$

satisfying integrability condition:

$$\int_0^{\bar{S}} \|\sigma(t)\|^2_{L_{HS}(l^2,H)} dt = \sum_{i=1}^{\infty} \int_0^{\bar{S}} \left(\int_0^{\bar{S}} \sigma^i(t,T)^2 dT \right) dt < \infty \qquad \boldsymbol{Q} - a.s.. \quad (4)$$

Conditions (3) and (4) guarantee that the forward curve $f(t,\cdot)$ is in H. If the discounted bond price process $\hat{P}$ takes its values in some function space G, then as strategies we take processes φ taking values in its dual G^* (for more motivation of this definition see [2] and [3]). Thus, our portfolios can contain infinite number of assets at any time and the classical hedging theory does not cover this case. We assume that the market is arbitrage free and the probability measure $\mathbf{Q}$ is a martingale measure. The discounted bond price given by

$$\hat{P}(t,T) = B^{-1}(t)P(t,T) = e^{-\int_0^T f(t,u)du}, \quad (5)$$

satisfies, under HJM condition, equation (see [6],[7])

$$d\hat{P}(t,T) = \hat{P}(t,T)\Big(\sum_{i=1}^{\infty} b^i(t,T) dW^i(t) \Big), \quad (6)$$

where

$$b^i(t,T) := -\int_0^T \sigma^i(t,u)du.$$

Again, since $\hat{P}(t,T)$ is a function of T for a fixed t, we will treat it as an element of some function space and sometimes write $\hat{P}(t)$ instead.

Taking into account (6) we say that a predictable, G^*-valued process φ is *stochastically integrable with respect to* $\hat{P}$ if and only if $((\hat{P} \circ b)^* \varphi)$ is integrable with respect to W, where " $\circ$" denotes the multiplication operator i.e. $(\hat{P}(t) \circ h)(T) := \hat{P}(t,T)h(T)$ for any $h \in G$. Then the stochastic integral with respect to the process $\hat{P}$ is defined as:

$$\int_0^S < \varphi(t), d\hat{P}(t) >_{G^*,G} := \int_0^S < (\hat{P}(t) \circ b(t))^* \varphi(t), dW(t) >_{l^2,l^2} . \quad (7)$$

We concentrate on the two choices of Hilbert spaces G, namely $G = L^2(0,\bar{S})$ and $G = H^1(0,\bar{S})$ with the norm given by

$$\| g \|^2_{H^1(0,\bar{S})} = | g(0) |^2 + \int_0^{\bar{S}} (g'(s))^2 ds.$$

We are interested in the problem of completeness of the bond market defined above. The completeness of the market we define in a standard way:

Definition 1 *Let $S \leq \bar{S}$. The bond market is complete if for any $\mathcal{F}_S$-measurable, bounded random variable ξ (contingent claim) there exists a process φ taking values in G^*, which is stochastically integrable with respect to $\hat{P}$, satysfying:*

$$\xi = C + \int_0^S < \varphi(t), d\hat{P}(t) >_{G^*,G} \tag{8}$$

for some constant C.

Below we present the idea which allows us to prove incompleteness.
The set of all final portfolio values has the following structure:

$$\left\{ \int_0^S < \Gamma^*(t)\varphi(t), dW(t) >_{l^2,l^2} : \ \varphi \in G^* \right\}$$

where $\Gamma^*(t) := (\hat{P}(t) \circ b(t))^*$. It turns out that the operator $\Gamma^*(t)$ is compact and thus as a compact operator with values in infinite dimensional Hilbert space, it is not surjective. We construct process ψ taking values in the space l^2 such that it is integrable with respect to W, takes values in $(Im(\Gamma(t)^*))^c$ and such that the random variable $\int_0^S < \psi(t), dW(t) >_{l^2,l^2}$ is bounded. Then

$$\int_0^S < \psi(t), dW(t) >_{l^2,l^2} \neq \int_0^S < \Gamma^*(t)\varphi(t), dW(t) >_{l^2,l^2} \quad \forall \varphi \in G^*$$

so $\int_0^S < \psi(t), dW(t) >_{l^2,l^2}$ is a bounded random variable which can not be replicated.
In the sequel we will use the following auxiliary theorem (see [5]).

Theorem 1 *Let X, Y, Z be three Hilbert spaces and $A : X \longrightarrow Z$, $B : Y \longrightarrow Z$ two linear bounded operators. Then $ImA \subseteq ImB$ if and only if there exists constant $c > 0$ such that $\|A^* f\| \leq c\|B^* f\|$ for all $f \in Z^*$.*

3 Problem of completeness in the case of $G = L^2(0, \bar{S})$

In this section we solve the problem of completeness in the case when the discounted bond price curves are elements of the space $G = L^2(0, \bar{S})$.

