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On a general density theorem

by

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To the 75th birthday of Henryk Iwaniec

Abstract. Following the pioneering work of Halász and Turán we prove a general zero-density theorem for a large class of Dirichlet series, containing the Riemann and Dedekind zeta functions. Owing to the application of an idea of Halász (contained in the above mentioned work) and a sharp Vinogradov-type estimate for the Riemann zeta function (due to Heath-Brown) the results are particularly sharp in the neighborhood of the boundary line $\operatorname{Re} s = 1$.

1. Introduction. More than 50 years ago Halász and Turán [HT1969] showed two important theorems about two significant approximations of the Riemann Hypothesis (RH). The Lindelöf Hypothesis (LH) asserts, with the notation ($s = \sigma + it$)

$$\mu_\zeta(\sigma_0) := \inf\{\mu;\ |\zeta(\sigma+it)| \le T^\mu \text{ for } \sigma \ge \sigma_0, 1 \le |t| \le T\}, \tag{1.1}$$

that

$$\mu_\zeta(1/2) = 0. \tag{1.2}$$

The *Density Hypothesis* (DH) asserts that the estimate

$$N(\sigma, T) := \sum_{\substack{\zeta(\beta+i\gamma)=0 \\ \beta\ge\sigma,\, |\gamma|\le T}} 1 \ll_\sigma T^{A(\sigma)(1-\sigma)} \log^C T \qquad (C > 0 \text{ absolute constant}), \tag{1.3}$$

or, in a slightly weaker form

$$N(1-\eta, T) \ll_{\eta,\varepsilon} T^{B(\eta)\eta+\varepsilon} \qquad (\varepsilon > 0 \text{ arbitrary}), \tag{1.4}$$

holds with

$$A(\sigma) \le A = 2, \quad \text{or equivalently} \quad B(\eta) \le 2, \tag{1.5}$$

for all $\sigma \ge 1/2$, respectively for all $\eta \le 1/2$.

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The significance of (DH) is that – together with a slight improvement of the classical zero-free region of de la Vallée Poussin – it implies for the difference of consecutive primes $\big(\mathcal{P} = \{p_i\}_{i=1}^{\infty}$ is the set of all primes$\big)$ the estimate

$$p_{n+1} - p_n \ll_\varepsilon p_n^{1-1/A+\varepsilon} \quad (\varepsilon > 0 \ \text{ arbitrary}), \tag{1.6}$$

while the slightly stronger relation, in case of $A = 2$,

$$p_{n+1} - p_n = o(p_n^{1/2} \log p_n), \tag{1.7}$$

is still undecided supposing (RH). In general, (DH) can often substitute (RH). On the other hand, (LH) was supposed to have a weaker connection with the distribution of zeros since Backlund [Bac1918/19] showed its equivalence to

$$N(\alpha, T+1) - N(\alpha, T) = o(\log T) \quad \text{ as } T \to \infty, \text{ for all } \alpha > 1/2. \tag{1.8}$$

Ingham [Ing1937] showed that much more is true, namely on (LH), Carlson's density theorem $A(\sigma) = 4\sigma$ [Car1921] holds in the stronger form with $A(\sigma) \le A$,

$$A = 2(1 + \mu(1/2)), \quad \text{or equivalently} \quad B(\eta) = 2(1 + \mu(1/2)), \tag{1.9}$$

which in particular shows that (LH) implies (DH). Another breakthrough came 30 years later when Halász and Turán [HT1969] proved that (LH) implies, for $\sigma = 1 - \eta > 3/4$,

$$N(\sigma, T) \ll T^{\varepsilon} \ (\varepsilon > 0 \text{ arbitrary}), \text{ or equivalently } B(\eta) = 0 \text{ for } \eta < 1/4. \tag{1.10}$$

In the same paper they showed unconditionally in a slightly improved form (see Turán's book [Tur1984, Theorem 38.2])

$$N(\sigma, T) \ll T^{1.2 \cdot 10^5 (1-\sigma)^{3/2}} \log^C T \quad (T > C), \tag{1.11}$$

which first proved unconditionally that (DH) is true in a nontrivial half-plane $\sigma > c_1$ with $c_1 < 1$ and even

$$B(\eta) \le 1.2 \cdot 10^5 \sqrt{\eta} = o(1) \quad \text{ as } \eta \to 0. \tag{1.12}$$

