Equivariant KK-theory and noncommutative index theory

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Contents

Introduction to KK-theory

Lecture given by Christian Voigt

Motivation and background

Atiyah and Hirzebruch defined topological K-theory in 1960. For a compact topological space X the K-theory group $\mathrm{K}^0(X)$ is the Grothendieck group of the semigroup of isomorphism classes of vector bundles over X. The definition can be extended to locally compact spaces. Using n-fold suspension $\mathbb{R}^n \times X$ one defines $\mathrm{K}^{-n}(X) := \mathrm{K}^0(\mathbb{R}^n \times X)$. Bott periodicity says that there is an isomorphism $\mathrm{K}^{-n-2}(X) \simeq \mathrm{K}^{-n}(X)$.

From the Serre-Swan theorem we know that the category of vector bundles over X is equivalent to the category of finitely generated projective modules over the ring of continuous functions $C(X)$. Thus $\mathrm{K}^0(X)$ can be identified with algebraic K_0-group $\mathrm{K}_0(C(X))$. Remark that $\mathrm{K}_0(C(X))$ uses only algebraic structure, with no topology on $C(X)$. Higher algebraic K-theory groups K_n use also topology.

The Atiyah-Singer index theorem gives a means to calculate the index $\mathrm{Index}(P) := \dim\ker P - \dim\operatorname{coker} P$, where P is an elliptic operator on a closed manifold M, in terms of topological information. More precisely, the symbol of P gives a class $[\sigma(P)] \in \mathrm{K}^0(T^*M)$. Atiyah and Singer defined two maps

$$a - \mathrm{Index},\ t - \mathrm{Index}\colon\ \mathrm{K}^0(T^*M) \to \mathbb{Z}$$

such that $a - \mathrm{Index}([\sigma(P)]) = \mathrm{Index}(P)$, and $t - \mathrm{Index}([\sigma(P)])$ is given in terms of topological data. Atiyah-Singer index theorem states that $a - \mathrm{Index} = t - \mathrm{Index}$. Using Chern character one can pass to cohomology.

K-theory is a generalized cohomology theory. There is a dual homology theory $\mathrm{K}_\bullet(X)$. Atiyah proposed an operator theoretic approach to K-homology based on "abstract elliptic operators". Let X be compact topological space and $\mathcal{H}$ a Hilbert space. The set $\mathrm{Ell}(X)$ consists of triples (ϕ_0, ϕ_1, T), where $\phi_i\colon C(X) \to \mathcal{L}(\mathcal{H})$ are *-homomorphisms, $T \in \mathcal{L}(\mathcal{H})$ is a Fredholm operator such that $\phi_1(f)T - T\phi_0(f)$ is compact for all $f \in C(X)$. Atiyah defined a map $\mathrm{Ell}(X) \to \mathrm{K}_0(X)$ and showed that it is surjective provided X is a finite complex. The problem was to describe explicitely the equivalence relation $\sim$ such that $\mathrm{Ell}(X)/\sim \simeq \mathrm{K}_0(X)$.

Consider an exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{H}) \to 0,$$

where $\mathcal{K}(\mathcal{H})$ is the ideal of compact operators on $\mathcal{H}$, and $\mathcal{Q}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra. An operator $T \in \mathcal{L}(\mathcal{H})$ is called essentially normal (selfadjoint) if $\pi(T)$ is normal (selfadjoint). Essential spectrum of T is the spectrum of $\pi(T)$. Weyl-von Neumann theorem states that if T is essentially selfadjoint, then $T = S + K$, where S is selfadjoint and K compact. One has $T = URU^* = K$, where U is unitary, K compact, if and only if T and

R have the same essential spectrum. Brown, Douglas and Filmore asked the following two questions. If T is essentionally normal, then

- under what conditions can one write $T = N + K$, where N is normal and K compact,
- under what conditions on R is $T = URU^* + K$, where U is unitary and K compact.

To answer these questions Brown, Douglas and Filmore studied extensions of C^*-algebras. We say that an algebra E is extansion of A by B if there exists an exact sequence

$$0 \to B \to E \to A \to 0.$$

If T is essentially normal, $X \subset \mathbb{C}$ its essential spectrum, then one has an extension

$$0 \to \mathcal{K}(\mathcal{H}) \to C^*(T, 1, \mathcal{K}(\mathcal{H})) \to \underbrace{C^*(\pi(T))}_{\simeq C(X)} \to 0.$$

If A is separable and nuclear, B is σ-unital, then one obtains an abelian group $\operatorname{Ext}(A, B)$ by considering extension of A by $B \otimes \mathcal{K}(\mathcal{H})$. For $A = C(X)$, $B = \mathbb{C}$ computation of $\operatorname{Ext}(C(X), \mathbb{C})$ yields the solution to the above questions.

Definition

If B is a C*-algebra, then a Hilbert B-module is a right B-module $\mathcal{E}$ with a positive definite sesquilinear form $\langle -, - \rangle \colon \mathcal{E} \times \mathcal{E} \to B$ such that for $\xi, \eta \in \mathcal{E}$, $b \in B$

$$\begin{aligned}
\langle \xi, \eta \cdot b \rangle &= \langle \xi, \eta \rangle \cdot b, \\
\langle \xi, \eta \rangle^* &= \langle \eta, \xi \rangle, \\
\langle \xi, \xi \rangle &\geq 0 \\
\langle \xi, \xi \rangle &= 0 \text{ iff } \xi = 0
\end{aligned}$$

and $\mathcal{E}$ is complete in the norm $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$.

For $B = \mathbb{C}$ Hilbert B-modules are just Hilbert spaces. For $B = C_0(X)$ Hilbert B-modules are continuous fields of Hilbert spaces over X. For each C*-algebra B, B itself is a Hilbert B-module with $\langle b, c \rangle := b^* c$. If $(\mathcal{E}_i)_{i \in I}$ is a family of Hilbert modules, then the completed direct sum $\bigoplus_{i \in I} \mathcal{E}_i$ is a Hilbert module. For a C*-algebra B we define a Hilbert B-module $\mathcal{H}_B := \bigoplus_{i=1}^{\infty} B$. Kasparov stabilization theorem states that if $\mathcal{E}_B$ is countably generates, then $\mathcal{E}_B \oplus \mathcal{H}_B = \mathcal{H}_B$.

Let $\mathcal{E}$, $\mathcal{F}$ be Hilbert B-modules. Denote by $\mathcal{L}(\mathcal{E}, \mathcal{F})$ the space of all maps $T \colon \mathcal{E} \to \mathcal{F}$ such that there exists $T^* \colon \mathcal{F} \to \mathcal{E}$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^* \eta \rangle$ for all $\xi \in \mathcal{E}$, $\eta \in \mathcal{F}$. All such maps are B-linear and bounded. $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is called finite rank operator if it is a finite sum of operators $|\eta\rangle\langle\xi|$, $|\eta\rangle\langle\xi|(\lambda) = \eta\langle \xi, \lambda \rangle$ for $\xi \in \mathcal{E}$, $\eta \in \mathcal{F}$. The set $\mathcal{K}(\mathcal{E}, \mathcal{F})$ of compact operators is the closed linear span of the finite rank operators. For $\mathcal{E} = \mathcal{F}$, $\mathcal{L}(\mathcal{E}, \mathcal{F})$ os a C*-algebra, $\mathcal{K}(\mathcal{E}, \mathcal{F}) = \mathcal{K}(\mathcal{E}) \subset \mathcal{L}(\mathcal{E})$ is an ideal.

If A, B are C*-algebras (separable for simplicity), then a Kasparov A-B-module is a triple $(\mathcal{E}, \phi, F)$, where $\mathcal{E}$ is countably generated Hilbert B-module with gradation $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, $\phi \colon A \to \mathcal{L}(E)$ is a *-homomorphism of degree 0

$$\phi(a) = \begin{pmatrix} \phi_+(a) & 0 \\ 0 & \phi_-(a) \end{pmatrix},$$

$F \in \mathcal{L}(\mathcal{E})$ is of degree one,

$$F = \begin{pmatrix} 0 & T \\ R & 0 \end{pmatrix}$$

such that all operators $[\phi(a), F]$, $\phi(a)(F - F^*)$, $\phi(a)(F^2 - \mathrm{Id})$ are compact for all $a \in A$.

A homotopy between Kasparow A-B-modules $\mathcal{E}_0$, $\mathcal{E}_1$ is a Kasparov A-$B \otimes C([0,1])$-module $\mathcal{E}$, such that

$$\mathrm{ev}\colon (\mathcal{E}, \phi, F) \simeq (\mathcal{E} \otimes_{\mathrm{ev}_i} B, \phi \otimes \mathrm{Id}, F \otimes \mathrm{Id}),$$

where $\mathrm{ev}_i\colon B \otimes C([0,1])$ is the evaluation at i, that is $(\mathcal{E}_i, \phi_i, F_i)$ for $i = 0, 1$. Let $\mathbb{E}(A, B)$ be the set of all Kasparov A-B-modules. There is a binary operation on $\mathcal{E}(A, B)$ given by direct sum. We define a KK-theory $\mathrm{KK}(A, B)$ to be the set of equivalence classes in $\mathbb{E}(A, B)$ with respect to homotopy. The set $\mathrm{KK}(A, B)$ is an abelian group with addition induced by direct sum. Zero element is a class of $0 = (0, 0, 0)$.

If $\phi\colon A \to B$ is a *-homomorphism, then $(B \oplus 0, \phi, 0)$ is a Kasparov A-B-module. Let M be a closed manifold, $P\colon \Gamma(E^+) \to \Gamma(E^-)$ be an elliptic pseudodifferential operator of order zero. Let $\mathcal{H} = L^2(E^+) \oplus L^2(E^-)$ and $\phi_i\colon C(M) \to \mathcal{L}(\mathcal{H})$ send function f to a multiplication operator by f. Then

$$\left(\mathcal{H}, \phi, \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}\right),$$

where Q is a parametrix for P, is a Kasparov A-B-module. Let A, B be Morita-Rieffel equivalent with equivalence bimodule ${}_A\mathcal{E}_B$. Then $({}_A\mathcal{E}_B, \phi, 0)$ is a Kasparov A-B-module.

Properties

One of the deepest theorems in KK-theory is that there is an associative natural product

$$\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \to \mathrm{KK}(A, C)$$

for all A, B, C.

The group $\mathrm{KK}(A, B)$ becomes a bifunctor, covariant in B, contravariant in A. It becomes also a category with C*-algebraa as objects, and $\mathrm{Mor}_{\mathrm{KK}}(A, B) = \mathrm{KK}(A, B)$. One defines $\mathrm{KK}_n(A, B) := \mathrm{KK}(C_0(\mathbb{R}^n) \otimes A, B)$.

We have $\mathrm{KK}_\bullet(\mathbb{C}, B) = \mathrm{K}_\bullet(B)$, $\mathrm{KK}_\bullet(A, \mathbb{C}) = \mathrm{K}^\bullet(A)$.

There is a Bott periodicity $\mathrm{KK}_2(A, B) \simeq \mathrm{KK}_0(A, B)$, natural for all A and B.

For an extension

$$0 \to K \to E \to Q \to 0$$

of C*-algebras and mild assumptions on A, there is a 6-term exact sequence in both variables

$$\begin{tikzcd}
\mathrm{KK}_0(A, K) \arrow[r] & \mathrm{KK}_0(A, E) \arrow[r] & \mathrm{KK}_0(A, Q) \arrow[d] \\
\mathrm{KK}_1(A, Q) \arrow[u] & \mathrm{KK}_1(A, E) \arrow[l] & \mathrm{KK}_1(A, K) \arrow[l]
\end{tikzcd}$$

If Q is nuclear, then every extension $0 \to K \to E \to Q \to 0$ yields an element $\partial([\mathrm{Id}]) \in \mathrm{KK}(Q, K)$. Actually $\mathrm{KK}_1(Q, K) \simeq \mathrm{Ext}(Q, K)$ in this case.

Applications and further development

In many cases the groups $\mathrm{KK}(A, B)$ are determined by $\mathrm{K}_\bullet(A)$, $\mathrm{K}_\bullet(B)$. Rosenberg-Schochet theorem states that if A is KK-equivalent to a commutative C*-algebra, then there is a short exact sequence of graded abelian groups

$$0 \to \mathrm{Ext}_\bullet(\mathrm{K}_{\bullet+1}(A), \mathrm{K}_\bullet(B)) \to \mathrm{KK}_\bullet(A, B) \to \mathrm{Hom}(\mathrm{K}_\bullet(A), \mathrm{K}_\bullet(B)) \to 0.$$

Using this for $A = C(X)$, $X \subset \mathbb{C}$, $B = \mathbb{C}$ one can use the universal coefficients theorem to prove the Brown-Douglas-Filmore theorem. It states that if $T \in \mathcal{L}(\mathcal{H})$ is essenitally normal with essenital spectrum $X \subset \mathbb{C}$, then T can be written as $T = N + K$, where N is normal and K is compact, if and only if $\mathrm{Index}(T - \lambda \mathrm{Id}) = 0$ for all $\lambda \in C(X)$. More generally $T = URU^* + K$, where U is unitary, K compact, if and only if $\mathrm{Index}(T - \lambda\mathrm{Id}) = \mathrm{Index}(R - \lambda\mathrm{Id})$ for all $\lambda \in \mathbb{C} \setminus X$.

Let M be a closed manifold. The cotangent bundle T^*M is an almost complex manifold. Hence there is the Dolbeault operator $D = \bar\partial + \bar\partial^*$ which gives a class $[\bar\partial_M]$ in $\mathrm{KK}(C_0(T^*M), \mathbb{C})$. If P is an elliptic pseudodifferential operator, $P\colon \Gamma(E^+) \to \Gamma(E^-)$ on M, then $[P] \in \mathrm{KK}(C(M), \mathbb{C})$. Its symbol $\sigma(P) \in \mathrm{Hom}(\pi^*E^+, \pi^*E^-)$ for $\pi\colon T^*M \to M$, and $[\sigma(P)] \in \mathrm{KK}_1(\mathbb{C}, C_0(T^*M))$. Furthermore $[[\sigma(P)]] \in \mathrm{KK}(C(M), C_0(T^*M))$ such that $[\sigma(P)] = 1 \cdot [[\sigma(P)]]$. Kasparov index theorem states that $[P] = [[\sigma(P)]] \cdot [\bar\partial_M]$. This implies the index theorem of Atiyah-Singer

$$a - \mathrm{Index}(P) = 1 \cdot [P] = 1 \cdot [[\sigma(P)]] \cdot [\bar\partial_M] = [\sigma(P)] \cdot [\bar\partial_M] = t - \mathrm{Index}([\sigma(P)]).$$

A functor F from the category of C*-algebras to an additive category $\mathcal{C}$ is called

- homotopy invariant if $F(f_0) = F(f_1)$ for f_0, f_1 homotopic *-homomorphisms,
- stable if $F(A \otimes \mathcal{K}(\mathcal{H})) \simeq F(A)$ (naturally),
- split exact if for every split extension

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & E \arrow[r, "\pi"] & Q \arrow[r] \arrow[l, bend right, "\sigma"'] & 0, \end{tikzcd}$$

where $\sigma\colon Q \to E$ is a *-homomorphism such that $\pi\sigma = \mathrm{id}$, there is a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F(K) \arrow[r] & F(E) \arrow[r] & F(Q) \arrow[r] \arrow[l, bend right, "\sigma"'] & 0. \end{tikzcd}$$

Theorem of Higson and Cuntz states that the obvious functor from the category of C*-algebras to KK-category is the universal split extact stable homotopy functor. It means that whenever $F'\colon C^* - \mathbf{Alg} \to \mathcal{C}$ is split exact stable homotopy invariant, then there exists a unique $F\colon \mathrm{KK} \to \mathbb{C}$ such that the following diagram commutes

$$\begin{tikzcd} C^* - \mathbf{Alg} \arrow[r] \arrow[dr, "F'"'] & \mathrm{KK} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$$

Further topics include

- Equivariant version of KK.
- Applications to Novikov conjecture and Baum-Connes conjecture.
- Applications in Kischberd-Philips classification of purely infinite simple C*-algebras.
- Generalizations of KK.

Chapter 1

C*-algebras

1.1 Definitions

Definition 1.1. *A **Banach algebra** (complex) is an algebra A which is a Banach space with norm satisfying the inequality*

$$\|ab\| \leq \|a\|\|b\|, \text{ for all } a, b \in A.$$

Assume that we have an involution on Banach algebra, $*\colon A \to A$ that is for all $a, b \in A$, $\lambda, \mu \in \mathbb{C}$

$$\begin{aligned} a^{**} &= a, \\ (\lambda a + \mu b)^* &= \bar{\lambda} a^* + \bar{\mu} b^*, \\ (ab)^* &= b^* a^*. \end{aligned}$$

Definition 1.2. *A **C*-algebra** is a Banach algebra A with involution $*\colon A \to A$ which satisfies the C*-identity*

$$\|a^* a\| = \|a\|^2$$

for all $a \in A$.

We say that A is unital if there exists $1 \in A$ such that $a \cdot 1 = 1 \cdot a = a$. The involution $*$ is an isometry

$$\|a\|^2 = \|a^* a\| \leq \|a^*\|\|a\|, \quad \|a\| \leq \|a^*\|.$$

The C*-identity forces a strong connection between algebra and analysis.

Theorem 1.3. *Let A, B be a C*-algebras (unital or not). If $\phi\colon A \to B$ is C*-homomorphism then*

1. *for all $a \in A$ we have $\|\phi(a)\| \leq \|a\|$, i.e. ϕ is continuous with norm $\|\phi\| \leq 1$.*
2. *$\phi(A)$ is closed in B, in particular $\phi(A)$ is a subalgebra of B and the induced homomorphism $A/\ker\phi \to \phi(A)$ is an isometry. An injective C*-homomorphism is an isometry.*

1.2 Examples

Example 1.4. Let X be a locally compact Hausdorff space, and $C_0(X)$ the algebra of functions vanishing at infinity. Then with respect to conjugation and norm $\|f\| = \sup_{x \in X} |f(x)|$, the algebra $C_0(X)$ is a C*-algebra.

Example 1.5. The matrix algebra $M_n(\mathbb{C})$ is a C*-algebra. Furthermore

Theorem 1.6. *Every finite dimensional C*-algebra A is of the form $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.*

More generally direct limits of finite dimensional C*-algebras are called **AF algebras**.

Example 1.7. Let $\mathcal{L}(\mathcal{H})$ be tha algebra of bounded operators on Hilbert space. It is not separable unless it is finite dimensional. If $\dim \mathcal{H} = n$, then $\mathcal{L}(\mathcal{H}) = M_n(\mathbb{C})$. If $\dim \mathcal{H} = \infty$, then there is a closed ideal of compact operators $\mathcal{K}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ which takes over the role of matrices. There is an extension

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to 0,$$

where the quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is denoted $\mathcal{Q}(\mathcal{H})$, and is called the **Calkin algebra**.

Theorem 1.8. *Every C*-algebra A admits a faithful representation on $\mathcal{H}$ i.e. there is an injective C*-homomorphism $\phi\colon A \to \mathcal{L}(\mathcal{H})$ for some $\mathcal{H}$. Then ϕ is an isometry, so A can be identified with a C*-subalgebra of $\mathcal{L}(\mathcal{H})$.*

Example 1.9. Let G be a discrete group (for simplicity). Its group ring $\mathbb{C}[G]$ is the ring of finitely supported functions $f\colon G \to \mathbb{C}$, $f = \sum_{g\in G} f_g \delta_g$, $f_g \in \mathbb{C}$, $\delta_g(s) = 1$ if $s = g$ and 0 otherwise. The multiplication is given by convolution

$$(f * g)(s) := \sum_{\alpha,\beta=s} f(\alpha)g(\beta) = \sum_{t\in G} f(st^{-1})g(t).$$

We have $\delta_s * \delta_t = \delta_{st}$. We will assume that G is countable and then $\{\delta_s\}_{s\in G}$ will provide a basis for $l^2(G)$. For fixed g the action of $\delta_g * -$ on $l^2(G)$ produces a permutation of $\{\delta_s\}_{s\in G}$ and so an operator $U_g\colon l^2(G) \to l^2(G)$,

$$(U_g\xi)(t) = (\delta_g * \xi)(t) = \xi(g^{-1}t)$$

The operator U_g is unitary $U_g^{-1} = U_{g^{-1}} = U_g^*$. Indeed

$$\begin{aligned}
\langle U_g\xi, \eta\rangle &= \sum_{t\in G} (U_g\xi)(t)\overline{\eta(t)} \\
&= \sum_{t\in G} \xi(g^{-1}t)\overline{\eta(t)} \\
&= \sum_{t'\in G} \xi(t')\overline{\eta(gt')} \\
&= \langle \xi, U_{g^{-1}}\eta\rangle
\end{aligned}$$

$$\|U_g\xi\|^2 = \sum_{t\in G} |\xi(g^{-1}t)|^2 = \sum_{t'\in G} |\xi(t')|^2 = \|\xi\|^2.$$

The left regular representation $\lambda\colon \mathbb{C}[G] \to \mathcal{L}(l^2(G))$

$$\lambda(f) = \sum_{g\in G} f_g U_g$$

$$\|\lambda(f)\| \le \sum_{g\in G} |f_g| = \|f\|_1$$

extends to $\lambda\colon l^1(G) \to \mathcal{L}(l^2(G))$.

Definition 1.10. *The* ***reduced group algebra*** $C^*_r(G)$ *of* G *is the norm closure* $\overline{\lambda(\mathbb{C}[G])} = \lambda(\overline{l^1(G)})$.