Proposition 1 *The discounted bond price curve $\hat{P}(t, \cdot)$ takes values in the space $L^2(0, \bar{S})$.*

Proof We have

$$\|b^i(t,\cdot)\|_H^2 = \int_0^{\bar{S}} \left(\int_0^T \sigma^i(t,u)du \right)^2 dT \leq \int_0^{\bar{S}} \left(T \int_0^T (\sigma^i(t,u))^2 du \right) dT \leq$$

$$\int_0^{\bar{S}} \left(\bar{S} \int_0^{\bar{S}} (\sigma^i(t,u))^2 du \right) dT = \bar{S}^2 \int_0^{\bar{S}} (\sigma^i(t,u))^2 du = \bar{S}^2 \|\sigma^i(t,\cdot)\|_H^2 \tag{9}$$

and thus P a.s.

$$\int_0^{\bar{S}} \|b(t)\|^2_{L_{HS}(l^2,H)}dt = \int_0^{\bar{S}} \sum_{i=1}^{\infty} \|b^i(t)\|^2_H dt \leq \bar{S}^2 \int_0^{\bar{S}} \|\sigma(t)\|^2_{L_{HS}(l^2,H)}dt < \infty.$$

Moreover, $\hat{P}(t,T) = e^{-\int_0^T f(t,u)du}$ is a continuous function of T and thus it is bounded on $[0, \bar{S}]$. Therefore $\hat{P}(t) \circ b(t)$ satisfies analogous condition as $\sigma(t)$ and this fact guarantees that $\hat{P}(t)$ is in G. ■

Theorem 2 *The bond market with the bond price curves in the space $G = L^2(0, \bar{S})$ is not complete.*

Proof First we find an explicit formula for the operator $\Gamma^*(t)$. For any function $h \in G^* = L^2(0, \bar{S})$ and $u \in l^2$ we have:

$$< \Gamma^*(t)h, u >_{l^2,l^2} = < h, \Gamma(t)u >_{L^2,L^2} = \int_0^{\bar{S}} h(T)\hat{P}(t,T)\sum_{i=1}^{\infty} b^i(t,T)u^i dT$$

$$= \sum_{i=1}^{\infty} u^i \int_0^{\bar{S}} h(T)\hat{P}(t,T)b^i(t,T)dT$$

and thus

$$\Gamma^*(t)h = \left(\int_0^{\bar{S}} h(T)\hat{P}(t,T)b^i(t,T)dT\right)_{i=1,2,\ldots} \in l^2.$$

Operator $\Gamma^*(t)$ is compact (almost surely). Indeed, from the fact that $\sum_{i=1}^{\infty} \| b^i(t) \|^2_H < \infty$ (see proof of Proposition 1) and since $\hat{P}(t)$ is a bounded function of T it follows that

$$\lim_{n\to\infty} \sup_{\|h\|\leq 1} \sum_{i=n}^{\infty} \int_0^{\bar{S}} h(T)\hat{P}(t,T)b^i(t,T)dT = 0.$$

As a compact operator with values in infinite dimensional Hilbert space, $\Gamma^*(t)$ is not surjective.
The next step is to construct a predictable process ψ integrable with respect to W such that $\psi(t)$ is outside of the set $Im(\Gamma^*(t))$. Then the equality

$$\int_0^S \psi(t)dW(t) = \int_0^S \Gamma^*(t)\varphi(t)dW(t)$$

couldn't hold for any process φ for which the last integral is well defined. Moreover, we require that the random variable $\int_0^S \psi(t)dW(t)$ is bounded.

Let us consider the self-adjoint operator $(\Gamma^*(t)\Gamma(t))^{\frac{1}{2}} : l^2 \longrightarrow l^2$ which is also compact. For any $u \in l^2$ we have

$$\| (\Gamma^*(t)\Gamma(t))^{\frac{1}{2}}u \|^2 = < (\Gamma^*(t)\Gamma(t))^{\frac{1}{2}}u, \Gamma^*(t)\Gamma(t))^{\frac{1}{2}}u >_{l^2,l^2} = < \Gamma^*(t)\Gamma(t)u, u >_{l^2,l^2}$$

$$= < \Gamma(t)u, \Gamma(t)u >_{L^2,L^2} = \| \Gamma(t)u \|^2_{L^2}$$

so by Theorem 1 it follows that $Im(\Gamma^*(t)) = Im((\Gamma^*(t)\Gamma(t))^{\frac{1}{2}})$.
By Proposition 1.8 in [5] the operator $(\Gamma^*(t)\Gamma(t))^{\frac{1}{2}}$ can be represented by the formula:

$$(\Gamma^*(t)\Gamma(t))^{\frac{1}{2}} = \sum_{i=1}^{\infty} \lambda_i(t)\ \ g_i(t) \otimes g_i(t)$$

where $\lambda_i(t)$ is a random variable and $g_i(t)$ is a l^2- valued random variable for i=1,2,... . Here "$\otimes$" denotes the linear operation: $(a \otimes b)h = a < b, h >$ for $a, b, h \in l^2$. Moreover, λ_i and g_i are predictable as functions of (ω, t) and $\lambda_i(t) \longrightarrow_{i\to\infty} 0$ by compactness of $(\Gamma^*(t)\Gamma(t))^{\frac{1}{2}}$.
Our aim now is to construct the process $(\tilde{\psi}(t))$ where $\tilde{\psi}(t) = (\tilde{\psi}^1(t), \tilde{\psi}^2(t), ...) \in l^2$ such that it is not of the form $\sum_{i=1}^{\infty} \lambda_i(t)g_i(t) < g_i(t), u >_{l^2,l^2}$ for any $u \in l^2$. Thus, the desired process must satisfy

$$\sum_{i=1}^{\infty} \left(\frac{\tilde{\psi}^i(t)}{\lambda_i(t)} \right)^2 = \infty$$

$$\sum_{i=1}^{\infty} (\tilde{\psi}^i(t))^2 < \infty.$$

Let us define the sequence (i_k) in the following way:

$$i_1 := \inf \left\{ i : \frac{1}{\lambda_i} \geq 1 \right\}$$

$$i_{k+1} := \inf \left\{ i > i_k : \frac{1}{\lambda_i} \geq k \right\}$$

and put

$$\psi^i(t) = \begin{cases} 0 & \text{if } i \neq i_k \\ \frac{1}{k} & \text{if } i = i_k. \end{cases}$$

Then we have $\sum_{i=1}^{\infty} \left(\frac{\tilde{\psi}^i(t)}{\lambda_i(t)} \right)^2 \geq \sum_{k=1}^{\infty} \frac{1}{k^2}\ k^2 = \infty$ and $\sum_{i=1}^{\infty} \tilde{\psi}^i(t)^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, so the process is bounded in l^2. It is also predictable since it is obtained by measurable operations on predictable elements. Thus, it is integrable with respect to W.
Now let us define a stopping time τ_M as:

$$\tau_M := \inf\{t > 0 : | \int_0^t < \tilde{\psi}(t), dW(t) > | \geq M\}$$

and take $\tilde{M} > 0$ such that $\tilde{\psi}(t)\mathbf{1}_{[0,\tau_{\tilde{M}})}(t) \not\equiv 0$.
Finally we define the required process as:

$$\psi(t) := \tilde{\psi}(t)\mathbf{1}_{[0,\tau_{\tilde{M}})}(t).$$

■

Corrollary 1 *From Theorem 2 with $S = \bar{S}$ it follows that the bond market with the bond price curves in the space $G = L^2(0, \bar{S})$ is not complete if trader can use natural strategies only.*

Remark 1 *In the paper [1] the authors use Musiela parametrization for the bond market description. Agent's portfolio is a measure-valued process given by the density φ : $\{\varphi(t,x), t \in [0,t_f], T \in [0,\hat{T}]\}$ satisfying integrability condition:*

$$\mathbf{E}\int_0^{t_f}\int_0^{\hat{T}} \varphi(t,x)^2 dxdt < \infty. \tag{10}$$

At any time $t \in [0,t_f]$ trader's portfolio can contain the bonds with time to maturity $x \in [0,\hat{T}]$. Notice, that each φ above can be written in our parametrization as a process $\tilde{\varphi}$ defined by:

$$\tilde{\varphi}(t,T) = \begin{cases} \varphi(t, T-t) & \text{if} \quad T \in [t, t+\hat{T}] \\ 0 & \text{if} \quad T \notin [t, t+\hat{T}], \end{cases}$$

where $t \in [0,S]$, $T \in [0,\bar{S}]$, $S = t_f$, $\bar{S} = t_f + \hat{T}$.
Below we construct a model for which condition (10) implies:

$$\int_0^S \parallel \Gamma^*(t)\tilde{\varphi}(t) \parallel_{l^2}^2 dt < \infty \tag{11}$$