Turán even conjectured that (LH) implies (1.10) for all $\sigma > 1/2$ but this has never been shown. The constant $1.2 \cdot 10^5$ was improved in the case of $\eta < c_2$ (a small positive absolute constant) in subsequent works of Montgomery [Mon1971], Ford [For2002] and Heath-Brown [Hea2017] to 4.95. Very recently the author [Pin2023] obtained a further slight improvement (using deep results of Heath-Brown's above-mentioned work), namely

$$B(\eta) \le (3\sqrt{2} + o(1))\sqrt{\eta} \quad \text{as } \eta \to 0, \tag{1.13}$$

with other similar improvements in the neighbourhood of the line $\operatorname{Re} s = 1$.

The author expresses thanks to Gábor Halász who suggested applying the above method to the investigation of the following problem. Halász and

Turán observed (see [Tur1984, Theorems 38.3–38.4]) that proven or hypothetical assumptions (like (LH)) on the vertical growth of Riemann's zeta-function can help to prove density theorems for *other* more general functions. The goal of the present work is to prove results in this direction. The class of functions we consider will be different from that considered in [Tur1984]. We will also separate the properties assumed for the general function $f(s)$ and those for $\zeta(s)$. Our method will be similar to that of our earlier paper [Pin2023] (but somewhat simpler). We can dispense with the power sum method of Turán [Tur1984] but will use a simple but ingenious idea of Halász [Hal1968] which in some form played a crucial role in all later density theorems.

As mentioned above, similarly to Halász and Turán [HT1969, Tur1984], in order to prove density theorems of type (1.3)–(1.4) for a general function $f(s)$ we need vertical growth conditions (cf. (1.1)) for *both* $f(s)$ and $\zeta(s)$, i.e., "the special function $\zeta(s)$ plays a role for general $f(s)$" [Tur1984, p. 368].

2. Results. We do not strive for full generality so will suppose that $f_1 \neq 0$ and

$$f(s) = \sum_{n=1}^{\infty} \frac{f_n}{n^s}, \quad M(s) = \frac{1}{f(s)} = \sum_{n=1}^{\infty} \frac{g_n}{n^s} \qquad \text{are analytic for } \sigma > 1 \tag{2.1}$$

(i.e. $M(s)f(s) = 1$ for $\sigma > 1$), and

$$f_n \ll n^{\Delta}, \quad g_n \ll n^{\Delta} \quad \text{for every } \Delta > 0. \tag{2.2}$$

Remark 1. If f_n is completely multiplicative as a function of n then $f_1 = 1$ and $g(n) = \mu(n) f_n$.

Further we suppose that $f(s)$ can be continued as an analytic function to the half-plane $\sigma \geq \alpha_f$, $\alpha_f < 1$, up to a simple pole at $s = 1$ with residue f_0 and (cf. (1.1)) with

$$\mu_f(\sigma_0) := \inf \{\mu; |f(\sigma + it)| \leq T^{\mu} \text{ for } \sigma \geq \sigma_0, 1 \leq |t| \leq T\} < \infty \tag{2.3}$$

for $\sigma_0 \geq \alpha_f$. This is clearly the analogue of Lindelöf's μ-function for $f(s)$ in place of $\zeta(s)$. The following technical definition will be useful in the formulation (and the proof) of our results. The function λ_f below depends on μ_f, and λ_ζ on μ_ζ. Let

$$\begin{aligned} \lambda_f(\eta) &:= \min_{0 \leq a;\, (a+1)\eta \leq 1-\alpha_f} \frac{\mu_f(1-(a+1)\eta)}{a\eta}, \\ \lambda_\zeta(\eta) &:= \min_{0 \leq b;\, (b+1)\eta \leq 1/2} \frac{\mu_\zeta(1-(b+1)\eta)}{b\eta}. \end{aligned} \tag{2.4}$$