If G is abelian, then $C^*_r(G) = C_0(\widehat{G})$, where $\widehat{G}$ is the **Pontryagin dual**, $\widehat{G} = \mathrm{Hom}(G, \mathrm{U}(1))$. There is a canonical trace on $\mathbb{C}[G]$

$$\tau \colon \sum_{g \in G} f_g \delta_g \mapsto f_e \in \mathbb{C}.$$

Proposition 1.11. *If* $\phi \colon G_1 \to G_2$ *is an injective group homomorphism, then there is an induced map* $\phi \colon C^*_r(G_1) \to C^*_r(G_2)$.

Let Π_U be the direct sum of all irreducible representations of G (up to unitary equivalence). The algebra $C^*(G)$ is defined as a closure of $\Pi_U(\mathbb{C}[G])$. Equivalently, if $\|f\| = \sup\{\|\pi(f)\| \mid f \in l^1(G)\}$, where the supremum is taken over all *-representations of $l^1(G)$, then $C^*(G)$ is the completion of $l^1(G)$ in this norm. Our λ extends to a C*-algebra homomorphism $\lambda \colon C^*(G) \to C^*_r(G)$. The following theorem holds for all locally compact groups.

Theorem 1.12. *The homomorphism* $\lambda \colon C^*(G) \to C^*_r(G)$ *is an isomorphism if and only if* G *is amenable.*

Proposition 1.13. *If* $\phi \colon G_1 \to G_2$ *is a group homomorphism, then there is an induced map* $\phi \colon C^*(G_1) \to C^*(G_2)$.

If X is a compact Hausdorff space, then $f \in C(X)$ is a projection if and only if $\bar{f} = f$, $f^2 = f$. It follows that $f(x) = 0$ or 1 for all $x \in X$. Denote $S_i := \{x \in X \mid f(x) = i\}$ for $i = 0, 1$. Then $S_0 \cap S_1 = \emptyset$, $S_0 \cup S_1 = X$. If F is continuous, integer valued, then δ_0, δ_1 are open and closed. So if f is a nontrivial projection, then X must be disconnected.

Hypothesis 1 (Idempotent conjecture). *If* G *is discrete, torsion free, then* $\mathbb{C}[G]$ *has no nontrivial idempotents.*

Hypothesis 2 (Strong idempotent conjecture, Kadison-Kaplansky conjecture). *If* G *is discrete, torsion free, then* $C^*_r(G)$ *has no nontrivial idempotents.*

Both conjectures follow from the Baum-Connes conjecture.

Example 1.14. If a locally compact group G acts on locally compact Hausdorff space X, then there is a crossed product algebra $C_0(X) \rtimes G$. When G acts freely, properly on X, then $C_0(X) \rtimes G$ is morita equivalent to $C_0(X/G)$. Remark that X/G is not a Hausdorff space in general.

Example 1.15. We will define a **Toeplitz algebra** as $\mathcal{T} := C^*(v)$, where $v^*v = 1$ (isometry), $vv^* \neq 1$ (not unitary). There is an isomorphism $C^*(v) \simeq C^*(S)$, where $S \colon l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is the shift operator

$$S(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots), \quad S^*(x_1, x_2, \ldots) := (x_2, x_3, \ldots).$$

Theorem 1.16 (Coburn). *The algebra* $C^*(S)$ *contains the compact operators* $\mathcal{K}$ *as an ideal and there is an extension*

$$0 \to \mathcal{K} \to C^*(S) \to C(S^1) \to 0,$$

where S^1 *is the circle.*

We can give another description using Hardy space $H^2 \subset L^2(S^1)$

$$H^2 = \text{span}\{z^n \mid n \geq 0\} \text{ (closed span)}.$$

Let $P\colon L^2(S^1) \to H^2$ be the orthogonal projection. For each $f \in C(S^1)$ define an operator $T_f\colon H^2 \to H^2$, $T_f(g) = P(fg)$ for $g \in H^2$. The operator T_z, where z is the identity funcion in $C(S^1)$, acts as a shift operator on H^2, so $C^*(T_z) \simeq \mathcal{T} \simeq C^*(S)$.

For $f \in C(S^1)$ let M_f be the operator of pointwise multiplication by f.

Exercise 1.17. $\|M_f\| = \|f\|$.

Consider the action of $[P, M_z]$ on the basis $\{z^n \mid n \in \mathbb{Z}\}$ of $L^2(S^1)$.

$$\begin{aligned} PM_z &\colon z^n \mapsto z^{n+1}, \quad n \geq -1 \\ M_zP &\colon z^n \mapsto z^{n+1}, \quad n \geq 0. \end{aligned}$$

Both operators are zero outside this range. It follows that $[P, M_z]$ is of rank one, and $[P, M_{z^n}]$ is of rank n on $L^2(S^1)$. If p is a polynomial in z, then $[P, M_p]$ is of finite rank.

For $f \in C(S^1)$ there exist a sequence of Laurent polynomials $p_n \to f$ such that

$$\|M_{p_n} - M_f\| = \|M_{p_n - f}\| = \|p_n - f\| \to 0, \text{ and so } M_{p_n} \to M_f.$$

From this we have that $[P, M_{p_n}] \to [P, M_f]$, so $[P, M_f]$ is compact.

For $f, g \in C(S^1)$

$$\begin{aligned} T_f T_g &= PM_f PM_g \\ &= P(PM_f - [P, M_f])M_g \\ &= PM_f M_g - P[P, M_f]M_g \\ &= T_{fg} + K, \end{aligned}$$

where K is compact operator. Denote

$$B := \{T_f + K \mid f \in C(S^1),\ K \in \mathcal{K}\}.$$

Theorem 1.18 (Coburn). *There is an isomorphism $B \simeq C^*(T_z) \simeq \mathcal{T}$.*

The map $f \mapsto \pi(T_f) \in Q$, where $\pi\mathcal{L}(\mathcal{H}) \to Q$ is a projection on Calkin algebra, gives an isomorphism $C(S^1) \simeq C^*(T_z)/\mathcal{K}$. Furthermore

$$\pi(T_f)\pi(T_g) = \pi(T_f T_g) = \pi(T_{fg} + K) = \pi(T_{fg}).$$

Consider the Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0.$$

We may ask whether there are other extensions

$$0 \to \mathcal{K} \to \mathcal{E} \to C(S^1) \to 0$$

not equivalent to the Toeplitz extension. The example is $\mathcal{E} = \mathcal{C}$, where

$$\mathcal{C} := \{M_f + K \mid f \in C(S^1),\ K \in \mathcal{K}\}.$$

There is no *-isomorphism $\mathcal{T} \to \mathcal{C}$. Now we can ask about the classification of such extensions. The answer was given by Brown, Douglas and Filmore, who introduced Ext-groups, which have relation with K-homology.

Example 1.19. More general construction than the Toeplitz algebra are the **Cuntz algebras** $\mathcal{O}_n$. These are generated by $S_1, \ldots, S_n$ such that $S_i^* S_i = 1$ (isometries), $\sum_{i=1}^n S_i S_i^* = 1$. The algebras O_n are unique up to isomorphism, simple, purely infinite for $n \geq 2$. There exist an extension $\mathcal{E}_n$

$$0 \to \mathcal{K} \to \mathcal{E}_n \to O_n \to 0.$$

We recall that:

Definition 1.20. *A projection $p \in A$ is* ***infinite*** *if p is equivalent to a proper subprojection of itself. Otherwise it is called* ***finite****.*

A simple C-algebra is* ***purely infinite*** *if and only if the closure of xAx contains an infinite projection for every positive $x \in A$.*

Example 1.21. Noncommutative Riemann surfaces. Let Γ_g be a fundamental group of compact oriented Riemann surface Σ_g of genus $g \geq 1$.

$$\Gamma_g = \{u_j v_j \mid j = 1, \ldots, g, \ \prod_{j=1}^{g} [u_j, v_j] = 1\},$$

$$\mathrm{B}\,\Gamma_g = \Sigma_g, \ \mathrm{H}^2(\Gamma_g; \mathrm{U}(1)) = \mathbb{R}/\mathbb{Z}.$$

For all $\theta \in [0, 1)$ there is a cocycle $\delta_\gamma * \delta_\mu = \sigma_\theta(\gamma, \mu)\delta_{\gamma\mu}$. By completion in operator norm we get $C_r^*(\Gamma_g, \sigma_\theta)$.

We can give an alternative description by unitaries u_j, v_j such that $\prod_{j=1}^g [u_j, v_j] = e^{2\pi i\theta}$. Noncommutative torus is a special case for $g = 1$.

1.3 Gelfand transform

Let A be a unital C*-algebra. For an element $a \in A$ we define its **spectrum** as

$$\mathrm{sp}_A(a) := \{\lambda \in \mathbb{C} \mid \lambda 1 - a \text{ is not invertible}\},$$

and the **resolvent** as

$$\rho_A(a) := \mathbb{C} \setminus \mathrm{sp}_A(a).$$

The **spectral radius** of an element is

$$r(a) := \sup\{|\lambda| \mid \lambda \in \mathrm{sp}_A(a)\}, \quad r(a) \leq \|a\|.$$

Proposition 1.22.

1. *If A is a Banach algebra, then for every $a \in A$*

$$\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = r(a).$$

2. *If A is a C*-algebra, and $a \in A$ is a normal element ($a^* a = aa^*$), then $r(a) = \|a\|$.*

3. *If A is a C*-algebra, then for every $a \in A$*

$$\|a\|^2 = r(a^* a).$$

Let B be a C*-algebra, $a \in B$. Consider C*-algebra $C^*(a)$ generated by a (when B is unital we assume $1 \in C^*(a)$). The algebra $C^*(a)$ is commutative if and only if a is normal. Define

$$\widehat{A} := \{\phi \colon A \to \mathbb{C} \mid \phi \text{ is a homomomorphism}, \|\phi\| \leq 1\}.$$

Definition 1.23. *Let A be a commutative C*-algebra.The* ***Gelfand transform*** *is the homomorphism*

$$A \to C_0(\widehat{A}), \quad a \mapsto \widehat{a},$$

$$\widehat{a}(\phi) := \phi(a).$$

Theorem 1.24 (Gelfand)**.** *If A is commutative, then the Gelfand transform is an isometric *-isomorphism form A to $C_0(\widehat{A})$.*

Corollary 1.25. *If a is normal element of a C*-algebra A, then the Gelfand transform gives an isometric *-isomorphism $C^*(a) \to C(\mathrm{sp}(a))$.*

Definition 1.26. *If a is a normal element in a unital C*-algebra A and $f \in C(\mathrm{sp}(a))$, then the inverse of Gelfand transform $f \mapsto f(a) \in C^*(a)$ is called the functional calculus for a.*

Example 1.27. Let A be a C*-algebra, $u \in A$ unitary element. Then $\mathrm{sp}(u) \subset S^1$. Assume $\mathrm{sp}(u) \subsetneq S^1$. Take a branch of logarithm defined on subset of S^1 containing $\mathrm{sp}(u)$. Use functional calculuc to define a family of unitary groups $u_t := \exp(t \log u)$, $t \in [0,1]$. This family constitutes a continuous path which connects u to the identity through unitaries.

There is also a **holomorphic functional calculus**. Let A be a unital Banach algebra, $a \in A$. Assume that f is a holomorphic sunction in on an open set containing $\mathrm{sp}(a)$. Choose a piecewise linear closed curve C in that set, but not intersecting $\mathrm{sp}(a)$. Then

$$f(a) := \frac{1}{2\pi i} \int_C f(z)(z-a)^{-1} dz$$

defines an element of A. If $H(a)$ is the set of holomorphic functions of this type, then this gives an algebra homomorphism $H(a) \to A$ - holomorphic functional calculus.

If A is a subalgebra of a Banach algebra B, and $\widetilde{A}$, $\widetilde{B}$ are unitizations, then we say that A is stable under holomorphic functional calculus if and only if for any $a \in A$, and f which is holomorphic in an open set containing $\mathrm{sp}_{\widetilde{B}}(a)$, we have $f(a) \in \widetilde{a}$.

Proposition 1.28. *Let A be a C*-algebra. Then for any $x \in A$ the following are equivalent*

1. $x = x^*$, $\mathrm{sp}(x) \subset \mathbb{R}_+$,
2. *there exists $y \in A$ such that $x = y^* y$,*
3. *there exists $y \in A$ such that $y = y^*$, $y^2 = x$.*

If x satisfies any of thers, then we say that it is ***positive*** *and write $x \geq 0$.*

If $x \geq 0$ and $-x \geq 0$ then $x = 0$. Positivity induces a partial order on elements of A. We say that $x \leq y$ if and only if $y - x \geq 0$. Positive elements form a cone $A_+ \subset A$. For projections p, q we have $p \leq q$ if and only if $pq = p$.

Now we will define tensor products of C*-algebras. Let A, B be C*-algebras and $A \odot B$ be the algebraic tensor product (as vector spaces). The vector space $A \odot B$ is a *-algebra

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb', \quad (a \otimes b)^* = a^* \otimes b^*.$$

C*-algebra norm on $A \odot B$ is a cross norm $\| - \|_\alpha$, $\|a \otimes b\|_\alpha = \|a\| \|b\|$, and satisfies

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha, \quad \|x^* x\|_\alpha = \|x\|_\alpha^2.$$

A completion of $A \odot B$ with respect to such norm is a C*-algebra $A \otimes_\alpha B$. Let $\pi\colon A \to \mathcal{L}(\mathcal{H})$, $\sigma\colon B \to \mathcal{L}(\mathcal{H}')$ be faithful representations. The algebraic tensor product gives a representation

$$\pi \odot \sigma\colon A \odot B \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}'),$$

$$((\pi \odot \sigma)(a \otimes b))(\xi \otimes \eta) = \pi(a)\xi \otimes \sigma(b)\eta.$$

Define a **minimal norm** $\|x\|_{min} := \|(\pi \odot \sigma)(x)\|_{\mathcal{L}(\mathcal{H}\otimes\mathcal{H}')}$. The theorem of Takesaki states that this definition does not depend on π, σ.

Definition 1.29. *A C*-algebra A is **nuclear** if and only if for any C*-algebra B there is a unique C*-norm on $A \odot B$.*

*A is **exact** if and only if the functor $B \mapsto A \otimes_{min} B$ is exact (i.e. sends exact sequences of C*-algebras to exact sequences).*

Theorem 1.30 (Kirchberg-Wassermann)**.** *A discrete group G is exact if and only if $C_r^*(G)$ is exact.*

Nuclear algebras are exact. For a free group on two generators F_2 the reduced group algebra $C_r^*(F_2)$ is exact but not nuclear. The full C*-subalgebra $C^*(F_2)$ of the nonabelian free group on two generators is not exact.

Proposition 1.31. *The reduced group algebra $C_r^*(G)$ is nuclear if and only if G is amenable.*

Maximal tensor product $\otimes_{max}$ has the following universal property. There is a natural bijection between non degenerate C*-homomorphisms

$$A_1 \otimes_{max} A_2 \to \mathcal{L}(\mathcal{H}),$$

and pairs of commuting non degenerate C*-homomorphisms

$$A_1 \to \mathcal{L}(\mathcal{H}), \quad A_2 \to \mathcal{L}(\mathcal{H}).$$

One can also replace $\mathcal{L}(\mathcal{H})$ be the multiplier algebra $\mathcal{M}(D)$ for any C*-algebra D.

There is a canonical C*-algebra homomorphism

$$A_1 \otimes_{max} A_2 \to A_1 \otimes_{min} A_2$$

for any C*-algebras A_1, A_2. We can give a second definition

Definition 1.32. *A C*-algebra A_1 is nuclear if this map is an isomorphism for any C*-algebra A_2.*

One can say that A_1 is K-nuclear if this map induces an isomorphism on K-theory for any C*-algebra A_2.

Chapter 2

K-theory

2.1 Definitions

Definition 2.1. *If A is a unital C^*-algebra, then $p \in A$ is a* ***projection*** *if and only if $p^* = p$, $p^2 = p$.*

Definition 2.2. *Let $p, q \in A$ be a projections. We say that they are*

1. ***Murray-von Neumann equivalent****, $p \sim_v q$, if there exist $v \in A$ such that $p = v^*v$, $q = vv^*$.*

2. ***unitarily equivalent****, $p \sim_u q$, if there exist a unitary $u \in A$ such that $upu^* = q$.*

3. ***homotopic****, $p \sim_h q$, if there exist a continuous map $\gamma\colon [0,1] \to A$ such that $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma(t)$ is a projection for all $t \in [0,1]$.*

In a general C*-algebra there are implications

$$p \sim_h q \implies p \sim_u q \implies p \sim_v q.$$

Let $M_\infty(A) = \bigcup_{n\geq 1} M_n(A)$. Then these three notions of equivalence coincide in $M_\infty(A)$.

Denote by $P(A)$ the set of projections in $M_\infty(A)$. We have the following structure:

- Semigroup, for $p \in M_n(A)$, $q \in M_n(A)$

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{n+m}(A).$$

- A projection $p \in M_n(A)$ is equivalent to $q \in M_m(A)$, $n \leq m$, if and only if $p \oplus 0_{m-n} \sim q$ in $M_m(A)$.

- Projections p and q are stably isomorphic if and only if $p \oplus r \sim q \oplus r$ for some projection $r \in P(A)$.

- The set of stable equivalence classes of projections in $P(A)$ with the addition induced from $P(A)$ is denoted by $[P(A)]$.

- Two pairs $([p_1],[p_2])$ and $([q_1],[q_2])$ are equivalent if and only if

$$[p_1] \oplus [q_2] = [p_2] \oplus [q_1].$$

Definition 2.3. *The set of equivalence classes of pairs* $([p_1],[p_2])$ *with componentwise addition is an abelian group denoted by* $\mathrm{K}_0(A)$*.*

Example 2.4. If $A = \mathbb{C}$, then two projections in $M_n(\mathbb{C})$ are homotopic if and only if they have the same rank. It follows that $\mathrm{K}_0(\mathbb{C}) = \mathbb{Z}$.

Example 2.5. If $\mathcal{H}$ is a separable Hilbert space, and $A = B(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$, then two projections $p, q \in B(\mathcal{H})$ are equivalent in the sense of Murray- von Neumann if and only if there exists a unitary isomorphism from the range of p to the range of q. The set of projections in $B(\mathcal{H})$ can be indexed by the dimension of the range (including 0 and ∞). Thus any two projections of infinite range are equivalent. If $p \in B(\mathcal{H})$ is any projection, then $p \oplus 1 \sim 0 \oplus 1$, $[p] + [1] = [0] + [1]$ in $\mathrm{K}_0(A)$, so $[p] = [0] = 0$ in $\mathrm{K}_0(A)$, and $\mathrm{K}_0(B(\mathcal{H})) = 0$.

Proposition 2.6.

1. *K_0 is a covariant functor. If $\phi\colon A \to B$ is a homomorphism of C^*-algebras, then there is an induced map $\phi_*\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$.*
2. *If $\phi_0, \phi_1\colon A \to B$ are homotopic homomorphisms then $\phi_{0*} = \phi_{1*}\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$.*
3. *If A is a unital C^*-algebra and $A_1 \subset A_2 \subset A_3 \subset \ldots$ is an increasing sequence of unital C^*-algebras whose union is dense in A then $\varinjlim \mathrm{K}_0(A_n) = \mathrm{K}_0(A)$.*

For any nonunital C*-algebra J there exists an unique (up to isomorphism) unital C*-algebra $\widetilde{J}$ which contains J as an ideal of codimension 1.

$$0 \to J \to \widetilde{J} \to \mathbb{C} \to 0.$$

Define $\mathrm{K}_0(J) := \ker(\mathrm{K}_0(\widetilde{J}) \to \mathrm{K}_0(\mathbb{C}))$. When J is unital, then $\mathrm{K}_0(\widetilde{J}) = \mathrm{K}_0(\mathbb{C}) \oplus \mathrm{K}_0(J)$.

2.2 Unitizations and multiplier algebras

There are at least two ways to adjoin a unit to a C*-algebra A.

1. Represent A on a Hilbert space $\mathcal{H}$. The image of A in $B(\mathcal{H}))$ may not contain 1, even if A is unital, as the following example shows

$$\mathbb{C} \to M_2(\mathbb{C}), \quad \mu \mapsto \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}.$$

   Let $\widetilde{A}$ be the C*-subalgebra of $B(\mathcal{H})$ generated by A and 1. It contains 1 as an ideal of codimension 1.

2. Use the left multiplication to represent A on the Banach space A. Regard $\widetilde{A}$ as generated by A and 1.

Is there a reasonable maximal unitization?

Definition 2.7. *A is an* ***essential ideal*** *in a C^*-algebra B if and only if for all $b \in B$ if $bA = \{0\}$ then $b = 0$.*

There is a unique (up to isomorphism) unital C*-algebra which contains A as an essential ideal and is maximal in the sense that it contains any other algebra with this property. This is the **multiplier algebra** $M(A)$.

We will give an interpretation of the two, minimal and maximal, unitizations, in the case of commutative C*-algebras. Let $A = C_0(X)$, and B a unital commutative C*-algebra, $B = C(Y)$ for a compact space Y. Then the inclusion

$$A = C_0(X) \hookrightarrow C(Y) = B$$

corresponds to inclusion of X as an open subset in Y, and is given by extension by 0. Then A is essential in B if and only if X is dense in Y, that is Y is a compactification of X. The minimal choice of compactification is the one-point compactification X^+. Then $B = \widetilde{A}$. The maximal choice is the Stone Čech compactification βX. Then $M(C_0(X)) = C(\beta X)$.

2.3 Stabilization

Stabilization map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

is an example of a nonunital C*-algebra morphism $A \to M_n(A)$ even when A is unital.

Proposition 2.8. *The stabilization map induces an isomorphism in K-theory for all n.*

Proof. For all k there is an isomorphism $M_k(M_n(A)) \simeq M_{kn}(A)$, so any matrix in $M_k(M_n(A))$ can be regarded as a projection in $M_{kn}(A)$ which provides the two-sided inverse to the stabilization map. $\square$

Example 2.9. Take $M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset M_8(\mathbb{C}) \subset \dots$. The direct limit $\bigcup_{n\geq 1} M_{2^n}(\mathbb{C})$ is dense in $\mathcal{K}$, so

$$\varinjlim \mathrm{K}_0(M_{2^n}(\mathbb{C})) = \mathrm{K}_0(\mathcal{K}) \implies \mathrm{K}_0(\mathcal{K}) = \mathbb{Z}.$$

By applying similar argument to $M_n(A)$ we get the following stability property.

