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## ON THE $(m,t)$ -EXTENSION DUAL COMPLEX FIBONACCI $p$ -NUMBERS

*Abstract.* We introduce  $(m,t)$ -extension dual complex Fibonacci  $p$ -numbers and we establish some of their properties. They are connected to complex Fibonacci numbers, complex Fibonacci  $p$ -numbers,  $m$ -extension dual complex Fibonacci  $p$ -numbers and dual complex Fibonacci  $p$ -numbers.

**1. Introduction.** The *Fibonacci numbers*  $F_n$  are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2$$

with initial terms

$$F_1 = F_2 = 1.$$

The Fibonacci numbers  $F_n$  and the golden mean (golden ratio)

$$\tau = \lim_{n \rightarrow \infty} \frac{F_n}{F_{n-1}} = \frac{1 + \sqrt{5}}{2}$$

have materialized in several areas of art, science, computer science, high energy physics, information and coding theory [3, 4, 6, 7, 17, 18].

The *Fibonacci  $p$ -numbers* [17] are defined by the recurrence relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with  $n > p+1$ , for a given integer  $p = 0, 1, 2, \dots$ , and with initial terms

$$F_p(1) = F_p(2) = \dots = F_p(p) = F_p(p+1) = 1.$$

The Fibonacci  $p$ -numbers coincide with the classical Fibonacci numbers for  $p = 1$ , e.g.  $F_1(n) = F_n$ .

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E. G. Kocer et al. [9] introduced the  $m$ -extension of Fibonacci  $p$ -numbers which satisfy the recurrence relation

$$F_{p,m}(n) = mF_{p,m}(n-1) + F_{p,m}(n-p-1)$$

with initial terms

$$F_{p,m}(1) = a_1, \quad F_{p,m}(2) = a_2, \quad F_{p,m}(3) = a_3, \dots, F_{p,m}(p+1) = a_{p+1},$$

where  $p (\geq 0)$  is an integer,  $m (> 0)$ ,  $n > p+1$  and  $a_1, \dots, a_{p+1}$  are arbitrary real or complex numbers.

Prasad et al. [16] introduced the  $(m, t)$ -extension of Fibonacci  $p$ -numbers which satisfy the recurrence relation

$$F_{p,m,t}(n) = mF_{p,m,t}(n-1) + tF_{p,m,t}(n-p-1)$$

with initial terms

$$F_{p,m,t}(1) = b_1, \quad F_{p,m,t}(2) = b_2, \quad F_{p,m,t}(3) = b_3, \dots, F_{p,m,t}(p+1) = b_{p+1},$$

where  $p (\geq 0)$  is an integer,  $m (> 0)$ ,  $t (> 0)$ ,  $n > p+1$  and  $b_1, \dots, b_{p+1}$  are arbitrary real or complex numbers.

The *complex Fibonacci numbers* [12] are defined by the recurrence relation

$$F_n^* = F_{n-1}^* + F_{n-2}^* \quad \text{for } n \geq 2$$

with initial terms

$$F_0^* = i, \quad F_1^* = 1 + i$$

where  $i$  is the imaginary unit and

$$F_n^* = F_n + iF_{n+1}.$$

Clifford algebra is a powerful mathematical tool that offers a natural and direct way to model geometric objects and their transformations. It makes geometric objects (points, lines and planes) into basic elements of computation and defines a few universal operators that are applicable to all types of geometric elements. In the 19th century, Clifford [5] defined a new number system by using the notion  $\varepsilon^2 = 0$  and  $\varepsilon \neq 0$ , i.e.  $\varepsilon$  is a nilpotent number. This number system is known as the *dual number system* and dual numbers are represented in the form  $A = a + \varepsilon a^*$  where  $a$  and  $a^*$  are real numbers and  $\varepsilon$  is the nilpotent number. Kotelnikov [10] and Study [19] found the first applications of dual numbers in mechanics. They also have applications in algebraic geometry, kinematics, number theory, theory of relativity etc. [2, 11, 14].

*Dual complex numbers* [11] are defined as follows:

$$W = \{w = z_1 + \varepsilon z_2 : z_1, z_2 \in \mathbb{C}, \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0\}.$$

If  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  then any dual complex number can be written as follows:

$$w = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2 \quad \text{where} \quad i^2 = -1, \varepsilon \neq 0, \varepsilon^2 = 0, (i\varepsilon)^2 = 0.$$

The real and dual quaternions form a division ring but dual complex numbers form a commutative ring with characteristic zero. The multiplication of dual complex numbers gives the structure of the two-dimensional complex Clifford algebra and the four-dimensional real Clifford algebra. The base elements of dual complex numbers satisfy the commutative multiplication scheme shown in Table 1.

**Table 1.** Multiplication scheme of dual complex numbers

$\times$	1	$i$	$\varepsilon$	$i\varepsilon$
1	1	$i$	$\varepsilon$	$i\varepsilon$
$i$	$i$	-1	$i\varepsilon$	$-\varepsilon$
$\varepsilon$	$\varepsilon$	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	$-\varepsilon$	0	0

Five different conjugation operators work on dual complex numbers [11] in the following manner: if  $w = z_1 + \varepsilon z_2 = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2$ ,  $z_2 \neq 0$ , then  
 $w^{*1} = (x_1 - iy_1) + \varepsilon(x_2 - iy_2) = z_1^* + \varepsilon z_2^*$  (complex conjugation),  
 $w^{*2} = (x_1 + iy_1) - \varepsilon(x_2 + iy_2) = z_1 - \varepsilon z_2$  (dual conjugation),  
 $w^{*3} = (x_1 - iy_1) - \varepsilon(x_2 - iy_2) = z_1^* - \varepsilon z_2^*$  (coupled conjugation),  
 $w^{*4} = (x_1 - iy_1)\left(1 - \frac{\varepsilon(x_2 + iy_2)}{(x_1 + iy_1)}\right) = z_1^*\left(1 - \frac{\varepsilon z_2}{z_1}\right)$  (dual complex conjugation),  
 $w^{*5} = (x_2 + iy_2) - \varepsilon(x_1 + iy_1) = z_2 - \varepsilon z_1$  (antidual conjugation).