Our result will express the density estimates

$$N_f(1-\eta, T) \ll_{\eta,\varepsilon} T^{B_f(\eta)\eta+\varepsilon} \tag{2.5}$$

as a function of λ_f *and* λ_ζ (in the case of $f = \zeta$ clearly λ_ζ is sufficient). For λ_ζ we have strong estimates using Korobov–Vinogradov's method and weaker classical ones by the method of Hardy–Littlewood–Weyl [Lit1922] or van der Corput [vdC1921, vdC1922], which imply that as $\eta \to 0$,

$$\mu_\zeta(1-\eta) = o(\eta), \quad \text{so} \quad \lambda_\zeta(\eta) = o(1). \tag{2.6}$$

Theorem 1. *Under conditions* (2.1)–(2.2) *and notation* (2.3)–(2.5) *we have, for* $\eta < \min(1-\alpha_f, 1/4)$,

$$B_f(\eta) \le \max(4\lambda_f(\eta), 3\lambda_\zeta(2\eta)). \tag{2.7}$$

Depending on the value

$$d_f(\eta) = \frac{\lambda_f(\eta)}{\lambda_\zeta(2\eta)} \tag{2.8}$$

we can improve Theorem 1 for $d_f(\eta) > 1$ and $d_f(\eta) < 1/2$ as follows (we take $d_f(\eta) = \infty$ for $\lambda_\zeta(2\eta) = 0$).

Theorem 2. *Under the assumptions of Theorem* 1 *we have*

$$B_f(\eta) \le \max(2\lambda_f(\eta), 4\lambda_\zeta(2\eta)) \quad \textit{if } d_f(\eta) > 1, \tag{2.9}$$

$$B_f(\eta) \le 2(\lambda_f(\eta) + \lambda_\zeta(2\eta)) \quad \textit{for arbitrary } d_f. \tag{2.10}$$

Remark 2. It is easy to see that (2.9) is stronger than both (2.7) and (2.10) if $d_f(\eta) > 1$.

Theorems 1–2 imply the following asymptotic result.

Corollary 1. *We have*

$$B_f(\eta) \le 2 \lim_{\eta\to 0} \lambda_f(\eta) + o(1) \quad \textit{as } \eta \to 0. \tag{2.11}$$

Proof. This follows from (2.10) since $\lambda_\zeta(2\eta) = o(1)$ as $\eta \to 0$. ∎

Corollary 2. *If there is a* $D_f > 0$ *such that for any* $\varepsilon > 0$,

$$|f(\sigma+it)| \le_{\varepsilon,\sigma} |t|^{D_f(1-\sigma)+\varepsilon} \quad \textit{for } |t| > 1,\ \sigma = 1-\eta \ge \alpha_f, \tag{2.12}$$

i.e., $\mu_f(1-\eta) \le D_f\eta$, *then*

$$B_f(\eta) \le \frac{2D_f}{1-\eta/(1-\alpha)}, \quad \textit{so } B_f(\eta) \le 2D_f + o(1) \textit{ if } \eta \to 0. \tag{2.13}$$

Proof. We have $\lambda_f(\eta) \le D_f(1-\alpha_f)/(1-\alpha_f-\eta)$ if we choose a with $(a+1)\eta = 1-\alpha_f$ in (2.4). ∎

Corollary 3. *If*

$$|f(1-\eta+it)| \le |t|^{o(\eta)} \quad \textit{for } |t| \ge 1 \textit{ as } \eta \to 0, \tag{2.14}$$

then

$$B_f(\eta) = o(1) \quad \textit{as } \eta \to 0. \tag{2.15}$$

Finally, in the case of $f = \zeta$ by Theorem 1 we are able to reach

COROLLARY 4. *We have*

$$B_\zeta(\eta) \le 3\sqrt{2\eta} + o(\sqrt{\eta}) \quad \textit{as } \eta \to 0. \tag{2.16}$$

Proof. Theorem 5 of Heath-Brown [Hea2017], together with the remark following it, shows that for any $\delta > 0$,

$$\mu_\zeta(1-\eta) \le \left(\frac{2}{3\sqrt{3}} + \delta\right)\eta^{3/2} \quad \text{if } \eta < \eta_0(\delta). \tag{2.17}$$

Choosing $b = 2$ in (2.4) we obtain, for any $\delta' > 0$,

$$\lambda_\zeta(\eta) \le \frac{\mu_\zeta(1-3\eta)}{2\eta} \le (1+\delta')\sqrt{\eta} \quad \text{if } \eta < \eta_0(\delta'). \tag{2.18}$$

In view of $3\sqrt{2} > 4$, (2.7) and (2.18) clearly imply (2.16). ∎

Finally, the conditional theorem of Halász–Turán also follows immediately for $f(s) = \zeta(s)$ or even for a larger class of functions as well, if we suppose (LH) for *both* functions $f(s)$ and $\zeta(s)$. This is contained in Corollary 2 as the limiting case $D_f = 0$.

