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Number of integers represented by families of binary forms (II): binomial forms

by

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Abstract. We consider some families of binary binomial forms $aX^d + bY^d$, with a and b integers. Under suitable assumptions, we prove that every rational integer m with $|m| \geq 2$ is only represented by a finite number of forms of this family (with varying d, a, b). Furthermore, the number of such forms of degree $\geq d_0$ representing m is bounded by $O(|m|^{1/d_0+\epsilon})$ uniformly for $|m| \geq 2$. We also prove that the integers in the interval $[-N, N]$ represented by one of the forms of the family of degree $d \geq d_0$ are almost all represented by some form of the family of degree $d = d_0$ if such forms of degree d_0 exist.

In a previous paper we investigated the particular case where the binary binomial forms are positive definite. We now treat the general case by using a lower bound for linear forms in logarithms.

1. Introduction. When d, a and b are rational integers different from 0, with $d \geq 3$, Theorem 1.1 of [SX] gives an asymptotic estimate for the number of rational integers in the interval $[-N, N]$ represented by the binary form $aX^d + bY^d$. This estimate has the shape

$$C_{a,b,d} N^{2/d} + O(N^{\beta}) \quad \text{as } N \to \infty,$$

where the exponent $\beta < 2/d$ is explicit and where the constant $C_{a,b,d} > 0$ is also explicit (it corresponds to the constant $C_F = A_F W_F$ in [SX, Corollary 1.3] associated with the binary form

$$F(X, Y) = F_{a,b,d}(X, Y) = aX^d + bY^d;$$

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for more details see §3 below). Here we consider the representation of integers by some element of families of such binary binomial forms.

For every integer $d \geq 3$, let $\mathcal{E}_d$ be a finite subset of $(\mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \setminus \{0\})$ and let $\mathcal{F}_d$ be the set of binary binomial forms $F_{a,b,d}(X,Y)$ with $(a,b) \in \mathcal{E}_d$. We are interested in the representation of integers $m \in \mathbb{Z}$ by some form of the family $\mathcal{F} = \bigcup_{d \geq 3} \mathcal{F}_d$. For $d \geq 3$ and m in $\mathbb{Z}$, we introduce the two sets

$$\mathcal{G}_{\geq d}(m) = \Big\{(d', a, b, x, y) \;\Big|\; m = ax^{d'} + by^{d'} \text{ with } d' \geq d,\ (a,b) \in \mathcal{E}_{d'},\ (x,y) \in \mathbb{Z}^2 \text{ and } \max\{|x|,|y|\} \geq 2\Big\}$$

and

$$\mathcal{R}_{\geq d} = \{m \in \mathbb{Z} \mid \mathcal{G}_{\geq d}(m) \neq \emptyset\}.$$

For N a positive integer, we denote

$$\mathcal{R}_{\geq d}(N) = \mathcal{R}_{\geq d} \cap [-N, N].$$

When we require that two different forms in $\mathcal{E}_d$ are not isomorphic, we will need to assume the following hypotheses (see §3 below):

(C1) For any distinct $(a,b), (a',b') \in \mathcal{E}_d$, at least one of the ratios a/a' and b/b' is not the dth power of a rational number.
(C2) For any distinct $(a,b), (a',b') \in \mathcal{E}_d$, at least one of the ratios a/b' and b/a' is not the dth power of a rational number.

These conditions are trivially satisfied when $\mathcal{E}_d$ has cardinality 0 or 1.

The exponent $\vartheta_d < 2/d$ is defined in [FW2, (2.1)]:

$$\vartheta_d = \begin{cases} \frac{24\sqrt{3}+73}{60\sqrt{3}+73} = \frac{2628\sqrt{3}-1009}{5471} = 0.6475\ldots & \text{for } d = 3, \\ \frac{2\sqrt{d}+9}{4d\sqrt{d}-6\sqrt{d}+9} & \text{for } 4 \leq d \leq 20, \\ \frac{1}{d-1} & \text{for } d \geq 21. \end{cases}$$

When the family $\mathcal{F}$ is given and $d \geq 3$, we define

$$d^\dagger := \begin{cases} \inf\{d' \mid d' > d,\ \mathcal{F}_{d'} \neq \emptyset\} & \text{if there exists } d' > d \text{ such that } \mathcal{F}_{d'} \neq \emptyset, \\ \infty & \text{if } \mathcal{F}_{d'} = \emptyset \text{ for all } d' > d. \end{cases}$$

We denote by $\sharp E$ the number of elements of a finite set E.

Our first result is the following.

Theorem 1.1 (Positive definite case). *Let $\mathcal{E}_d \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ with $\mathcal{E}_d = \emptyset$ for odd d. Furthermore, suppose that*

$$\frac{1}{d}\log(\sharp\mathcal{E}_d + 1) \to 0 \quad \text{as } d \to \infty. \tag{1.1}$$

Then:

(a) *For all $m \in \mathbb{Z} \setminus \{0,1\}$ and all $d \geq 4$, the set $\mathcal{G}_{\geq d}(m)$ is finite. Furthermore, for all $d \geq 4$ and all $\epsilon > 0$, we have, as $|m| \to \infty$,*

$$\sharp\mathcal{G}_{\geq d}(m) = O_{d,\epsilon}(|m|^{1/d+\epsilon}).$$

(b) *Let $d \geq 4$ be an integer such that conditions* (C1) *and* (C2) *hold. For all $\epsilon > 0$, we have, as $N \to \infty$,*

$$\sharp\mathcal{R}_{\geq d}(N) = \Big(\sum_{(a,b)\in\mathcal{E}_d} C_{a,b,d}\Big) N^{2/d} + O_{d,\epsilon}(N^{\max\{\vartheta_d+\epsilon, 2/d^\dagger\}}).$$

(c) *In the above formula of* (b), *we have $C_{a,b,d} = A_{F_{a,b,d}} W_{F_{a,b,d}}$, where*

$$A_{F_{a,b,d}} = \iint_{|ax^d+by^d|\leq 1} dx\, dy$$

and where the values of the rational positive numbers $W_{F_{a,b,d}}$ are given in Proposition 3.3.

