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Systems of bihomogeneous forms of small bidegree

by

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Abstract. We use the circle method to count the number of integer solutions to systems of bihomogeneous equations of bidegree $(1,1)$ and $(2,1)$ of bounded height in lopsided boxes. Previously, adjusting Birch's techniques to the bihomogeneous setting, Schindler showed an asymptotic formula provided the number of variables grows at least quadratically with the number of equations considered. Based on recent methods by Rydin Myerson we weaken this assumption and show that the number of variables only needs to satisfy a linear bound in terms of the number of equations.

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1. Introduction. Studying the number of rational solutions of bounded height to a system of equations is a fundamental tool in order to understand the distribution of rational points on varieties. A longstanding result by Birch [3] establishes an asymptotic formula for the number of integer points of bounded height that are solutions to a system of homogeneous forms of the same degree in a general setting, provided the number of variables is sufficiently large relative to the singular locus of the variety defined by the system of equations. This was recently improved upon by Rydin Myerson [22, 23] whenever the degree is 2 or 3. These results may be used in order to prove Manin's conjecture for certain Fano varieties, which arise as complete intersections in projective space.

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Analogous to Birch's result, Schindler [24] studied systems of bihomogeneous forms. Using the hyperbola method, Schindler [26] established Manin's conjecture for certain bihomogeneous varieties as a result. The aim of this paper is to improve Schindler's result by applying the ideas of Rydin Myerson to the bihomogeneous setting. While the results presented only hold for bidegree $(1, 1)$ and $(2, 1)$, they follow from Theorem 2.1, which deals with general bidegree. This theorem could in principle be used to improve Schindler's result for general bidegree. However, one would run into problems that are very similar to the ones appearing in [21].

Consider a system of bihomogeneous forms

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = (F_1(\boldsymbol{x}, \boldsymbol{y}), \dots, F_R(\boldsymbol{x}, \boldsymbol{y}))$$

with integer coefficients in variables $\boldsymbol{x} = (x_1, \dots, x_{n_1})$ and $\boldsymbol{y} = (y_1, \dots, y_{n_2})$. We assume that all of the forms have the same bidegree, which we denote by (d_1, d_2) for non-negative integers d_1, d_2. By this we mean that for any scalars $\lambda, \mu \in \mathbb{C}$ we have

$$F_i(\lambda \boldsymbol{x}, \mu \boldsymbol{y}) = \lambda^{d_1} \mu^{d_2} F_i(\boldsymbol{x}, \boldsymbol{y}), \qquad i = 1, \dots, R.$$

This system defines a biprojective variety $V \subset \mathbb{P}_{\mathbb{Q}}^{n_1-1} \times \mathbb{P}_{\mathbb{Q}}^{n_2-1}$. One can also interpret the system in the affine variables $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$ and thus $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})$ also defines an affine variety which we will denote by $V_0 \subset \mathbb{A}_{\mathbb{Q}}^{n_1+n_2}$. We are interested in studying the set of integer solutions to this system of bihomogeneous equations. Consider two boxes $\mathcal{B}_i \subset [-1, 1]^{n_i}$ where each edge is of side length at most 1 and they are all parallel to the coordinate axes. In order to study the questions from an analytic point of view, for $P_1, P_2 > 1$ we define the following counting function:

$$N(P_1, P_2) = \#\{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \colon \boldsymbol{x}/P_1 \in \mathcal{B}_1,\ \boldsymbol{y}/P_2 \in \mathcal{B}_2,\ \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}\}.$$

Generalising the work of Birch [3], Schindler [24] used the circle method to achieve an asymptotic formula for $N(P_1, P_2)$ as $P_1, P_2 \to \infty$ provided certain conditions on the number of variables are satisfied, to be described below. Before we can state Schindler's result, consider the varieties V_1^* and V_2^* in $\mathbb{A}_{\mathbb{Q}}^{n_1+n_2}$ to be defined by

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j} < R \quad \text{and} \quad \operatorname{rank}\left(\frac{\partial F_i}{\partial y_j}\right)_{i,j} < R$$

respectively. Assume that V_0 is a complete intersection, which means that $\dim V_0 = n_1 + n_2 - R$. Write $b = \max\left\{\frac{\log(P_1)}{\log(P_2)}, 1\right\}$ and $u = \max\left\{\frac{\log(P_2)}{\log(P_1)}, 1\right\}$. If $n_i > R$ for $i = 1, 2$ and

$$n_1 + n_2 - \dim V_i^* > 2^{d_1+d_2-2} \max\{R(R+1)(d_1+d_2-1), R(bd_1+ud_2)\} \tag{1.1}$$

is satisfied, then Schindler showed the asymptotic formula

(1.2)
$$N(P_1,P_2)=\sigma P_1^{n_1-Rd_1}P_2^{n_2-Rd_2}+O(P_1^{n_1-Rd_1}P_2^{n_2-Rd_2}\min\{P_1,P_2\}^{-\delta}),$$

for some $\delta>0$ and where σ is positive if the system $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0}$ has a smooth p-adic zero for all primes p, and the variety V_0 has a smooth real zero in $\mathcal{B}_1\times\mathcal{B}_2$. In the case when the equations $F_1(\boldsymbol{x},\boldsymbol{y}),\dots,F_R(\boldsymbol{x},\boldsymbol{y})$ define a smooth complete intersection V, and the bidegree is $(1,1)$ or $(2,1)$, the goal of this paper is to improve the restriction on the number of variables (1.1) and still show the asymptotic formula (1.2). In particular, if the bidegree is $(1,1)$ or $(2,1)$ we will show that if

$$bd_1+ud_2<\frac{R+1}{2},$$

our result will require fewer variables.