COROLLARY 5. (LH) *implies* (DH) *for* $\sigma > 3/4$. *More generally, if*

$$\alpha_f \le 3/4, \qquad \mu_f(3/4) = \mu_\zeta(1/2) = 0, \tag{2.19}$$

then

$$B_f(\eta) = 0 \quad \textit{for } \eta < 1/4. \tag{2.20}$$

Proof. (2.19) implies $\lambda_f(\eta) = \lambda_\zeta(2\eta) = 0$ for $\eta > 1/4$ by (2.4). ∎

REMARK 3. It is interesting to note that we need only a weaker assumption for $f(s)$ than for $\zeta(s)$ (i.e. in the range $\sigma > 3/4$) to obtain

$$N_f(\sigma, T) \ll T^\varepsilon \quad \text{for } \sigma > 3/4 \quad (\varepsilon > 0, \text{ arbitrary}). \tag{2.21}$$

The same phenomenon was observed by Halász and Turán (see [Tur1984, p. 367]) for the class of functions $f(s)$ they investigated.

3. Notation and preparation. We will consider a maximal number K of zeros $\varrho_j = \beta_j + i\gamma_j := 1 - \eta_j + i\gamma_j$ of $f(s)$ with $|\gamma_j| \in [T/2, T]$, $|\gamma_j - \gamma_\nu| \ge 1$ for $\nu \ne j$ $(j, \nu \in [1, K])$ and $\beta_j := 1 - \eta_j \ge \sigma := 1 - \eta$. Let ε and Δ be sufficiently small, positive with $\Delta, \varepsilon < c_0(\eta, f, T)$; ε may be different at different occurrences and may depend on η and f. Analogously, C will be a positive constant which may depend on η and $f(s)$ and might be different at different occurrences. Further, let μ be the Möbius function, let

$$
\begin{aligned}
&Y = T^{\lambda_f(\eta)+\Delta}, \quad Z = T^{\lambda_\zeta(2\eta)+\Delta}, \quad X = T^{\Delta^2},\\
&a_n = \sum_{\substack{d\mid n\\ n/d\le X}} f_d g_{n/d}, \quad M_X(s) = \sum_{n\le X} g_n n^{-s},\\
&\lambda = \log Y, \quad \mathcal{L} = \max(\lambda, \log T), \quad \eta < 1/4, \quad Y_1 = Ye^3.
\end{aligned} \tag{3.1}
$$

We note that $\mathcal{L} \ll \log T$; note that the constants implied by o, O and $\ll$ may always depend on η, Δ and $f(s)$.

As a preparation we will show a lemma (actually an application of Perron's formula) which shows that sufficiently long partial sums of the zeta-function are of size $o(1)$. An alternative possibility would be (cf. [Pin2023]) to use a weighted sum as in [Mon1971, Appendix II].

Lemma 1. *Suppose that $\delta > 0$, $\sigma_0 = 1 - \eta_0 \in [1/2, 1]$, $1 < |t| < T$, $N \ll T$, $[N_1, N_2] = I(N) \subseteq [N, 2N)$, and $N \geq T^{\lambda_\zeta(\eta_0)+\delta}$. Then (with $s = \sigma_0 + it$)*

$$
S := \sum_{n\in I(N)} n^{-s} \ll \frac{\mathcal{L}N^{1-\sigma_0}}{|t|} + O(T^{-C\delta}), \tag{3.2}
$$

with a C depending on η_0.