Remark 1.2. Consider the family of forms

$$\mathcal{F} := \{F_d(X,Y) := (r_d+1)X^d + Y^d \mid d \text{ even } \geq 4\}$$

where we write $d = s_d(s_d+1) + 2r_d$ with $s_d \geq 1$ and $0 \leq r_d \leq s_d$ (this decomposition is unique). Considering the values of these forms at the points $(1,0)$ and $(1,1)$, we see that if, in the definition of $\mathcal{G}_{\geq d}(m)$, one eliminates the condition $\max\{|x|,|y|\} \geq 2$, then the set $\mathcal{G}_{\geq d}(m)$ becomes infinite for all $m \geq 1$. This shows that the condition $\max\{|x|,|y|\} > 1$ is necessary for the validity of Theorem 1.1 (and also Theorems 1.4 and 4.1 below).

Remark 1.3. In [SX, Corollary 1.3] one finds explicit values, in terms of the Γ-function, of the fundamental area $A_{F_{a,b,d}}$.

The hypothesis $\sharp\mathcal{E}_d \leq d^{A_1}$ in [FW2, Theorem 1.13], which implies condition (iii) in Definition 2.3 below of an $(A, A_1, d_0, d_1, \kappa)$-regular family, is replaced here by (1.1) which cannot be omitted: for $d \geq 3$ and $N = 2^{2d}+1$, each of the 2^d integers of the form $a2^d + 1$, $a = 1, 2, 3, \dots, 2^d$, is represented by one of the forms $aX^d + Y^d$ with the choice $x = 2$, $y = 1$.

Our second result is

Theorem 1.4 (General case). *Let $\epsilon > 0$. There exists a constant $\eta > 0$ depending only on ϵ with the following property. Suppose that there exists $d_0 > 0$ such that, for all $d \geq d_0$,*

$$\max_{(a,b)\in\mathcal{E}_d} \{|a|,|b|\} \leq \exp(\eta d/\log d).$$

Then:

(a) *For all* $m \in \mathbb{Z} \setminus \{-1,0,1\}$ *and all* $d \geq 3$, *the set* $\mathcal{G}_{\geq d}(m)$ *is finite. Furthermore, for all* $d \geq 3$, *we have, as* $|m| \to \infty$,

$$\sharp\mathcal{G}_{\geq d}(m) = O_{d,\epsilon}(|m|^{1/d+\epsilon}).$$

(b) *Let* $d \geq 3$ *be an integer such that* (C1) *and* (C2) *hold. Then, as* $N \to \infty$,

$$\sharp\mathcal{R}_{\geq d}(N) = \Big(\sum_{(a,b)\in\mathcal{E}_d} C_{a,b,d}\Big)N^{2/d} + O_{d,\epsilon}(N^{\max\{\vartheta_d+\epsilon, 2/d^\dagger\}}).$$

(c) *The properties of the constant* $C_{a,b,d}$ *are the same as in Theorem* 1.1(c).

We will prove the result with the choice

$$\eta = \epsilon\, 2^{-81}3^{-15},$$

corresponding to the right-hand side of (4.1) for $\lambda = 2+\epsilon$.

In both Theorems 1.1 and 1.4, the proof of the bound for $\sharp\mathcal{G}_{\geq d}(m)$ is based on the explicit estimate (2.4). The fact that $\mathcal{G}_{\geq d}(m)$ is finite for all $m \notin \{-1,0,1\}$ is not a consequence of the bound for $\sharp\mathcal{R}_{\geq d}(N)$ (see Example 2.5).

Compared to [FW2], our new tool is a lower bound for linear forms in logarithms; the finiteness of the number of representations of a given integer m depends on this estimate. As we will show in Section 7, the *abc* conjecture would give an estimate very close to what would be deduced from conjectures on linear forms in logarithms.

2. A more general setting. Let $\mathcal{F}$ be a family of distinct binary forms, with non-zero discriminants and with degrees ≥ 3. We are interested in the following counting function of the set of values taken by some form $F \in \mathcal{F}$ of degree $\geq d$:

$$\begin{aligned}\mathcal{R}_{\geq d}(\mathcal{F}, N, A) := \sharp\big\{&m : 0 \leq |m| \leq N, \text{ there is } F \in \mathcal{F} \text{ with } \deg F \geq d\\ &\text{and } (x,y) \in \mathbb{Z}^2 \text{ with } \max\{|x|,|y|\} \geq A \text{ such that } F(x,y) = m\big\}.\end{aligned}$$

We study this function as N tends to infinity. The introduction of the parameter $A \geq 1$ is necessary to avoid situations without interest (see the comment after Definition 2.2 below). Several statements below are based on the positive constants A_F and W_F associated with the form F. These constants are defined in [SX, Theorem 1.2]. Finally, $\mathcal{F}_d$ is the subset of forms $F \in \mathcal{F}$ with $\deg F = d$ and the classical notion of $\mathrm{GL}(2,\mathbb{Q})$-isomorphism between binary forms is recalled at the beginning of §3 below.

2.1. The case of a finite family $\mathcal{F}$. The first case to consider is the situation when $\mathcal{F}$ is finite. We have the following result, which is trivial when d is larger than the degrees of all the forms of $\mathcal{F}$.

Theorem 2.1. *Let $\mathcal{F}$ be a finite family of distinct binary forms with degrees ≥ 3 and with discriminants different from zero. Furthermore, suppose that two forms of the family $\mathcal{F}$ are* $\mathrm{GL}(2,\mathbb{Q})$*-isomorphic if and only if they are equal. Then for every $d \geq 3$, every positive ε, and every $A \geq 1$,*

$$\mathcal{R}_{\geq d}(\mathcal{F}, N, A) = \Big(\sum_{F \in \mathcal{F}_d} A_F W_F \Big) \cdot N^{2/d} + O_{\mathcal{F},A,d,\varepsilon}(N^{\vartheta_d + \varepsilon}) + O_{\mathcal{F},A,d}(N^{2/d^\dagger}),$$

uniformly as $N \to \infty$.