The norms of dual complex numbers are defined in the following manner:

$$N_{w^{*1}} = \|w \times w^{*1}\| = \sqrt{|z_1^2| + 2\varepsilon \operatorname{Re}(z_1 z_2^*)},$$

$$N_{w^{*2}} = \|w \times w^{*2}\| = \sqrt{z_1^2},$$

$$N_{w^{*3}} = \|w \times w^{*3}\| = \sqrt{|z_1^2| - 2i\varepsilon \operatorname{Im}(z_1 z_2^*)},$$

$$N_{w^{*4}} = \|w \times w^{*4}\| = \sqrt{|z_1^2|},$$

$$N_{w^{*5}} = \|w \times w^{*5}\| = \sqrt{z_1 z_2 + \varepsilon(z_2^2 - z_1^2)}.$$

In 2017, Güngör and Azak [8] defined the *dual complex Fibonacci numbers* as follows:

$$DF_n^* = F_n + iF_{n+1} + \varepsilon(F_{n+2} + iF_{n+3}),$$

where the basis  $\{1, i, \varepsilon, i\varepsilon\}$  satisfies the conditions

$$i^2 = -1, \quad \varepsilon \neq 0, \quad \varepsilon^2 = 0, \quad (i\varepsilon)^2 = 0.$$

In 2018, Aydin [1] defined the *dual complex k-Fibonacci numbers* as follows:

$$DCF_{k,n} = F_{k,n} + iF_{k,n+1} + \varepsilon(F_{k,n+2} + iF_{k,n+3}),$$

where the basis  $\{1, i, \varepsilon, i\varepsilon\}$  satisfies the above condition.

In 2019, Prasad [12] introduced *complex Fibonacci p-numbers* by the following recurrence relation:

$$F_p^*(n) = F_p^*(n-1) + F_p^*(n-p-1)$$

with  $n > p+1$  and with initial terms

$$F_p^*(0) = i, \quad F_p^*(1) = F_p^*(2) = \cdots = F_p^*(p) = 1+i.$$

In 2021, Prasad [13] introduced *dual complex Fibonacci p-numbers* by the following recurrence relation:

$$\begin{aligned} DF_p^*(n) &= F_p^*(n-1) + \varepsilon F_p^*(n-p-1) \\ &= F_p(n-1) + iF_p(n) + \varepsilon(F_p(n-p-1) + iF_p(n-p)) \end{aligned}$$

with  $n > p+1$  and with initial terms

$$F_p^*(0) = i, \quad F_p^*(1) = F_p^*(2) = \cdots = F_p^*(p) = 1+i,$$

and the basis  $\{1, i, \varepsilon, i\varepsilon\}$  satisfying the above condition.

In 2022, Prasad [15] introduced *m-extension dual complex Fibonacci p-numbers* by the recurrence relation

$$(1.1) \quad \begin{aligned} DF_{p,m}^*(n) &= mF_{p,m}^*(n-1) + \varepsilon F_{p,m}^*(n-p-1) \\ &= m(F_{p,m}(n-1) + iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p)) \end{aligned}$$

with  $n > p+1$  and with initial terms

$$F_{p,m}^*(n) = m^{n-1}, \quad n = 1, \dots, p+1,$$

and the basis  $\{1, i, \varepsilon, i\varepsilon\}$  satisfying the same condition.

In this paper, we introduce  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers (see Definition 2.1 below) and establish some of their properties.

## 2. $(m, t)$ -extension dual complex Fibonacci $p$ -numbers

DEFINITION 2.1.  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers are given by the recurrence relation

$$(2.1) \quad \begin{aligned} DF_{p,m,t}^*(n) &= mF_{p,m,t}^*(n-1) + t\varepsilon F_{p,m,t}^*(n-p-1) \\ &= m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + t\varepsilon(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)) \end{aligned}$$

with  $n > p+1$  and with initial terms

$$F_{p,m,t}^*(1) = c_1, \quad F_{p,m,t}^*(2) = c_2, \dots, F_{p,m,t}^*(p+1) = c_{p+1},$$

where  $p$  ( $\geq 0$ ) is an integer,  $m$  ( $> 0$ ),  $t$  ( $> 0$ ),  $n > p + 1$  and  $c_1, \dots, c_{p+1}$  are arbitrary real or complex numbers and the basis  $\{1, i, \varepsilon, i\varepsilon\}$  satisfies the condition  $i^2 = -1$ ,  $\varepsilon \neq 0$ ,  $\varepsilon^2 = 0$ ,  $(i\varepsilon)^2 = 0$ .