The result by Schindler generalises a well-known result by Birch [3], which deals with systems of homogeneous equations: Let $\mathcal{B}\subset[-1,1]^n$ be a box containing the origin with side lengths at most 1 and edges parallel to the coordinate axes. Given homogeneous equations $G_1(\boldsymbol{x}),\dots,G_R(\boldsymbol{x})$ with rational coefficients of common degree $d>1$ define the counting function

$$N(P)=\#\{\boldsymbol{x}\in\mathbb{Z}^n\colon \boldsymbol{x}/P\in\mathcal{B},\, G_1(\boldsymbol{x})=\dots=G_R(\boldsymbol{x})=0\}.$$

Write $V^*\subset\mathbb{A}^n_{\mathbb{Q}}$ for the variety defined by

$$\operatorname{rank}\left(\frac{\partial G_i}{\partial x_j}\right)_{i,j}<R,$$

commonly referred to as the *Birch singular locus.* Assuming that $G_1,\dots,G_R$ define a complete intersection $X\subset\mathbb{P}^{n-1}_{\mathbb{Q}}$ and that the number of variables satisfies

$$n-\dim V^*>R(R+1)(d-1)2^{d-1},\tag{1.3}$$

Birch showed

$$N(P)=\tilde{\sigma}P^{n-dR}+O(P^{n-dR-\varepsilon}),\tag{1.4}$$

where $\tilde{\sigma}>0$ if the system $\boldsymbol{G}(\boldsymbol{x})$ has a smooth p-adic zero for all primes p and the variety X has a smooth real zero in $\mathcal{B}$.

Building on the ideas of Müller [16, 17] on quadratic Diophantine inequalities, Rydin Myerson improved Birch's theorem. He weakened the assumption on the number of variables in the cases $d=2,3$ (see [22, 23]) whenever R is reasonably large. Assuming that $X\subset\mathbb{P}^{n-1}_{\mathbb{Q}}$ defines a complete intersection, he was able to replace the condition in (1.3) by

$$n-\sigma_{\mathbb{R}}>d2^dR,\tag{1.5}$$

where

$$\sigma_{\mathbb{R}} = 1 + \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}} \dim \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{G}),$$

and where $\mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{G})$ is the pencil defined by $\sum_{i=1}^R \beta_i G(\boldsymbol{x})$ in $\mathbb{P}_{\mathbb{Q}}^{n-1}$. We note at this point that several other authors have replaced the Birch singular locus condition with weaker assumptions, such as Schindler [25] and Dietmann [9], who also considered dimensions of pencils, and very recently Yamagishi [31], who replaced the Birch singular locus with a condition regarding the Hessian of the system. Returning to Rydin Myerson's result, if X is non-singular then one can show

$$\sigma_{\mathbb{R}} \leq R - 1$$

and in this case if $n \geq (d2^d + 1)R$ then one obtains the desired asymptotic. Notably, the work of Rydin Myerson showed the number of variables n thus only has to grow linearly in the number of equations R, whereas R appeared quadratically in Birch's work. If $d \geq 4$ he showed that for *generic* systems of forms it suffices to assume (1.5) for the asymptotic (1.4) to hold. Generic here means that the set of coefficients is required to lie in some non-empty Zariski open subset of the parameter space of coefficients of the equations.

Our goal in this paper is to generalise the results obtained by Rydin Myerson to the case of bihomogeneous varieties whenever the bidegree of the forms is $(1,1)$ or $(2,1)$. Those two cases correspond to degrees 2 and 3 in the homogeneous case, respectively. We call a bihomogeneous form *bilinear* if the bidegree is $(1,1)$. Given a bilinear form $F_i(\boldsymbol{x}, \boldsymbol{y})$ we may write it as

$$F_i(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^T A_i \boldsymbol{x},$$

for some $n_2 \times n_1$-dimensional matrices A_i with rational entries. Given $\boldsymbol{\beta} \in \mathbb{R}^R$ write

$$A_{\boldsymbol{\beta}} = \sum_{i=1}^R \beta_i A_i.$$

Regarding $A_{\boldsymbol{\beta}}$ as a map $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and and $A_{\boldsymbol{\beta}}^T$ as a map $\mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ we define the quantities

$$\sigma_{\mathbb{R}}^{(1)} := \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}} \dim \ker(A_{\boldsymbol{\beta}}) \quad \text{and} \quad \sigma_{\mathbb{R}}^{(2)} := \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}} \dim \ker(A_{\boldsymbol{\beta}}^T).$$

We state our first theorem for systems of bilinear forms. Since the situation is completely symmetric with respect to the $\boldsymbol{x}$ and $\boldsymbol{y}$ variables if the forms are bilinear, we may without loss of generality assume $P_1 \geq P_2$ in the counting function, and still obtain the full result.