Proof. The proof mimics the proof of [FW2, Theorem 1.11] which concerned an infinite family $\mathcal{F}$. No need to write this proof in full detail in this simpler situation. It is sufficient to recall that it is based on the following four points:

- an application of the inclusion-exclusion formula which produces a finite number of terms,
- an asymptotic formula for the number of integers m, $|m| \leq N$ which are the images of a fixed binary form F (see [SX, Theorem 1.1]),
- an upper bound for the number of integers m, $|m| \leq N$, which are the images of two fixed non-$\mathrm{GL}(2,\mathbb{Q})$-isomorphic binary forms F and G (see [FW2, Theorem 1.1]),
- an application of the easy bound

$$\sharp\big\{m \mid m = F(x,y) \text{ for some } F \in \mathcal{F}, \text{ with } (x,y) \in \mathbb{Z}^2 \text{ and } \max\{|x|,|y|\} \leq A\big\} \leq (2A+1)^2 \cdot (\sharp\mathcal{F}). \quad \blacksquare$$

2.2. The case of infinite family $\mathcal{F}$ and a new definition of a regular family. We are now concerned with infinite families $\mathcal{F}$. This case is more delicate, since it requires some condition of uniform growth on the forms $F \in \mathcal{F}$ (see Definitions 2.2(ii) and 2.3(v)).

Definition 2.2. Let $\mathcal{F}$ be an infinite set of distinct binary forms with discriminants different from zero and of degrees ≥ 3. We assume that for each $d \geq 3$, the subset $\mathcal{F}_d$ of $\mathcal{F}$ of forms of degree d is finite. We will say this set $\mathcal{F}$ is *regular* if there exists a positive integer A satisfying the following two conditions:

(i) Two forms of the family $\mathcal{F}$ are $\mathrm{GL}(2,\mathbb{Q})$-isomorphic if and only if they are equal.

(ii) For all $\epsilon > 0$, there exist two positive integers $N_0 = N_0(\epsilon)$ and $d_0 = d_0(\epsilon)$ such that, for all $N \geq N_0$, the number of integers m in the interval $[-N, N]$ for which there exist $d \in \mathbb{Z}$, $(x,y) \in \mathbb{Z}^2$ and $F \in \mathcal{F}_d$ satisfying

$$d \geq d_0, \quad \max\{|x|,|y|\} \geq A \quad \text{and} \quad F(x,y) = m$$

is bounded by N^ϵ.

For the truth of Theorem 2.6 below, one cannot drop the parameter A, as one sees by considering the family of cyclotomic forms [FW1] where hypothesis (ii) is satisfied with $A = 2$ but not with $A = 1$.

Recall the Definition 1.10 of an $(A, A_1, d_0, d_1, \kappa)$-regular family, introduced in [FW2].

DEFINITION 2.3. Let A, A_1, d_0, d_1 be integers and let κ be a real number such that

$$A \geq 1, \qquad A_1 \geq 1, \qquad d_1 \geq d_0 \geq 0, \qquad 0 < \kappa < A.$$

Let $\mathcal{F}$ be a set of distinct binary forms with integral coefficients and with discriminants different from zero. We say that $\mathcal{F}$ is $(A, A_1, d_0, d_1, \kappa)$-*regular* if it satisfies the following conditions:

(i) The set $\mathcal{F}$ is infinite.
(ii) All the forms of $\mathcal{F}$ have their degrees ≥ 3.
(iii) For all $d \geq 3$, we have $\sharp\mathcal{F}_d \leq d^{A_1}$.
(iv) Two forms of $\mathcal{F}$ are isomorphic if and only if they are equal.
(v) For any $d \geq \max\{d_1, d_0 + 1\}$, the following holds:

$$\left.\begin{array}{l} F \in \mathcal{F}_d, \\ (x, y) \in \mathbb{Z}^2 \text{ and } F(x, y) \neq 0, \\ \max\{|x|, |y|\} \geq A, \end{array}\right\} \Rightarrow \max\{|x|, |y|\} \leq \kappa\, |F(x, y)|^{\frac{1}{d-d_0}}.$$

These two definitions are not independent since we have

LEMMA 2.4. *If a family of binary forms is* $(A, A_1, d_0, d_1, \kappa)$*-regular in the sense of Definition* 2.3 *then it is also regular in the sense of Definition* 2.2.

Proof. Suppose that the family $\mathcal{F}$ satisfies condition (v) of Definition 2.3. Let $\epsilon > 0$, let N_0 be sufficiently large and let $d_2 > 2/\epsilon$. We use d_0 and d_1 as in Definition 2.3 and we replace d_0 by $\max\{d_1, d_0 + 1\} + d_2$ in condition (ii) of Definition 2.2. Let $N \in \mathbb{Z}$, $d \in \mathbb{Z}$, $m \in \mathbb{Z}$, $(x, y) \in \mathbb{Z}^2$ and $F \in \mathcal{F}_d$ be such that

$$N \geq N_0, \quad d \geq \max\{d_1, d_0 + 1\} + d_2, \quad |m| \leq N,$$
$$X \geq A \quad \text{and} \quad F(x, y) = m$$

with $X := \max\{|x|, |y|\}$. From condition (v) in Definition 2.3, we deduce

$$A^{d-d_0} \leq X^{d-d_0} \leq \kappa^{d-d_0}|m| \leq \kappa^{d-d_0}N.$$

From these inequalities we deduce on the one hand

$$(d - d_0)\log(A/\kappa) \leq \log N,$$

which is

$$d \leq d_0 + \frac{\log N}{\log(A/\kappa)},$$

and on the other hand

$$X \le \kappa N^{1/(d-d_0)} \le \kappa N^{1/d_2}.$$

Condition (iii) of Definition 2.3 states that the family $\mathcal{F}$ contains at most d^{A_1} forms of degree d. One deduces that the number of (d, x, y, F) (such that $F(x, y) = m$ with degree of F equal to d) and also the number of m, are bounded by $O(N^{2/d_2}(\log N)^{A_1+1})$. ■

Example 2.5. Let $(\ell_d)_{d\ge 3}$ be a sequence of positive integers. Let $\mathcal{F}$ be the family obtained by considering the sequence of binary forms $F_d(X, Y) = (X - dY)^{2d} + \ell_d Y^{2d}$. We have the equalities

$$F_d(d, 1) = \ell_d \quad \text{and} \quad F_d(d-1, 1) = F_d(d+1, 1) = \ell_d + 1. \tag{2.1}$$

We then check that this family is regular in the sense of Definition 2.2 if and only if, when N tends to infinity, we have

$$\frac{1}{\log N} \log \sharp\bigl\{\{\ell_d \mid d \ge 3\} \cap [1, N]\bigr\} \to 0.$$

Choosing $(\ell_d)_{d\ge 3}$ to be the sequence $(1, 2, 4,\ 1, 2, 4, 8,\ 1, 2, 4, 8, 16,\ \dots)$ defined by the formula

$$\ell_d = 2^j \quad \text{when} \quad d = \frac{k(k+1)}{2} + j = 1 + 2 + \cdots + k + j, \quad k \ge 2, \quad 0 \le j \le k,$$

we obtain an example of a regular family for which there exists an infinite set of integers m with infinitely many representations of the form $m = F_d(x, y)$.