**DEFINITION 2.2.** Addition and subtraction of  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers are defined by

$$\begin{aligned}
 DF_{p,m,t}^*(n_1) &\pm DF_{p,m,t}^*(n_2) \\
 &= mF_{p,m,t}^*(n_1 - 1) + t\varepsilon F_{p,m,t}^*(n_1 - p - 1) \\
 &\quad \pm (mF_{p,m,t}^*(n_2 - 1) + t\varepsilon F_{p,m,t}^*(n_2 - p - 1)) \\
 &= m(F_{p,m,t}(n_1 - 1) + iF_{p,m,t}(n_1)) \\
 &\quad + t\varepsilon(F_{p,m,t}(n_1 - p - 1) + iF_{p,m,t}(n_1 - p)) \\
 &\quad \pm (m(F_{p,m,t}(n_2 - 1) + iF_{p,m,t}(n_2)) \\
 &\quad + t\varepsilon(F_{p,m,t}(n_2 - p - 1) + iF_{p,m,t}(n_2 - p))) \\
 &= m(F_{p,m,t}(n_1 - 1) \pm F_{p,m,t}(n_2 - 1)) + im(F_{p,m,t}(n_1) \pm F_{p,m,t}(n_2)) \\
 &\quad + t\varepsilon(F_{p,m,t}(n_1 - p - 1) \pm F_{p,m,t}(n_2 - p - 1)) \\
 &\quad + it\varepsilon(F_{p,m,t}(n_1 - p) \pm F_{p,m,t}(n_2 - p)).
 \end{aligned}$$

**DEFINITION 2.3.** Multiplication of  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers by real scalars  $\lambda$  is defined by

$$\begin{aligned}
 \lambda DF_{p,m,t}^*(n) \\
 &= \lambda mF_{p,m,t}(n - 1) + i\lambda mF_{p,m,t}(n) + \varepsilon\lambda tF_{p,m,t}(n - p - 1) + i\varepsilon\lambda tF_{p,m,t}(n - p).
 \end{aligned}$$

**DEFINITION 2.4.** Multiplication of  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers is defined by

$$\begin{aligned}
 &DF_{p,m,t}^*(n_1)DF_{p,m,t}^*(n_2) \\
 &= [mF_{p,m,t}(n_1 - 1) + imF_{p,m,t}(n_1) + t\varepsilon F_{p,m,t}(n_1 - p - 1) + it\varepsilon F_{p,m,t}(n_1 - p)] \\
 &\quad \times [mF_{p,m,t}(n_2 - 1) + imF_{p,m,t}(n_2) + t\varepsilon F_{p,m,t}(n_2 - p - 1) + it\varepsilon F_{p,m,t}(n_2 - p)] \\
 &= m^2[F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 - 1) - F_{p,m,t}(n_1)F_{p,m,t}(n_2)] \\
 &\quad + im^2[F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2 - 1)] \\
 &\quad + m\varepsilon t[F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 - p - 1) + F_{p,m,t}(n_1 - p - 1)F_{p,m,t}(n_2 - 1) \\
 &\quad \quad \quad - F_{p,m,t}(n_1)F_{p,m,t}(n_2 - p) - F_{p,m,t}(n_2)F_{p,m,t}(n_1 - p)] \\
 &\quad + im\varepsilon t[F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 - p) + F_{p,m,t}(n_1)F_{p,m,t}(n_2 - p - 1) \\
 &\quad \quad \quad + F_{p,m,t}(n_2)F_{p,m,t}(n_1 - p - 1) + F_{p,m,t}(n_1 - p)F_{p,m,t}(n_2 - 1)] \\
 &= DF_{p,m,t}^*(n_2)DF_{p,m,t}^*(n_1).
 \end{aligned}$$

Since five kinds of conjugation can be defined for dual complex numbers [11], conjugation of  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers

can be defined in five different ways:

$$(2.2) \quad DF_{p,m,t}^{*1}(n) = m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) \\ + \varepsilon t(F_{p,m,t}(n-p-1) - iF_{p,m,t}(n-p)) \\ \text{(complex conjugation),}$$

$$(2.3) \quad DF_{p,m,t}^{*2}(n) = m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) \\ - \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)) \\ \text{(dual conjugation),}$$

$$(2.4) \quad DF_{p,m,t}^{*3}(n) = m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) \\ - \varepsilon t(F_{p,m,t}(n-p-1) - iF_{p,m,t}(n-p)) \\ \text{(coupled conjugation),}$$

$$(2.5) \quad DF_{p,m,t}^{*4}(n) = m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) \\ \times \left( 1 - \varepsilon t \frac{F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)}{m(F_{p,m,t}(n-1) + iF_{p,m,t}(n))} \right) \\ \text{(dual complex conjugation),}$$

$$(2.6) \quad DF_{p,m,t}^{*5}(n) = m(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)) \\ - \varepsilon t(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) \\ \text{(antidual conjugation).}$$

Some properties of these conjugations are given in the following theorems.

**THEOREM 2.1.** *We have the following relations:*

$$(2.7) \quad DF_{p,m,t}^*(n)DF_{p,m,t}^{*1}(n) \\ = m^2[F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)] \\ + 2m\varepsilon t[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) + F_{p,m,t}(n)F_{p,m,t}(n-p)],$$

$$(2.8) \quad DF_{p,m,t}^*(n)DF_{p,m,t}^{*2}(n) \\ = m^2[F_{p,m,t}^2(n-1) - F_{p,m,t}^2(n)] + i2m^2F_{p,m,t}(n-1)F_{p,m,t}(n),$$

$$(2.9) \quad DF_{p,m,t}^*(n)DF_{p,m,t}^{*3}(n) \\ = m^2[F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)] \\ + i2m\varepsilon t[F_{p,m,t}(n-p)F_{p,m,t}(n-1) - F_{p,m,t}(n)F_{p,m,t}(n-p-1)],$$