Proof. Using Perron's formula [Per1908] in the form given in [MV2007, Corollary 5.3] we have

$$
S = \frac{1}{2\pi i} \int_{1-\sigma_0+1/\mathcal{L}-2iT}^{1-\sigma_0+1/\mathcal{L}+2iT} \zeta(s+w)\frac{N_2^w - N_1^w}{w}\, dw + O(N^{-\sigma_0}) + o\biggl(\frac{N^{1-\sigma_0}\mathcal{L}}{T}\biggr). \tag{3.3}
$$

Let us denote by b' the value of b for which the minimum in the definition of $\lambda_\zeta(\eta)$ is attained.

Moving the line of integration to the vertical line segment $\operatorname{Re} w = -b'\eta_0$, $\operatorname{Im} w = [-2T, 2T]$ along the horizontal segments $\operatorname{Im} w = -2T$ and $\operatorname{Im} w = 2T$, and denoting the new integration line by J, we obtain, from the pole of ζ at $w = 1 - s$,

$$
S = \frac{1}{2\pi i}\int_J \zeta(s+w)\frac{N_2^w - N_1^w}{w}\, dw + O\biggl(\frac{\mathcal{L}N^{1-\sigma_0}}{|t|}\biggr) + O(N^{-\sigma_0}). \tag{3.4}
$$

The value of the integral along the vertical segment is

$$
S_1 \ll \frac{\mathcal{L}T^{\mu_\zeta(1-(b'+1)\eta_0)+\varepsilon}}{N^{b'\eta_0}} \ll \mathcal{L}T^{-b'\delta\eta_0/2} \quad \text{if } \varepsilon \le \delta b'\eta_0/2. \tag{3.5}
$$

The contribution of the integral along the horizontal segments is, in view of $\mu_\zeta(1/2) \le 1/4$,

$$S_2 \ll T^{\mu_\zeta(1/2)+\varepsilon+1-\sigma_0-1} \ll T^{-1/5}. \tag{3.6}$$

Formulae (3.3)–(3.6) prove Lemma 1. ∎

REMARK 4. We can even work with the simpler bound $\mu(0) = 1/2$ if we change the definition (2.4), or even with $\mu(1/2) \le 1/2$ if $\sigma_0 > 1/2$.

REMARK 5. Lemma 1 is also true if $N > CT$, by the simple Theorem 4.11 of Titchmarsh [Tit1951].

4. Proof of Theorems 1–2. Let a' denote the value of a for which the minimum is attained in the definition of λ_f in (2.4). Our starting formula will be, similarly to [Pin2023],

$$\begin{aligned} I_j &:= \frac{1}{2\pi i} \int\limits_{(3)} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+\varrho_j}} \frac{e^{s^2/\mathcal{L}+\lambda s}}{s}\, ds \\ &= \frac{1}{2\pi i} \int\limits_{(3)} M_X(s+\varrho_j) f(s+\varrho_j) \frac{e^{s^2/\mathcal{L}+\lambda s}}{s}\, ds \\ &= \frac{1}{2\pi i} \int\limits_{(\eta_j-(a'+1)\eta)} M_X(s+\varrho_j) \frac{f(s+\varrho_j)}{s} e^{s^2/\mathcal{L}+\lambda s}\, ds + O\Bigg(\frac{XY^{\eta_j}}{|\gamma_j|} e^{-\gamma_j^2/\mathcal{L}}\Bigg) \\ &\ll X \int\limits_{-\infty}^{\infty} \frac{|\gamma_j+t|^{\mu_f(1-(a'+1)\eta)+\varepsilon}}{a'\eta+|t|} e^{-|\gamma_j+t|^2/\mathcal{L}} Y^{-a'\eta}\, dt + O(e^{-T^2/5\mathcal{L}}) \\ &\ll \mathcal{L} \frac{T^{\mu_f(1-(a'+1)\eta)+\varepsilon}}{Y^{a'\eta}} + O(e^{-T^2/(5\mathcal{L})}) = O(T^{-a'\eta\delta/3}) = o(1) \end{aligned} \tag{4.1}$$

if $\varepsilon < \delta a'\eta/2$, where the error term represents the contribution of the pole of ζ at $s = 1 - \varrho_j = \eta_j - i\gamma_j$ if we use the trivial estimate $M_X(s) \ll X$.