Proposition 2.10. *For any C*-algebra A and the algebra of compact operators $\mathcal{K}$ there is an isomorphism*

$$\mathrm{K}_0(A) = \mathrm{K}_0(A \otimes \mathcal{K}).$$

2.4 Higher K-theory

Let A be a unital C*-algebra. Define the **cone** of A as a C*-algebra

$$CA := \{f\colon [0,1] \to A \mid f \text{ is continuous, } f(0) = 0\}.$$

This is a contractible algebra, and a map $\phi_s\colon CA \to CA$ given by

$$\phi_s(f)(t) = f(ts), \quad s \in [0,1]$$

gives a homotopy between $\mathrm{id}\colon A \to A$ $(s = 1)$ and $0\colon A \to 0$ $(s = 0)$.

Define the **suspension** of A as a C*-algebra

$$SA := \{f \in CA \mid f(1) = 0\}.$$

There is a suspension extension

$$0 \to SA \to CA \to A \to 0.$$

Definition 2.11. *The **higher K-theory** groups are defined by*

$$\mathrm{K}_1(A) := \mathrm{K}_0(SA) = \mathrm{K}_0(C_0(\mathbb{R}) \otimes A)$$
$$\mathrm{K}_p(A) := \mathrm{K}_0(S^pA) = \mathrm{K}_0(C_0(\mathbb{R}^p) \otimes A)$$

2.5 Excision and relative K-theory

Let J be an ideal in a C*-algebra A,

$$0 \to J \to A \to A/J \to 0.$$

Then the induced sequence of K_0-groups

$$\mathrm{K}_0(J) \to \mathrm{K}_0(A) \to \mathrm{K}_0(A/J)$$

is exact in the middle (half-exactness). If the sequence is split-exact, then K_0 is additive, $\mathrm{K}_0(A) = \mathrm{K}_0(J) \oplus \mathrm{K}_0(A/J)$.

Definition 2.12. *A **relative cycle** is a triple (p, q, x), where p, q are projections in $M_n(A)$ for some n, and $x \in M_n(A)$ is such that $\pi(x) \in M_n(A/J)$ for $\pi\colon A \to A/J$ is a partial isometry implementing the Murray-von Neumann equivalence between $\pi(p)$ and $\pi(q)$.*

*Such a triple is **nondegenerate** if and only if x provides the Murray-von Neumann equivalence between p and q.*

Definition 2.13. ***Relative K-theory** group $\mathrm{K}_0(A, A/J)$ is the abelian group with one generator $[p, q, x]$ for each relative cycle modulo homotopy equivalence and degeneracy.*

If J is an ideal in a unita algebra A, then $\widetilde{J}$ may be regarded as a subalgebra of A. The **excision map** is a homomorphism

$$\mathrm{K}_0(J) = \mathrm{K}_0(\widetilde{J}, \mathbb{C}) \to \mathrm{K}_0(A, A/J).$$

Theorem 2.14. *The excision map $\mathrm{K}_0(J) \to \mathrm{K}_0(A, A/J)$ is an isomorphism.*

Example 2.15. Let D be the open unit disc in $\mathbb{R}^2$, $A = C(\overline{D})$. Let $J = C_0(D)$ - continuous functions on $\overline{D}$ which vanish on ∂D. Then $A/J = C(\partial D)$.

The inclusion $\overline{D} \hookrightarrow \mathbb{C}$ can be regarded as an element of A. The triple $(1, 1, \bar{z})$ defines a relative K-cycle in $\mathrm{K}_0(C(\overline{D}), C(\partial D))$. By excision this gives an element of $\mathrm{K}_0(C_0(D))$. Since $D \simeq \mathbb{R}^2$ we have an element $b \in \mathrm{K}_0(C_0(\mathbb{R}^2))$. This is the **Bott generator**. Under the isomorhism $\mathrm{K}_0(C_0(\mathbb{R}^2)) \simeq \mathbb{Z}$, the Bott generator b is mapped to $1 \in \mathbb{Z}$.

Definition 2.16. *The **mapping cone** of a surjective morphism $\pi\colon A \twoheadrightarrow B$ of C*-algebras is the C*-algebra*

$$C(A, B) := \{(a, f) \mid a \in A,\ f\colon [0,1] \to B \textit{ is continuous },\ f(0) = 0,\ f(1) = \pi(a)\}.$$

If $\pi = \mathrm{id}\colon A \to A$ then $C(A, A) = CA$. This construction is useful in the following situation. If J is an ideal in A, $\pi\colon A \to A/J$, we get $C(A, A/J)$. There is a map $C(A, A/J) \to A$, $(a, f) \mapsto a$. An element (a, f) is in the kernel of this map if and only if $a = 0$ and $f(1) = 0$. Since $f(0) = 0$ by definition, this means that $f \in S(A/J)$. Thus we have the following exact sequence

$$0 \to S(A/J) \to C(A, A/J) \to A \to 0,$$

where the first map is given by $f \mapsto (0, f)$.

There is also a homomorphism $J \to C(A, A/J)$ given by $a \mapsto (a, 0)$.

Proposition 2.17. *Excision map $\mathrm{K}_0(J) \to \mathrm{K}_0(C(A, A/J))$ is an isomorphism.*

By applying K_0 to the above exact sequence we get

$$0 \to \mathrm{K}_0(S(A/J)) \to \mathrm{K}_0(C(A, A/J)) \to \mathrm{K}_0(A) \to 0.$$

Using the definition of K_1 and the isomorphism in proposition we can write a sequence

$$\mathrm{K}_1(A/J) \to \mathrm{K}_0(J) \to \mathrm{K}_0(A) \to \mathrm{K}_0(A/J),$$

which is exact at $\mathrm{K}_0(J)$ and $\mathrm{K}_0(A)$. By iterating this we obtain

Proposition 2.18. *Let $0 \to J \to A \to A/J \to 0$ be a short exact sequence of C^*-algebras. Then there is a natural exact sequence of abelian groups.*

$$\ldots \mathrm{K}_{n+1}(A/J) \to \mathrm{K}_n(J) \to \mathrm{K}_n(A) \to \mathrm{K}_n(A/J) \to \mathrm{K}_{n-1}(J) \to \ldots \to \mathrm{K}_0(A/J).$$

Example 2.19. Consider a Hilbert space $\mathcal{H}$ and an exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) \to 0,$$

where $\mathcal{Q}(\mathcal{H})$ is the Calkin algebra. Take $T \in B(\mathcal{H})$ such that $T^*T - 1 \in \mathcal{K}(\mathcal{H})$ and $TT^* - 1 \in \mathcal{K}(\mathcal{H})$ (T is **essenitally unitary**). Then $(1, 1, T)$ is a relative K-cycle for $(B(\mathcal{H}), \mathcal{Q}(\mathcal{H}))$,

$$\pi(T)^*\pi(T) = 1, \quad \pi(T)\pi(T)^* = 1.$$

By excision and computation of $\mathrm{K}_0(\mathcal{K}(\mathcal{H}))$ we have

$$\mathrm{K}_0(B(\mathcal{H}), \mathcal{Q}(\mathcal{H})) = \mathrm{K}_0(\mathcal{K}(\mathcal{H})) = \mathbb{Z}, \quad [T] \mapsto m \in \mathbb{Z}.$$

Let p be an orthogonal projection onto $\ker T$, and q an orthogonal projection onto $\ker T^*$. Then

$$(1, 1, T) = (p, q, 0) + (1 - p, 1 - q, T(1 - p)).$$

The second cycle is degenerated because T restricts to an invertible operator from $\mathrm{im}(1 - p)$ to $\mathrm{im}(1 - q)$. The cycle $(p, q, 0) \in \mathrm{K}_0(\widehat{\mathcal{K}}, \mathbb{C})$ corresponds to

$$\dim \mathrm{im}\, p - \dim \mathrm{im}\, q = \mathrm{Index}(T).$$

To summarise, the relative K-theory leads to half-exactness of K-theory and the cone construction provides the connecting homomorphism ∂ in the and long exact sequence in K-theory. Bott periodicity provides a six term exact sequence

$$\begin{tikzcd}
\mathrm{K}_0(J) \arrow[r] & \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A/J) \arrow[d] \\
\mathrm{K}_1(A/J) \arrow[u] & \mathrm{K}_1(A) \arrow[l] & \mathrm{K}_1(J) \arrow[l]
\end{tikzcd}$$

We will give a more explicit description of $\mathrm{K}_1(A)$.

Definition 2.20. *Let A be a unital C^*-algebra. Denote by $\mathrm{K}_1^u(A)$ the abelian group with one generator for each unitary matrix in $\mathrm{GL}_n(A)$, subject to the following relations.*

1. *If $u, v \in \mathrm{GL}_n(A)$ can be joined by a path of unitaries in $\mathrm{GL}_n(A)$ then $[u] = [v]$.*
2. $[1] = [0]$.

3. $[u] + [v] = [u \oplus v]$

For unitaries $u, v \in \mathrm{GL}_n(A)$ we write $u \sim v$ if u and v can be joined by a path of unitaries. Then $u \oplus 1 \sim 1 \oplus u$ by using

$$R_t \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R_t^*, \quad R_t = \begin{pmatrix} \cos \frac{\pi t}{2} & \sin \frac{\pi t}{2} \\ -\sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix}.$$

Furthermore

$$u \oplus v \sim uv \oplus 1 \sim vu \oplus 1, \quad u \oplus u^* \sim 1 \oplus 1$$

$$[u] + [v] = [u \oplus v] = [uv \oplus 1] = [uv],$$

so addition in $\mathrm{K}_1^u(A)$ corresponds to matrix product.

Proposition 2.21. *For a unital C*-algebra A*

$$\mathrm{K}_1^u(A) \simeq \mathrm{K}_0(SA) = \mathrm{K}_1(A).$$

2.6 Products

For any unital C*-algebras A_1, A_2 there exists a bilinear associative product

$$\times\colon\ \mathrm{K}_i(A_1) \times \mathrm{K}_j(A_2) \to \mathrm{K}_{i+j}(A_1 \otimes_{min} A_2)$$

defined as follows.

1. If q_1, q_2 are projections in $M_k(A_1)$, $M_p(A_2)$, then $q_1 \otimes q_2$ is a projection in $M_{kp}(A_1 \otimes_{min} A_2)$ using $M_k(\mathbb{C}) \otimes M_p(\mathbb{C}) \simeq M_{kp}(\mathbb{C})$.
2. This gives rise to the product
$$\mathrm{K}_0(A_1) \otimes \mathrm{K}_0(A_2) \to \mathrm{K}_0(A_1 \otimes A_2).$$
3. This extends to nonunital algebras.
4. Now use suspension and the isomorphism $S^i A_1 \otimes S^j A_2 \simeq S^{i+j}(A_1 \otimes A_2)$ to get
$$\mathrm{K}_i(A_1) \otimes \mathrm{K}_j(A_2) \to \mathrm{K}_{i+j}(A_1 \otimes A_2).$$

2.7 Bott periodicity

Let $b \in \mathrm{K}_2(\mathbb{C}) = \mathrm{K}_0(C_0(\mathbb{R}^2))$ be the Bott generator. Taking the exterior product with b defines a map

$$\beta_A\colon\ \mathrm{K}_0(A) \to \mathrm{K}_0(A \otimes C_0(\mathbb{R}^2)) = \mathrm{K}_2(A).$$

Theorem 2.22 (Bott periodicity)**.** *For every C*-algebra A, the map β_A is an isomorphism.*

Proof. We shall use the Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0$$

Proposition 2.23. *The tensor product of a short exact sequence*

$$0 \to \mathcal{T}_1 \to A_1 \to A_1/\mathcal{T}_1 \to 0$$

with a C-algebra A_2 i.e. a sequence*

$$0 \to \mathcal{T}_1 \otimes A_2 \to A_1 \otimes A_2 \to A_1/\mathcal{T}_1 \otimes A_2 \to 0$$

remains exact if either

1. *the surjection $A_1 \to A_1/\mathcal{T}_1$ has completely positive section $s\colon A_1/\mathcal{T}_1 \to A_2$, or*

2. *A_2 is nuclear.*

A linear map $f\colon A \to B$ of C*-algebras is **positive** if and only if $f(x) \geq 0$ for all $x \geq 0$. It is **completely positive** if and only if $f_n\colon M_n(A) \to M_n(B)$, $(a_{ij}) \mapsto (f(a_{ij}))$ is positive for all n.

Proposition 2.24. *The Toeplitz extension has completely positive section $C(S^1) \to \mathcal{T}$, $f \mapsto T_f$.*

Remark that the map $f \mapsto T_f$ is not an algebra homomorphism.

Using the two propositions above we get that for every C*-algebra A ther is an exact sequence.

$$0 \to \mathcal{K} \otimes A \to \mathcal{T} \otimes A \to C(S^1) \otimes A \to 0.$$

The boundary map of this sequence is

$$\partial\colon \mathrm{K}_1(C(S^1) \otimes A) \to \mathrm{K}_0(\mathcal{K} \otimes A) \simeq \mathrm{K}_0(A).$$

Regard S^1 as a one-point compactification of $\mathbb{R}$. Restrict to $C_0(\mathbb{R}) \otimes A$. Then we have

$$\alpha_A\colon \mathrm{K}_2(A) = \mathrm{K}_1(C_0(\mathbb{R}) \otimes A) \to \mathrm{K}_0(A).$$

We will prove, after Atiyah, that α_A is an inverse to β_A with respect to the exterior product. The proof depends on the following formal properties of α_A

1. $\alpha_{\mathbb{C}}(b) = 1$. If u is a unitary-valued function on S^1, then $\alpha_{\mathbb{C}}\colon [u] \to T_u$ is the minus winding number of u. Furthermore $b = (1, 1, \bar{z}) \mapsto 1$.

2. for all A, B the following diagram is commutative

$$\begin{tikzcd}
\mathrm{K}_2(A) \otimes \mathrm{K}_0(B) \arrow[r] \arrow[d, "\alpha_A \otimes \mathrm{id}"'] & \mathrm{K}_2(A \otimes B) \arrow[d, "\alpha_{A\otimes B}"] \\
\mathrm{K}_0(A) \otimes \mathrm{K}_0(B) \arrow[r] & \mathrm{K}_0(A \otimes B)
\end{tikzcd}$$

(α_A is right linear over $\mathrm{K}_0(B)$, $\alpha_{A\otimes B}(x \times y) = \alpha_A(a) \times Y$).

We have from (1) that $\alpha_A \beta_A = \mathrm{id}$ for $A = \mathbb{C}$. In general if $x \in \mathrm{K}_0(A)$ then from (2)

$$\alpha_A \beta_A(x) = \alpha_A(b \times x) = \alpha_A(b \times x) = \alpha_{\mathbb{C}}(b) \times x = 1 \times x = x.$$

$$\alpha_{\mathbb{X}\otimes A}(b \times x) = \alpha_{\mathbb{C}}(b) \times x = 1 \times x = x.$$

Thus β_A is injective. The idea of Atiyah's proof is to use $\alpha_A\beta_A = \mathrm{id}$ to prove that $\beta_A\alpha_A = \mathrm{id}$. Consider two flip isomorphisms:

$$\begin{aligned}\sigma &\colon A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes A \\ \tau &\colon C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2)\end{aligned}$$

which interchange the first and last terms in the tensor products.

For any $y \in \mathrm{K}_0(A \otimes C_0(\mathbb{R}^2))$

$$\tau_*(b \times y) = \sigma_*(y) \times b.$$

The map induced by τ on K-theory is the identity. Now

$$y = \alpha_{A\otimes C_0(\mathbb{R}^2)}(\beta_{A\otimes C_0(\mathbb{R}^2)}(y)) = \alpha_{A\otimes C_0(\mathbb{R}^2)}(b \times y) = \alpha_{A\otimes C_0(\mathbb{R}^2)}(\sigma_*(y) \times b) = \alpha_A(\sigma_*(y)) \times b$$

Applying σ_* to both sides we obtain

$$\sigma_*(y) = \sigma_*\beta_A\alpha_A\sigma_*(y).$$

But $\sigma_*^2 = \mathrm{id}$ and y was arbitrary, so $\beta_A\alpha_A = \mathrm{id}$. □

2.8 Cuntz's proof of Bott periodicity

We will give another proof of Bott periodicity, due to Cuntz. Let E be a functor on some class of C*-algebras which is

1. homotopy invariant,
2. half exact,
3. stable.

Then one can define higher E-functors E_n, $n \geq 0$. Moreover E is additive, that is if $\phi_1, \phi_2 \colon A \to B$ are C*-algebra morphisms such that $\phi_1(A)\phi_2(A) = 0$ then $\phi_1 + \phi_2 \colon A \to B$ is a C*-algebra morphism and $E(\phi_1 + \phi_2) = E(\phi_1) + E(\phi_2)$.

Theorem 2.25 (Cuntz)**.** *Let E be a functor with these properties. Then E satisfies Bott periodicity $E_2(A) \simeq E_0(A)$ for every C*-algebra for which E is defined.*

Proof. We start with Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\sigma} C(S^1) \to 0$$

Define $p\colon \mathcal{T} \to \mathbb{C}$ as the composition

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{\sigma} & C(S^1) & \xrightarrow{\varepsilon_1} & \mathbb{C} \\ & & & & \\ T_f & \longmapsto & f & \longmapsto & f(1) \end{array}$$

Then p has a right inverse $j\colon \mathbb{C} \to \mathcal{T}$. We want to prove, that $E(p)\colon E(\mathcal{T}) \to E(\mathbb{C})$, $E(j)\colon E(\mathbb{C}) \to E(\mathcal{T})$ are inverses of each other. The easy part is $\mathrm{id} = E(p \circ j) = E(p) \circ E(j)$ because $p \circ j\colon \mathbb{C} \to \mathbb{C}$ is the identity map.

Proposition 2.26. *The maps $E(j)\colon E(\mathbb{C}) \to E(\mathcal{T})$ and $E(p)\colon E(\mathcal{T}) \to E(\mathbb{C})$ estabilish an isomorphism $E(\mathbb{C}) \simeq E(\mathcal{T})$. Moreover for any C^*-algebra the maps*

$$\begin{aligned} &\mathrm{id}_A \otimes j\colon A = A \otimes \mathbb{C} \to A \otimes \mathcal{T} \\ &\mathrm{id}_A \otimes p\colon A \otimes \mathcal{T} \to A \end{aligned}$$

estabilish an isomorphism $E(A) \simeq E(A \otimes \mathcal{T})$.

Granted the proposition, the proof proceeds as follows. The extension

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T}_0 \arrow[r, "i"] & \mathcal{T} \arrow[r, "p"] & \mathbb{C} \arrow[r] \arrow[l, bend left, "j"] & 0 \end{tikzcd}$$

where by definition $\mathcal{T}_0 = \ker p$, is split and the sequence

$$0 \to A \otimes \mathcal{K} \to A \otimes \mathcal{T}_0 \to A \otimes C(S^1) \to 0$$

is exact. By proposition $E(\mathcal{T}_0 \otimes A) = 0$, so $E_0(A \otimes \mathcal{K}) \simeq E_1(A \otimes C_0(\mathbb{R})) = E_2(A)$. □

2.9 The Mayer-Vietoris sequence

Assume we have the pull-back diagram

$$\begin{tikzcd} A \arrow[r, "q_2"] \arrow[d, "q_1"'] & A_2 \arrow[d, "p_2"] \\ A_1 \arrow[r, "p_1"'] & B \end{tikzcd}$$

$$A = \{(a_1, a_2) \in A_1 \oplus A_2 \mid p_1(a_1) = p_2(a_2)\}.$$

Then there is an exact sequence

$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A_1) \oplus \mathrm{K}_0(A_2) \arrow[r] & \mathrm{K}_0(B) \arrow[d] \\ \mathrm{K}_1(B) \arrow[u] & \mathrm{K}_1(A_1) \oplus \mathrm{K}_1(A_2) \arrow[l] & \mathrm{K}_1(A) \arrow[l] \end{tikzcd}$$

We have only to assume that at least one of p_1, p_2 is surjective.

Example 2.27. For $n \geq 2$ the K-theory of Cuntz algebra O_n is

$$\begin{aligned} &\mathrm{K}_0(O_n) = \mathbb{Z}/(n-1)\mathbb{Z}, \\ &\mathrm{K}_1(O_n) = 0. \end{aligned}$$

From these computations it follows that $O_n \not\simeq O_m$.

Example 2.28. Noncommutative torus A_θ has the following K-theory

$$\begin{aligned} \mathrm{K}_0(A_\theta) = \mathbb{Z} \oplus \mathbb{Z} \\ \mathrm{K}_1(A_\theta) = \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

Example 2.29. For the free group on two generators F_2 the map

$$C^*(F_2) \to C^*_r(F_2)$$

induces an isomorphism in K-theory (K-amenability) which gives $\mathrm{K}_0(C^*_r(F_2))$, $\mathrm{K}_1(C^*_r(F_2))$.