THEOREM 1.1. *Let $F_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, F_R(\boldsymbol{x}, \boldsymbol{y})$ be bilinear forms with integer coefficients such that the biprojective variety $\mathbb{V}(F_1, \ldots, F_R) \subset \mathbb{P}_{\mathbb{Q}}^{n_1-1} \times \mathbb{P}_{\mathbb{Q}}^{n_2-1}$ is a complete intersection. Let $P_1 \geq P_2 > 1$, write $b = \frac{\log(P_1)}{\log(P_2)}$ and assume* *further that*

$$n_i - \sigma_{\mathbb{R}}^{(i)} > (2b+2)R \tag{1.6}$$

for $i = 1, 2$. Then there exists some $\delta > 0$ depending at most on b, $\boldsymbol{F}$, R and n_i such that

$$N(P_1, P_2) = \sigma P_1^{n_1-R} P_2^{n_2-R} + O(P_1^{n_1-R} P_2^{n_2-R-\delta}),$$

where $\sigma > 0$ if the system $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ has a smooth p-adic zero for all primes p and if the variety V_0 has a smooth real zero in $\mathcal{B}_1 \times \mathcal{B}_2$.

Moreover, if we assume $\mathbb{V}(F_1, \dots, F_R) \subset \mathbb{P}_{\mathbb{Q}}^{n_1-1} \times \mathbb{P}_{\mathbb{Q}}^{n_2-1}$ to be smooth the same conclusions hold if we assume

$$\min\{n_1, n_2\} > (2b+2)R \quad \textit{and} \quad n_1 + n_2 > (4b+5)R$$

instead of (1.6).

We now move on to systems of forms $F_1(\boldsymbol{x}, \boldsymbol{y}), \dots, F_R(\boldsymbol{x}, \boldsymbol{y})$ of bidegree $(2,1)$. We may write such a form $F_i(\boldsymbol{x}, \boldsymbol{y})$ as

$$F_i(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T H_i(\boldsymbol{y}) \boldsymbol{x},$$

where $H_i(\boldsymbol{y})$ is a symmetric $n_1 \times n_1$ matrix whose entries are linear forms in the variables $\boldsymbol{y} = (y_1, \dots, y_{n_2})$. Similarly to above, given $\boldsymbol{\beta} \in \mathbb{R}^R$ we write

$$H_{\boldsymbol{\beta}}(\boldsymbol{y}) = \sum_{i=1}^{R} \beta_i H_i(\boldsymbol{y}).$$

Given $\ell \in \{1, \dots, n_2\}$ write $\boldsymbol{e}_\ell \in \mathbb{R}^{n_2}$ for the standard unit basis vectors. Write

$$\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} = \mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_1)\boldsymbol{x}, \dots, \boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_{n_2})\boldsymbol{x}) \subset \mathbb{P}_{\mathbb{Q}}^{n_1-1}$$

for this intersection of zero loci, and define

$$s_{\mathbb{R}}^{(1)} := 1 + \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}} \dim \mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2}. \tag{1.7}$$

Further write $\mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})$ for the biprojective variety defined by the system of equations

$$\mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) = \mathbb{V}((H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})_1, \dots, (H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})_{n_1}) \subset \mathbb{P}_{\mathbb{Q}}^{n_1-1} \times \mathbb{P}_{\mathbb{Q}}^{n_2-1}$$

and define

$$s_{\mathbb{R}}^{(2)} := \left\lfloor \frac{\max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}} \dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})}{2} \right\rfloor + 1, \tag{1.8}$$

where $\lfloor x \rfloor$ denotes the largest integer m such that $m \leq x$.

THEOREM 1.2. *Let $F_1(\boldsymbol{x}, \boldsymbol{y}), \dots, F_R(\boldsymbol{x}, \boldsymbol{y})$ be bihomogeneous forms with integer coefficients of bidegree $(2,1)$ such that the biprojective variety $\mathbb{V}(F_1, \dots, F_R) \subset \mathbb{P}_{\mathbb{Q}}^{n_1-1} \times \mathbb{P}_{\mathbb{Q}}^{n_2-1}$ is a complete intersection. Let $P_1, P_2 > 1$ be* *real numbers. Write* $b = \max\left\{\frac{\log(P_1)}{\log(P_2)}, 1\right\}$ *and* $u = \max\left\{\frac{\log(P_2)}{\log(P_1)}, 1\right\}$. *Assume further that*

$$n_1 - s_{\mathbb{R}}^{(1)} > (8b+4u)R \quad \textit{and} \quad \frac{n_1+n_2}{2} - s_{\mathbb{R}}^{(2)} > (8b+4u)R. \tag{1.9}$$

Then there exists some $\delta > 0$ *depending at most on* b, u, R, n_i *and* $\boldsymbol{F}$ *such that*

$$N(P_1, P_2) = \sigma P_1^{n_1-2R} P_2^{n_2-R} + O(P_1^{n_1-2R} P_2^{n_2-R} \min\{P_1, P_2\}^{-\delta}), \tag{1.10}$$

where $\sigma > 0$ *if the system* $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ *has a smooth p-adic zero for all primes p, and if the variety* V_0 *has a smooth real zero in* $\mathcal{B}_1 \times \mathcal{B}_2$.

If we assume that $\mathbb{V}(F_1, \dots, F_R) \subset \mathbb{P}_{\mathbb{Q}}^{n_1-1} \times \mathbb{P}_{\mathbb{Q}}^{n_2-1}$ *is smooth, then the same conclusions hold if we assume*

$$n_1 > (16b+8u+1)R \quad \textit{and} \quad n_2 > (8b+4u+1)R \tag{1.11}$$

instead of (1.9).