Let us assume, that σ satisfies the following conditions:

$$0 \le \sigma^i(t,T) \le K \quad i = 1,2,,, ,(t,T) \in [0,S]\times[0,\bar{S}] \text{ and some } K > 0 \tag{12}$$

$$\parallel \sigma^i(t) \parallel \le \frac{1}{i^2} \quad i = 1,2,,, \tag{13}$$

and define a new operator $\tilde{\sigma}$ as:

$$\tilde{\sigma}^i(t,T) = \begin{cases} \sigma^i(t,T) & \text{if} \quad \sum\int_0^t \sigma^i(s,T)dW^i(s) \ge 0 \\ 0 & \text{if} \quad \sum\int_0^t \sigma^i(s,T)dW^i(s) < 0. \end{cases} \tag{14}$$

Let the coefficient $\tilde{\alpha}$ be given by the HJM condition:

$$\tilde{\alpha}(t,T) = \sum \tilde{\sigma}^i(t,T)\int_t^T \tilde{\sigma}^i(t,s)ds.$$

It follows from (12),(13),(14) that coefficients $\tilde{\alpha}$ and $\tilde{\sigma}$ satisfy (3) and (4). Assume that the initial forward rate curve satisfies: $\tilde{f}(0,T) \geq 0$ for $T \in [0,\bar{S}]$. Then

$$\tilde{f}(t,T) = \tilde{f}(0,T) + \int_0^t \tilde{\alpha}(s,T)ds + \sum \int_0^t \tilde{\sigma}^i(s,T)dW^i(s) \geq 0, \quad (t,T) \in [0,S] \times [0,\bar{S}].$$

and thus $\hat{P}(t,T) = e^{-\int_0^T \tilde{f}(t,u)du} \leq 1$. It follows from condition (13) that:

$$\sum \int_0^{\bar{S}} \tilde{b}^i(t,T)^2 dT = \sum \int_0^{\bar{S}} \left(\int_0^T \tilde{\sigma}^i(t,u)du \right)^2 dT$$
$$\leq \sum \int_0^{\bar{S}} \left(T \int_0^T \tilde{\sigma}^i(t,u)^2 du \right) dT \leq \bar{S}^2 \sum \frac{1}{i^2}$$

As a consequence we obtain the following inequalities:

$$\int_0^S \| \Gamma^*(t)\tilde{\varphi}(t) \|_{l^2}^2 \, dt = \int_0^S \sum \left(\int_0^{\bar{S}} \tilde{\varphi}(t,T)\hat{P}(t,T)\tilde{b}^i(t,T)dT \right)^2 dt$$
$$\leq \int_0^S \left(\int_0^{\bar{S}} \tilde{\varphi}(t,T)^2 dT \sum \int_0^{\bar{S}} \tilde{b}^i(t,T)^2 dT \right) dt \leq \bar{S}^2 \sum \frac{1}{i^2} \int_0^S \int_0^{\bar{S}} \tilde{\varphi}(t,T)^2 dT dt$$
$$= \bar{S}^2 \sum \frac{1}{i^2} \int_0^{t_f} \int_0^{\hat{T}} \varphi(t,x)^2 dx dt < \infty$$

and thus (10) implies (11). We conclude that in our model investor can use wider class of strategies than in model considered in [1] but in spite of this fact the market is not complete. So, neither is the bond market presented in [1].
This example shows, that Theorem 4.1 in [1] is false.

4 Problem of completeness in the case of $G = H^1(0,\bar{S})$

Here we will study the problem of bond market completeness but allowing the investor to use strategies which are elements of the dual to $H^1(0,\bar{S})$. This is a larger class then $L^2(0,\bar{S})$ and one can expect that this can cause the market to be complete. Unfortunately, in this case the market still remains incomplete what is shown below.

Proposition 2 *The discounted bond price curve $\hat{P}(t,\cdot)$is an element of $H^1(0,\bar{S})$.*

Proof By (5) we obtain:

$$\frac{d}{dT}\hat{P}(t,T) = -\hat{P}(t,T)f(t,T)$$

and thus

$$|| \hat{P}(t,\cdot) ||_{H^1(0,\bar{S})}^2 = \hat{P}(t,0)^2 + \int_0^{\bar{S}} \left(\hat{P}(t,T)f(t,T) \right)^2 dT$$
$$\leq \hat{P}(t,0)^2 + C(t) \int_0^{\bar{S}} f(t,T)^2 dT < \infty.$$

The inequality $\hat{P}^2(t,T) \leq C(t)$ holds since $\hat{P}(t,T)$ is bounded as a continuous function of T on the interval $[0,\bar{S}]$ (for a fixed ω). ∎

Theorem 3 *The bond market with the bond price curves in the space $G = H^1(0,\bar{S})$ is not complete.*