$$(2.10) \quad DF_{p,m,t}^*(n)DF_{p,m,t}^{*4}(n) = m^2[F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)],$$

$$(2.11) \quad DF_{p,m,t}^*(n)DF_{p,m,t}^{*5}(n) \\ = m^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) - F_{p,m,t}(n)F_{p,m,t}(n-p)] \\ + im^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p) + F_{p,m,t}(n)F_{p,m,t}(n-p-1)] \\ + m\varepsilon t[F_{p,m,t}^2(n-p-1) + F_{p,m,t}^2(n) - F_{p,m,t}^2(n-1) - F_{p,m,t}^2(n-p)] \\ + i2m\varepsilon t[F_{p,m,t}(n-p-1)F_{p,m,t}(n-p) - F_{p,m,t}(n-1)F_{p,m,t}(n)],$$

$$(2.12) \quad DF_{p,m,t}^*(n) + DF_{p,m,t}^{*1}(n) = 2[mF_{p,m,t}(n-1) + \varepsilon t F_{p,m,t}(n-p-1)],$$

$$(2.13) \quad DF_{p,m,t}^*(n) + DF_{p,m,t}^{*2}(n) = 2m[F_{p,m,t}(n-1) + iF_{p,m,t}(n)],$$

$$(2.14) \quad DF_{p,m,t}^*(n) + DF_{p,m,t}^{*3}(n) = 2[mF_{p,m,t}(n-1) + i\varepsilon t F_{p,m,t}(n-p)],$$

$$(2.15) \quad (F_{p,m,t}(n-1) + iF_{p,m,t}(n))DF_{p,m,t}^{*4}(n) \\ = [F_{p,m,t}(n-1) - iF_{p,m,t}(n)]DF_{p,m,t}^{*2}(n),$$

$$(2.16) \quad \varepsilon t DF_{p,m,t}^*(n) + mDF_{p,m,t}^{*5}(n) = m^2[F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)],$$

$$(2.17) \quad mDF_{p,m,t}^*(n) - \varepsilon t DF_{p,m,t}^{*5}(n) = m^2[F_{p,m,t}(n-1) + iF_{p,m,t}(n)],$$

$$(2.18) \quad DF_{p,m,t}^*(n) + DF_{p,m,t}^*(n+p) = DF_{p,m,t}^*(n+p+1).$$

*Proof.* We will prove the above relations one by one. By (2.1) and (2.2), we get

$$\begin{aligned} & DF_{p,m,t}^*(n)DF_{p,m,t}^{*1}(n) \\ &= [m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p))] \\ &\quad \times [m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) - iF_{p,m,t}(n-p))] \\ &= m^2[F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)] \\ &\quad + 2m\varepsilon t[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) + F_{p,m,t}(n)F_{p,m,t}(n-p)]. \end{aligned}$$

By (2.1) and (2.3), we get

$$\begin{aligned} & DF_{p,m,t}^*(n)DF_{p,m,t}^{*2}(n) \\ &= [m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p))] \\ &\quad \times [m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) - \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p))] \\ &= m^2[F_{p,m,t}^2(n-1) - F_{p,m,t}^2(n)] + i2m^2F_{p,m,t}(n-1)F_{p,m,t}(n). \end{aligned}$$

By (2.1) and (2.4), we get

$$\begin{aligned} & DF_{p,m,t}^*(n)DF_{p,m,t}^{*3}(n) \\ &= [m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p))] \\ &\quad \times [m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) - \varepsilon t(F_{p,m,t}(n-p-1) - iF_{p,m,t}(n-p))] \\ &= m^2[F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)] \\ &\quad + i2m\varepsilon t[F_{p,m,t}(n-p)F_{p,m,t}(n-1) - F_{p,m,t}(n)F_{p,m,t}(n-p-1)]. \end{aligned}$$

By (2.1) and (2.5), we get

$$\begin{aligned} & DF_{p,m,t}^*(n)DF_{p,m,t}^{*4}(n) \\ &= [m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p))] \\ &\quad \times \left[ m(F_{p,m,t}(n-1) - iF_{p,m,t}(n)) \left( 1 - \varepsilon t \frac{F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)}{m(F_{p,m,t}(n-1) + iF_{p,m,t}(n))} \right) \right] \\ &= m^2[F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)]. \end{aligned}$$

By (2.1) and (2.6), we get

$$\begin{aligned}
DF_{p,m,t}^*(n)DF_{p,m,t}^{*5}(n) &= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))] \\
&\quad \times [m(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))-\varepsilon t(F_{p,m,t}(n-1)+iF_{p,m,t}(n))] \\
&= m^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)-F_{p,m,t}(n)F_{p,m,t}(n-p)] \\
&\quad + im^2[F_{p,m,t}(n-1)F_{p,m,t}(n-p)+F_{p,m,t}(n)F_{p,m,t}(n-p-1)] \\
&\quad + m\varepsilon t[F_{p,m,t}^2(n-p-1)+F_{p,m,t}^2(n)-F_{p,m,t}^2(n-1)-F_{p,m,t}^2(n-p)] \\
&\quad + i2m\varepsilon t[F_{p,m,t}(n-p-1)F_{p,m,t}(n-p)-F_{p,m,t}(n-1)F_{p,m,t}(n)].
\end{aligned}$$

By (2.1) and (2.2), we get

$$\begin{aligned}
DF_{p,m,t}^*(n) + DF_{p,m,t}^{*1}(n) &= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))] \\
&\quad + [m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)-iF_{p,m,t}(n-p))] \\
&= 2[mF_{p,m,t}(n-1)+\varepsilon tF_{p,m,t}(n-p-1)].
\end{aligned}$$