Both of the results above are a consequence of Theorem 2.1. This theorem states that if one is able to estimate the number of solutions to an associated multilinear counting problem then one obtains the asymptotic formula (1.2). However, whenever $d_1 + d_2 > 3$ the associated multilinear problem becomes very difficult and one encounters problems as in [21].

We remark that we preferred to give conditions in terms of the geometry of the variety regarded as a biprojective variety, as opposed to an affine variety. The reason for this is the potential application of this result to proving Manin's conjecture for this variety, which will be addressed in due course.

Compared to the result by Schindler we thus basically remove the assumption that the number of variables needs to grow at least quadratically in R. In particular, our results require fewer variables than Schindler's if

$$bd_1 + ud_2 < \frac{R+1}{2} \tag{1.12}$$

is satisfied, in the cases $(d_1, d_2) = (1,1)$ or $(2,1)$. To see this, note firstly that (1.12) implies that Schindler's condition on the number of variables becomes

$$n_1 + n_2 - \dim V_i^* > 2^{d_1+d_2-2} R(R+1)(d_1+d_2-1).$$

Further, similarly to [22, Lemma 1.1] one may show

$$\dim V_1^* \geq \begin{cases} n_2 + \sigma_1^{(\mathbb{R})} & \text{if } (d_1, d_2) = (1,1), \\ n_2 + s_1^{(\mathbb{R})} & \text{if } (d_1, d_2) = (2,1), \end{cases}$$

and

$$\dim V_2^* \geq \begin{cases} n_1 + \sigma_2^{(\mathbb{R})} & \text{if } (d_1, d_2) = (1,1), \\ 2s_2^{(\mathbb{R})} & \text{if } (d_1, d_2) = (2,1). \end{cases}$$

Comparing Schindler's condition with the conditions on the number of variables appearing in Theorems 1.1 and 1.2 the claim now transpires.

In particular, if R is large this means our result provides significantly more flexibility in the choice of u and b.

One cannot hope to achieve the asymptotic formula (1.2) in general where a condition of the shape $n_i > R(bd_1 + ud_2)$ is not present. To see this, note that the counting function satisfies

$$N(P_1, P_2) \gg P_1^{n_1} + P_2^{n_2},$$

coming from the solutions when $x_1 = \cdots = x_{n_1} = 0$ and $y_1 = \cdots = y_{n_2} = 0$. The asymptotic formula (1.2) thus implies

$$P_i^{n_i} \ll P_1^{n_1 - d_1 R} P_2^{n_2 - d_2 R}$$

for $i = 1, 2$. Noting that $P_1^u = P_2$ if $u > 1$ and $P_2^b = P_1$ if $b > 1$ and comparing the exponents one necessarily finds that $n_i > R(bd_1 + ud_2)$.

If the forms are diagonal then one can take boxes $\mathcal{B}_i$ which avoid the coordinate axes in order to remedy this obstruction. In fact, this is the approach taken by Blomer and Brüdern [4] and they proved an asymptotic formula of a system of multihomogeneous equations without a restriction on the number of variables similar to the type described above.

If the forms are not diagonal the problem still persists, even if one were to take boxes avoiding the coordinate axes. In general there may be 'bad' vectors $\boldsymbol{y}$ away from the coordinate axes such that

$$\#\{\boldsymbol{x} \in \mathbb{Z}^{n_1} : \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0},\ |\boldsymbol{x}| \le P_1\} \gg P_1^{n_1 - a},$$

where $a < d_1 R$ for example. This is in contrast to the diagonal case, where the only vectors $\boldsymbol{y}$ where this occurs lie on at least one coordinate axis. It would be interesting to consider a modified counting function where one excludes such vectors $\boldsymbol{y}$, and analogously 'bad' vectors $\boldsymbol{x}$. In a general setting it seems difficult to control the set of such vectors. In particular, it is not clear how one would deal with the Weyl differencing step if one were to consider such a counting function.

1.1. Manin's conjecture. Let $V \subset \mathbb{P}_{\mathbb{Q}}^{n_1 - 1} \times \mathbb{P}_{\mathbb{Q}}^{n_2 - 1}$ be a non-singular complete intersection defined by a system of forms $F_i(\boldsymbol{x}, \boldsymbol{y})$, $i = 1, \dots, R$, of common bidegree (d_1, d_2). Assume $n_i > d_i R$ so that V is a Fano variety, which means that the inverse of the canonical bundle in the Picard group, the *anticanonical bundle*, is very ample. For a field K, write $V(K)$ for the set of K-rational points of V. In the context of Manin's conjecture we define this to be the set of K-morphisms

$$\mathrm{Spec}(K) \to V_K,$$

where V_K denotes the base change of V to the field K. For a subset $U(\mathbb{Q}) \subset V(\mathbb{Q})$ and $P \geq 1$ consider the counting function

$$N_U(P) = \#\{(\boldsymbol{x}, \boldsymbol{y}) \in U(\mathbb{Q})\colon H(\boldsymbol{x}, \boldsymbol{y}) \leq P\},$$

where $H(\cdot,\cdot)$ is the *anticanonical height* induced by the anticanonical bundle and a choice of global sections. In our case one such height may be explicitly given as follows. If $(\boldsymbol{x}, \boldsymbol{y}) \in U(\mathbb{Q})$ we may pick representatives $\boldsymbol{x} \in \mathbb{Z}^{n_1}$ and $\boldsymbol{y} \in \mathbb{Z}^{n_2}$ such that $(x_1, \dots, x_{n_1}) = (y_1, \dots, y_{n_2}) = 1$ and we define

$$H(\boldsymbol{x}, \boldsymbol{y}) = \Big(\max_i |x_i|\Big)^{n_1 - Rd_1} \Big(\max_i |y_i|\Big)^{n_2 - Rd_2}.$$