The family $\mathcal{F}$ is regular in the sense of Definition 2.3 only if

$$\ell_d \ge (d/\kappa)^{d-d_0} \tag{2.2}$$

for all $d \ge \max\{d_1, d_0 + 1\}$. This follows from (2.1). For instance condition (2.2) is not satisfied when the sequence $(\ell_d)_{d\ge 2}$ is bounded.

We now turn our attention to the statement of [FW2, Theorem 1.11] when one considers a family which satisfies the new notion of regularity. The conclusion of that theorem remains true when one replaces the assumption that the family is $(A, A_1, d_0, d_1, \kappa)$-regular by the assumption that the family is regular in the sense of Definition 2.2.

Theorem 2.6. *Let $\mathcal{F}$ be a regular family of distinct binary forms in the sense of Definition 2.2. Then for every $d \ge 3$ and every positive ε, we have*

$$\mathcal{R}_{\ge d}(\mathcal{F}, N, A) = \Bigl(\sum_{F\in\mathcal{F}_d} A_F W_F\Bigr) \cdot N^{2/d} + O_{\mathcal{F},A,d,\varepsilon}(N^{\vartheta_d+\varepsilon}) + O_{\mathcal{F},A,d}(N^{2/d^\dagger}),$$

uniformly as $N \to \infty$.

We do not need the assumption $d \ge d_1$ which occurred in [FW2, Theorem 1.11]. Notice that if a family does not satisfy condition (ii) of Definition

2.2, then it does not satisfy the conclusion of Theorem 2.6 — condition (ii) in Definition 2.2 is essentially optimal.

Proof of Theorem 2.6. We fix $\varepsilon > 0$ and we first assume $d \geq d_0 = d_0(\varepsilon)$ (see Definition 2.2).

We use the notation

$$\mathcal{Z}_A = \mathbb{Z}^2 \setminus ([-A, A] \times [-A, A])$$

introduced in [FW2]. Conditions (iii) and (v) of Definition 2.3 appear in [FW2] when considering (3.5) and (3.7) there, to show that the cardinality of the set

$$\{(n, F, x, y) \mid n > d^\dagger + d_0,\ F \in \mathcal{F}_n,\ (x, y) \in \mathcal{Z}_A,\ |F(x, y)| \leq B\}$$

is bounded by $o_{\mathcal{F}}(B^{2/d^\dagger})$. Firstly we remark that it suffices to bound the cardinality of the set

$$\{m \mid 0 \leq m \leq B, \text{ there exists } (n, F, x, y),\\ n > d^\dagger + d_0,\ F \in \mathcal{F}_n,\ (x, y) \in \mathcal{Z}_A,\ |F(x, y)| = m\}.$$

The claimed bound immediately follows from assumption (ii) of Definition 2.2.

It remains to consider the case $3 \leq d < d_0$. We start from the double inequality

$$\begin{aligned} \mathcal{R}_{\geq d}\Big(\bigcup_{d' < d_0} \mathcal{F}_{d'}, N, A\Big) &\leq \mathcal{R}_{\geq d}(\mathcal{F}, N, A) \\ &\leq \mathcal{R}_{\geq d}\Big(\bigcup_{d' < d_0} \mathcal{F}_{d'}, N, A\Big) + \mathcal{R}_{\geq d_0}(\mathcal{F}, N, A). \end{aligned} \tag{2.3}$$

Since the family $\bigcup_{d' < d_0} \mathcal{F}_{d'}$ is finite, we can appeal to Theorem 2.1. The last term $\mathcal{R}_{\geq d_0}(\mathcal{F}, N, A)$ in (2.3) has just been treated by Theorem 2.6 in the particular case $d = d_0$. Comparing the exponents of the different terms, we complete the proof of Theorem 2.6 in all the cases. ∎

The following lemma is easy. It will be used several times

LEMMA 2.7. *Let $\theta > 0$. Suppose that there exists $d_0 \geq 3$ such that, for m and d in $\mathbb{Z}$ with $|m| \geq 2$ and $d \geq d_0$, the conditions*

$$d' \geq d, \quad (a, b) \in \mathcal{E}_{d'}, \quad \max\{|x|, |y|\} \geq 2 \quad \textit{and} \quad m = ax^{d'} + by^{d'}$$

imply the inequality

$$X^{d'} \leq |m|^\theta$$

with $X := \max\{|x|, |y|\}$. Also suppose that condition (1.1) is satisfied. Then

(a) *For every* $m \in \mathbb{Z} \setminus \{-1, 0, 1\}$ *and every* $d \geq 3$, *the set* $\mathcal{G}_{\geq d}(m)$ *is finite. In addition for every* $d \geq d_0$ *and every* $\epsilon > 0$, *we have, as* $|m| \to \infty$,

$$\sharp\mathcal{G}_{\geq d}(m) = O_{\theta,d,\epsilon}(|m|^{(\theta+\epsilon)/d}).$$

(b) *For every* $d \geq d_0$ *and every* $\epsilon > 0$, *there exists* N_0 *such that, for* $N \geq N_0$,

$$\sharp\mathcal{R}_{\geq d}(N) \leq N^{(2\theta+\epsilon)/d}.$$

Proof. Let $|m| \geq 2$, $d \geq d_0$ and $d' \geq d$. The inequalities

$$2^{d'} \leq X^{d'} \leq |m|^{\theta}$$

imply

$$d' \leq \frac{\theta \log |m|}{\log 2} \quad \text{and} \quad X \leq |m|^{\theta/d'}.$$