Chapter 3

Hilbert modules

3.1 Definitions

Suppose that A is a commutative unital C*-algebra, that is $A = C(X)$ for some compact topological space X. If X happens to be a manifold then suppose that E is a Hermitian vector bundle over X. For instance we can take a fixed inner product space H and for all $t \in X$ let $H_t \subset H$ be a subspace. Then we can put

$$E := \{\xi\colon X \to H \mid \text{for all } t \in X,\ \xi(t) \in H_t\}.$$

Then E is a $C(X)$-module and has a $C(X)$-valued inner product

$$\langle \xi,\, \eta\rangle(t) \in \langle \xi(t),\, \eta(t)\rangle_H.$$

Definition 3.1. *If A is a C*-algebra (not necessarily unital or commutative), then an **inner product A-module** is a right A-module E with a compatible scalar multiplication*

$$\lambda(xa) = (\lambda x)a = x(\lambda a),\ \lambda \in \mathbb{C},\ x \in E,\ a \in A$$

together with a map (inner product) $E \times E \to A$ such that

1. $\langle x,\, \alpha y + \beta z\rangle = \alpha\langle x,\, y\rangle + \beta\langle x,\, z\rangle$
2. $\langle z,\, \alpha y\rangle = \langle x,\, y\rangle\alpha$
3. $\langle y,\, x\rangle = \langle x,\, y\rangle^*$
4. $\langle x,\, x\rangle \geq 0$ *(in A) and if* $\langle x,\, x\rangle = 0$ *then* $x = 0$.

There is a Cauchy-Schwartz inequality for $x, y \in E$

$$\langle y,\, x\rangle\langle x,\, y\rangle \leq \|\langle x,\, x\rangle\|\langle y,\, y\rangle.$$

Define a norm of $x \in E$ by $\|x\| := \|\langle x,\, x\rangle\|^{\frac{1}{2}}$. Then there is an inequality

$$\|\langle x,\, y\rangle\| \leq \|x\|\|y\|.$$

Definition 3.2. *If an inner product A-module E is complete with respect to $\|\cdot\|$ then it is called a Hilbert A-module.*

Example 3.3. A is a Hilbert A-module with respect to

$$\langle x,\, y\rangle = x^*y,\ \|x\|_H = \|x\|_A.$$

Similarly A^n is a Hilbert A-module with respect to

$$\langle x,\, y\rangle = \sum_{i=1}^{n} x_i^* y_i.$$

Example 3.4. If $\{E_i\}_{i=1}^n$ is a finite family of Hilbert A-modules, then $\bigoplus_{i=1}^n E_i$ is a Hilbert A-module with respect to

$$\langle x,\, y\rangle = \sum_{i=1}^{n} x_i^* y_i.$$

If $\{E_i\}_{i\in I}$ is an arbitrary family of Hilbert A-modules, then $\bigoplus_{i\in I} E_i$ is the space of sequences $(x_i)_{i\in I}$ such that $\sum_{i\in I}\langle x_i,\, x_i\rangle$ converges in A. Then

$$\langle x,\, y\rangle = \sum_{i\in I} x_i^* y_i.$$

converges by Cauchy-Schwartz inequality.

Example 3.5. If $\mathcal{H}$ is a Hilbert space, then the algebraic tensor product $\mathcal{H}\otimes_{alg} A$ has A-valued inner product

$$\langle \xi\otimes a,\, \eta\otimes b\rangle = \langle \xi,\, \eta\rangle a^*b,\ \xi,\eta\in H,\ a,b\in A.$$

The completion with respect to a Hilbert A-module norm is a Hilbert A-module denoted by $\mathcal{H}\otimes A$. If $\{e_i\}$ is an orthonormal basis for $\mathcal{H}$, then $\mathcal{H}\otimes A \simeq \bigoplus A_i$. When $\mathcal{H}$ is infinite dimensional, separable, then $\mathcal{H}\otimes A$ is denoted by $\mathcal{H}_A$.

Suppose E, F are Hilbert A-modules. Denote by $\mathcal{L}(E,F)$ the set of bounded, adjointable maps $t\colon E\to F$ that is such that there exists $t^*\colon F\to E$ for which

$$\langle tx,\, y\rangle_F = \langle x,\, t^*y\rangle_E,\ x\in E,\ y\in F.$$

For this to make sense, t needs to be A-linear, $t(xa) = t(x)a$. Not every bounded A-linear map has an adjoint (for example the inclusion $\{f\in C([0,1]) \mid f(1)=0\}\hookrightarrow C([0,1])$).

There is a composition

$$\mathcal{L}(E,F)\times\mathcal{L}(F,G)\to\mathcal{L}(E,G),$$

$$(t,s)\mapsto s\circ t.$$

It follows that $\mathcal{L}(E,E)$ is a C*-algebra.

Let E, F be Hilbert A-modules, $x\in E$, $y\in F$. Define for $z\in F$

$$\theta_{x,y}\colon F\to E,\quad \theta_{x,y}(z) = x\langle y,\, z\rangle.$$

Then $\theta_{x,y}\in\mathcal{L}(E,F)$, $(\theta_{x,y})^* = \theta_{y,x}$ and $\theta_{x,y}\theta_{u,v} = \theta_{x\langle x,\, y\rangle v} = \theta_{x,v\langle u,\, y\rangle}$. For $t\in\mathcal{L}(E,G)$, $s\in\mathcal{L}(G,F)$

$$t\theta_{x,y} = \theta_{tx,y},\quad \theta_{x,y}s = \theta_{x,s^*y}.$$

Denote by $\mathcal{K}(E,F)$ the closed linear span of $\{\theta_{x,y}\}$. We write $\mathcal{K}(E)$ for $\mathcal{K}(E,E)$, which is an analogue of compact operators.

Example 3.6. If $E = A$, then $\mathcal{K}(A) = A$ and the isomorphism is given by

$$\theta_{a,b} \mapsto m_{ab^*} \text{ (left multiplication)}$$

$$\theta_{1,1} = \mathrm{id}\colon A \to A.$$

If A is unital, then $\mathcal{K}(A) \simeq \mathcal{L}(A)$ and every $t \in \mathcal{L}(A)$ acts by $t(1)$.

Example 3.7. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{K}(H \otimes A) = \mathcal{K}(H) \otimes A$, where $\mathcal{K}(\mathcal{H})$ is the usual space of compact operators. Apply

Proposition 3.8. *Assume A is unital, E a Hilbert A-module. then the following are equivalent*

1. *E is a finitely generated projective A-module.*
2. *$\mathcal{K}(E) \simeq \mathcal{L}(E)$.*
3. *The identity map on E is compact.*
4. *$\mathrm{id}\colon E \to E$ is of finite rank.*

Proposition 3.9. *Let A, B, C be C*-algebras such that A is an ideal in B and let E be a Hilbert C-module. Suppose that $\alpha\colon A \to \mathcal{L}(E)$ is a nondegenerate *-homomorphism ($A \cdot E$ is dense in E). Then α extends uniqualy to a *-homomorphism $\bar{\alpha}\colon B \to \mathcal{L}(E)$. If α is injective and A is essential in B, then $\bar{\alpha}$ is injective.*

Proof. Let e_j be an approximate unit for A. For $b \in B$, $a_1, \ldots, a_n \in A$, $\xi_1, \ldots, \xi_n \in E$

$$\begin{aligned}
\|\sum_{i=1}^n \alpha(ba_i)\xi_i\| &\leq \lim_j \|\sum_{i=1}^n \alpha(be_j a_i)\xi_i\| \\
&= \lim_j \|\alpha(be_j)\sum_{i=1}^n \alpha(a_i)\xi_i\| \\
&\leq \|b\|\|\sum_{i=1}^n \alpha(a_i)\xi_i\|.
\end{aligned}$$

The map

$$\sum_{i=1}^n \alpha(a_i)\xi_i \mapsto \sum_{i=1}^n \alpha(ba_i)\xi_i$$

is well defined and continuous.

Since α is non-degenerate, it extends by continuity to a bounded map $\bar{\alpha}(b)$ on E. Similar argument shows that $\bar{\alpha}(b^*)$ is an adjoint for $\bar{\alpha}(b)$. □

Apply this when $C = E = A$, and $\alpha\colon A \to \mathcal{L}(A)$ is the canonical embedding. Then any C*-algebra B which contains A as an essential ideal embeds in $\mathcal{L}(A)$.

If B is a maximal essential extension of A (A is an essential ideal in B and if A is also an essential ideal in C, then $\mathrm{id}\colon A \to A$ extends to an embedding $\beta\colon C \to B$), then there is an injection $\beta\colon \mathcal{L}(A) \to B$ whose restriction to A is the identity map.

By proposition, the canonical embedding $\alpha\colon A \to \mathcal{L}(A)$ has an injective extension $\bar{\alpha}\colon B \to \mathcal{L}(A)$. We can apply the proposition again to A as an ideal in $\mathcal{L}(A)$. Then α has a unique extension to a *-homomorphism $\mathcal{L}(A) \to \mathcal{L}(A)$. There are two maps

$$\mathrm{id}, \bar{\alpha}\beta\colon \mathcal{L}(A) \to \mathcal{L}(A)$$

and $\bar{\alpha}\beta = \mathrm{id}$, so $\bar{\alpha}$ is surjective. $\mathcal{L}(A)$ is a unique maximal essential extension of A so $\mathcal{L}(A) = M(A)$.

Theorem 3.10. *Let A be a C^*-algebra. Then*

1. *$\mathcal{L}(A)$ is an essential extension of $\mathcal{K}(A)$ which is maximal in the above sense.*
2. *If a C^*-algebra B is maximal essential extension of A, then we have a *-isomorphism $B \xrightarrow{\simeq} \mathcal{L}(A)$ whose restriction to A is the canonical map $A \mapsto \mathcal{K}(A)$.*

Proposition 3.11. *Let A, C be C^*-algebras and E a Hilbert c-module. Suppose $\alpha\colon A \to \mathcal{L}(E)$ is a nondegenerate injective *-homomorphism and let B be the idealiser of α in $\mathcal{L}E$,*

$$B := \{s \in \mathcal{L}(E) \mid s\mathcal{L}(A) \subseteq \mathcal{L}(A),\ \mathcal{L}(A)s \subseteq \mathcal{L}(A)\}.$$

*Then α extends to a *-isomomorphism*

$$M(A) \xrightarrow{\simeq} B.$$

Theorem 3.12 (Kasparov)**.** *If E is a Hilbert module then $\mathcal{L}(E) \simeq M(\mathcal{K}(E))$.*

Proof. The inclusion map $i\colon \mathcal{K}(E) \to \mathcal{L}(E)$ is nondegenerate and the idealiser of $\mathcal{K}(E)$ is $\mathcal{L}(E)$. ☐

Example 3.13. For $A = \mathbb{C}$ we have $M(\mathcal{K}(\mathcal{H})) = \mathcal{L}(\mathcal{H})$ and an exact sequence

$$0 \to \mathcal{K}(A) \to M(A) \to M(A)/sK(A) \to 0.$$

We call $Q(A) := M(A)/A$ the **outer multiplier algebra**.

Definition 3.14. *The* ***stable multiplier algebra***

$$M^s(A) := M(A \otimes \mathcal{K}),$$

and the quotient

$$Q^s(A) := M(A \otimes \mathcal{K})/A \otimes \mathcal{K}$$

is the ***stable outer multiplier algebra****.*

Proposition 3.15. *For any C^*-algebra A*

$$\mathrm{K}_0(M^s(A)) = \mathrm{K}_1(M^s(A)) = 0.$$

Proof. Let v_i be a sequence of projections in $1 \otimes \mathcal{L}(\mathcal{H})$ with orthogonal ranges. If p is any projection in $M^s(A)$, then let $q := \sum_i v_i p v_i^*$

$$w := \begin{pmatrix} 0 & 0 \\ v_1 & \sum_i v_{i+1} v_i^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

The sums $\sum_i v_i p v_i^*$ and $\sum_i v_{i+1} v_i^*$ converge in $A \otimes \mathcal{K}$.

$$w^* w = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \quad w w^* = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

so $[p] + [q] = [q]$ in $\mathrm{K}_0(M^s(A))$.

For K_1 there is a similar argument and the Cuntz-Higson theorem that $\mathrm{U}(M^s(A))$ is contractible. ☐

For any C*-algebra A there is an isomorphism

$$\mathrm{K}_i(A) \xrightarrow{\simeq} \mathrm{K}_{i-1}(Q^s(A)).$$

3.2 Kasparov stabilization theorem

A Hilbert B-module E is countably generated if there exists a countable subset $X \subset E$ such that the smallest closed submodule of E containing X is E.

Theorem 3.16. *For every countably generated Hilbert B-module E there is an isomorphism*

$$\mathcal{H}_B \oplus B \simeq \mathcal{H}_B.$$

Proof. A variant of Gram-Schmidt orthogonalization. There exists $u \in \mathcal{L}(\mathcal{H}_B \oplus B, \mathcal{H}_B)$ such that $u^*u = 1_{\mathcal{H}_B \oplus B}$, $uu^* = 1_{\mathcal{H}_B}$. It implies that for every countably generated B-module H there exists a porjection $p \in \mathcal{L}(\mathcal{H}_B)$ such that $E \simeq p\mathcal{H}_B$. $\square$

3.3 Morita equivalence

Recall:

- A C*-algebra A is stable if and only if $A \simeq A \otimes \mathcal{K}$.
- Two C*-algebras A, B are stably isomorphic if and only if $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$.
- A Hilbert A-module E is full if and only if $\langle E, E\rangle$ is dense in A.

Suppose we have a C*-algebra, E, F are Hilbert A-modules. The space of compact operators $\mathcal{K}(E, F)$ from E to F is a right $\mathcal{K}(E)$-module and a left $\mathcal{K}(F)$-module with respect to the natural composition of maps.

$$\begin{tikzcd} E \arrow[r, "\beta"] \arrow[rr, bend left, "f\beta"] & E \arrow[r, "f"] \arrow[rr, bend right, "\alpha f"'] & F \arrow[r, "\alpha"] & E \end{tikzcd}$$

Let $B = \mathcal{K}(E)$, $G = \mathcal{K}(E, F)$. Then G is a right B-module and has a B-valued inner product

$$\langle s, t\rangle_B := s^*t, \quad s, t \in G.$$

Proposition 3.17. *Let A be a C*-algebra and E, F Hilbert A-modules. If E is full, then*

$$\mathcal{K}_B(G) \simeq \mathcal{K}_A(F),$$
$$\mathcal{L}_B(G) \simeq \mathcal{L}_A(F).$$

Proof. Let $t \in \mathcal{L}_A(F)$. The map $\alpha(t)\colon u \mapsto tu$, $u \in G$, is adjointable

$$\langle tu, v\rangle_B = (tu)^*v = u^*t^*v = \langle u, t^*v\rangle_B$$

so $\alpha(t) \in \mathcal{L}_B(G)$. Thus the left $\mathcal{L}_A(F)$-module structure on G provides a map $\alpha\colon \mathcal{L}_A(F) \to \mathcal{L}_B(G)$ which is a *-homomorphism. If $\alpha(t) = 0$ then $tu = 0$ for all $u \in G$. In particular

$$t\theta_{z,x}(y) = 0, \quad x, y \in E, \ z \in F$$

so $tz\langle x, y\rangle = 0$.

Now suppose E is full, so

$$\overline{F\langle E, E\rangle} = \overline{FA} = F.$$

Since $tF\langle E, E\rangle = \{0\}$ implies $t = 0$, we have that α is surjective.

Let $x, y \in E$, $z, w \in F$, $s = \theta_{z,x}$, $t = \theta_{w,y}$. Then $s, t \in G$ and $\alpha(\theta_{z\langle x, y\rangle, w}) = \theta_{s,t}$. Since G is generated as a normed linear space by the elements of the form s, t, and *-homomorphisms between C*-algebras have closed range, it follows that $\alpha(\mathcal{K}_A(F)) \supset \mathcal{K}_B(G)$.

On the other hand if E is full then elements of the form $\theta_{z\langle x, y\rangle, w}$ generate $\mathcal{K}_A(F)$, so $\alpha(\mathcal{K}_A(F)) \subset \mathcal{K}_B(G)$. We can now restrict α to $\mathcal{K}_A(F)$ to get $\mathcal{K}_A(F) \simeq \mathcal{K}_B(G)$.

For the second statement we use the fact that if algebras are isomorphic, then their multiplier algebras are also isomorphic. $\square$

Definition 3.18. *Two C*-algebras are* ***Morita equivalent****,* $A \sim_M B$ *if and only if there is a full Hilbert A-module* E *such that* $B \simeq \mathcal{K}_A(E)$ *(strong Morita equivalence due to Rieffel).*

Proposition 3.19. *Morita equivalence is an equivalence relation.*

Proof.

1. Reflexive: $A \simeq \mathcal{K}_A(A)$.

2. Symmetric: by proposition ($F = A$) if $B \simeq \mathcal{K}_A(E)$ and $G = \mathcal{K}_A(E, A)$ as B-modules, then $A \simeq \mathcal{K}_B(G)$.

3. Transitive: suppose $B \simeq \mathcal{K}_A(E)$, $C \simeq \mathcal{K}_B(F)$, E-full Hilbert A-module, F-fill Hilbert B-module. If $\iota\colon B \to \mathcal{L}_A(E)$ let $G := F \otimes_i E$. Then G is a full Hilbert A-module and $\iota_*\colon C \xrightarrow{\simeq} \mathcal{K}_A(G)$.

$\square$

Theorem 3.20. *Two σ-unital C*-algebras are Morita equivalent if and only if they are stably isomorphic.*

Proof. For any C*-algebra A

$$\mathcal{K}_A(\mathcal{H}_A) = \mathcal{K}_A(\mathcal{H} \otimes A) \simeq \mathcal{K}_{\mathbb{C}}(\mathcal{H}) \otimes \mathcal{K}_A(A) = \mathcal{K} \otimes A$$

so $A \sim_M \mathcal{K} \otimes A$. If A and B are stably isomorphic then

$$A \sim_M \mathcal{K} \otimes A \simeq \mathcal{K} \otimes B \sim_M B$$

so $A \sim B$ (we do not need σ-unitality here).

Suppose that $A \sim_M B$ and let $B \simeq \mathcal{K}_A(E)$. Then if A, B are σ-unital

$$\mathcal{K}_{\otimes} B \simeq \mathcal{K}_A(\mathcal{H} \otimes E) \simeq \mathcal{K}_A(\mathcal{H}_A) \simeq \mathcal{K} \otimes A.$$

$\square$

3.4 Tensor products of Hilbert modules

1. **Outer tensor products** For $i = 1, 2$ let B_i be a C*-algebras and E_i a Hilbert B_i-module. The Hilbert $B_1 \otimes_{min} B_2$-module $E_1 \otimes E_2$ is by definition the completion of the algebraic tensor product $E_1 \otimes_{alg} E_2$ in the norm $\|\xi\| := \|\langle \xi, \xi\rangle\|^{\frac{1}{2}}$, where for $\xi_i, \eta_i \in E_i$

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle := \langle \xi_1, \eta_1\rangle \otimes \langle \xi_2, \eta_2\rangle.$$

2. **Inner tensor products** Let A, B be two C*-algebras, E_1 a Hilbert A-module, E_2 a Hilbert B-module, and $\pi\colon A \to \mathcal{L}(E_2)$ a *-homomorphism. The Hilbert B-module $E_1 \otimes_\pi E_2$ (also denoted by $E_1 \otimes_A E_2$) is the Hausdorff completion of the algebraic tensor product $E_1 \otimes_{alg} E_2$ with respect to the norm $\|\xi\| := \|\langle \xi,\, \xi\rangle\|^{\frac{1}{2}}$, where for $\xi_i, \eta_i \in E_i$
$$\langle \xi_1 \otimes \xi_2,\, \eta_1 \otimes \eta_2\rangle := \langle \xi_2,\, \pi(\langle \xi_1,\, \eta_1\rangle)\eta_1\rangle.$$
The action of B is given by $(\xi_1 \otimes \xi_2)b := \xi_1 \otimes \xi_2 b$. Note that for $a \in A$, $\xi_1 \in E_1$, $\xi_2 \in E_2$ we have $\xi_1 \otimes \pi(a)\xi_2 = \xi_1 a \otimes \xi_2$.

Chapter 4

Fredholm modules and Kasparov's K-homology

4.1 Fredholm modules

For the two bounded operators P, Q on Hilbert space we write $P \sim Q$ if and only if they differ by a compact operator. We assume that A is a separable C*-algerba, not necessarily unital.

Definition 4.1. *An (ungraded)* ***Fredholm module*** *over A is given by the following data:*

1. *a separable Hilbert space $\mathcal{H}$,*

2. *a representation $\rho\colon A \to B(\mathcal{H})$,*

3. *an operator F on $\mathcal{H}$ such that for all $a \in A$*

$$\begin{aligned}(F^2 - 1)\rho(a) &\sim 0\\ (F - F^*)\rho(a) &\sim 0\\ F\rho(a) - \rho(a)F &\sim 0.\end{aligned}$$

The representation ρ is not required to be non-degenerate.

Definition 4.2. *Aa* $\mathbb{Z}_2$***-graded Fredholm module*** *over A is given by the same data as in definition (4.1) plus the following additional structure:*

1. *the Hilbert space is equipped with the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$,*

2. *for each $a \in \mathcal{H}$, $\rho(a)$ is even, $\rho(a) = \rho^+(a) \oplus \rho^-(a)$,*

$$\rho(a) = \begin{pmatrix} \rho^+(a) & 0 \\ 0 & \rho^-(a) \end{pmatrix}$$

   where $\rho^\pm$ is a representation on $\mathcal{H}^\pm$,

3. *F is odd,*

$$F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}, \quad u\colon \mathcal{H}^+ \to \mathcal{H}^-,\ v\colon \mathcal{H}^- \to \mathcal{H}^+.$$

The operators u, v are not independent: V is essentially the adjoint of u. We can rewrite the conditions of the original definition as follows

$$\begin{aligned}(uv-1)\rho^-(a) &\sim 0\\ (vu-1)\rho^+(a) &\sim 0\\ (u-v^*)\rho^+(a) &\sim 0\\ u\rho^+(a) &\sim \rho^-(a)u.\end{aligned}$$

Let $p \in \mathbb{N}$.