Manin's Conjecture in this context states that, provided V is a Fano variety such that $V(\mathbb{Q}) \subset V$ is Zariski dense, there exists a subset $U(\mathbb{Q}) \subset V(\mathbb{Q})$ where $(V \setminus U)(\mathbb{Q})$ is a *thin* set such that

$$N_U(P) \sim cP(\log P)^{\rho - 1},$$

where ρ is the Picard rank of the variety V and c is a constant as predicted and interpreted by Peyre [18]. We briefly recall the definition of a thin set, according to Serre [28]. First recall a set $A \subset V(K)$ is of type

(C_1) if $A \subseteq W(K)$, where $W \subsetneq V$ is Zariski closed,
(C_2) if $A \subseteq \pi(V'(K))$, where V' is irreducible such that $\dim V = \dim V'$, where $\pi\colon V' \to V$ is a generically finite morphism of degree at least 2.

Now a subset of the K-rational points of V is *thin* if it is a finite union of sets of type (C_1) or (C_2). Originally Batyrev–Manin [1] conjectured that it suffices to assume that $V \setminus U$ is Zariski closed, but there have been found various counterexamples to this, the first one being due to Batyrev–Tschinkel [2].

In [26] Schindler showed an asymptotic formula of the shape above, if V is smooth and $d_1, d_2 \geq 2$ and

$$n_i > 3 \cdot 2^{d_1 + d_2} d_1 d_2 R^3 + R$$

is satisfied for $i = 1, 2$. If $R = 1$ she moreover verified that the constant obtained agrees with the one predicted by Peyre, and thus proved Manin's conjecture for bihomogeneous hypersurfaces when the conditions above are met. The proof uses the asymptotic (1.2) established in [24] along with uniform counting results on fibres. That is, for a vector $\boldsymbol{y} \in \mathbb{Z}^{n_2}$ one may consider the counting function

$$N_{\boldsymbol{y}}(P) = \#\{\boldsymbol{x} \in \mathbb{Z}^{n_1}\colon \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0},\ |\boldsymbol{x}| \leq P\},$$

and to understand its asymptotic behaviour uniformly means to understand the dependence of $\boldsymbol{y}$ on the constant in the error term. Similarly she considered $N_{\boldsymbol{x}}(P)$ for 'good' $\boldsymbol{x}$ and combined the three resulting estimates to obtain an asymptotic formula for the number of solutions $\widetilde{N}(P_1, P_2)$ to the system $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$, where $|\boldsymbol{x}| \leq P_1$, $|\boldsymbol{y}| \leq P_2$ and $\boldsymbol{x}, \boldsymbol{y}$ are 'good'. Considering only 'good' tuples essentially removes a closed subset from V, and thus, after an application of a slight modification of the hyperbola method developed as in [4] she obtained an asymptotic formula for $N_U(P)$ of the desired shape.

In forthcoming work the result established in Theorem 1.2 will be used in verifying Manin's Conjecture for V, when $(d_1, d_2) = (2, 1)$ in fewer variables than would be expected using Schindler's method as described above. Further, since the Picard rank of V is strictly greater than 1, it would be interesting to consider the *all heights approach* as suggested by Peyre [19, Question V.4.8]. As noted by Peyre himself, in the case when a variety has Picard rank 1, the answer to his Question V.4.8 follows provided one can prove Manin's conjecture with respect to the height function induced by the anticanonical bundle.

Schindler's results have been improved upon in a few special cases. Browning and Hu [6] showed Manin's conjecture in the case of smooth biquadratic hypersurfaces in $\mathbb{P}_{\mathbb{Q}}^{n-1} \times \mathbb{P}_{\mathbb{Q}}^{n-1}$ if the number of variables satisfies $n > 35$. If the bidegree is $(2, 1)$ then Hu [13] showed that $n > 25$ suffices in order to obtain Manin's conjecture. Systems of bilinear varieties are flag varieties and thus Manin's conjecture follows from the result for flag varieties, which was proven by Franke, Manin and Tschinkel [10] using the theory of Eisenstein series. The same result was later proven by Thunder [30] using arguments from the geometry of numbers. In the special case when the variety is defined by $\sum_{i=0}^{s} x_i y_i = 0$, Robbiani [20] showed how one may use the circle method to establish Manin's conjecture if $s \geq 3$, which was later improved to $s \geq 2$ by Spencer [29].

Building up on Schindler's papers, Teddy Mignot proved Manin's conjecture for certain triprojective hypersurfaces [14], and he also established Manin's conjecture for certain hypersurfaces inside toric varieties [15]. Inspired by Schindler's techniques, Brandes [5] proved an asymptotic formula for the number of lines of bounded height lying on a hypersurface of degree at least 5 of sufficiently large dimension.