The cardinality of the set $\mathcal{G}_{\geq d}(m)$ is less than

$$4|m|^{\theta/d} \sum_{d'=d}^{\lfloor \frac{\theta \log |m|}{\log 2} \rfloor} \sharp\mathcal{E}_{d'},$$

since, when one unknown is fixed in the equation $m = ax^{d'} + by^{d'}$, the other unknown takes two values at most. The fact that $\mathcal{G}_{\geq d}(m)$ is a finite set for $d \geq 3$ and $|m| \geq 2$ follows from the fact that $\bigcup_{3 \leq d' < d} \mathcal{F}_{d'}$ is also finite. Thus assertion (a) is a consequence of (1.1). Finally, (b) follows from

$$\sharp\mathcal{R}_{\geq d}(N) \leq 4N^{2\theta/d} \sum_{d'=d}^{\lfloor \frac{\theta \log N}{\log 2} \rfloor} \sharp\mathcal{E}_{d'}. \ \blacksquare \tag{2.4}$$

Proof of Theorem 1.1. The equality $ax^d + by^d = m$ with a and $b > 0$ and $d \geq 4$ even implies $X^d \leq m$. Lemma 2.7 applied with $\theta = 1$ proves part (a) of Theorem 1.1. We also check condition (ii) in Definition 2.2 of a regular family for the value $A = 2$. To prove assertion (b) it remains to apply Theorem 2.6 since part (i) of Definition 2.2 is fulfilled by Corollary 3.2 below. $\blacksquare$

3. Isomorphisms between binomial binary forms and their automorphisms. We recall the action of the group of matrices $\mathrm{GL}(2, \mathbb{Q})$ on the set $\mathrm{Bin}(d, \mathbb{Q})$ of binary forms with degree d, with rational coefficients and with non-zero discriminant. If $F = F(X, Y)$ and $\gamma = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ respectively belong to $\mathrm{Bin}(d, \mathbb{Q})$ and $\mathrm{GL}(2, \mathbb{Q})$, we define

$$(F \circ \gamma)(X, Y) = F(a_1X + a_2Y, a_3X + a_4Y).$$

By definition, we say that two forms F and G are *isomorphic* if there exists $\gamma \in \mathrm{GL}(2, \mathbb{Q})$ such that $F \circ \gamma = G$. The group of automorphisms of a form F

is

$$\mathrm{Aut}(F,\mathbb{Q}) = \{\gamma \in \mathrm{GL}(2,\mathbb{Q}) \mid F\circ\gamma = F\}.$$

PROPOSITION 3.1. *Let $d \geq 3$ and a, b, a' and b' be integers different from zero. Then the two binary forms $aX^d + bY^d$ and $a'X^d + b'Y^d$ are isomorphic if and only if at least one of the following two conditions holds:*

(1) *the ratios a/a' and b/b' are both dth powers of a rational number,*
(2) *the ratios a/b' and b/a' are both dth powers of a rational number.*

Proof. The proof is an extension of the proof of [FW2, Lemma 1.14] which worked under the restrictions that d is an even integer and a, b, a' and b' are all positive. We quickly give the necessary modifications to obtain Proposition 3.1. Indeed, the beginning of the proof of [FW2, Lemma 1.14] does not require these restrictions. They are only used at the very last item of the proof where we prove that if $\gamma = \left(\begin{smallmatrix} a_1 & a_2 \\ a_2 & a_4\end{smallmatrix}\right)$ with $a_1a_2a_3a_4 \neq 0$, then the equality

$$F_{a,b,d}\circ\gamma = F_{a',b',d}$$

cannot hold. Indeed, if it holds, a computation leads to the equalities

$$\frac{a_1}{a_3} = -\frac{b}{a}\left(\frac{a_4}{a_2}\right)^{d-1}, \qquad \left(\frac{a_1}{a_3}\right)^2 = -\frac{b}{a}\left(\frac{a_4}{a_2}\right)^{d-2}.$$

Dividing the second equality by the first one we obtain $a_1/a_3 = a_2/a_4$. This is impossible since $\det\gamma \neq 0$. ∎

The following corollary is straightforward.

COROLLARY 3.2. *Let $\mathcal{F}$ be a family of binomial forms $F_{a,b,d}$ with $d \geq 3$, $(a,b) \in \mathcal{E}_d$, where $\mathcal{E}_d$ satisfies conditions* (C1) *and* (C2) *of* §1. *Then $\mathcal{F}$ satisfies part* (i) *of Definition* 2.2 *and part* (iv) *of Definition* 2.3.

We now recall the values of the constants $W_{F_{a,b,d}}$. More generally, for any binary form F, the constant W_F is a rational number only depending on the group $\mathrm{Aut}(F,\mathbb{Q})$, more precisely on lattices defined by some subgroups of $\mathrm{Aut}(F,\mathbb{Q})$. The constant W_F has a rather intricate definition but in the case of binomial forms, the corresponding group of automorphisms is rather simple (see [SX, Lemma 3.3]). The following proposition is the first part of [SX, Corollary 1.3].

PROPOSITION 3.3. *Let $F_{a,b,d}(X,Y) = aX^d + bY^d$ be a binary binomial form with $ab \neq 0$ and with $d \geq 3$.*

- *If a/b is not a dth power of a rational number, then*

$$W_{F_{a,b,d}} = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ 1/4 & \text{if } d \text{ is even.}\end{cases}$$

- *If a/b is a dth power of a rational number say $a/b = (A/B)^d$, then*

$$W_{F_{a,b,d}} = \begin{cases} 1 - 1/(2|AB|) & \text{if } d \text{ is odd,} \\ (1 - 1/(2|AB|))/4 & \text{if } d \text{ is even.} \end{cases}$$

4. On the integers represented by binary binomial forms with large degree. The following result gives an asymptotic upper bound for the number of integers represented by binary forms with high degree and for the number of representations of such integers.

THEOREM 4.1. *Let $d_0 \geq 3$ be an integer. Let λ and μ be real numbers such that $\lambda > 2$ and*

$$0 < \mu < 2^{-81} 3^{-15} \frac{\lambda - 2}{\lambda}. \tag{4.1}$$

Suppose that

$$\mathcal{E}_d \subset \big\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid ab \neq 0,\ \max\{|a|, |b|\} \leq \exp(\mu d/\log d)\big\}$$

for all $d \geq d_0$. Then:

(a) *For every $m \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and every $d \geq 3$, the set $\mathcal{G}_{\geq d}(m)$ is finite. Furthermore, for every $\epsilon > 0$ and as $|m| \to \infty$,*

$$\sharp\mathcal{G}_{\geq d}(m) = O_{\lambda,\mu,d,\epsilon}(|m|^{\epsilon + \lambda/(2d)}).$$

(b) *For every $d \geq 3$, there exists $N_0 > 0$ such that, for every $N \geq N_0$,*

$$\sharp\mathcal{R}_{\geq d}(N) \leq N^{\lambda/d}.$$

The set of m's that we are considering in assertion (b) contains the set of m's for which the hypotheses are satisfied with $(a, b) \in \mathcal{E}_d$. By the result of [SX] (see [BDW] for the particular case of binary binomial forms), each of these forms with degree d contributes $N^{2/d}$ to this number of m's, up to some positive constant. The hypothesis $\lambda > 2$ is thus natural.