Definition 4.3. *A p-graded Fredholm module is a Fredholm module (H, ρ, F) as above for which there exist operators $\varepsilon_1, \ldots, \varepsilon_p$ such that*

$$\varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad , \varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0, ; i \neq j.$$

Example 4.4. Fredholm modules over $\mathbb{C}$. Assume that $\rho\colon \mathbb{C} \to B(\mathcal{H})$ is the unique unital representation. Then an ungraded Fredholm module is given by an essentially selfadjoint Fredholm operator F. This characterisation follows from Atkinson's theorem. Recall we defined a Fredholm operator to be an operator F such that $\ker F$, $\ker F^*$ are finite dimensional.

Theorem 4.5 (Atkinson). *Let $F \in B(\mathcal{H})$. Then then the following are equivalent*

1. *F is Fredholm.*
2. *The image of F in $\mathcal{Q}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is invertible.*
3. *There exist $G \in B(\mathcal{H})$ such that $1 - FG$, $1 - GF$ are compact.*

A graded Fredholm module is given by an essentially selfadjoint operator F of the form

$$F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$$

where u and v are Fredholm and $u \sim v^*$. By definition $\mathrm{Index}(F) = \mathrm{Index}(u)$.

Example 4.6. The pseudodifferential operator extension. Let M be a smooth manifold without boundary (not necessarily compact). Let S^*M be the cosphere bundle of M: take the cotangent bundle of M, delete the zero section (zero cotangent vectors), identify non-zero cotangent vectors which differ only by multiplication by a positive scalar (if M is equipped qith a Riemannian metric then S^*M can be identified with the space of unit length cotangent vectors).

There is an extension

$$0 \to \mathcal{K}(L^2(M)) \to \Psi DO(M) \to C_0(S^*M) \to 0$$

The outline of the construction is as follows. If M is an opent subset of $\mathbb{R}^n$, then suppose that σ is a complex valued function on T^*M which has the property (homogenity):

$$\sigma(x, t\xi) = \sigma(x, \xi), \ t \geq 1, \ |\xi| \geq 1.$$

Assume that σ is compactly supported in the M-direction, i.e. $\sigma(x, \xi)$ vanishes when x is outside some compact subset of M. Then the linear map $D_\sigma\colon C_c^\infty(M) \to C_c^\infty(M)$ given by the integral formula

$$D_\sigma f(x) := \frac{1}{(2\pi)^n} \int \sigma(x, \xi)\hat{f}(\xi)e^{i\langle x, \xi\rangle} d\xi,$$

where $\hat{f}$ denotes the Fourier transform of f, is an example of a pseudodifferential operator. Because σ is homogeneous, it defines a function σ_0 on the cosphere bundle S^*M, which is called the **symbol** of the operator D_σ.

*Proposition 4.7. The operator D_σ extends by continuity to a bounded linear operator on $L^2(M)$. The map which associates to each D_σ its symbol σ_0 extends to a *-homomorphism form the C*-algebra $\Psi DO(M)$ generated by all D_σ onto the C*-algebra $C_0(S^*M)$.*

The map $\Psi DO(M) \to C_0(S^*M)$ is called the **symbol map**.

This proposition gives the extension when M is an open subset of $\mathbb{R}^n$. The extension to manifolds is done as follows. If $M \subseteq \mathbb{R}^n$ is open and $g \in C_c^\infty(M)$ then the multiplication operator M_g is a pseudodifferential operator associated with the function $\sigma(x,\xi) = g(x)$, so $M_g \in \Psi DO(M)$. Next we use the invariance of pseudodifferential operators under smooth changes of coordinates. If $\Psi\colon M \to M'$ is a diffeomorphism of open sets in $\mathbb{R}^n$, then the transform under Ψ of an operator in $\Psi DO(M)$ with symbol σ_0 is an operator in $\Psi DO(M')$ with symbol $\Phi_*(\sigma_0)$ ($\Phi\colon M \to M'$ induces $C_c(M') \to C_c(M)$ by composition. Get unitary $u\colon L^2(M') \to L^2(M)$ by multiplying by $\sqrt{\mathrm{Jac}(f)}$ and then $T \in B(L^2(M)) \mapsto u^*TU \in B(L^2(M')))$. So we can define $\Psi DO(M)$ for any smooth manifold (using invariance plus partition of unity) to be a C*-algebra consisting of those $T \in B(L^2(M))$ such that

1. $\lim \|TM_{g_n} - T\| = 0 = \lim \|M_{g_n}T - T\|$ for some approximate unit g_n for $C_0(M)$
2. T commutes with $C_0(M)$ modulo compact operators
3. for each coordinate chart U and each $g \in C_0(U)$, the operator M_gTM_g belongs to $\Psi DO(M)$.

Symbol of T is well defined as an element of $C_0(S^*M)$.

The operator D_σ is Fredholm if and only if σ_0 is nowhere zero. Let $D \in M_k(\Psi DO(M))$ be a system of psedudodifferential operators whose symbol is a unitary matrix-valued function on S^*M. Then

$$\mathcal{H} = L^2(M)^k \oplus L^2(M)^k,\ F = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

together with a representation $\rho\colon C(M) \to B(L^2(M))$ by multiplication operators is a graded Fredholm module over $C(M)$.

This construction generalises Atiyah's definition of Ell. There is a pairing with K-theory. For a projection $p \in M_k(C(M))$

$$F_p := \begin{pmatrix} 0 & \rho(p)D^* \\ \rho(p)D & 0 \end{pmatrix}$$

is an operator on $\mathcal{H} = \rho(p)L^2(M)^k \oplus \rho(p)L^2(M)^k$, and

$$\langle [p],\, [D] \rangle := \mathrm{Index}(\rho(p)D\rho(p)).$$

Let A be a C*-algebra and $(\mathcal{H}, F)$ a Fredholm module over A. It extends to $M_n(A)$ and $\mathcal{H}_n := \mathcal{H} \otimes \mathbb{C}^n$, $F_n := F \otimes \mathrm{id}_n$.

Proposition 4.8. *Let $(\mathcal{H}, F)$ be a Fredholm module over A. There exists a unique additive map $\phi\colon \mathrm{K}_0(A) \to \mathbb{Z}$ such that for every projection $e \in M_n(A)$ we have $\phi([e]) = \mathrm{Index}(T)$, where $T\colon e\mathcal{H}_n^+ \to e\mathcal{H}_n^-$ is defined by $Tx = eF_nx$ for all $x \in \mathcal{H}_n^+$.*

4.2 Commutator conditions

In the definition of Fredholm module $(\mathcal{H}, F)$ we have a condition $[F, \rho(a)] \in \mathcal{K}$ for all $a \in A$. In Kasparov K-homology A has to be a separable C*-algebra. For more subtle invariants, Connes allows Fredholm modules over *-algebras $\mathcal{A}$, not necessarily C*-algebras. Most useful condition is that $[F, a] \in \mathcal{L}^p(\mathcal{H})$ for some $p \geq 1$. There is a fine balance to be struck here: the class of algebras we allow for Fredholm modules should still have a meaningful K-theory, fairly close to the K-theory for C*-algebras. Ideally we want K-theory with the same formal properties as K-theory for C*-algebras. Note that the K-theory for such algebras needs to be developed from scratch. A sensible class of C*-algebras may be determined using the following

Proposition 4.9 (Connes). *Let $\mathcal{A}$ be an involutive algebra, $(\mathcal{H}, F)$ an $(n+1)$-summable Fredholm module over $\mathcal{A}$ with the parity of n. Let A be the C*-algebra closure of $\mathcal{A}$ (in its action on $\mathcal{H}$). Let $\bar{\mathcal{A}}$ be the smallest involutive subalgebra of A containing $\mathcal{A}$ and stable under holomorphic functional calculus. Then $(\mathcal{H}, F)$ is an $(n+1)$-summable Fredholm module over $\bar{\mathcal{A}}$.*

From this one deduces that it is sufficient to restrict attention to local C*-algebras (pre C*-algebras).

Proposition 4.10. *Let $\mathcal{A}$ be a pre C*-algebra (local C*-algebra). Then*

1. *Any Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$ extends by continuity to a Fredholm module over the associated C*-algebra A.*
2. *The inclusion $\mathcal{A} \hookrightarrow A$ is an isomorphism on K-theory.*

Proposition 4.11 (Connes). *Suppose that $(\mathcal{H}, F)$ is a 1-summable Fredholm module and γ is an involution im[plementing the $\mathbb{Z}/2$-grading on $\mathcal{H}$. Then the map*

$$\tau \colon A \to \mathbb{C}, \; a \mapsto \tfrac{1}{2} \operatorname{Tr}(\gamma F[F, a])$$

is a trace on A.

Proof. Define

$$\mathcal{A} := \{a \in A \mid [F, a] \in \mathcal{L}^1(\mathcal{H})\}, \; \mathcal{A} \subset A.$$

We have

$$\begin{aligned}
\gamma F[F, a] &= \gamma a - \gamma F a F \\
&= \gamma a + F a \gamma F \\
&= a \gamma F^2 + F a \gamma F - F \gamma F + F a \gamma F \\
&= [F, a] \gamma F,
\end{aligned}$$

where we use $F^2 = 1$ or the equalities are modulo $\mathcal{K}$. Next

$$\begin{aligned}
\tau(ab) &= \tfrac{1}{2} \operatorname{Tr}(\gamma F[F, ab]) \\
&= \tfrac{1}{2} \operatorname{Tr}(\gamma F[F, a]b + \gamma F a[F, b]) \\
&= \tfrac{1}{2} \operatorname{Tr}([F, a]\gamma F b + [F, b]\gamma F a) \\
&= \tau(ba).
\end{aligned}$$

□

We call τ the **character** of the Fredholm module $(\mathcal{H}, F)$.

Theorem 4.12 (Connes). *Let A be a unital C^*-algebra equipped with a faithful positive trace τ, $\tau(1) = 1$. Let $(\mathcal{H}, F)$ be a Fredholm module over A such that*

$$\mathcal{A} := \{a \in A \mid [F, a] \in \mathcal{L}^1(\mathcal{H})\}$$

is a dense subalgebra of A and the restriction of τ to $\mathcal{A}$ is the character of the Fredholm module $(\mathcal{H}, F)$. Then A contains no nontrivial idempotents.

Proof. $\mathcal{A}$ is a subalgebra of A stable under holomorphic functional calculus. The inclusion $\mathcal{A} \hookrightarrow A$ induces an isomorphism $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(A)$. The trace τ takes integer values (this is the index map). If e is a projection then $\tau(e) = 0, 1$. Because τ is faithful $e = 0, 1$. □

Example 4.13. Let F_2 be the nonabelian free group on two generators. It acts on a tree (1-dimensional simplicial complex with no loops). Let Δ_0 be the set of vertices and Δ_1 be the set of edges. Denote by $[x, y]$ for $x, y \in \Delta_0$ the set of vertices on the unique path from x to y, and by x_0 the origin. For all $x \in \Delta_0 \setminus \{x_0\}$ let $\beta(x) \in \Delta_1$ be the unique edge containing x in $[x, x_0]$.

Lemma 4.14.

1. *The map $\beta\colon \Delta_0 \to \Delta_1$ is a bijection.*
2. *For a fixed $g \in F_2$, the set of $x \in \Delta_0$ such that $g\beta(g^{-1}x) \neq \beta(x)$ is finite and equals $[x_0, gx_0]$.*

Proof. 1. The inverse is given by

$$\beta^{-1}(\text{edge } u) := \text{vertex of } u \text{ further from } x_0.$$

2. $g\beta(g^{-1}x)$ is the edge containing x and lying in $[gx_0, x]$. Suppose $x \notin [gx_0, x]$.

□

Define a map $\mathcal{F}\colon l^2(\Delta_0) \to l^2(\Delta_1)$ by

$$\mathcal{F}\delta_x := \begin{cases} \delta_{\beta_x} & \text{for } x \neq x_0 \\ 0 & \text{for } x = x_0 \end{cases}$$

Proposition 4.15.

1. *$\mathcal{F}$ is an operator of index 1, $\mathcal{F}\mathcal{F}^* = 1$, $\mathcal{F}^*\mathcal{F} = 1 - p_{x_0}$, where $p_{x_0}\colon l^2(\Delta_0) \to \mathbb{C}\delta_{x_0}$.*
2. *Let π_0, π_1 be actions of F_2 on $l^2(\Delta_0)$, $l^2(\Delta_1)$. For all $g \in F_2$ the operator $\pi_1(g)\mathcal{F} - \mathcal{F}\pi_0(g)$ is of finite rank and $(l^2(\Delta_0) \oplus l^2(\Delta_1), \mathcal{F})$ is a Fredholm module.*

Let

$$\mathcal{A} := \{a \in C_r^*(F_2) \mid [\mathcal{F}, a] \in l^2(\Delta_0) \oplus l^2(\Delta_1)\}.$$

By the proposition $\mathbb{C}[F_2] \subset \mathcal{A}$, so $\mathcal{A}$ is dense in $C_r^*(F_2)$. Now from the Connes theorem one obtains the proof of the Kadison-Kaplansky conjecture

Theorem 4.16. *The algebra $C_r^*(F_2)$ has no nontrivial idempotents.*

Fredholm modules of this type can be constructed for any locally compact group acting on a tree (Julg, Vallette).

Theorem 4.17. *Let G be any locally compact group acting on a tree such that the stabiliser of any vertes is amenable. Then G is K-amenable.*

Remark 4.18. (Christian Voigt)

4.3 Quantised calculus of one variable

Let f be a function on $\mathbb{R}$. Find function algebras for which $df := [F, f]$ has a given regularity. Take $\mathcal{H} = L^2(\mathbb{R})$. The Hilbert transform is given by

$$(F\xi)(s) = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{|s-t|>\varepsilon} \frac{\xi(t)}{s-t} dt.$$

We have $F^2 = 1$ and $[F, f]$ is the operator on $L^2(\mathcal{H})$ associated to the kernel

$$k(s,t) = \frac{f(s) - f(t)}{s-t}$$

This can be transported to S^1 by some conformal map. Then we obtain a Fredholm module given by the data

$$\mathcal{H} = L^2(S^1), \quad F = 2P - 1,$$

where $P\colon L^2(S^1) \to H^2(S^1)$ is the orthogonal projection onto the Hardy space.

For any $f \in L^\infty(S^1)$

- $[F, f]$ is a finite rank operator if and only if f is a rational function.
- $[F, f]$ is compact if and only if f is of vanishing mean oscillation, that is for

$$M_a f := \sup_{|I| \leq a} \frac{1}{|I|} \int_I |f - I(t)| dt,$$

  where $I(t) = \frac{1}{|I|} \int f dx$, we have $\lim_{a \to 0} M_a f = 0$.

- $[F, f]$ is in $\mathcal{L}^p(\mathcal{H})$ if and only if f is in Besov space $B_p^{\frac{1}{p}}$, that is

$$\int \int |f(x+t) - 2f(x) + f(x-t)|^p t^{-2} dx dt < \infty.$$

4.4 Quantised differential calculus

Let $(\mathcal{A}, H, F)$ be a Fredholm module over an involutive algebra $\mathcal{A}$, n integer ≥ 0. We assume that the Fredholm module is even for n even and odd for n odd. In either case it is $(n+1)$-summable: $[F, a] \in \mathcal{L}^{n+1}(\mathcal{H})$ for all $a \in \mathcal{H}$.

For $k = 0$, put $\Omega^0 = \mathcal{A}$. For $k > 0$

$$\Omega^k := \operatorname{span}\{a^0[F, a^1] \dots [F, a^k] \mid a^j \in \mathcal{A}\}.$$

By Hölder inequality, $\Omega^k \subset \mathcal{L}^{\frac{n+1}{k}}(\mathcal{H})$. Put $\Omega^\bullet := \bigoplus_{k \geq 0} \Omega^k$. The product in $\Omega^\bullet$ is the operator product. We use the Leibniz rule for $[F, -]$ to check that if $\omega \in \Omega^k$ and $\omega \in \Omega^{k'}$ then $\omega\omega' \in \Omega^{k+k'}$

$$a^0[F, a^1] \dots [F, a^k] a^{k+1} = \sum_{j=1}^{k-1} (-1)^{k-j} a^0 [F, a^1] \dots [F, a^j a^{j+1}] \dots [F, a^{k+1}] +$$
$$+ (-1)^k a^0 a^1 [F, a^2] \dots [F, a^{k+1}].$$

It is a differential graded algebra (DGA) with differential $d\colon \Omega^k \to \Omega^{k+1}$ given by the graded commutator

$$d\omega = [F, \omega] = F\omega - (-1)^{|\omega|} \omega F.$$

It is a graded derivation, that is

$$d(\omega_1 \omega_2) = (d\omega_1)\omega_2 + (-1)^{|\omega_1|} d\omega_2, \; d^2 = 0.$$

4.5 Closed graded trace

We will define a supertrace $\mathrm{Tr}_s \colon \Omega^n \to \mathbb{C}$. If T is an operator on $\mathcal{H}$ such that $FT+TF \in \mathcal{L}^1(\mathcal{H})$ then put

$$\mathrm{Tr}'(T) := \tfrac{1}{2}\,\mathrm{Tr}(F(FT+TF)).$$

If $T \in \mathcal{L}^1(\mathcal{H})$, then put

$$\mathrm{Tr}'(T) := \mathrm{Tr}(T).$$

Now define $\mathrm{Tr}_s(\omega)$ for $\omega \in \Omega^n$

$$\mathrm{Tr}_s(\omega) := \begin{cases} \mathrm{Tr}'(\omega) & \text{for } n \text{ odd} \\ \mathrm{Tr}'(\gamma\omega) & \text{for } n \text{ even} \end{cases}$$

where γ is the involution implementing the $\mathbb{Z}/2$-grading on $\mathcal{H}$.

Proposition 4.19. Tr_s *is a closed graded trace.*

1. $\mathrm{Tr}_s(d\omega) = 0$
2. *If* $\omega \in \Omega^k$, $\omega' \in \Omega^{k'}$, $k+k'=n$, *then*

$$\mathrm{Tr}_s(\omega\omega') = (-1)^{kk'}\,\mathrm{Tr}_s(\omega'\omega)$$

Proof. In the odd case

$$F\omega + \omega F = [F,\omega] = d\omega,$$

and in the even case

$$F\gamma\omega + \gamma\omega F = -\gamma F\omega + \gamma\omega F = -\gamma[F,\omega] = -\gamma d\omega,$$

so for $\omega = d\eta$, $\mathrm{Tr}_s(\omega) = 0$.

For the trace condition, take $\omega \in \Omega^k$, $\omega' \in \Omega^{k'}$, $k+k'=n$, n odd.

$$\begin{aligned}
\mathrm{Tr}_s(\omega\omega') &= \tfrac{1}{2}\,\mathrm{Tr}(Fd(\omega\omega')) \\
&= \tfrac{1}{2}\,\mathrm{Tr}(F(d\omega)\omega' + (-1)^{|\omega|}F\omega(d\omega)) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|}(d\omega)F\omega' + (-1)^{|\omega|}F\omega d\omega') \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+1}F\omega' d\omega + (-1)^{|\omega|}F\omega) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+1}F\omega' d\omega + (-1)^{|\omega|+|\omega'|+1}F(d\omega')\omega) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+1}F\omega' d\omega + (-1)^{|\omega|+|\omega'|+1}F(d(\omega'\omega) + (-1)^{|\omega'|+1}\omega'(d\omega))) \\
&= \tfrac{1}{2}\,\mathrm{Tr}((-1)^{|\omega|+|\omega'|+1}Fd(\omega'\omega)) \\
&= \tfrac{1}{2}\,\mathrm{Tr}(Fd(\omega'\omega)).
\end{aligned}$$

If $|\omega| + |\omega'| = n$ and n is odd, then $|\omega|$, ω' cannot both be odd, so $|\omega||\omega'| = 0 \mod 2$ and

$$\mathrm{Tr}_s(\omega\omega') = (-1)^{|\omega||\omega'|}\,\mathrm{Tr}_s(\omega'\omega) = \mathrm{Tr}_s(\omega'\omega).$$

The even case is very similar, use the extra condition $F\gamma = -\gamma F$. ∎

Definition 4.20. *The character of the cycle* $(A, \mathcal{H}, F)$ *is the cyclic cocycle*

$$\tau_n(a^0, a^1, \dots, a^n) = \mathrm{Tr}_s\, a^0[F,a^1]\dots[F,a^n].$$

Difficult problem: provide an explicit formula for this cocycle in terms of data defining the Fredholm module. The cyclic cocycle seems to depend on n, but in fact there is no problem here.

Recall Connes' periodicity operator $S\colon\ \mathrm{HC}^n(\mathcal{A}) \to \mathrm{HC}^{n+2}(\mathcal{A})$,

$$\dots \mathrm{HH}^{n-1}(\mathcal{A},\mathcal{A}) \to \mathrm{HC}^n(\mathcal{A}) \to \mathrm{HC}^{n+2}(\mathcal{A}) \to \mathrm{HH}^{n+2}(\mathcal{A},\mathcal{A}) \to \mathrm{HC}^{n+1}(\mathcal{A}) \to \dots$$

Proposition 4.21. *The characters τ_{n+2q} satisfy*

$$\tau_{m+2} = -\frac{2}{m+2} S\tau_m,\ m = n+2q,$$

so the cocycles together determine a class in periodic cyclic cohomology

$$\mathrm{HP}^\bullet(\mathcal{A}) = \varinjlim(\mathrm{HC}^n(\mathcal{A}), S)$$

Definition 4.22. *Let $(\mathcal{A},\mathcal{H},F)$ be a finitely summable Fredholm module over an involutive algebra $\mathcal{A}$. The **Chern character** $\mathrm{ch}_\bullet(\mathcal{H},F) \in \mathrm{HP}^\bullet(\mathcal{A})$ is the periodic cyclic cohomology class whose components are the following cyclic cocycles for large enough n:*

$$(-1)^{\frac{n(n-1)}{2}}\Gamma\left(\frac{n}{2}+1\right)\mathrm{Tr}'(\gamma a^0[F,a^1]\dots[F,a^n])$$

for n even (even Fredholm module), and

$$\sqrt{2i}(-1)^{\frac{n(n-1)}{2}}\Gamma\left(\frac{n}{2}+1\right)\mathrm{Tr}'(\gamma a^0[F,a^1]\dots[F,a^n])$$

for n odd.