Conventions. The symbol $\varepsilon > 0$ is an arbitrarily small value, which we may redefine whenever convenient, as is usual in analytic number theory. Given forms g_ℓ, $\ell = 1, \ldots, k$, we write $\mathbb{V}(g_\ell)_{\ell=1,\ldots,k}$ or sometimes just $\mathbb{V}(g_\ell)_\ell$ for the intersection $\mathbb{V}(g_1, \ldots, g_k)$. Further, we may sometimes consider a vector of forms $\boldsymbol{h} = (h_1, \ldots, h_k)$ and we similarly write $\mathbb{V}(\boldsymbol{h})$ for the intersection $\mathbb{V}(h_1, \ldots, h_k)$.

For $x \in \mathbb{R}$ we will write $e(x) = e^{2\pi i x}$. We will use Vinogradov's notation $O(\cdot)$ and $\ll$.

We shall repeatedly use the convention that the dimension of the empty set is -1.

2. Multilinear forms. Both Theorems 1.1 and 1.2 follow from a more general result. If we have control over the number of 'small' solutions to the associated linearised forms then we can show that the asymptotic (1.2) holds. More explicitly, given a bihomogeneous form $F(\boldsymbol{x},\boldsymbol{y})$ with integer coefficients of bidegree (d_1,d_2) for positive integers d_1,d_2, we may write it as

$$F(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{j}}\sum_{\boldsymbol{k}} F_{\boldsymbol{j},\boldsymbol{k}} x_{j_1}\cdots x_{j_{d_1}} y_{k_1}\cdots y_{k_{d_2}},$$

where the coefficients $F_{\boldsymbol{j},\boldsymbol{k}} \in \mathbb{Q}$ are symmetric in $\boldsymbol{j}$ and $\boldsymbol{k}$. We define the associated multilinear form

$$\Gamma_F(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}) := d_1!d_2!\sum_{\boldsymbol{j}}\sum_{\boldsymbol{k}} F_{\boldsymbol{j},\boldsymbol{k}} x^{(1)}_{j_1}\cdots x^{(d_1)}_{j_{d_1}} y^{(1)}_{k_1}\cdots y^{(d_2)}_{k_{d_2}},$$

where $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(d_1)})$ and $\widetilde{\boldsymbol{y}} = (\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(d_2)})$ for vectors $\boldsymbol{x}^{(i)}$ of n_1 variables and vectors $\boldsymbol{y}^{(i)}$ of n_2 variables. Write further $\widehat{\boldsymbol{x}} = (\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(d_1-1)})$ and $\widehat{\boldsymbol{y}} = (\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(d_2-1)})$. Given $\boldsymbol{\beta} \in \mathbb{R}^R$ we define the auxiliary counting function $N_1^{\mathrm{aux}}(\boldsymbol{\beta};B)$ to be the number of integer vectors satisfying $\widehat{\boldsymbol{x}} \in (-B,B)^{(d_1-1)n_1}$ and $\widetilde{\boldsymbol{y}} \in (-B,B)^{d_2n_2}$ such that

$$|\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\widehat{\boldsymbol{x}},\boldsymbol{e}_\ell,\widetilde{\boldsymbol{y}})| < \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty B^{d_1+d_2-2},$$

for $\ell = 1,\ldots,n_1$, where $\|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty := \frac{1}{d_1!d_2!}\max_{\boldsymbol{j},\boldsymbol{k}}\Big|\frac{\partial^{d_1+d_2}(\boldsymbol{\beta}\cdot\boldsymbol{F})}{\partial x_{j_1}\cdots\partial x_{j_{d_1}}\partial y_{k_1}\cdots\partial y_{k_{d_2}}}\Big|$. We define $N_2^{\mathrm{aux}}(\boldsymbol{\beta};B)$ analogously.

The technical core of this paper is the following theorem.

Theorem 2.1. *Assume that $n_1,n_2 > (d_1+d_2)R$ and let $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = (F_1(\boldsymbol{x},\boldsymbol{y}),\ldots,F_R(\boldsymbol{x},\boldsymbol{y}))$ be a system of bihomogeneous forms with integer coefficients of common bidegree (d_1,d_2) such that the variety $\mathbb{V}(\boldsymbol{F}) \subset \mathbb{P}^{n_1-1}_{\mathbb{Q}} \times \mathbb{P}^{n_2-1}_{\mathbb{Q}}$ is a complete intersection. Let $P_1,P_2 > 1$ and write $b = \max\{\log(P_1)/\log(P_2),1\}$ and $u = \max\{\log(P_2)/\log(P_1),1\}$.*

Assume there exist $C_0 \geq 1$ and $\mathscr{C} > (bd_1+ud_2)R$ such that for all $\boldsymbol{\beta} \in \mathbb{R}^R\setminus\{\boldsymbol{0}\}$ and all $B>0$ we have

$$N_i^{\mathrm{aux}}(\boldsymbol{\beta};B) \leq C_0 B^{d_1n_1+d_2n_2-n_i-2^{d_1+d_2-1}\mathscr{C}} \tag{2.1}$$

for $i=1,2$. There exists some $\delta>0$ depending on b, u, C_0, R, d_i and n_i such that

$$N(P_1,P_2) = \sigma P_1^{n_1-d_1R}P_2^{n_2-d_2R} + O(P_1^{n_1-d_1R}P_2^{n_2-d_2R}\min\{P_1,P_2\}^{-\delta}).$$