By (2.1) and (2.3), we get

$$\begin{aligned}
DF_{p,m,t}^*(n) + DF_{p,m,t}^{*2}(n) &= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))] \\
&\quad + [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))] \\
&= 2m[F_{p,m,t}(n-1)+iF_{p,m,t}(n)].
\end{aligned}$$

By (2.1) and (2.4), we get

$$\begin{aligned}
DF_{p,m,t}^*(n) + DF_{p,m,t}^{*3}(n) &= [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))] \\
&\quad + [m(F_{p,m,t}(n-1)-iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)-iF_{p,m,t}(n-p))] \\
&= 2[mF_{p,m,t}(n-1)+i\varepsilon tF_{p,m,t}(n-p)].
\end{aligned}$$

By (2.3) and (2.5), we get

$$\begin{aligned}
(F_{p,m,t}(n-1)+iF_{p,m,t}(n))DF_{p,m,t}^{*4}(n) &= (F_{p,m,t}(n-1)+iF_{p,m,t}(n)) \\
&\quad \times \left[ m(F_{p,m,t}(n-1)-iF_{p,m,t}(n)) \left( 1 - \varepsilon t \frac{F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p)}{m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))} \right) \right] \\
&= [F_{p,m,t}(n-1)-iF_{p,m,t}(n)] \\
&\quad \times [m(F_{p,m,t}(n-1)+iF_{p,m,t}(n))-\varepsilon t(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p))] \\
&= [F_{p,m,t}(n-1)-iF_{p,m,t}(n)]DF_{p,m,t}^{*2}(n).
\end{aligned}$$

By (2.1) and (2.6), we get

$$\begin{aligned}
\varepsilon tDF_{p,m,t}^*(n) + mDF_{p,m,t}^{*5}(n) &= m\varepsilon t(F_{p,m,t}(n-1)+iF_{p,m,t}(n))+m^2(F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p)) \\
&\quad - m\varepsilon t(F_{p,m,t}(n-1)+iF_{p,m,t}(n)) \\
&= m^2[F_{p,m,t}(n-p-1)+iF_{p,m,t}(n-p)].
\end{aligned}$$

By (2.1) and (2.6), we get

$$\begin{aligned} mDF_{p,m,t}^*(n) - \varepsilon t DF_{p,m,t}^{*5}(n) \\ = m[m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p))] \\ - \varepsilon t[m(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p)) - \varepsilon t(F_{p,m,t}(n-1) + iF_{p,m,t}(n))] \\ = m^2[F_{p,m,t}(n-1) + iF_{p,m,t}(n)]. \end{aligned}$$

We know that  $DF_{p,m,t}^*(n) = DF_{p,m,t}^*(n-1) + DF_{p,m,t}^*(n-p-1)$ . Putting  $n = n+p+1$ , we get

$$DF_{p,m,t}^*(n+p+1) = DF_{p,m,t}^*(n+p) + DF_{p,m,t}^*(n) = DF_{p,m,t}^*(n) + DF_{p,m,t}^*(n+p).$$

Therefore, the norm of  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers can be defined in five different ways as follows:

$$\begin{aligned} NDF_{p,m,t}^{*1}(n) &= \|DF_{p,m,t}^*(n)DF_{p,m,t}^{*1}(n)\| \\ &= m^2(F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)) \\ &\quad + 2m\varepsilon t(F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) + F_{p,m,t}(n)F_{p,m,t}(n-p)) \end{aligned}$$

by (2.1) and (2.2);

$$\begin{aligned} NDF_{p,m,t}^{*2}(n) &= \|DF_{p,m,t}^*(n)DF_{p,m,t}^{*2}(n)\| \\ &= [(m^2(F_{p,m,t}^2(n-1) - F_{p,m,t}^2(n)))^2 + 4m^4F_{p,m,t}^2(n)F_{p,m,t}^2(n-1)]^{1/2} \\ &= m^2(F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)) \end{aligned}$$

by (2.1) and (2.3);

$$\begin{aligned} NDF_{p,m,t}^{*3}(n) &= \|DF_{p,m,t}^*(n)DF_{p,m,t}^{*3}(n)\| = m^2(F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n)) \\ \text{by (2.1) and (2.4);} \end{aligned}$$

$$NDF_{p,m,t}^{*4}(n) = \|DF_{p,m,t}^*(n)DF_{p,m,t}^{*4}(n)\| = m^2(F_{p,m,t}^2(n-1) + F_{p,m,t}^2(n))$$

by (2.1) and (2.5);

$$\begin{aligned} (NDF_{p,m,t}^{*5}(n))^2 &= (\|DF_{p,m,t}^*(n)DF_{p,m,t}^{*5}(n)\|)^2 \\ &= m^4[F_{p,m,t}^2(n-1)F_{p,m,t}^2(n-p-1) + F_{p,m,t}^2(n)F_{p,m,t}^2(n-p) \\ &\quad + F_{p,m,t}^2(n-1)F_{p,m,t}^2(n-p) + F_{p,m,t}^2(n)F_{p,m,t}^2(n-p-1)] \\ &\quad + 2m^3\varepsilon t[F_{p,m,t}(n-1)F_{p,m,t}^3(n-p-1) - F_{p,m,t}^3(n-1)F_{p,m,t}(n-p-1) \\ &\quad + F_{p,m,t}(n)F_{p,m,t}^3(n-p) - F_{p,m,t}^3(n)F_{p,m,t}(n-p) \\ &\quad + F_{p,m,t}(n-1)F_{p,m,t}(n-p-1)F_{p,m,t}^2(n-p) \\ &\quad - F_{p,m,t}^2(n)F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) \\ &\quad + F_{p,m,t}(n)F_{p,m,t}(n-p)F_{p,m,t}^2(n-p-1) \\ &\quad - F_{p,m,t}^2(n-1)F_{p,m,t}(n)F_{p,m,t}(n-p)] \end{aligned}$$