Remark 4.23. Let ΩA be the universal differential graded algebra, $\mathbb{N}$-graded. It is also $\mathbb{Z}/2$-graded algebra with respect to the Fedosov product

$$\omega_1 \circ \omega_2 = \omega_1\omega_2 + (-1)^{|\omega_1|} d\omega_1 d\omega_2.$$

Supertraces $\mathrm{Tr}\colon \Omega A \to \mathbb{C}$ are linear maps which satisfy the suspension conditions.

Theorem 4.24 (Connes, Cuntz-Quillen). *There is one-to-one correspondence between (harmonic) periodic cocycles and supertraces on ΩA.*

$$(\Omega A, b, B) \to \text{ (entire) cyclic type homology theories.}$$

4.6 Index pairing formula

From Atiyah and Kasparov we have the following result:

Proposition 4.25. *Let $\mathcal{A}$ be an involutive algebra, $(\mathcal{H},F)$ a Fredholm module over $\mathcal{A}$. For $q \in \mathbb{N}$ let $(\mathcal{H}_q,F_q)$ be the Fredholm module over $M_q(\mathcal{A}) = \mathcal{A}\otimes M_q(\mathbb{C})$, $\mathcal{H}_q = \mathcal{H}\otimes\mathbb{C}^q$, $F_q = F\otimes \mathrm{id}_q$. Extend the action of $\mathcal{A}$ on $\mathcal{H}$ to a unital action of $\widetilde{\mathcal{A}}$.*

1. *In the even case: let γ be the involution for $\mathbb{Z}/2$-grading and $e \in M_q(\widetilde{\mathcal{A}})$ be a projection. Then the operator*

   $$\pi_q^-(e)F_q\pi_q^+(e)\colon \pi_q^+(e)\mathcal{H}_q^+ \to \pi_q^-(e)\mathcal{H}_q^-$$

   is Fredholm. There is an additive map

   $$\varphi\colon\ \mathrm{K}_0(\mathcal{A}) \to \mathbb{Z}$$

   given by

   $$\varphi([e]) := \mathrm{Index}(\pi_q^-(e)F_q\pi_q^+(e)).$$

2. In the odd case: let $u \in \mathrm{GL}_q(\widetilde{\mathcal{A}})$, $E = \frac{1+F}{2}$. *Then*

$$E_q \pi_q(u) E_q \colon E_q \mathcal{H}_q \to E_q \mathcal{H}_q$$

is Fredholm. There is an additive map

$$\varphi([u]) := \mathrm{Index}(E_q \pi_q(u) E_q).$$

When A is a C*-algebra, $\mathrm{K}_1(A)$ in 2. is the topological K-theory $\mathrm{K}_1^{top}(A)$ as defined before.

In both cases, the index map depends only on the class

$$[(\mathcal{H}, F)] \in \mathrm{KK}_i(A, \mathbb{C}) = \mathrm{K}^i(A),$$

the K-homology of A. This can be regarded as a pairing

$$\mathrm{K}_i(A) \times \mathrm{K}^i(A) \to \mathbb{Z}.$$

Proposition 4.26. *For* $x \in \mathrm{K}_i(A)$

$$\varphi(x) = \langle x,\, \mathrm{ch}_*(\mathcal{H}, F)\rangle = \langle \mathrm{ch}(x),\, \mathrm{ch}(\mathcal{H}, F)\rangle.$$

On the right hand side in the proposition we have a pairing between K-theory and cyclic sohomology. A more symmetric formula would use a complementary Chern character on K-homology. Since Connes' construction, formulae were given for Chern characters in K-theory with values in $\mathrm{HP}_\bullet(\mathcal{A})$.

The pairing has simple definition. Let $\tau \in \mathrm{HC}^n(\mathcal{A})$. Take $\tau \otimes \mathrm{Tr} \colon M_k(\mathcal{A}) \to \mathbb{C}$ for every k,

$$\tau \otimes \mathrm{Tr}(a^0 \otimes T^0, a^1 \otimes T^1, \dots, a^n \otimes T^n) = \tau(a^0, \dots, a^n)\,\mathrm{Tr}(T^0, \dots . T^n).$$

Then

$$\langle [e],\, [\tau]\rangle = \frac{1}{m!}(\tau \otimes \mathrm{Tr})(e, e, \dots, e).$$

All this is explained in Quillen's higher traces paper.

4.7 Kasparov's K-homology

Let $(\rho, \mathcal{H}, F)$ be a Fredholm module, $U \colon \mathcal{H}' \to \mathcal{H}$ be a unitary isomorphism (preserving the grading if there is one). Then $(U^* \rho U, \mathcal{H}', U^* F U)$ is also a Fredholm module unitarily equivalent to $(\rho, \mathcal{H}, F)$.

Definition 4.27. *Suppose* $(\rho, \mathcal{H}, F_t)$ *is a family of Fredholm modules parametrized by* $t \in [0, 1]$*,* $\mathcal{H}$ *is fixed Hilbert space, and* F_t *varies with* t*. If the function* $t \mapsto F_t$ *is norm continuous, then we say that the family defines an* ***operator homotopy*** *between the Fredholm modules* $(\rho, \mathcal{H}, F_0)$ *and* $(\rho, \mathcal{H}, F_1)$ *an that these two Fredholm modules are* ***Operator homotopic****.*

Definition 4.28. *If* $(\rho, \mathcal{H}, F)$ *and* $(\rho, \mathcal{H}, F')$ *are Fredholm modules on* $\mathcal{H}$ *and* $(F - F')\rho(a)$ *is compact for all* $a \in A$*, then we call* F *a* ***compact perturbation*** *of* F'*.*

Compact perturbation impliest operator homotopy - the linear path from F to F' defines an operator homotopy.

One can perform a compact perturbation to make F exactly self adjoint, $F \mapsto \frac{1}{2}(F + F^*)$.

Definition 4.29. ***K-homology*** *of a C^*-algebra A, $\mathrm{K}^p(A)$, is an abelian group with one generator $[x]$ for each unitary equivalence class of graded Fredholm modules over A with the following relations:*

1. *If x_0 and x_1 are operator homotopic graded Fredholm modules, then $[x_0] = [x_1] \in \mathrm{K}^p(A)$.*
2. *If x_0 and x_1 are graded Fredholm modules then $[x_0 \oplus x_1] = [x_0] + [x_1]$ in $\mathrm{K}^p(A)$, where $x_0 \oplus x_1 = (A, \mathcal{H}_0 \oplus \mathcal{H}_1, \rho_0 \oplus \rho_1, F_0 \oplus F_1)$.*

We have $p = 0$ for graded, and $p = 1$ for ungraded Fredholm modules.

Remark 4.30. p-graded Fredholm modules give rise to lower K-homology $\mathrm{K}^{-p}(A)$ for all $p \in \mathbb{N}$.

$\mathrm{K}^p(A)$ is a contravariant functor in A. If $\alpha\colon A' \to A$ is a *-homomorphism, and $(\rho, \mathcal{H}, F)$ is a Fredholm A-module, then $(\rho \circ \alpha, \mathcal{H}, F)$ is a Fredholm A'-module. We have an induced map

$$\alpha^*\colon \mathrm{K}^p(A) \to \mathrm{K}^p(A').$$

Definition 4.31. *A Fredholm module $(\rho, \mathcal{H}, F)$ is degenerate if and only if*

$$\begin{aligned}
[\rho(a), F] &= 0 \\
\rho(a)(F^2 - 1) &= 0 \\
\rho(a)(F - F^*) &= 0
\end{aligned}$$

for all $a \in A$.

Proposition 4.32. *The class of a degenerate Fredholm module is zero in $\mathrm{K}^p(A)$.*

Proof. Let $x = (\rho, \mathcal{H}, F)$ be a degenerate Fredholm module. Then

$$x' := (\rho', \mathcal{H}', F'), \quad \mathcal{H}' := \bigoplus_{i=1}^{\infty}, \quad F' := \bigoplus_{i=0}^{\infty} F, \quad \rho' := \bigoplus_{i=0}^{\infty} \rho.$$

This is a Fredholm module, since x is degenerate. But $x \oplus x'$ is unitarily equivalent to x', so $[x \oplus x'] = [x] + [x'] = [x']$ and $[x] = 0$. ☐

Lemma 4.33. *For a graded Fredholm module $(\rho, \mathcal{H}, F)$ denote by $(\rho^{op}, \mathcal{H}^{op}, -F^{op})$ the Fredholm module with the opposite grading. This is the additive inverse to $(\rho, \mathcal{H}, F)$.*

Proof. Let

$$F_t := \begin{pmatrix} \cos(\frac{\pi}{2}t)F & \sin(\frac{\pi}{2}t)\mathrm{Id} \\ \sin(\frac{\pi}{2}t)\mathrm{Id} & -\cos(\frac{\pi}{2}t)F \end{pmatrix},$$

$$F_0 = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is the operator homotopy on $\mathcal{H} \oplus \mathcal{H}^{op}$ from $F_0 = F \oplus (-F^{op})$ to the degenerate $F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. ☐

Example 4.34. $\mathrm{K}^0(\mathbb{C}) = \mathbb{Z}$. If $(\rho, \mathcal{H}, F)$ is a Fredholm module over $\mathbb{C}$, then $\rho(1) =: p$ is a projection in $B(\mathcal{H})$ and up to compact perturbation $(\rho, \mathcal{H}, F)$ is the direct sum of $(\rho, p\mathcal{H}, pFp)$ and $(\rho, (1-p)\mathcal{H}, (1-p)F(1-p))$. The second module carries the zero action of $\mathbb{C}$. The first is determined by pFp. Put

$$(\rho, \mathcal{H}, F) \mapsto \mathrm{Index}(pFp).$$

This gives a homomorphism $\mathrm{K}^0(\mathbb{C}) \to \mathbb{Z}$. Since an essentially unitary operator with index zero is a compact perturbation of a unitary, this map is an isomorphism.

Lemma 4.35. *Let $(\rho, \mathcal{H}, F)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm module. Assume that there exists a self adjoint odd-graded involution $E \colon \mathcal{H} \to \mathcal{H}$ which commutes with ρ (the action of A) and anticommutes with F. Then the Fredholm module $(\rho, \mathcal{H}, F)$ represents the zero element in $\mathrm{K}^0(A)$.*

Proof. Let $F_t := \cos(t)F + \sin(t)E$. This is an operator homotopy from F to the degenerate operator E. □

In particular putting tho ungraded Fredholm modules together produces a degenerate Fredholm module. Conversely, if we ignore $\mathbb{Z}/2\mathbb{Z}$-grading on an even Fredholm module then the resulting odd Fredholm module represents the zero element. A possible argument is as follows. A $\mathbb{Z}/2\mathbb{Z}$-graded module is given by the data $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$,

$$F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho(a) & 0 \\ 0 & \rho(a) \end{pmatrix}.$$

We construct an operator homotopy

$$F_t = \begin{pmatrix} \cos(\frac{\pi}{2}t)\mathrm{Id} & \sin(\frac{\pi}{2}t)v \\ \sin(\frac{\pi}{2}t)u & \cos(\frac{\pi}{2}t)\mathrm{Id} \end{pmatrix}$$

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}.$$

For this we need to assume that F_1 is an involution.

Chapter 5

Boundary maps in K-homology

5.1 Relative K-homology

Definition 5.1. *Let J be an ideal in A. A* ***relative Fredholm module*** *for $(A, A/J)$ is a triple $(\rho, \mathcal{H}, F)$, where*

1. *$\mathcal{H}$ is a separable Hilbert space*
2. *$\rho\colon A \to B(\mathcal{H})$ is a *-representation*
3. *for all $a \in A$, $j \in J$*

$$\begin{aligned}(F^2 - 1)\rho(j) &\sim 0\\ (F - F^*)\rho(j) &\sim 0\\ [F, \rho(a)] &\sim 0\end{aligned}$$

One defines also the graded version.

The relative Fredholm modules generate **relative K-homology** $\mathrm{K}^p(A, A/J)$. The natural map

$$\mathrm{K}^p(A, A/J) \to \mathrm{K}^p(J)$$

is an isomorphism (excision).

To any extension of separable C*-algebras one can associate an exact sequence of lenght six

$$\begin{tikzcd} \mathrm{K}^1(A/J) \arrow[r] & \mathrm{K}^1(A) \arrow[r] & \mathrm{K}^1(J) \arrow[d] \\ \mathrm{K}^0(J) \arrow[u] & \mathrm{K}^0(A) \arrow[l] & \mathrm{K}^0(A/J) \arrow[l] \end{tikzcd}$$

We can give an explicit description of the boundary maps in this six term exact sequence.

5.2 Semisplit extensions

There is ono-to-one correspondence between extensions of A by $\mathcal{K}(\mathcal{H})$ and unitary equivalence classes of *-homomorphisms $\phi\colon A \to \mathcal{Q}(\mathcal{H})$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] & B(\mathcal{H}) \arrow[r, "\pi"] & \mathcal{Q}(\mathcal{H}) \arrow[r] & 0 \end{tikzcd}$$

Definition 5.2. *A unital injective extension $\phi\colon A \to \mathcal{Q}(\mathcal{H})$ is semisplit if there is another unital extension $\phi'\colon A \to \mathcal{Q}(\mathcal{H})$ such that $\phi \oplus \phi'$ is split extension.*

Definition 5.3. *Let the extension*

$$0 \to J \to A \to A/J \to 0$$

be semisplit by a completely positive map $\widetilde{A/J} \to \widetilde{A}$. Let $\rho\colon A \to B(\mathcal{H})$ be a representation of A on a separable Hilbert space $\mathcal{H}$. A ***Stinespring dilation*** *associates to the above data is a *-homomorphism*

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} : A/J \to B(\mathcal{H} \oplus \mathcal{H}'),$$

where $\mathcal{H}'$ is a separable Hilbert space and $\psi_{11}(x) = \rho(s(x))$.

The existence of such extension follows from Stinespring's theorem.

Theorem 5.4 (Stinespring)**.** *A unital linear map $\sigma\colon A \to B(\mathcal{H})$ is absolutely positive if and only if there are*

1. *an isometry $v\colon \mathcal{H} \to \mathcal{H}$*

2. *a nondegenerate representation $\rho\colon A \to B(\mathcal{H})$ such that $\sigma(a) = v^*\rho(a)v$*

In $\mathbb{Z}/2\mathbb{Z}$-graded case one applies this to each component separately.

Proposition 5.5. *Take an extension as above. Let $(\rho, \mathcal{H}, F)$ be a selfadjoint relative Fredholm module (graded or ungraded). Let $\psi\colon A/J \to B(\mathcal{H} \oplus \mathcal{H}')$ be a Stinespring dilaton. Then the boundary maps are given by*

1. *$\partial\colon \mathrm{K}^0(A, A/J) \to \mathrm{K}^1(A/J)$: the cycle $(\rho, \mathcal{H}, F)$ is graded Fredholm module. Assume that F^2 is a projection (this can always be done). Let $Q_\pm$ be the components of the projection $1 - F^2$ on $\mathcal{H}^\pm$. Then the projections*

$$\begin{pmatrix} Q_\pm & 0 \\ 0 & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H}')$$

commute modulo compacts with $\psi(x)$ for all $x \in A/J$ and so define ungraded Fredholm modules. Their difference represents a class of $\partial[\rho, \mathcal{H}, F]$ (if $\rho\colon A \to B(\mathcal{H})$, $P \in B(\mathcal{H})$ is a projection such that $[P, \rho(a)] \in \mathcal{K}$ for all $a \in A$, then $(\rho, \mathcal{H}, F = 2P - 1)$ is an ungraded Fredholm module over A).

2. *$\partial\colon \mathrm{K}^1(A, A/J) \to \mathrm{K}^0(A/J)$: the cycle $(\rho, \mathcal{H}, F)$ is ungraded Fredholm module. Then*

$$\begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H}')$$

is unitary on $\mathcal{H} \oplus \mathcal{H}'$ commuting with ψ. The corresponding Fredholm module represents a class of $\partial[\rho, \mathcal{H}, F]$.

5.3 Schrödinger pairs

Recall that we call an operator $X \in B(\mathcal{H})$ **contractive** if and only if $\|X\| \leq 1$. For a selfadjoint contractive operator X we define X^b to be a commutative C*-subalgebra of $B(\mathcal{H})$ consisting of all $\psi(X)$ for $\psi \in C_0(-1,1)$.

Definition 5.6. *Let X, Y be contractive operators on $\mathcal{H}$. The pair (X,Y) is a **Schrödinger pair** if and only if*

1. *Y commutes with X^b modulo $\mathcal{K}(\mathcal{H})$.*
2. *$X^b \cdot Y^b \subseteq \mathcal{K}(\mathcal{H})$.*

*We call (X,Y) a **graded Schrödinger pair** if the commutator in 1 is graded.*

*We call (X,Y) a **strong Schrödinger pair** if Y commutes with X.*

Example 5.7. Let $\mathcal{H} = L^2(\mathbb{R})$, X multiplies by $x \mapsto \frac{x}{\sqrt{1+x^2}}$, and Y multiplies the Fourier transform by $\xi \mapsto \frac{\xi}{\sqrt{1+\xi^2}}$

$$(Xf)(x) := \frac{x}{\sqrt{1+x^2}} f(x)$$
$$(Y\hat{f})(\xi) := \frac{\xi}{\sqrt{1+\xi^2}} \hat{f}(\xi)$$

These are the position and momentum operators in quantum mechanics and (X,Y) is a strong Schrödinger pair.

Example 5.8. Let $(\rho, \mathcal{H}, F)$ be an ungraded Fredholm module over J, where J is an ideal in some C*-algebra A and ρ extends to A. If a is an element of A such that $a^2 - 1 \in J$, then $X = \rho(a)$, $Y = F$ constitute Schrödinger pair.

If the extension of ρ makes $(\rho, \mathcal{H}, F)$ into a relative Fredholm module (i.e. $[F, \rho(a)] \sim 0$ for all $a \in A$) then (X,Y) is a strong Schrödinger pair.

Example 5.9. Let $\mathcal{H} = L^2([-1,1])$. Define operators

$$(T\psi)(x) = \psi(-x),$$
$$(S\psi)(x) = x\psi(x).$$

If $f \in C_0(-1,1)$ then $f(T) = 0$ and (T,S) is a Schrödinger pair. If f is an odd function on $[-1,1]$ such that $f(-1) = f(1) = 0$, then

$$[f(S), T] = -2f(S)T$$

which is not compact. Thus (S,T) is not a Schrödinger pair.

Definition 5.10. *Let (X,Y) be a Schrödinger pair. The **Schrödinger operator** is given by*

$$V(X,Y) := \varepsilon X + (1 - X^2)^{\frac{1}{2}} Y,$$

where $\varepsilon = i$ in the ungraded, and $\varepsilon = 1$ in the graded case.

Proposition 5.11. *Let (X,Y) be a (graded) strong Schrödinger pair. Then*

1. *in the ungraded case the Schrödinger operator*

$$V(X,Y) = iX + (1-X^2)^{\frac{1}{2}}Y$$

is essentially unitary and so Fredholm,

2. *in the graded case the Schrödinger operator*

$$V(X,Y) = X + (1-X^2)^{\frac{1}{2}}Y$$

is essentially selfadjoint graded and Fredholm.

Proposition 5.12.

1. *In the ungraded case*

$$\operatorname{Index} V(X,Y) = \operatorname{Index} V(Y,X).$$

2. *In the graded case*

$$\operatorname{Index} V(X,Y) = -\operatorname{Index} V(Y,X).$$

Proposition 5.13. *If X, Y commute modulo compacts with a representation of C^*-algebra B then*

1. *in the graded case*

$$[V(X,Y)] = [V(Y,X)] \in \mathrm{K}^0(B),$$

2. *in the ungraded case*

$$[V(X,Y)] = -[V(Y,X)] \in \mathrm{K}^1(B).$$

Proof.

$$\begin{aligned} V(X,Y)^2 - 1 &\sim X^2 + (1-X^2)Y^2 - 1 \\ &= -(1-X^2)(1-Y^2) \in \mathcal{K}(\mathcal{H}), \end{aligned}$$

because $XY + YX \sim 0$. Next

$$V(X,Y)V(Y,X) + V(Y,X)V(X,Y) \sim 2(Y(1-X^2)^{\frac{1}{2}}Y + X(1-Y^2)^{\frac{1}{2}}X) \geq 0,$$

so Fredholm modules associated with $V(X,Y)$ and $V(Y,X)$ are homotopic, which is a consequence of the following

Proposition 5.14. *If $(\rho, \mathcal{H}, F_0)$ and $(\rho, \mathcal{H}, F_1)$ are (graded) Fredholm modules such that $\rho(a)(F_0F_1 + F_1F_0)\rho(a^*)$ are positive modulo compacts for all $a \in A$, then F_0 and F_1 are operator homotopic.*

$\square$

Recall the index map on K-homology

$$\operatorname{Index}\colon \mathrm{K}^p(A) \to \mathbb{Z}, \quad (\rho, \mathcal{H}, F) \mapsto \inf F.$$

If $F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$, then $\operatorname{Index} F = \operatorname{Index} u$.

Lemma 5.15. *Let (X, Y) be a Schrödinger pair on an ungraded Hilbert space $\mathcal{H}$. Put $P_Y := \frac{1}{2}(1+Y)$. Then the operator*

$$W_1(X, Y) := e^{i\pi X} P_Y - (1 - P_Y)$$

is essentially unitary and Fredholm. Furthermore

$$\operatorname{Index} W_1(X, Y) = \operatorname{Index} V(X, Y).$$

Proof. Denote for convinience $S := \sin(\frac{\pi}{2} X)$. From the definition of Schrödinger pair we know that

- Y commutes with S^b modulo compacts,
- $(1 - S^2)(1 - Y^2)$ is compact.