The factor $\sigma = \mathfrak{I}\mathfrak{S}$ is the product of the singular integral $\mathfrak{I}$ and the singular series $\mathfrak{S}$, as defined in (5.11) *and* (5.8), *respectively. Moreover, if the system $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{0}$ has a non-singular real zero in $\mathcal{B}_1\times\mathcal{B}_2$ and a non-singular p-adic zero for every prime p, then $\sigma>0$.*

While showing that (2.1) holds is rather straightforward when the bidegree is $(1,1)$ it becomes significantly more difficult when the bidegree increases. In fact, in Rydin Myerson's work a similar upper bound on a similar auxiliary counting function needs to be shown. He was successful in doing so when the degree is 2 or 3 and the system defines a complete intersection, but for higher degrees he was only able to show this upper bound for generic systems. Our strategy is as follows. We will establish Theorem 2.1 in Sections 4 and 5 and then use this to show Theorem 1.1 and in Section 6 and Theorem 1.2 in Section 7.

3. Geometric preliminaries

LEMMA 3.1 ([26, Lemma 2.2]). *Let W be a smooth variety that is complete over some algebraically closed field and consider a closed irreducible subvariety $Z \subseteq W$ such that $\dim Z \geq 1$. Given an effective divisor D on W the dimension of every irreducible component of $D \cap Z$ is at least $\dim Z - 1$. If D is moreover ample, then $D \cap Z$ is non-empty.*

In particular the following corollary will be very useful.

COROLLARY 3.2. *Let $V \subseteq \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ be a closed variety such that $\dim V \geq 1$. Consider $H = \mathbb{V}(f)$ where $f(\boldsymbol{x}, \boldsymbol{y})$ is a polynomial of bidegree at least $(1,1)$ in the variables $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$. Then*

$$\dim(V \cap H) \geq \dim V - 1;$$

in particular, $V \cap H$ is non-empty.

Proof. Since the bidegree of f is at least $(1,1)$ we see that H defines an effective and ample divisor on $\mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$. We apply Lemma 3.1 with $W = \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$, $D = H$ and Z any irreducible component of V. ■

LEMMA 3.3. *Let $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})$ be a system of R bihomogeneous equations of the same bidegree (d_1, d_2) with $d_1, d_2 \geq 1$. Assume that $\mathbb{V}(\boldsymbol{F}) \subset \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ is a smooth complete intersection. Given $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}$ we have*

$$\dim \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \leq R - 2,$$

where we write $\boldsymbol{\beta} \cdot \boldsymbol{F} = \sum_i \beta_i F_i$.

Proof. The singular locus of $\mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F})$ is given by

$$\operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) = \mathbb{V}\left(\frac{\partial(\boldsymbol{\beta} \cdot \boldsymbol{F})}{\partial x_j}\right)_{j=1,\dots,n_1} \cap \mathbb{V}\left(\frac{\partial(\boldsymbol{\beta} \cdot \boldsymbol{F})}{\partial y_j}\right)_{j=1,\dots,n_2}.$$

Assume without loss of generality $\beta_R \neq 0$ so that $\mathbb{V}(\boldsymbol{F}) = \mathbb{V}(F_1, \dots, F_{R-1}, \boldsymbol{\beta} \cdot \boldsymbol{F})$. We claim that

$$\mathbb{V}(F_1, \dots, F_{R-1}) \cap \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \subseteq \operatorname{Sing} \mathbb{V}(\boldsymbol{F}). \tag{3.1}$$

To see this, note first that $\mathbb{V}(F_1, \dots, F_{R-1}) \cap \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \subseteq \mathbb{V}(\boldsymbol{F})$. Further, the Jacobian matrix $J(\boldsymbol{F})$ of $\boldsymbol{F}$ is given by

$$J(\boldsymbol{F}) = \left(\frac{\partial F_i}{\partial z_j} \right)_{ij},$$

where $i = 1, \ldots, R$ and z_j ranges through $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$. Now if the equations

$$\frac{\partial(\boldsymbol{\beta} \cdot \boldsymbol{F})}{\partial x_j} = \frac{\partial(\boldsymbol{\beta} \cdot \boldsymbol{F})}{\partial y_j} = 0,$$

are satisfied then this implies that the rows of $J(\boldsymbol{F})$ are linearly dependent. Since $\mathbb{V}(\boldsymbol{F})$ is a complete intersection we deduce the claim.

Assume now for a contradiction that $\dim \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \geq R-1$. Applying Corollary 3.2 $R-1$ times with $V = \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F})$, noting that the bidegree of F_i is at least $(1,1)$, we find that

$$\mathbb{V}(F_1, \ldots, F_{R-1}) \cap \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \neq \emptyset.$$

This contradicts (3.1) since $\operatorname{Sing} \mathbb{V}(\boldsymbol{F}) = \emptyset$ by assumption. ∎

Lemma 3.4. *Let $n_1 \leq n_2$ be two positive integers. For $i = 1, \ldots, n_2$ let $A_i \in \mathrm{M}_{n_1 \times n_1}(\mathbb{C})$ be symmetric matrices. Consider the varieties $V_1 \subset \mathbb{P}_{\mathbb{C}}^{n_1-1}$ and $V_2 \subset \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ defined by*

$$V_1 = \mathbb{V}(\boldsymbol{t}^T A_i \boldsymbol{t})_{i=1,\ldots,n_2}, \qquad V_2 = \mathbb{V}\Big(\sum_{i=1}^{n_2} y_i A_i \boldsymbol{x} \Big).$$

Then

$$\dim V_2 \leq \dim V_1 + n_2 - 1.$$

In particular, if $V_1 = \emptyset$ then $\dim V_2 \leq n_2 - 2$.