by (2.1) and (2.6). ■

**THEOREM 2.2.** *Let  $n_1, n_2 \geq 0$ . The Honsberger identity for  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers is given by*

$$\begin{aligned}
& DF_{p,m,t}^*(n_1)DF_{p,m,t}^*(n_2) + DF_{p,m,t}^*(n_1+1)DF_{p,m,t}^*(n_2+1) \\
&= m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-1) - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2+1)] \\
&\quad + im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-1) \\
&\quad \quad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2+1)] \\
&\quad + \varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2-1) \\
&\quad \quad - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p+1) - F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2+1)] \\
&\quad + i\varepsilon mt[F_{p,m,t}(n_1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p) \\
&\quad \quad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2)] \\
&\quad \quad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-p+1) \\
&\quad \quad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2+1) + F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2)]. 
\end{aligned}$$

*Proof.* We get

$$\begin{aligned}
& DF_{p,m,t}^*(n_1)DF_{p,m,t}^*(n_2) + DF_{p,m,t}^*(n_1+1)DF_{p,m,t}^*(n_2+1) \\
&= [m(F_{p,m,t}(n_1-1)+iF_{p,m,t}(n_1))+\varepsilon tF_{p,m,t}(n_1-p-1)+i\varepsilon tF_{p,m,t}(n_1-p)] \\
&\quad \times [m(F_{p,m,t}(n_2-1)+iF_{p,m,t}(n_2))+\varepsilon tF_{p,m,t}(n_2-p-1)+i\varepsilon tF_{p,m,t}(n_2-p)] \\
&\quad + [m(F_{p,m,t}(n_1)+iF_{p,m,t}(n_1+1))+\varepsilon tF_{p,m,t}(n_1-p)+i\varepsilon tF_{p,m,t}(n_1-p+1)] \\
&\quad \times [m(F_{p,m,t}(n_2)+iF_{p,m,t}(n_2+1))+\varepsilon tF_{p,m,t}(n_2-p)+i\varepsilon tF_{p,m,t}(n_2-p+1)] \\
&= m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-1) - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2+1)] \\
&\quad + im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-1) \\
&\quad \quad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2) + F_{p,m,t}(n_1)F_{p,m,t}(n_2+1)] \\
&\quad + \varepsilon mt[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2-1) \\
&\quad \quad - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p+1) - F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2+1)] \\
&\quad + i\varepsilon mt[F_{p,m,t}(n_1)F_{p,m,t}(n_2-p-1) + F_{p,m,t}(n_1-1)F_{p,m,t}(n_2-p) \\
&\quad \quad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2-1) + F_{p,m,t}(n_1-p-1)F_{p,m,t}(n_2)] \\
&\quad \quad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-p) + F_{p,m,t}(n_1)F_{p,m,t}(n_2-p+1) \\
&\quad \quad + F_{p,m,t}(n_1-p)F_{p,m,t}(n_2+1) + F_{p,m,t}(n_1-p+1)F_{p,m,t}(n_2)]
\end{aligned}$$

in view of (2.1) and the properties of  $(m, t)$ -extension Fibonacci  $p$ -numbers. ■

**THEOREM 2.3.** *Let  $n_1, n_2 \geq 0$ . D’Ocagne’s identity for  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers is given by*

$$\begin{aligned}
& DF_{p,m,t}^*(n_1)DF_{p,m,t}^*(n_2+1) - DF_{p,m,t}^*(n_1+1)DF_{p,m,t}^*(n_2) \\
&= m^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1)F_{p,m,t}(n_2+1) \\
&\quad + F_{p,m,t}(n_1+1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1)F_{p,m,t}(n_2-1)] \\
&\quad + im^2[F_{p,m,t}(n_1-1)F_{p,m,t}(n_2+1) - F_{p,m,t}(n_1+1)F_{p,m,t}(n_2-1)]
\end{aligned}$$

$$\begin{aligned}
 & + \varepsilon m t [F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 - p) - F_{p,m,t}(n_1)F_{p,m,t}(n_2 - p + 1) \\
 & \quad + F_{p,m,t}(n_1 - p - 1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1 - p)F_{p,m,t}(n_2 + 1) \\
 & \quad - F_{p,m,t}(n_1)F_{p,m,t}(n_2 - p - 1) + F_{p,m,t}(n_1 + 1)F_{p,m,t}(n_2 - p) \\
 & \quad - F_{p,m,t}(n_1 - p)F_{p,m,t}(n_2 - 1) + F_{p,m,t}(n_1 - p + 1)F_{p,m,t}(n_2)] \\
 & + i\varepsilon m t [F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 - p + 1) + F_{p,m,t}(n_1 - p - 1)F_{p,m,t}(n_2 + 1) \\
 & \quad - F_{p,m,t}(n_1 + 1)F_{p,m,t}(n_2 - p - 1) - F_{p,m,t}(n_1 - p + 1)F_{p,m,t}(n_2 - 1)].
 \end{aligned}$$