Write

$$\begin{aligned}
e^{-i\frac{\pi}{2}X} W_1(X, Y) &= e^{i\frac{\pi}{2}X} P_Y - e^{-i\frac{\pi}{2}X}(1 - P_Y) \\
&= (\cos(\frac{\pi}{2}X) + i\sin(\frac{\pi}{2}X))P_Y - (\cos(\frac{\pi}{2}X) - i\sin(\frac{\pi}{2}X))(1 - P_Y) \\
&= ((1 - S^2)^{\frac{1}{2}} + iS)P_Y - ((1 - S^2)^{\frac{1}{2}} - iS)(1 - P_Y) \\
&= ((1 - S^2)^{\frac{1}{2}} + iS)(\tfrac{1}{2}(1 + Y)) - ((1 - S^2)^{\frac{1}{2}} - iS)(\tfrac{1}{2}(1 - Y)) \\
&= iS + (1 - S^2)^{\frac{1}{2}} Y \\
&= V(S, Y).
\end{aligned}$$

Thus the operator $W_1(X, Y)$ is essentially unitary and homotopic to $V(S, Y)$ through the path

$$t \mapsto e^{-it\frac{\pi}{2}X} W_1(X, Y),\ t \in [0, 1],$$

so $[W_1(X, Y)] = [V(S, Y)]$. But S is homotopic to X via a path $X_t := tX + (1 - t)S$, and (X_t, Y) are a Schrödinger pairs for all t. This gives a homotopy $V(X_t, Y)$ and $[V(X, Y)] = [V(S, Y)] = W_1(X, Y)$. $\square$

Lemma 5.16. *Let (X, Y) be a graded Schrödinger pair on a graded Hilbert space $\mathcal{H}$. Suppose also that*

1. *$Q_X = 1 - X^2$ is a projection*
2. *there exists a self adjoint involution Y_0 on $\mathcal{H}$ which commutes with Q_X modulo the compacts.*

Then the operator $W_2(X, Y) = YQ_X + Y_0(1 - Q_X)$ is essentially self-adjoint, graded, Fredholm, and

$$\operatorname{Index} W_2(X, Y) = \operatorname{Index} V(X, Y).$$

Proof. Let

$$X_t := \cos\left(\frac{\pi}{2}t\right) X + \sin\left(\frac{\pi}{2}t\right) Y_0(1 - Q_X).$$

Then for all $t \in [0,1]$ we have

$$\begin{aligned} X_t(YQ_X) &\sim 0 \\ (YQ_X)X_t &\sim 0 \\ X_t^2 &\sim 1 - Q_X \end{aligned}$$

Indeed:

$$\begin{aligned} X_t^2 &= \cos^2\left(\frac{\pi}{2}t\right) X^2 + \sin^2\left(\frac{\pi}{2}t\right) Y_0(1-Q_X)Y_0(1-Q_X) \\ &\quad + \cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right) XY_0(1-Q_X) + \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right) Y_0(1-Q_X)X \\ &= \cos^2\left(\frac{\pi}{2}t\right) X^2 + \sin^2\left(\frac{\pi}{2}t\right)(1-Q_X)^2. \end{aligned}$$

The operator Q_X is a projection onto $\ker X$. The path $t \mapsto YQ_X + X_t$ gives an operator homotopy from $V(X,Y)$ to $W_2(X,Y)$. Indeed:

$$Q_X X_t = \cos\left(\frac{\pi}{2}t\right) \underbrace{Q_X X}_{0} + \sin\left(\frac{\pi}{2}t\right) Q_X Y_0(1-Q_X) \in \mathcal{K}(\mathcal{H}).$$

Recall that $V(X,Y) = X + (1-X^2)^{\frac{1}{2}}Y$. Thus for

$$\begin{aligned} t = 0: &\quad YQ_X + X_t = YQ_x + X \sim V(X,Y), \\ t = 1: &\quad YQ_X + Y_0(1-Q_X) = W_2(X,Y). \end{aligned}$$

□

5.4 The index pairing

Proposition 5.17 (odd case)**.** *Let A be a unital C^*-algebra and suppose given*

1. *an ungraded unital Fredholm module $(\rho, \mathcal{H}, F)$ over A,*
2. *a unitary u in a matrix algebra $M_k(A)$ over A.*

Let $P_k = 1 \otimes \frac{1}{2}(1+F)\colon \mathcal{H}^k \to \mathcal{H}^k$ and $U = (1 \otimes \rho)(u)\colon \mathcal{H}^k \to \mathcal{H}^k$ be a unitary operator. Then:

1. *the operator $W := P_kUP_k - (1-P_k)\colon \mathcal{H}^k \to \mathcal{H}^k$ is essentially unitary, so Fredholm.*
2. *The Fredholm index of $W = P_kUP_k - (1-P_k)$ depends only on $[U] \in \mathrm{K}_1(A)$ and $[P] \in \mathrm{K}^1(A)$. This index gives a pairing*

$$\begin{aligned} \mathrm{K}_1(A) \times \mathrm{K}^1(A) &\to \mathbb{Z} \\ ([u],[F]) &\mapsto \mathrm{Index}(P_kUP_k - (1-P_k)). \end{aligned}$$

Assume that $F^2 = 1$ so that P_k is a projection. Then $\mathcal{H}^k = \operatorname{im} P_k \oplus \operatorname{im}(1-P_k)$ and $W = P_kUP_k \oplus (-\mathrm{Id})$ with respect to this decomposition. The second summand has index zero, and this is precisely the pairing that was defined before.

By definition of Fredholm module P_k and U commute modulo compacts $\mathcal{K}(\mathcal{H}^k)$ and

$$W^*W \sim P_kU^*UP_k + (1-P_k) \sim 1.$$

Thus $W^*W \sim 1$ and similarly $WW^* \sim 1$. The map $([u],[F]) \mapsto \operatorname{Index} W$ is additive and stable under compact perturbations and homotopies.

Proposition 5.18 (even case)**.** *Let $(\rho, \mathcal{H}, F)$ be a graded unital Fredholm module over A and let $p \in M_k(A)$ be a projection. Put $P = 1 \otimes \rho(p) \colon \mathcal{H}^k \to \mathcal{H}^k$ (projection) and write*

$$F = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$$

relative to the graded decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Then

1. *the operator $P(1 \otimes u)P \colon P(\mathbb{C}^n \otimes \mathcal{H}^+) \to P(\mathbb{C}^n \otimes \mathcal{H}^-)$ is essentially unitary and so Fredholm.*

2. *the pairing*
$$(p, F) \mapsto \operatorname{Index} P(1 \otimes u)P$$
depends only on the K-theory class of $[p]$ and K-homology class of $(\rho, \mathcal{H}, F)$.

Example 5.19. Let $\alpha \colon A \to \mathbb{C}$ be a *-homomorphism. Define $(\rho, \mathcal{H}, F)$ by $\mathcal{H} = \mathbb{C} \oplus 0$, $\rho = \alpha \oplus 0$, $F = 0$. The index pairing with this Fredholm module gives a homomorphism $i_\alpha \colon \mathrm{K}_0(A) \to \mathbb{Z}$ which by definition sends a projection p to the index of the zero operator $0 \colon \operatorname{im} \alpha(p) \to 0$, hence the index pairing gives the same map as

$$\alpha_* \colon \mathrm{K}_0(A) \to \mathrm{K}_0(\mathbb{C}).$$

Theorem 5.20. *Let J be an ideal in a separable C^*-algebra A for which the quotient map $A \to A/J$ is semi-split. We denote by ∂_0, ∂_1 the connecting homomorphisms in K-theory and by ∂^0, ∂^1 the connecting homomorphisms in K-homology.*

If $x \in \mathrm{K}_0(A/J)$ and $y \in \mathrm{K}^1(J)$ then

$$\langle \partial_0 x, y, \, y \rangle = -\langle x, \, \partial^1 y \rangle.$$

If $x \in \mathrm{K}_1(A/J)$ and $y \in \mathrm{K}^0(J)$ then

$$\langle \partial_1 x, \, y \rangle = \langle x, \, \partial^0 y \rangle.$$

Proof. The six term exact sequences in K-theory and K-homology:

$$\begin{tikzcd}
\mathrm{K}_0(J) \arrow[r] & \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A/J) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(A/J) \arrow[u, "\partial_1"] & \mathrm{K}_1(A) \arrow[l] & \mathrm{K}_1(J) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{K}^0(A/J) \arrow[r] & \mathrm{K}^0(A) \arrow[r] & \mathrm{K}^0(J) \arrow[d, "\partial^0"] \\
\mathrm{K}^1(J) \arrow[u, "\partial^1"] & \mathrm{K}^1(A) \arrow[l] & \mathrm{K}^1(A/J) \arrow[l]
\end{tikzcd}$$

We shall assume that A/J is unital. The strategy is to construct a Schrödinger operator V and show using two diefferent deformation arguments that

$$\operatorname{Index} V = \langle \partial_{0,1} x, \, y \rangle,$$
$$\operatorname{Index} V = \mp \langle x, \, \partial^{1,0} y \rangle$$

Case 1.

Step 1. Suppose we are given a short exact sequence of separable C*-algebras

$$0 \to J \to A \to A/J \to 0.$$

Let $(\rho, \mathcal{H}, F)$ be an ungraded Fredholm module for J. Let $p \in M_k(A/J)$ be a projection and let $a \in M_n(A)$ be a lift of p. Then

$$X = (1 \otimes \rho)(2a - 1), \quad Y = 1 \otimes F$$

form a Schrödinger pair. If $(\rho, \mathcal{H}, F)$ is a relative Fredholm module for $(A, A/J)$ then (X, Y) is a strong Schrödinger pair. The map

$$(p, F) \mapsto \operatorname{Index} V(X, Y)$$

defines pairings

$$\mathrm{K}_0(A/J) \otimes \mathrm{K}^1(J) \to \mathbb{Z}$$

$$\mathrm{K}_0(A/J) \otimes \mathrm{K}^1(A, A/J) \to \mathbb{Z}$$

which are compatible with the excision isomorphism $\mathrm{K}^1(A, A/J) \xrightarrow{\simeq} \mathrm{K}^1(J)$. For $x \in \mathrm{K}_0(A/J)$, $y \in \mathrm{K}^1(J)$ denote this pairing by $x \cdot y$.

Step 2. $x \cdot y = -\langle x, \partial^1 y\rangle$.
Assume that $x = [p]$ with $p \in A/J$ (similar arguments works for matrices) and $y = [(\rho, \mathcal{H}, F)]$.

$$\partial^1 y = \left[\left(\psi, \mathcal{H} \oplus \mathcal{H}', \begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix}\right)\right],$$

where $\psi \colon A/J \to B(\mathcal{H} \oplus \mathcal{H}')$ is a representation obtained from a completely positive section $s \colon \widetilde{A/J} \to \widetilde{A}$ by composing with ρ and then applying Stinespring's dilation. Put

$$\widehat{X} = \psi(2p - 1) \in B(\mathcal{H} \oplus \mathcal{H}'), \quad \widehat{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Then $X_{11} = \rho(2a - 1)$, $a \in A$ is a lift of $p \in A/J$. This is the operator which appears in the definition of Schrödinger pairing.
If $Y = F$ then $X = X_{11}$ and Y form a strong Schrödinger pair and $x \cdot y = \operatorname{Index} V(X, Y)$. Now

$$\widehat{Y} = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}, \quad \widehat{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

form a Schrödinger pair (not strong). Now

$$(\widehat{X})^2 = \psi((2p - 1)^2) = 1$$

$$X_{12} X_{21} = 1 - X_{11}^2 = 1 - X^2 \in \rho(J).$$

Furthermore $X = \rho(2a - 1)$, and $2a - 1$ lifts $2p - 1$, $(2p - 1)^2 = 1$, so $(2a - 1)^2 = 1 + j$, where $j \in J$. Now $X^2 = \rho((2a - 1)^2) = 1 + \rho(j)$ and we get the required compactness conditions from the definition of Fredholm module. Indeed, $1 - Y^2 = 1 - F^2$, so

$$(1 - F^2)\rho(j) = (1 - F^2)(1 - X^2) \in \mathcal{K}(\mathcal{H}).$$

Essentially the same calculation will show that

$$V(\widehat{Y},\widehat{X}) \sim \begin{pmatrix} V(X,Y) & 0 \\ 0 & 1 \end{pmatrix}.$$

By the proposition

$$\operatorname{Index} V(\widehat{Y},\widehat{X}) = \operatorname{Index} V(Y,X) = -\operatorname{Index} V(X,Y) = -x\cdot y.$$

If $P_{\widehat{X}} = \frac{1}{2}(\widehat{X}+1) = \psi(p)$ then using the formula for $\partial^1 y$

$$\begin{aligned}\langle x,\, \partial^1 y\rangle &= \operatorname{Index}\left(\begin{pmatrix} e^{i\pi F} & 0 \\ 0 & -1 \end{pmatrix} P_{\widehat{X}} - (1-P_{\widehat{X}})\right) \\ &= \operatorname{Index}(e^{i\pi F} P_{\widehat{X}} - (1-P_{\widehat{X}})) \\ &= \operatorname{Index} W_1(\widehat{Y},\widehat{X}).\end{aligned}$$

But we have seen that $\operatorname{Index}(W_1(\widehat{Y},\widehat{X})) = \operatorname{Index}(V(\widehat{Y},\widehat{X}))$ so

$$\langle x,\, \partial^1 y\rangle = \operatorname{Index} W_1(\widehat{Y},\widehat{X}) = \operatorname{Index}(V(\widehat{Y},\widehat{X})) = \operatorname{Index}(V(X,Y)) = -x\cdot y.$$

Step 3. As before, assume that a projection $p \in A/J$ has a lift to a self adjoint $a \in A$, and that y is represented by $(\rho,\mathcal{H},F)$. Put $X := \rho(2a-1)$, $Y := F$. The boundary map in K-theory gives

$$\partial_0 x = [e^{2\pi i a}] \in M_k(\widetilde{J}).$$

The index pairing is

$$\langle \partial_0 x,\, y\rangle = \operatorname{Index}((1\otimes\rho)e^{2\pi i a}P_Y + (1-P_Y)),$$

where $P_Y = \frac{1}{2}(1+Y)$. Put $T := -e^{-i\pi X}P_Y + (1-P_Y) = -W_1(X,Y)$. Then

$$\langle \partial_0 x,\, y\rangle = \operatorname{Index}(T) = \operatorname{Index}(W_1(X,Y)) = \operatorname{Index}(V(X,Y)) = x\cdot y,$$

so

$$\langle \partial_0 x,\, y\rangle = -\langle x,\, \partial^1 y\rangle.$$

Case 2.

Step 1. In the graded case take a short exact sequence

$$0 \to J \to A \to A/J \to 0$$

and a graded Fredholm module $(\rho,\mathcal{H},F)$ for $(A, A/J)$. Let $u \in M_n(A/J)$ be a unitary matrix and $a \in M_n(A)$ a lift of u to A. Then

$$X = \begin{pmatrix} 0 & (1\otimes\rho)(a^*) \\ (1\otimes\rho)(a) & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1\otimes F & 0 \\ 0 & -1\otimes F \end{pmatrix}$$

on a graded Hilbert space $\mathbb{C}^n\otimes\mathcal{H}\oplus\mathbb{C}^n\otimes\mathcal{H}^{op}$ form a strong Schrödinger pair. Furthermore

$$(u,F) \mapsto \operatorname{Index}(V(X,Y))$$

defines a bilinear pairing $\mathrm{K}_1(A/J)\times\mathrm{K}^0(A,A/J) \to \mathbb{Z}$ denoted again by $x\cdot y$.

Step 2. For $x \in \mathrm{K}_1(A/J)$, $y \in \mathrm{K}^0(A, A/J)$ we have $x \cdot y = \langle x, \partial^0 y\rangle$. Assume that $y = [(\rho, \mathcal{H}, F)]$ is a graded relative module for $\mathrm{K}^0(A, A/J)$. Use the descritption of boundary map $\partial^0 \colon \mathrm{K}^0(J) \to \mathrm{K}^1(A/J)$, so assume that $(\rho, \mathcal{H}, F)$ is paritally isometric i.e. $Q := 1 - F^2$ is a projection with graded components $Q^{\pm}$. Then $\partial^0[y] = [Q^+] - [Q^-]$.

Assume that u is a unitary in A/J representing x. Define

$$X := \begin{pmatrix} 0 & \rho(a^*) \\ \rho(a) & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}^{op}$. Then by definition $x \cdot y = \mathrm{Index}(V(X, Y)) = \mathrm{Index}(V(Y, X))$. Put

$$Q_Y := 1 - Y_2 = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

The operator $X_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is odd and commutes modulo compacts with Y^2. By lemma (**??**) for $W_2(Y, X) := XQ_Y + X_0(1 - Q_Y)$

$$\mathrm{Index}(V(Y, X)) = \mathrm{Index}(W_2(Y, X)) = \langle x, [Q^+]\rangle - \langle x, [Q^-]\rangle = \langle x, \partial^0 y\rangle.$$

Step 3. Assume that A is unital. Let $y \in \mathrm{K}^0(A, A/J)$ be represented by a graded relative Fredholm module $(\rho, \mathcal{H}, F)$, and $x \in \mathrm{K}_1(A/J)$ represented by a unitary $u \in A/J$. Then u lifts to a partial isometry $a \in A$. Put

$$X := \begin{pmatrix} 0 & \rho(a^*) \\ \rho(a) & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}.$$

Then $x \cdot y = \mathrm{Index}(V(X, Y))$, and

$$Q_X := 1 - X^2 = \begin{pmatrix} \rho(1 - a^*a) & 0 \\ 0 & \rho(1 - aa^*) \end{pmatrix}$$

is a projection onto $\ker X$. We have furthermore

$$V(X, Y) = X + (1 - X^2)Y = (1 - Q_X)X(1 - Q_X) + Q_X Y Q_X,$$

$$\mathrm{Index}(V(X, Y)) = \mathrm{Index}(Q_X Y Q_X).$$

Using the boundary formula for the boundary map in K-theory

$$\partial_1 x = [1 - a^*a] - [1 - aa^*] \in \mathrm{K}_0(J)$$

we get

$$\mathrm{Index}(Q_X Y Q_X) = \langle \partial_1 x, y\rangle.$$

$\square$

5.5 Product of Fredholm operators

The construction of the index pairing by means of Schrödinger operators is a special case of the Kasparov product.

Let F_1, F_2 be graded Fredholm operators, $F_i = \begin{pmatrix} 0 & U_i^* \\ U_i & 0 \end{pmatrix}$, on a graded Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. A graded Fredholm operator F on $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ (graded Hilbert space product) is **aligned** with F_i if $F(F_i \widehat{\otimes} 1) + (F_i \widehat{\otimes} 1)F \geq 0$ modulo compacts.

Proposition 5.21. *Let F_i be graded Fredholm operator on $\mathcal{H}_i$, $i = 1, 2$. There exist graded Fredholm operators on $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ which are simultaneously aligned with F_1 and F_2. Any two such operators have the same index.*

Proof. Define $F \in B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ by $F := F_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} F_2$. Then

$$F(F_i \widehat{\otimes} 1) + (F_i \widehat{\otimes} 1)F = 2(F_i^2 \otimes 1) \geq 0,$$

so F is aligned with both F_1, F_2. Moreover

$$\mathrm{Index}(F) = \mathrm{Index}(F_1) \cdot \mathrm{Index}(F_2),$$

so F is a good model for the product of F_i, but we need $F^2 - 1 \sim 0$.

Lemma 5.22. *Let F_i are graded Fredholm operators on $\mathcal{H}_i$, and N_i a positive operators on $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ such that*

1. $N_1^2 + N_2^2 = 1$,
2. $[N_i, F_j \widehat{\otimes} 1] \sim 0$, $i, j = 1, 2$,
3. $N_i (F_i \widehat{\otimes} 1)^2 \sim N_i$.

Then

$$F := N_1^{\frac{1}{2}} (F_1 \widehat{\otimes} 1) N_1^{\frac{1}{2}} + N_2^{\frac{1}{2}} (1 \widehat{\otimes} f_2) N_2^{\frac{1}{2}}$$

is an odd Fredholm operator aligned with F_1, F_2. Moreover $F^2 \sim 1$.

□

Let $(\rho_i, \mathcal{H}_i, F_i)$ be graded Fredholm modules over C*-algebras A_i, $i = 1, 2$. Define a representation of $A_1 \otimes A_2$ on $B(\mathcal{H})$, $\mathcal{H} = \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$:

$$\rho \colon A_1 \otimes A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2), \quad \rho(a_1 \otimes a_2) := \rho_1(a_1)\rho_2(a_2).$$

We say that Fredholm module $(\rho, \mathcal{H}, F)$ is aligned with $(\rho_i, \mathcal{H}_i, F_i)$ if

$$\rho(a)(F(F_i \widehat{\otimes} 1) + (F_i \widehat{\otimes} 1)F)\rho(a^*) \geq 0 \mod \mathcal{K}(\mathcal{H})$$

for all $a \in A_1 \otimes A_2$, and if $\rho(a)F$ derives $\mathcal{K}(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ for all $a \in A_1 \otimes A_2$, that is

$$[\rho(a)F, K_1 \otimes T_2] \in \mathcal{K}_1(\mathcal{H}_1) \otimes B(\mathcal{H}_2), \quad \forall K_1 \otimes T_2 \in \mathcal{K}(\mathcal{H}_1) \otimes B(\mathcal{H}_2).$$

Proposition 5.23. *Let $(\rho_i, \mathcal{H}_i, F_i)$ be graded Fredholm modules over separable C*-algebras A_1 and A_2. There exist Fredholm modules F which are aligned with F_1 and F_2. Moreover the operator homotopy class of such an F is determined uniquely by the operator homotopy classes of F_1 and F_2.*

The hard part is to prove existence of such Fredholm modules.