One cannot drop the condition $a \neq 0$. Indeed, if $a = 0$, then every $m \leq N$ is represented by some form (take $y = 1$, $b = m$ and d sufficiently large). Similarly, one cannot drop the condition $b \neq 0$.

One cannot omit the hypothesis $\max\{|x|, |y|\} \geq 2$: if $x = 1$ and $y = -1$, then every integer m in the interval $1 \leq m \leq N$ satisfies the equality $m = a - b$ with d, a, b satisfying the conditions of Theorem 4.1.

One cannot replace the condition $\max\{|a|, |b|\} \leq \exp(\mu d/\log d)$ by $\max\{|a|, |b|\} \leq 2^d$, as can be seen by the example $x = 2$, $y = a = 1$, $b = m - 2^d$, $m = 1, \ldots, 2^d - 1$. In §7 we will see to what extent one can hope to weaken this hypothesis by assuming either Conjecture 1 of [L, p. 212] or the *abc* conjecture. In this connection, in [FW2, Theorem 1.13], there is no hypothesis concerning $\max\{|a|, b|\}$ when (a, b) is in the set $\mathcal{E}_d$: the only condition deals with the number of elements which must be less than d^{A_1}

for [FW2, Theorem 1.13], and must satisfy condition (1.1) for Theorem 1.1. The example of the family $X^d + (d - 2^d)Y^d$ shows that such a result cannot be extended to the case where the binary form has real zeroes.

5. A Diophantine result. The central tool in the proof of Theorem 4.1 is a lower bound coming from the theory of linear forms in logarithms, more precisely [W, Corollary 9.22]. The usual height of the rational number p/q, written under its irreducible form, is defined by $\mathrm{H}(p/q) = \max\{|p|, q\}$ and its logarithmic height is

$$\mathrm{h}(p/q) = \log \mathrm{H}(p/q) = \log \max\{|p|, q\}.$$

We write e for exp(1).

Proposition 5.1. *Let a_1, a_2 be rational numbers, b_1, b_2 be positive integers, A_1, A_2, B be real positive numbers. Suppose for $j = 1, 2$ that*

$$B \geq \max\{\mathrm{e}, b_1, b_2\}, \quad \log A_j \geq \max\{\mathrm{h}(a_j), 1\}.$$

If $a_1^{b_1} a_2^{b_2} \neq 1$, then

$$|a_1^{b_1} a_2^{b_2} - 1| \geq \exp\{-C(\log B)(\log A_1)(\log A_2)\}$$

with $C = 2^{79}3^{15}$.

This lower bound follows from [W, Corollary 9.22, p. 308] by taking

$$D = 1, \quad m = 2, \quad \alpha_1 = a_1, \quad \alpha_2 = a_2$$

and the constant $C(m)$ defined in [W, p. 252].

Corollary 5.2. *Let d, a, b, x and y be rational integers. Let*

$$\mathcal{A} := \max\{|a|, |b|\}, \quad X := \max\{|x|, |y|\}.$$

Suppose $d \geq 2$, $\mathcal{A} \geq 2$, $X \geq 2$ and $ax^d + by^d \neq 0$. Then

$$|ax^d + by^d| \geq \max\{|ax^d|, |by^d|\} \exp\{-4C(\log d)(\log X)(\log \mathcal{A})\}.$$

The conclusion is obviously false when one of the parameters d, X, $\mathcal{A}$ equals 1.

Proof. By symmetry one can suppose that $|ax^d| \leq |by^d|$. We use Proposition 5.1 with

$$b_1 = d, \quad b_2 = 1, \quad a_1 = \frac{x}{y}, \quad a_2 = -\frac{a}{b},$$

$$B = \begin{cases} d & \text{if } d \geq 3, \\ \mathrm{e} & \text{if } d = 2, \end{cases} \quad A_1 = \begin{cases} X & \text{if } X \geq 3, \\ \mathrm{e} & \text{if } X = 2, \end{cases} \quad A_2 = \begin{cases} \mathcal{A} & \text{if } \mathcal{A} \geq 3, \\ \mathrm{e} & \text{if } \mathcal{A} = 2. \end{cases}$$

We conclude the proof using the inequality $1/(\log 2)^3 < 4$. ∎

Corollary 5.2 implies the lower bound

$$|ax^d + by^d| \geq X^d \exp\{-4C(\log d)(\log X)(\log \mathcal{A})\},$$

which we write as

$$|ax^d + by^d| \geq X^{d-4C(\log d)(\log \mathcal{A})}. \tag{5.1}$$

6. Proofs of Theorems 4.1 and 1.4

Proof of Theorem 4.1. Let λ' in the interval $2 < \lambda' < \lambda$ be such that

$$\mu = \frac{\lambda' - 2}{4C\lambda'},$$

where C is defined in Proposition 5.1. Let $m = ax^{d'} + by^{d'}$ with $|m| \geq 2$, $d' \geq d$, $(a, b) \in \mathcal{E}_{d'}$ and $X \geq 2$. When $ax^{d'}$ and $by^{d'}$ have the same sign, we have $|m| \geq X^{d'}$ and we use Lemma 2.7 with $\theta = 1$. When $ax^{d'}$ and $by^{d'}$ have opposite signs, in order to use Lemma 2.7, we can suppose that $m \geq 2$, $a, x, y > 0$ and $b < 0$.