*Proof.* We get

$$\begin{aligned}
 & DF_{p,m,t}^*(n_1)DF_{p,m,t}^*(n_2 + 1) - DF_{p,m,t}^*(n_1 + 1)DF_{p,m,t}^*(n_2) \\
 & = [m(F_{p,m,t}(n_1 - 1) + iF_{p,m,t}(n_1)) + \varepsilon t F_{p,m,t}(n_1 - p - 1) + i\varepsilon t F_{p,m,t}(n_1 - p)] \\
 & \quad \times [m(F_{p,m,t}(n_2) + iF_{p,m,t}(n_2 + 1)) + \varepsilon t F_{p,m,t}(n_2 - p) + i\varepsilon t F_{p,m,t}(n_2 - p + 1)] \\
 & \quad - [m(F_{p,m,t}(n_1) + iF_{p,m,t}(n_1 + 1)) + \varepsilon t F_{p,m,t}(n_1 - p) + i\varepsilon t F_{p,m,t}(n_1 - p + 1)] \\
 & \quad \times [m(F_{p,m,t}(n_2 - 1) + iF_{p,m,t}(n_2)) + \varepsilon t F_{p,m,t}(n_2 - p - 1) + i\varepsilon t F_{p,m,t}(n_2 - p)] \\
 & = m^2 [F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1)F_{p,m,t}(n_2 + 1) \\
 & \quad + F_{p,m,t}(n_1 + 1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1)F_{p,m,t}(n_2 - 1)] \\
 & + im^2 [F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 + 1) - F_{p,m,t}(n_1 + 1)F_{p,m,t}(n_2 - 1)] \\
 & + \varepsilon m t [F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 - p) - F_{p,m,t}(n_1)F_{p,m,t}(n_2 - p + 1) \\
 & \quad + F_{p,m,t}(n_1 - p - 1)F_{p,m,t}(n_2) - F_{p,m,t}(n_1 - p)F_{p,m,t}(n_2 + 1) \\
 & \quad - F_{p,m,t}(n_1)F_{p,m,t}(n_2 - p - 1) + F_{p,m,t}(n_1 + 1)F_{p,m,t}(n_2 - p) \\
 & \quad - F_{p,m,t}(n_1 - p)F_{p,m,t}(n_2 - 1) + F_{p,m,t}(n_1 - p + 1)F_{p,m,t}(n_2)] \\
 & + i\varepsilon m t [F_{p,m,t}(n_1 - 1)F_{p,m,t}(n_2 - p + 1) + F_{p,m,t}(n_1 - p - 1)F_{p,m,t}(n_2 + 1) \\
 & \quad - F_{p,m,t}(n_1 + 1)F_{p,m,t}(n_2 - p - 1) - F_{p,m,t}(n_1 - p + 1)F_{p,m,t}(n_2 - 1)]
 \end{aligned}$$

in view of (2.1) and the properties of  $(m, t)$ -extension Fibonacci  $p$ -numbers. ■

**THEOREM 2.4.** For  $n \geq 1$ , Cassini's identity for  $DF_{p,m,t}^*(n)$  is given by

$$\begin{aligned}
 & DF_{p,m,t}^*(n - 1)DF_{p,m,t}^*(n + 1) - D^2 F_{p,m,t}^*(n) \\
 & = m^2 [F_{p,m,t}^2(n) - F_{p,m,t}^2(n - 1) + F_{p,m,t}(n - 2)F_{p,m,t}(n) \\
 & \quad - F_{p,m,t}(n + 1)F_{p,m,t}(n - 1)] \\
 & + im^2 [F_{p,m,t}(n + 1)F_{p,m,t}(n - 2) - F_{p,m,t}(n)F_{p,m,t}(n - 1)] \\
 & + \varepsilon m t [F_{p,m,t}(n - 2)F_{p,m,t}(n - p) - F_{p,m,t}(n - 1)F_{p,m,t}(n - p + 1) \\
 & \quad + F_{p,m,t}(n)F_{p,m,t}(n - p - 2) - F_{p,m,t}(n - p - 1)F_{p,m,t}(n + 1) \\
 & \quad - 2F_{p,m,t}(n - 1)F_{p,m,t}(n - p - 1) + 2F_{p,m,t}(n)F_{p,m,t}(n - p)] \\
 & + i\varepsilon m t [F_{p,m,t}(n - 2)F_{p,m,t}(n - p + 1) - F_{p,m,t}(n - 1)F_{p,m,t}(n - p) \\
 & \quad + F_{p,m,t}(n - p - 2)F_{p,m,t}(n + 1) - F_{p,m,t}(n - p - 1)F_{p,m,t}(n)].
 \end{aligned}$$