Proof First, we will show that the H^1-norm of $\hat{P}(t,\cdot)b^i(t,\cdot)$ is bounded above by the L^2-norm of $\sigma^i(t,\cdot)$. We will use the fact, that $\hat{P}(t,T)$ is bounded (by $k(t)$) with respect to T on the interval $[0,\bar{S}]$ (for a fixed ω) since it is a continuous function. Since

$$\frac{d}{dT}[\hat{P}(t,T)b^i(t,T)] = b^i(t,T)\frac{d}{dT}\hat{P}(t,T) + \hat{P}(t,T)\frac{d}{dT}b^i(t,T)$$
$$= f(t,T)\hat{P}(t,T)\int_0^T \sigma^i(t,u)du - \hat{P}(t,T)\sigma^i(t,T),$$

we have

$$\| \hat{P}(t,\cdot)b^i(t,\cdot) \|^2_{H^1(0,\bar{S})} = \int_0^{\bar{S}} \hat{P}(t,T)^2 \left(f(t,T)\int_0^T \sigma^i(t,u)du - \sigma^i(t,T) \right)^2 dT$$
$$\leq k^2(t)\int_0^{\bar{S}} 2\left(\left(f(t,T)\int_0^T \sigma^i(t,u)du \right)^2 + \sigma^i(t,T)^2 \right) dT$$
$$\leq k^2(t)\left(2\int_0^{\bar{S}} f(t,T)^2 \left(T\int_0^T \sigma^i(t,u)^2 du \right) dT + 2 \| \sigma^i(t,\cdot) \|^2_{L^2(0,\bar{S})} \right)$$
$$\leq k^2(t)\left(2\int_0^{\bar{S}} f(t,T)^2 \bar{S} \| \sigma^i(t,\cdot) \|^2_{L^2(0,\bar{S})} dT + 2 \| \sigma^i(t,\cdot) \|^2_{L^2(0,\bar{S})} \right)$$
$$\leq 2k^2(t) \| \sigma^i(t,\cdot) \|^2_{L^2(0,\bar{S})} \left(\bar{S}\int_0^{\bar{S}} f(t,T)^2 dT + 1 \right)$$
$$= 2k^2(t) \| \sigma^i(t,\cdot) \|^2_{L^2(0,\bar{S})} \left(\bar{S} \| f(t,\cdot) \|^2_{L^2(0,\bar{S})} + 1 \right)$$
$$= K(t) \| \sigma^i(t,\cdot) \|^2_{L^2(0,\bar{S})} . \qquad (15)$$

Now, we will show that the operator $\Gamma(t)$ is compact. For any orthonormal basis $(e_n)_{n=1,2,...}$ in $H^1(0,\bar{S})$ and $u \in l^2$ s.t. $\| u \|^2_l \leq 1$ we have:

$$< \Gamma(t)u, e_n >^2_{H^1(0,\bar{S})} = \left(\Gamma(t)u(0)e_n(0) + \int_0^{\bar{S}} ((\hat{P}(t)\circ b(t))u)'(T)e_n'(T)dT \right)^2$$
$$= \left(\sum_{i=1}^{\infty} u_i \int_0^{\bar{S}} (\hat{P}(t,T)b_i(t,T))'e_n'(T) + \Gamma(t)u(0)e_n(0) \right)^2$$
$$= \left(\sum_{i=1}^{\infty} u_i < \hat{P}(t,T)b_i(t,T), e_n >_{H^1(0,\bar{S})} \right)^2$$

and thus

$$\sum_{n=1}^{\infty} < \Gamma(t)u, e_n >^2_{H^1(0,\bar{S})} \leq \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} < \hat{P}(t,T)b_i(t,T), e_n >^2_{H^1(0,\bar{S})}$$

$$= \sum_{i=1}^{\infty} \parallel \hat{P}(t,T)b_i(t,T) \parallel^2_{H^1(0,\bar{S})} < \infty$$

where the last inequality follows from (15) and assumption (4). Thus, operator $\Gamma(t)$ is compact since

$$\lim_{N\to\infty} \sup_{\|u\|_{l^2}\leq 1} \sum_{n=N}^{\infty} < \Gamma(t)u, e_n >^2_{H^1(0,\bar{S})} = 0.$$

The rest of the proof is the same as in the proof of Theorem 2. ■

We have analogous corrollary as in the previous section.

Corrollary 2 *From Theorem 3 with $S = \bar{S}$ it follows that the bond market with the bond price curves in the space $H^1(0, \bar{S})$ is not complete if trader can use natural strategies only.*

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