We are first interested in the pairs $(a, b) \in \bigcup_{d' \geq d} \mathcal{E}_{d'}$ satisfying $\max\{|a|, |b|\} = 1$. By our hypotheses, we have $a = 1$ and $b = -1$. It is no restriction to suppose that $d \geq \lambda'/(\lambda' - 2)$, since when $m \neq 0$ is given, the equation $x^d - y^d = m$ has at most $(d - 1) \cdot \sharp\{k \mid k|m\} = O_d(|m|^\epsilon)$ solutions. For $d' \geq d$, we write

$$\begin{aligned} m = x^{d'} - y^{d'} &= (x - y)(x^{d'-1} + x^{d'-2}y + \cdots + xy^{d'-2} + y^{d'-1}) \\ &> X^{d'-1} \geq X^{2d'/\lambda'}. \end{aligned}$$

Thus we can use Lemma 2.7 with $\theta = \lambda'/2$.

We now consider the pairs $(a, b) \in \bigcup_{d' \geq d} \mathcal{E}_{d'}$ such that $\mathcal{A} := \max\{|a|, |b|\}$ satisfies $\mathcal{A} \geq 2$. Since we have supposed that $\mathcal{A} \leq \exp(\mu d'/\log d')$, we have

$$\frac{d'}{(\log d')(\log \mathcal{A})} \geq \frac{1}{\mu} = \frac{4C\lambda'}{\lambda' - 2}.$$

Let $X := \max\{|x|, |y|\}$. We deduce from (5.1) the inequality

$$X^{d'-4C(\log d')(\log \mathcal{A})} \leq m$$

with

$$d' - 4C(\log d')(\log \mathcal{A}) \geq d'(1 - 4C\mu) = \frac{2d'}{\lambda'},$$

which allows us to use Lemma 2.7 with the choice $\theta = \lambda'/2$. To conclude the proof, we add the three values $m = 0$ and $m = \pm 1$. ∎

Proof of Theorem 1.4. It mimics the proof of Theorem 1.1 in Section 2.2: combining Corollary 3.2 and Theorem 4.1(b), one deduces that the family $\mathcal{F}$ satisfies the conditions of Definition 2.2 of a regular family. ∎

7. Conjectures. Let X_0 be an integer ≥ 2. We introduce the following subset of $\mathcal{R}_{\geq d}$:

$$\mathcal{R}_{\geq d,X_0} = \bigl\{m \in \mathbb{Z} \bigm| \text{there exists } (d', a, b, x, y) \text{ such that } m = ax^{d'} + by^{d'} \\ \text{with } d' \geq d,\ (a,b) \in \mathcal{E}_{d'},\ (x,y) \in \mathbb{Z}^2 \text{ and } \max\{|x|,|y|\} \geq X_0\bigr\},$$

so that $\mathcal{R}_{\geq d} = \mathcal{R}_{\geq d,2}$. For N a positive integer we also denote

$$\mathcal{R}_{\geq d,X_0}(N) = \mathcal{R}_{\geq d,X_0} \cap [-N, N].$$

After the statement of Theorem 4.1, we gave the example of the equation $2^d - b = m$ to show that one cannot replace the condition $\max\{|a|,|b|\} \leq \exp(\mu d/\log d)$ by $\max\{|a|,|b|\} \leq 2^d$. Introducing a parameter X_0 and assuming $\max\{|x|,|y|\} \geq X_0$, one might expect a more general result to hold. It is interesting to notice that a conjecture on lower bounds for linear forms in logarithms and the *abc* conjecture would produce very similar results.

7.1. Conjecture 1 of [L]. We state Conjecture 1 of [L, Introduction to Chapters X and XI, p. 212] as follows.

Conjecture 7.1. *Let $\epsilon > 0$. There exists a constant $C(\epsilon) > 0$ only depending on ϵ such that if $a_1, \dots, a_n$ are rational positive numbers and $b_1, \dots, b_n$ are integers, and we define*

$$B_j = \max\{|b_j|, 1\}, \quad A_j = \max\{\mathrm{e}^{\mathrm{h}(a_j)}, 1\}, \quad B = \max_{1 \leq j \leq n} B_j$$

and suppose that $b_1 \log a_1 + \dots + b_n \log a_n \neq 0$, then

$$|b_1 \log a_1 + \dots + b_n \log a_n| > \frac{C(\epsilon)^n B}{(B_1 \cdots B_n A_1^2 \cdots A_n^2)^{1+\epsilon}}.$$

Actually, we will only use a weak form of this conjecture: we will suppose the existence of a number $\epsilon > 0$ for which Conjecture 7.1 holds.

Theorem 7.2. *Let $\epsilon > 0$ be such that Conjecture 7.1 is satisfied. Let $\lambda > 2$. Let d_0 be a sufficiently large integer and let $X_0 \geq 2$. Suppose*

$$\mathcal{E}_d \subset \bigl\{(a,b) \in \mathbb{Z} \times \mathbb{Z} \bigm| ab \neq 0,\ \max\{|a|,|b|\} \leq X_0^{d/d_0}\bigr\}$$

for all $d \geq d_0$. Then for every $d \geq d_0$, we have

$$\sharp\mathcal{R}_{\geq d,X_0}(N) \leq N^{\lambda/d}.$$

Proof. Conjecture 7.1 with $n = 2$, $B = B_1 = B_2 = d$, $A_1 = X$, $A_2 = \mathcal{A}$ allows us to replace the conclusion of Corollary 5.2 by

$$|ax^d + by^d| \geq X^d \exp\{-c(\epsilon) - (1+\epsilon)\log d - (2+\epsilon)\log X - (2+\epsilon)\log \mathcal{A}\}$$

with $c(\epsilon) > 0$ depending only on ϵ. Let $2 < \lambda' < \lambda$ and let d_0 be sufficiently large that

$$1 - \frac{2(2+\epsilon)}{d_0} - \frac{c(\epsilon) + (1+\epsilon)\log d_0}{d_0 \log 2} > \frac{2}{\lambda'}.$$

For $d' \geq d$ and $m = ax^{d'} + by^{d'}$, the resulting upper bound

$$X^{d'} \leq |m|^{\lambda'/2}$$

allows us to use Lemma 2.7. ∎

7.2. The *abc* conjecture. Let $R(m)$ be the *radical* of a positive integer m:

$$R(m) = \prod_{p \,\mathrm{prime},\, p|m} p.$$

The well known *abc* conjecture (see for example [W, §1.2]) asserts that *for all $\epsilon > 0$, there exists a constant $\kappa(\epsilon)$ such that if a, b, c are coprime positive integers such that $a + b = c$, then*

$$c \leq \kappa(\epsilon) R(abc)^{1+\epsilon}.$$

As in Section 7.1, we will only assume the existence of a number $\epsilon > 0$ for which the property holds.