*Proof.* We get

$$\begin{aligned}
& DF_{p,m,t}^*(n-1)DF_{p,m,t}^*(n+1) - D^2F_{p,m,t}^*(n) \\
&= [m(F_{p,m,t}(n-2) + iF_{p,m,t}(n-1)) + \varepsilon t(F_{p,m,t}(n-p-2) + iF_{p,m,t}(n-p-1))] \\
&\quad \times [m(F_{p,m,t}(n) + iF_{p,m,t}(n+1)) + \varepsilon t(F_{p,m,t}(n-p) + iF_{p,m,t}(n-p+1))] \\
&\quad - [m(F_{p,m,t}(n-1) + iF_{p,m,t}(n)) + \varepsilon t(F_{p,m,t}(n-p-1) + iF_{p,m,t}(n-p))]^2 \\
&= m^2[F_{p,m,t}^2(n) - F_{p,m,t}^2(n-1) + F_{p,m,t}(n-2)F_{p,m,t}(n) \\
&\quad \quad \quad - F_{p,m,t}(n+1)F_{p,m,t}(n-1)] \\
&\quad + im^2[F_{p,m,t}(n+1)F_{p,m,t}(n-2) - F_{p,m,t}(n)F_{p,m,t}(n-1)] \\
&\quad + \varepsilon mt[F_{p,m,t}(n-2)F_{p,m,t}(n-p) - F_{p,m,t}(n-1)F_{p,m,t}(n-p+1) \\
&\quad \quad \quad + F_{p,m,t}(n)F_{p,m,t}(n-p-2) - F_{p,m,t}(n-p-1)F_{p,m,t}(n+1) \\
&\quad \quad \quad - 2F_{p,m,t}(n-1)F_{p,m,t}(n-p-1) + 2F_{p,m,t}(n)F_{p,m,t}(n-p)] \\
&\quad + i\varepsilon mt[F_{p,m,t}(n-2)F_{p,m,t}(n-p+1) - F_{p,m,t}(n-1)F_{p,m,t}(n-p) \\
&\quad \quad \quad + F_{p,m,t}(n-p-2)F_{p,m,t}(n+1) - F_{p,m,t}(n-p-1)F_{p,m,t}(n)]
\end{aligned}$$

in view of (2.1) and the properties of  $(m, t)$ -extension Fibonacci  $p$ -numbers. ■

**THEOREM 2.5.** For  $n \geq 1$ , Catalan's identity for  $DF_{p,m,t}^*(n+r)$  is given by

$$\begin{aligned}
& DF_{p,m,t}^*(n+r-1)DF_{p,m,t}^*(n+r+1) - D^2F_{p,m,t}^*(n+r) \\
&= m^2[F_{p,m,t}^2(n+r) - F_{p,m,t}^2(n+r-1) + F_{p,m,t}(n+r-2)F_{p,m,t}(n+r) \\
&\quad \quad \quad - F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-1)] \\
&\quad + im^2[F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-2) - F_{p,m,t}(n+r)F_{p,m,t}(n+r-1)] \\
&\quad + \varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p) - F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p+1) \\
&\quad \quad \quad + F_{p,m,t}(n+r)F_{p,m,t}(n+r-p-2) - F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r+1) \\
&\quad \quad \quad - 2F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p-1) + 2F_{p,m,t}(n+r)F_{p,m,t}(n+r-p)] \\
&\quad + i\varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p+1) - F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p) \\
&\quad \quad \quad + F_{p,m,t}(n+r-p-2)F_{p,m,t}(n+r+1) - F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r)].
\end{aligned}$$

*Proof.* We get

$$\begin{aligned}
& DF_{p,m,t}^*(n+r-1)DF_{p,m,t}^*(n+r+1) - D^2F_{p,m,t}^*(n+r) \\
&= [m(F_{p,m,t}(n+r-2) + iF_{p,m,t}(n+r-1)) \\
&\quad \quad \quad + \varepsilon t(F_{p,m,t}(n+r-p-2) + iF_{p,m,t}(n+r-p-1))] \\
&\quad \times [m(F_{p,m,t}(n+r) + iF_{p,m,t}(n+r+1)) \\
&\quad \quad \quad + \varepsilon t(F_{p,m,t}(n+r-p) + iF_{p,m,t}(n+r-p+1))] \\
&\quad - [m(F_{p,m,t}(n+r-1) + iF_{p,m,t}(n+r))] \\
&\quad \quad \quad + \varepsilon t(F_{p,m,t}(n+r-p-1) + iF_{p,m,t}(n+r-p))]^2
\end{aligned}$$

$$\begin{aligned}
&= m^2[F_{p,m,t}^2(n+r) - F_{p,m,t}^2(n+r-1) + F_{p,m,t}(n+r-2)F_{p,m,t}(n+r) \\
&\quad - F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-1)] \\
&+ im^2[F_{p,m,t}(n+r+1)F_{p,m,t}(n+r-2) - F_{p,m,t}(n+r)F_{p,m,t}(n+r-1)] \\
&+ \varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p) - F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p+1) \\
&\quad + F_{p,m,t}(n+r)F_{p,m,t}(n+r-p-2) - F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r+1) \\
&\quad - 2F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p-1) + 2F_{p,m,t}(n+r)F_{p,m,t}(n+r-p)] \\
&+ i\varepsilon mt[F_{p,m,t}(n+r-2)F_{p,m,t}(n+r-p+1) - F_{p,m,t}(n+r-1)F_{p,m,t}(n+r-p) \\
&\quad + F_{p,m,t}(n+r-p-2)F_{p,m,t}(n+r+1) - F_{p,m,t}(n+r-p-1)F_{p,m,t}(n+r)],
\end{aligned}$$

in view of (2.1) and properties of  $(m, t)$ -extension Fibonacci  $p$ -numbers. ■

**3. Conclusion.** In this paper, we introduced  $(m, t)$ -extension dual complex Fibonacci  $p$ -numbers and established some of their properties, including the Honsberger identity, D’Ocagne’s identity, Cassini’s identity and Catalan’s identity. I hope that these results will be useful in applied mathematics, quantum mechanics, quantum physics, Lie groups, number theory, kinematics and differential equations, just as are dual complex  $k$ -Fibonacci numbers [1], dual complex Fibonacci  $p$ -numbers [13], dual-complex numbers and their holomorphic functions [11] and dual-complex Jacobsthal quaternions [2] etc.

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