Definition 5.24. *The module F from the proposition is the* ***product*** *of F_1 and F_2.*

Chapter 6

Equivariant K-homology of spaces

Assume X is a Hausdorff space on which a discrete group Γ acts by homeomorphisms.

Definition 6.1. *The action of Γ is **proper** if and only if for every $x, y \in X$ there exist neighbourhoods U_x, U_y such that the set*

$$\{\gamma \in \Gamma \mid \gamma U_x \cap U_y \neq \emptyset\}$$

is finite.

If Γ is discrete, this definition is equivalent to the following one:

Definition 6.2. *The action of Γ is proper if and only if the quotient X/Γ is Hausdorff and for every $x \in X$ there exists (U, H, ρ) such that U is a Γ-invariant neighbourhood of x, H is a finite subgroup of Γ, and $\rho\colon U \to \Gamma/H$ is a Γ-equivariant map.*

Another definition is:

Definition 6.3. *The action $X \times \Gamma \to X$ is proper if and only if the map*

$$X \times \Gamma \to X \times X, \quad (x, \gamma) \mapsto (x, x\gamma)$$

is proper (preimage of a compact set is compact).

Examples 6.4.

1. If $p\colon X \to Y$ is a locally trivial covering space with group Γ, then the Γ-action on X is proper.

2. Any action by a finite group is proper.

3. If Γ acts simplicially on a simplicial complex X, then the action is proper if and only if the vertex stabilizers are finite.

Let X be a locally compact space equipped with a proper action by a discrete group Γ.

Definition 6.5. *A **generalized elliptic** Γ**-equivariant operator** on X is a triple (U, π, F) such that*

- *U is a unitary representation of Γ on some Hilbert space $\mathcal{H}$,*

- *π is a *-representation of $C_0(X)$ by bounded operators on $\mathcal{H}$ which is covariant, that is*

$$\pi(f \circ \gamma^{-1}) = U_\gamma \pi(f) U_\gamma$$

for all $f \in C_0(X)$,

- *F is bounded self adjoint operator which is Γ-equivariant, that is $U_\gamma F = F U_\gamma$, and*

$$\pi(f)(F^2 - 1), \quad [\pi(f), F]$$

are compact for all $f \in C_0(X)$.

Definition 6.6. *Two cycles $\alpha_0 = (U_0, \pi_0, F_0)$, $\alpha_1 = (U_1, \pi_1, F_1)$ are* ***operator homotopic****, $\alpha_0 \sim_h \alpha_1$, if and only if $U_0 = U_1$, $\pi_0 = \pi_1$ and there exists a path $t \mapsto F_t$, $t \in [0,1]$ such that $\alpha_t = (U_0, \pi_0, F_t)$ is a Γ-elliptic operator.*

We say that α_0, α_1 are equivalent, $\alpha_0 \sim \alpha_1$, if and only if there exist degenerate operators β_0, β_1 such that $\alpha_0 \oplus \beta_0 \sim_h \alpha_1 \oplus \beta_1$, up to unitary equivalence.

Definition 6.7. *The* ***equivariant K-homology*** *groups of X are defined by*

- $\mathrm{K}_0^\Gamma(X) =$ *equivalence classes of $\mathbb{Z}/2\mathbb{Z}$-graded Γ-elliptic operators, that is*

$$U_\gamma = \begin{pmatrix} U_\gamma^+ & 0 \\ 0 & U_\gamma^- \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi^+ & 0 \\ 0 & \pi^- \end{pmatrix}, \quad F = \begin{pmatrix} 0 & p^* \\ p & 0 \end{pmatrix},$$

- $\mathrm{K}_1^\Gamma(X) =$ *equivalence classes of Γ-elliptic operators.*

Remark 6.8. Kasparov uses a weaker form of homotopy. He allows the representations to vary as well, but proves that the resulting theory is isomorphic to the one defined here.

This construction is functorial with respect to Γ-equivariant proper maps between Γ-spaces. If $h\colon X \to Y$ is such a map, then it induces $h^*\colon C_0(Y) \to C_0(X)$, $h^*(f) = f \circ h$. The induced map $h_*\colon \mathrm{K}_0^\Gamma(X) \to \mathrm{K}_0^\Gamma(Y)$ sends a cycle (U, π, F) over $C_0(X)$ to the cycle $(U, \pi \circ h^*, F)$, so the theory is covariant.

Proposition 6.9 (Kasparov)**.** *If $f, g\colon X \to Y$ are proper Γ-homotopic maps, then*

$$f_* = g_*\colon \mathrm{K}_j^\Gamma(X) \to \mathrm{K}_j^\Gamma(Y).$$

Example 6.10. Let $X = \mathbb{R}$, $\Gamma = \mathbb{Z}$ act on X by translations $(x, m) \mapsto x + m$. Let $\mathcal{H} = L^2(\mathbb{R})$, π a representation of $C_0(\mathbb{R})$ on $L^2(\mathbb{R})$ by pointwise multiplication. The Fourier transform sends the unbounded operator $D = -i\frac{d}{dt}$ to the multiplication by the dual variable λ on $L^1(\widehat{\mathbb{R}})$. Let G be the operator of multiplication by $\mathrm{sign}(\lambda)$ and let F be the operator obtained by the inverse Fourier transform (Hilbert transform)

$$(Hf)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x - t} dt,$$

where the integral is considered in the sense of principal value. Then $F^2 = 1$, $FU_n = U_n F$ for all $n \in \mathbb{Z}$, and $[\pi(f), F]$ is compact. This data gives a generator for $\mathrm{K}_1^\Gamma(X)$.

Example 6.11. Let $\Gamma = \{e\}$, $X = S^1$. Denote $e_n := e^{2\pi i n\theta}$. Then

$$F(e_n) = \begin{cases} e_n & n > 0, \\ -e_n & n < 0, \\ 0 & n = 0, \end{cases}$$

and $1 - F^2$ is a rank one projection onto the subspace of constant functions in $L^2(S^1)$. Let

$$A = \{f \in C(S^1) \mid [\pi(f), F] \text{ is compact}\}.$$

Then $A = C(S^1)$. Indeed, $[\pi(e_1), F]$ is an operator of rank 2, so A contains the *-subalgebra generated by e_1, which is the algebra of trigonometric polynomials, and these are dense in $C(S^1)$.

The operator F is the sign of the unbounded operator $D = -i\frac{d}{d\theta}$, $D(e_n) = 2\pi n e_n$. That is

$$F = \operatorname{sign} D = \frac{D}{|D|} \text{ on } \mathbb{C}^{\perp}.$$

This data gives a generator for $\mathrm{K}_1(S^1)$. There is a descent map

$$\mathrm{K}_j^{\mathbb{Z}}(\mathbb{R}) \to \mathrm{K}_j(S^1).$$

In degree one it sends the generator of $\mathrm{K}_1^{\mathbb{Z}}(\mathbb{R})$ to the one just described.

Proposition 6.12. *If* Γ *acts freely and properly then*

$$\mathrm{K}_j^{\Gamma}(X) \simeq \mathrm{K}_j(X/\Gamma)(= \mathrm{KK}(C_0(X/\Gamma)), \mathbb{C})).$$

Proof. (sketch) Use Green-Julg theorem to identify

$$\mathrm{K}_j^{\Gamma}(X) \simeq \mathrm{K}_j(C_0(X) \rtimes \Gamma)$$

which is an example of descent map. Then use freenes to prove the Morita equivalence

$$C_0(X) \rtimes \Gamma \sim_M C_0(X/\Gamma).$$

□

Example 6.13. If $X = \mathrm{pt}$, Γ is finite, then $\mathrm{K}_0^{\Gamma}(\mathrm{pt})$ is the additive group of the representation ring $\mathrm{R}(\Gamma)$, $\mathrm{K}_1^{\Gamma}(\mathrm{pt})$.

$$[U_0] - [U_1] \in \mathrm{R}(\Gamma) \mapsto (U_0 \oplus U_1, \mathbb{C}, 0),$$

$$(U, \mathbb{C}, F = \begin{pmatrix} 0 & p^* \\ p & 0 \end{pmatrix}) \mapsto \mathrm{Index}_{\Gamma}(F) = \ker(p) - \ker(p^*)$$

regarded as an element of $\mathrm{R}(\Gamma)$.

If Y is a topological Hausdorff space with proper Γ-action, then we define

$$\mathrm{K}_j^{\Gamma} := \varinjlim_{X \subset Y} \mathrm{K}_j^{\Gamma}(X),$$

where the limit is taken over an inductive system of Γ-compact subsets of Y (i.e. with compact quotient X/Γ). This is Γ**-equivariant K-homology with compact supports**.

Chapter 7

KK-theory

7.1 Kasparov's bifunctor

Definition 7.1. *Let A, B be C^*-algebras. An A-B-bimodule is a pair $(\mathcal{E}, \pi)$ where $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert bimodule acted upon through *-homomorphism $\pi\colon A \to \mathcal{L}(\mathcal{E})$ such that for all $a \in A$, $\pi(a)$ is of degree 0.*

Denote by $E(A, B)$ the set of triples $(\mathcal{E}, \pi, F)$, where $(\mathcal{E}, \pi)$ is an (A, B)-bimodule, $F \in \mathcal{L}(E)$ is homogenous of degree 1, and for all $a \in A$

$$\pi(a)(F^2 - 1) \in \mathcal{K}(\mathcal{E}), \quad [\pi(a), F] \in \mathcal{K}(\mathcal{E}).$$

The triple $(\mathcal{E}, \pi, F)$ is **degenerate** if for all $a \in A$ $\pi(a)(F^2 - 1) = 0$, $[\pi(a), F] = 0$. Denote by $D(A, B)$ the set of degenerate triples.

The **addition** on $E(A, B)$ is defined by

$$(\mathcal{E}, \pi, F) + (\mathcal{E}', \pi', F') = (\mathcal{E} \oplus \mathcal{E}', \pi \oplus \pi', F \oplus F').$$

A **homotopy** in $E(A, B)$ is an element of $E(A, B[0, 1])$. In some sense it is a map $[0, 1] \to E(A, B)$. A homotopy in which the bimodule $(\mathcal{E}, \pi)$ is fixed and the operator F varies in a norm continuous way is called an operator homotopy. It is a stronger notion then.

Let $(\mathcal{E}, \pi, F) \in E(A, B)$. Denote by $-\mathcal{E}$ the same Hilbert module, but with opposite grading $(-\mathcal{E})^{(i)} = \mathcal{E}^{1-i}$. Then $(-\mathcal{E}, \pi, -F)$ is the **opposite** element to $(\mathcal{E}, \pi, F)$.

Definition 7.2. *The group* $\mathrm{KK}(A, B)$ *is defined to be the set of homotopy classes in* $E(A, B)$ *modulo degenerate elements.*

The construction is functorial in both variables, that is

- If $f\colon A_1 \to A_2$, $(\mathcal{E}, \pi, F) \in E(A_2, B)$, then $(\mathcal{E}, \pi \circ f, F) \in E(A_1, B)$ and this induces

$$f^*\colon \mathrm{KK}(A_2, B) \to \mathrm{KK}(A_1, B).$$

- If $g\colon B_1 \to B_2$, $(\mathcal{E}, \pi, F) \in E(A, B_1)$, then define $\pi \otimes 1\colon A \to \mathcal{L}(\mathcal{E} \otimes_g B_2)$,

$$(\pi \otimes 1)(a)(\xi \otimes b) = \pi(a)\xi \otimes b,$$

  then $(\mathcal{E} \otimes_g B_2, \pi \otimes 1, F \otimes 1) \in E(A, B_2)$. This induces

$$g_*\colon \mathrm{KK}(A, B_1) \to \mathrm{KK}(A, B_2).$$

7.2 Equivariant KK-theory

A Γ-C*-algebra is a C*-algebra equipped with an action of Γ by *-automorphisms.

Definition 7.3. *Let A, B be Γ-C*-algebras. A* ***cycle*** *over the pair (A, B) is a triple (U, π, F), where*

- *U is a representation of Γ on some Hilbert module $\mathcal{E}$ over B, which is unitary in the sense that*

$$\langle U_\gamma \xi,\, U_\gamma \eta\rangle_B = \gamma\langle \xi,\, \eta\rangle_B$$

  for all $\gamma \in \Gamma$, $\xi, \eta \in \mathcal{E}$,

- *$\pi\colon A \to \mathcal{L}_B(\mathcal{E})$ is a covariant *-homomorphism*

$$U_\gamma \pi(a) U_{\gamma^{-1}} = \pi(\gamma \cdot a),$$

  for all $\gamma \in \Gamma$, $a \in A$,

- *F is a selfadjoint operator in $\mathcal{L}_B(\mathcal{E})$.*

We also require that the operators

$$\pi(a)(F^2 - 1), \quad [\pi(a), F], \quad [U_\gamma, F]$$

are compact in the sense of Hilbert modules for all $a \in A$, $\gamma \in F$. This is the ***odd cycle****. For an* ***even cycle*** *we assume that $\mathcal{E}$ is $\mathbb{Z}/2\mathbb{Z}$ graded, with U, π even, F odd.*

A cycle is ***degenerate*** *if and only if the operators $\pi(a)(F^2 - 1)$, $[\pi(a), F]$, and $[U_\gamma, F]$ are zero for all $a \in A$, $\gamma \in \Gamma$.*

By Kasparov's stabilisation theorem (**??**) $\mathcal{E}$ can be taken to be $\mathcal{H}_B = l^2(\mathbb{N}) \otimes B$.

Definition 7.4. *Two cycles $\alpha_0 = (U_0, \pi_0, F_0)$, $\alpha_1 = (U_1, \pi_1, F_1)$ are homotopic, $\alpha_0 \sim_h \alpha_1$, if and only if $U_0 = U_1$, $\pi_0 = \pi_1$ and there exists a norm continuous path F_t, $t \in [0, 1]$ connecting F_0 to F_1 such that for all $t \in [0, 1]$, $\alpha_t = (U_0, \pi_0, F_t)$ is a cycle.*

Two cycles are equivalent, $\alpha_0 \sim \alpha_1$ if and only if there exist degenerate cycles β_0, β_1 such that (up to unitary equivalence) $\alpha_0 \oplus \beta_0 \sim_h \alpha_1 \oplus \beta_1$.

We write $\mathrm{KK}^\Gamma_j(A, B)$ for the set of equivalence classes of cycles over (A, B).

Again Kasparov originally used a weaker form of homotopy, where one was obliged to provide a path joining π_0 and π_1.

When Γ is trivial, we get non-equivariant KK-theory.

$\mathrm{KK}(A, B)$ is covariant in B and contravariant in A. If $\alpha = (U, \pi, F) \in \mathrm{KK}^\Gamma_j(A, B)$ and

- if $\theta\colon C \to A$ then $\theta^*\alpha = (U, \pi \circ \theta, F) \in \mathrm{KK}^\Gamma_j(C, B)$
- if $\theta\colon B \to C$ then for a Hilbert C-module $\mathcal{E}_B C$ we have $\theta_*\alpha = (U \otimes 1, \pi \otimes 1, F \otimes 1) \in \mathrm{KK}^\Gamma_j(A, C)$.

Example 7.5. If Γ is a discrete group, X is a locally compact proper Γ-space, then

$$\mathrm{K}^\Gamma_i(X) = \mathrm{KK}^\Gamma_i(C_0(X), \mathbb{C}).$$

This follows from the fact that when Γ acts properly one may assume in the definition of KK-cycles that $[U_\gamma, F] = 0$ for all $\gamma \in \Gamma$ (Valette).

Example 7.6. When Γ is trivial then

$$\mathrm{K}^i(A) = \mathrm{KK}_i(A, \mathbb{C}),$$
$$\mathrm{K}_j(B) = \mathrm{KK}_j(\mathbb{C}, B).$$

First equality follows from definition.

For the second one assume B is unital. Then for

- $j = 0$: let $x = [e_0] - [e_1]$, and e_0, e_1 be idempotents in $M_n(B)$. Then $e_0 \oplus 0$, $e_1 \oplus 0$ are finite rank operators on $\mathcal{H}_B = l^2(\mathbb{N}) \otimes B$. For $\lambda \in \mathbb{C}$ define

  $$\pi(\lambda) = \begin{pmatrix} \lambda(e_1 \oplus 0) & 0 \\ 0 & \lambda(e_1 \oplus 1) \end{pmatrix} : \mathcal{H}_B \oplus \mathcal{H}_B \to \mathcal{H}_B \oplus \mathcal{H}_B.$$

  Take $F = 0$. Since π is a representation by compact operators, $\pi(\lambda)(F^2 - 1) = -\pi(\lambda)$ is compact. This defines a map

  $$\mathrm{K}_0(B) \to \mathrm{KK}_0(\mathbb{C}, B)$$

  which is an isomorphism (see Lafforgues's thesis).

- $j = 1$: denote by $\mathcal{G}$ the group of all invertible operators V on $\mathcal{H}_B$ such that $V - 1$ is compact. Then $\mathrm{K}_1(B) \simeq \pi_0(\mathcal{G})$, where $\mathrm{K}_1(B) \to \pi_0(\mathcal{G})$ is given by

  $$s \mapsto \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}, \quad s \in \mathrm{GL}_n(B).$$

  This uses the definition $\mathrm{K}_n(A) = \pi_{n-1}(\mathrm{GL}_\infty(A))$.

  The map

  $$\mathrm{KK}_1(\mathbb{C}, B) \to \mathrm{K}_1(B)$$

  is defined as follows. Take $F = F^*$ on $\mathcal{E}$, such that $F^2 - 1$ is compact. By the stabilization theorem one may assume that $\mathcal{E} = \mathcal{H}_B$. Then $e^{-i\pi F} \in \mathcal{G}$ which gives a map

  $$\mathrm{KK}_1(\mathbb{C}, B) \to \pi_0(\mathcal{G}) \simeq \mathrm{K}_1(B).$$

7.3 Kasparov product

Theorem 7.7 (Kasparov). *Let A, B, C be separable Γ-C^*-algebras. Then there is a biadditive **pairing** for $i, j \in \mathbb{Z}_2$*

$$\mathrm{KK}^\Gamma_i(A, B) \times \mathrm{KK}^\Gamma_j(B, C) \to \mathrm{KK}^\Gamma_{i+j}(A, C)$$
$$(x, y) \mapsto x \otimes_B y.$$

*If D is a separable Γ-C^*-algebra, then there is the **extension of scalars** homomorphism*

$$\tau_D \colon KK^\Gamma_i(A, B) \to KK^\Gamma_i(A \otimes D, B \otimes D)$$
$$(U, \pi, F) \mapsto (U \otimes 1, \pi \otimes 1, F \otimes 1).$$

*and the **descent homomorphism***

$$j_\Gamma \colon KK^\Gamma_i(A, B) \to KK_i(A \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

These are functorial in all variables:

- *If* $\alpha\colon A \to B$ *is* Γ*-equivariant *-homomorphism and* $y \in \mathrm{KK}_j^\Gamma(B, C)$ *then*

$$[\alpha] \otimes_B y = \alpha^*(y) \in \mathrm{KK}_j^\Gamma(A, C)$$

- *If* $\beta\colon B \to C$ *is a* Γ*-equivariant *-homomorphism and* $x \in KK_j^\Gamma(A, B)$*, then*

$$x \otimes_B [\beta] = \beta_*(x) \in KK_j^\Gamma(A, C).$$

- *If* $x = (U, \pi, F)$*,* $y = (V, \rho, G)$ *are cycles over* (A, B)*,* (B, C) *respectively. Let* $\mathcal{E}_B$ *be the underlying Hilbert* B*-module for* x*, and* $\mathcal{E}_C$ *the underlying Hilbert* C*-module for* y*. Then* $\mathcal{E} = \mathcal{E}_B \otimes_B \mathcal{E}_C$ *(*B *acts via* ρ *on* C*) is a Hilbert* C*-module and*

$$x \otimes_B y = (U \otimes V, \pi \otimes 1, F \otimes 1 + 1 \otimes G).$$

- *For* $x \in KK_i^\Gamma(A, B)$*,* $y \in KK_0^\Gamma(\mathbb{C}, \mathbb{C})$

$$\tau_A(y) \otimes_A x = x \otimes_B \tau_B(y),$$

where

$$\tau_A\colon KK_0^\Gamma(\mathbb{C}, \mathbb{C}) \to \mathrm{KK}_0^\Gamma(A, A) \; \tau_B\colon KK_0^\Gamma(\mathbb{C}, \mathbb{C}) \to \mathrm{KK}_0^\Gamma(B, B)$$

are extensions of scalars.

- *If* $(U, \pi, F) \in \mathrm{KK}_i^\Gamma(A, B)$ *with corresponding Hilbert* B*-module* $\mathcal{E}_B$*. Take* $\widetilde{\mathcal{E}} = \mathcal{E}_B \otimes \mathbb{C}[\Gamma] = C(\Gamma, \mathcal{E}_B)$*, which is a Hilbert* $B \rtimes_r \Gamma$*-module with* $B \rtimes_r \Gamma$*-scalar product given by*

$$\langle \xi,\, \eta \rangle(\gamma) = \sum_{s \in \Gamma} s\langle \xi(s),\, \eta(s^{-1}\gamma)\rangle_B.$$

For $a \in C(\Gamma, A)$*,* $\xi \in \widetilde{\mathcal{E}}$*,* $\gamma \in \Gamma$

$$(\widetilde{\pi}(a)\xi)(\gamma) := \sum_{s \in \Gamma} \pi(a(s)) U_s(\xi(s^{-1}\gamma))$$

$$(\widetilde{F}\xi)(\gamma) := (F\xi)(\gamma).$$

We set

$$j_\Gamma(U, \pi, F) = (\widetilde{\pi}, \widetilde{F}).$$