The RHS of (4.1) can be evaluated term by term for every n. The contribution of the term $n = 1$ can be obtained by moving the line of integration to $\operatorname{Re} s = -4$. The pole at $s = 0$ contributes $a_1 = 1$ and the integral is $O(\mathcal{L}Y^{-4}) = o(1)$. Further, we have $a_n = 0$ for $1 < n \le X$. For the terms with $n > Y_1 = Ye^3$ we can shift the line of integration to $\operatorname{Re} s = \mathcal{L}$. Using ($\varepsilon > 0$ arbitrary) $a_n \ll \tau(n)n^\varepsilon \ll n^{\beta_j}$ and $\sum_{n>M} n^{-u} \ll M^{-(u-1)}$ for $u \ge 2$, $M \ge 1$, we obtain

(4.2)

$$\int\limits_{(\mathcal{L})} \sum_{n>e^{\lambda+3}} \frac{a_n}{n^{s+\varrho_j}} e^{s^2/\mathcal{L}+\lambda s}\, ds \ll e^{-(\lambda+3)(\mathcal{L}-1)+\lambda\mathcal{L}} \int\limits_{-\infty}^{\infty} e^{(\mathcal{L}^2-t^2)/\mathcal{L}}\, dt = o(1).$$

Summarizing, we get

$$\sum_{X<n<Y_1} \frac{a_n}{n_j^{\varrho}} h(n) = 1 + o(1), \tag{4.3}$$

where since $\lambda \le \mathcal{L}$ we have, for every $n \ge 1$,

$$\begin{aligned} h(n) &:= \frac{1}{2\pi i} \int\limits_{(3)} \frac{e^{s^2/\mathcal{L}+(\lambda-\log n)s}}{s}\, ds = \frac{1}{2\pi i} \int\limits_{(1/\mathcal{L})} \frac{e^{s^2/\mathcal{L}+(\lambda-\log n)s}}{s}\, ds \\ &\ll \int\limits_{-\infty}^{\infty} \frac{e^{-t^2/\mathcal{L}}}{|1/\mathcal{L}+it|}\, dt \ll \mathcal{L}\log\mathcal{L}. \end{aligned} \tag{4.4}$$

From this we obtain, by a dyadic subdivision of (X, Y_1), for some $U \in [X, Y_1]$ and $I(U) \subseteq [U, 2U)$,

$$\sum_{j=1}^{K} \Big| \sum_{n\in I(U)} a_n^* n^{-\varrho_j} \Big| \gg \frac{K}{\mathcal{L}} \qquad \text{with } a_n^* = a_n h(n). \tag{4.5}$$

Next we will raise the Dirichlet polynomial $\sum_{n\in I(U)} a_n^* n^{-\varrho_j}$ to a minimal integral power $h_0 \ge 1$ such that $(2U)^{h_0} \ge Z = T^{\lambda_\zeta(2\eta)+\varDelta}$. Since $U \ge X$ and $\lambda_\zeta(2\eta) \le 1$, we have $h_0 \le 1 + \frac{\log Y_1}{\log X} \ll 1$. The resulting polynomial will have coefficients $b_n^* \ll \tau_{h_0}(n) n^{\varepsilon} \mathcal{L}^{h_0} (\log\mathcal{L})^{h_0} = T^{o(1)}$ by (4.4)–(4.5) for any $\varepsilon > 0$.

Further, with a suitable value $M \in [U^h, (2U)^h)$, (4.5) can be substituted using Hölder's inequality by

$$\sum_{j=1}^{K} \Big| \sum_{n\in I(M)} b_n^* n^{-\varrho_j} \Big| \gg \frac{K}{\mathcal{L}^{h_0}}. \tag{4.6}$$

If $U^2 \ge T^{\lambda_\zeta(2\eta)+\varDelta} = Z$ we take $h_0 = 2$. If $X \le U \le T^{(\lambda_\zeta(2\eta)+\varDelta)/2}$ we can choose the minimal integer $h \ge 2$, i.e. $h = h_0$ with

$$U^{h_0} \in \big[T^{\lambda_\zeta(2\eta)+\varDelta}, T^{(3/2)(\lambda_\zeta(2\eta)+\varDelta)}\big] = [Z, Z^{3/2}].$$

Since $U \le Y_1$, in both cases we have $U^{h_0} \ll \max(Y^2, Z^{3/2})$.