LEMMA 7.3. *Let $\epsilon > 0$ be such that the abc conjecture holds. Then under the hypotheses of Corollary* 5.2, *we have*

$$X^{d-2-2\epsilon} \leq \kappa(\epsilon) \mathcal{A}^{1+2\epsilon} |m|^{1+\epsilon}.$$

Proof. Let $m = ax^d + by^d$. Without loss of generality, one can suppose $|ax^d| \geq |by^d|$. If $|m| \geq |ax^d|$, the conclusion is obvious. Now suppose that $|ax^d| > |m|$. After a possible change of signs, we can also suppose that $a, x, y > 0$ and $b < 0$.

Let $\mathit{\Delta}$ be the greatest common divisor of ax^d and $|b|y^d$ and let P be the set of prime divisors of $\mathit{\Delta}$. For $p \in P$, we write

$$\alpha_p = v_p(a), \quad \beta_p = v_p(b), \quad \xi_p = v_p(x), \quad \eta_p = v_p(y), \quad \delta_p = v_p(\mathit{\Delta}).$$

Thus

$$\delta_p = \min\{\alpha_p + d\xi_p,\, \beta_p + d\eta_p\}.$$

We also define

$$a = \tilde{a} \prod_{p\in P} p^{\alpha_p}, \qquad |b| = \tilde{b} \prod_{p\in P} p^{\beta_p}, \qquad x = \tilde{x} \prod_{p\in P} p^{\xi_p}, \qquad y = \tilde{y} \prod_{p\in P} p^{\eta_p},$$

so that, for $p \in P$, we have $v_p(\tilde{a}) = v_p(\tilde{b}) = v_p(\tilde{x}) = v_p(\tilde{y}) = 0$.

Let $\tilde{m} = \mathit{\Delta}^{-1} m$; we have

$$\tilde{m} = \mathit{\Delta}^{-1} a x^d - \mathit{\Delta}^{-1} |b| y^d$$

with

$$\Delta^{-1}ax^d = \tilde{a}\tilde{x}^d \prod_{p\in P} p^{\alpha_p+d\xi_p-\delta_p},$$
$$\Delta^{-1}|b|y^d = \tilde{b}\tilde{y}^d \prod_{p\in P} p^{\beta_p+d\eta_p-\delta_p}.$$

The radical of Δ is $\tilde{\Delta} := \prod_{p\in P} p$. The integers $\Delta^{-1}ax^d$ and $\Delta^{-1}|b|y^d$ are coprime, so the radical of their product is less than $\tilde{\Delta}\tilde{a}\tilde{b}\tilde{x}\tilde{y}$. We use the *abc* conjecture for

$$c = \Delta^{-1}ax^d = \tilde{m} + \Delta^{-1}|b|y^d.$$

It gives

$$\Delta^{-1}ax^d \le \kappa(\epsilon)(\tilde{\Delta}\tilde{a}\tilde{b}\tilde{x}\tilde{y}\tilde{m})^{1+\epsilon},$$

which is

$$ax^d \le \kappa(\epsilon)(a|b|xym)^{1+\epsilon}\Delta \prod_{p\in P} p^{(1+\epsilon)(1-\alpha_p-\beta_p-\xi_p-\eta_p-\delta_p)}.$$

Since $\delta_p \ge 1$ we have $\alpha_p+\eta_p \ge 1$, $\beta_p+\xi_p \ge 1$ and we obtain

$$ax^d \le \kappa(\epsilon)(a|b|xym)^{1+\epsilon},$$

which we write as

$$x^{d-1-\epsilon} \le \kappa(\epsilon)a^{\epsilon}(|b|ym)^{1+\epsilon}. \tag{7.1}$$

We now use the bound $|b|y^d \le ax^d$ written as

$$y \le (a/|b|)^{1/d}x. \tag{7.2}$$

Then we have

$$y^{1+\epsilon} \le (a/|b|)^{(1+\epsilon)/d}x^{1+\epsilon}$$

and (7.1) gives

$$\begin{aligned} x^{d-2-2\epsilon} &\le \kappa(\epsilon)a^{\epsilon}(a/|b|)^{(1+\epsilon)/d}(|b|\,m)^{1+\epsilon} \\ &= \kappa(\epsilon)a^{(1/d)+\epsilon+(\epsilon/d)}|b|^{1-(1/d)+\epsilon-(\epsilon/d)}m^{1+\epsilon}. \end{aligned} \tag{7.3}$$

We again use (7.1) and (7.2) to obtain

$$\begin{aligned} y^{d-2-2\epsilon} &\le \kappa(\epsilon)(a/|b|)^{1-2/d-2\epsilon/d}a^{1/d+\epsilon+\epsilon/d}|b|^{(1-1/d+\epsilon-\epsilon/d)}m^{1+\epsilon} \\ &= \kappa(\epsilon)a^{(1-1/d+\epsilon-\epsilon/d)}|b|^{1/d+\epsilon+\epsilon/d}m^{1+\epsilon}. \end{aligned}$$

Thanks to (7.3) we conclude the proof of Lemma 7.3. ∎

THEOREM 7.4. *Let $\epsilon > 0$ be such that the abc conjecture is satisfied. Let $\lambda > 2+2\epsilon$, let d_0 be a sufficiently large integer and let $X_0 \ge 2$. Suppose*

$$\mathcal{E}_d \subset \left\{(a,b)\in\mathbb{Z}\times\mathbb{Z} \;\middle|\; ab\ne 0,\ \max\{|a|,|b|\} \le X_0^{d/d_0}\right\}$$

for all $d \geq d_0$. Then for every $d \geq d_0$, we have

$$\sharp\mathcal{R}_{\geq d, X_0}(N) \leq N^{\lambda/d}.$$

Proof. Let $2 < \lambda' < \lambda/(1+\epsilon)$ and let d_0 be a sufficiently large integer such that

$$1 - \frac{3+4\epsilon}{d_0} - \frac{\log \kappa(\epsilon)}{d_0 \log 2} > \frac{2}{\lambda'}.$$

Let $d' \geq d$ and $m = ax^{d'} + by^{d'}$. From Lemma 7.3 we deduce the bound $X^{d'} \leq |m|^\theta$ with $\theta = \lambda'(1+\epsilon)/2$, which allows us to apply Lemma 2.7. ∎

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