Let us now define the numbers φ_j with $|\varphi_j| = 1$ so that

$$\Big| \sum_{n\in I(M)} b_n^* n^{-\varrho_j} \Big| = \varphi_j \sum_{n\in I(M)} b_n^* n^{-\varrho_j} \qquad (j = 1, \ldots, K). \tag{4.7}$$

Halász's idea is to square the LHS of (4.7), interchange the order of summation over j and n and use the Cauchy–Schwarz inequality for the sum when n runs through $I(M)$ with

$$b_n^* n^{-\varrho_j} = b_n^* n^{-1/2-a'\eta} \cdot n^{-1/2+a'\eta+\eta_j-i\gamma_j} =: d_n \cdot e_n^{(j)} \quad (n \in I(M)). \tag{4.8}$$

Separating the diagonal terms (those with $j = \nu$), we deduce, from Lemma 1 and (4.6)–(4.8) for any $\varepsilon > 0$,

$$
\begin{aligned}
\frac{K^2}{\mathcal{L}^{2h_0}} &\ll \Big(\sum_{j=1}^{K} \varphi_j \sum_{n\in I(M)} b_n^* n^{-\varrho_j}\Big)^2 = \Big(\sum_{n\in I(M)} d_n \sum_{j=1}^{K} \varphi_j e_n^{(j)}\Big)^2 \\
&\ll \Big(\sum_{n\in I(M)} \frac{|b_n^*|^2}{n^{1+2a'\eta}}\Big)\Big(\sum_{j=1}^{K}\sum_{\nu=1}^{K} \varphi_j\overline{\varphi_\nu} \sum_{n\in I(M)} \frac{1}{n^{1-2a'\eta-\eta_j-\eta_\nu+i(\gamma_j-\gamma_\nu)}}\Big) \\
&\ll T^{o(1)} M^{-2a'\eta}\Big\{K(K-1)o(1) \\
&\quad + M^{2(a'+1)\eta} \sum_{\substack{j=1 \\ j\neq\nu}}^{K}\sum_{\nu=1}^{K} \frac{1}{|\gamma_j-\gamma_\nu|} + KM^{2(a'+1)\eta}\Big\} \\
&\ll o\big(K^2\mathcal{L}^{-2h_0}\big) + KT^{o(1)}M^{2\eta}.
\end{aligned}
\tag{4.9}
$$

Since $M \asymp U^{h_0} \ll \max(Y^2, T^{3/2(\lambda_\zeta(2\eta)+\Delta)})$ and (3.1) we deduce

$$B_f(\eta) \le \max(4\lambda_f(\eta),\ 3\lambda_\zeta(2\eta)), \tag{4.10}$$

because Δ can be chosen arbitrarily small with $\Delta < c_0(f,\eta,T)$.

The proof of Theorem 2 runs completely analogously with the following small change. To show (2.10) we can take

$$u = \frac{\log U}{\log T}, \quad h_1 = \left[\frac{\lambda_\zeta(2\eta)+\Delta}{u}\right] + 1. \tag{4.11}$$

In this case we can choose $h = h_0 = h_1$ to obtain

$$U^{h_1} \ge Z, \tag{4.12}$$

and so (since Δ can be chosen arbitrarily small) since $U \le Y_1$ we have

$$B_f(\eta) \le 2h_1 u \le 2(\lambda_\zeta(2\eta) + \lambda_f(\eta)). \tag{4.13}$$

To prove (2.9) we distinguish three cases.

Case 1. If $U \ge Z$ we choose $h_0 = 1$ to obtain

$$B_f(\eta) \le 2u \le 2\lambda_f(\eta). \tag{4.14}$$

Case 2. If $U \in (\sqrt{Z}, Z)$ we choose $h_0 = 2$ to obtain

$$B_f(\eta) \le 4u \le 4\lambda_\zeta(2\eta). \tag{4.15}$$

Case 3. If $U \le \sqrt{Z}$ we choose $h_0 = h_1$ as in (4.11) to obtain

$$B_f(\eta) \le 2h_1 u \le 2\lambda_\zeta(2\eta) + 2u \le 3\lambda_\zeta(2\eta). \tag{4.16}$$

Inequalities (4.14)–(4.16) prove (2.9).

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