Proof. Consider the variety $V_3 \subset \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_1-1}$ defined by

$$V_3 = \mathbb{V}(\boldsymbol{z}^T A_i \boldsymbol{x})_{i=1,\ldots,n_2}.$$

Further, for $\boldsymbol{x} = (x_1, \ldots, x_{n_1})^T$ consider

$$A_{\boldsymbol{x}} = (A_1 \boldsymbol{x} \cdots A_{n_2} \boldsymbol{x}) \in \mathrm{M}_{n_1 \times n_2}(\mathbb{C})[x_1, \ldots, x_{n_1}].$$

We may write $V_2 = \mathbb{V}(A_{\boldsymbol{x}} \boldsymbol{y})$ and $V_3 = \mathbb{V}(\boldsymbol{z}^T A_{\boldsymbol{x}})$. Our first goal is to relate the dimensions of the varieties above as follows:

$$\dim V_2 \leq \dim V_3 + n_2 - n_1. \tag{3.2}$$

For $r = 0, \ldots, n_1$ define the quasi-projective varieties $D_r \subset \mathbb{P}_{\mathbb{C}}^{n_1-1}$ given by

$$D_r = \{ \boldsymbol{x} \in \mathbb{P}_{\mathbb{C}}^{n_1-1} \colon \operatorname{rank}(A(\boldsymbol{x})) = r \}.$$

These are quasiprojective since they may be written as the intersection of the vanishing sets of all $(r+1) \times (r+1)$ minors of $A_{\boldsymbol{x}}$ with the complement of the intersection of the vanishing sets of all $r \times r$ minors. For each r let

$$D_r = \bigcup_{i \in I_r} D_r^{(i)}$$

be a decomposition into finitely many irreducible components. Since $\bigcup_r D_r = \mathbb{P}_{\mathbb{C}}^{n_1-1}$ we have

$$\dim V_2 = \max_{\substack{0 \le r < n_2 \\ i \in I_r}} \dim((D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2).$$

Note that $r = n_2$ does not play a role here, since the intersection $(D_{n_2}^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2$ is empty. Similarly we get

$$\dim V_3 = \max_{\substack{0 \le r < n_2 \\ i \in I_r}} \dim((D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_1-1}) \cap V_3).$$

For $0 \le r < n_2$ and $i \in I_r$ consider now the surjective projection maps

$$\pi_{2,r,i} \colon (D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2 \to D_r^{(i)}, \qquad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x},$$

and

$$\pi_{3,r,i} \colon (D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_1-1}) \cap V_3 \to D_r^{(i)}, \qquad (\boldsymbol{x}, \boldsymbol{z}) \mapsto \boldsymbol{x}.$$

We note that by the way $D_r^{(i)}$ was constructed here, the fibres of both of these projection morphisms have constant dimension for fixed r. By the rank-nullity theorem we find that the dimensions of the fibres are related as follows:

$$\dim \pi_{2,r,i}^{-1}(\boldsymbol{x}) = \dim \pi_{3,r,i}^{-1}(\boldsymbol{x}) + n_2 - n_1. \tag{3.3}$$

We claim that the morphism $\pi_{2,r,i}$ is proper. For this note that the structure morphism $\mathbb{P}_{\mathbb{C}}^{n_1-1} \to \operatorname{Spec} \mathbb{C}$ is proper whence $D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_1-1} \to D_r^{(i)}$ must be proper too, as properness is preserved under base change. As $(D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2$ is closed inside $D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_1-1}$, the restriction $\pi_{2,r,i}$ must also be proper. By an analogous argument it follows $\pi_{3,r,i}$ is also proper.

Further note that the fibres of $\pi_{2,r,i}$ are irreducible since they define linear subspaces of $(D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2$, and similarly the fibres of $\pi_{3,r,i}$ are irreducible. Since $D_r^{(i)}$ is irreducible by construction and all the fibres have constant dimension, it follows that $(D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2$ is irreducible. Similarly $(D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_1-1}) \cap V_3$ is irreducible.

Hence all the conditions of Chevalley's upper semicontinuity theorem [11, Théorème 13.1.3] are satisfied, so that for any $\boldsymbol{x} \in D_r^{(i)}$ we obtain

$$\dim \pi_{2,r,i}^{-1}(\boldsymbol{x}) = \dim((D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2) - \dim D_r^{(i)}, \tag{3.4}$$

and

$$\dim \pi_{3,r,i}^{-1}(\boldsymbol{x}) = \dim((D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_1-1}) \cap V_3) - \dim D_r^{(i)}. \tag{3.5}$$

Hence (3.4) and (3.5) together with (3.3) yield

$$\dim((D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2) = \dim((D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_1-1}) \cap V_3) + n_2 - n_1.$$

Choosing r and i such that $\dim V_2 = \dim((D_r^{(i)} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap V_2)$ the claim (3.2) now follows.

Thus it is enough to find an upper bound for $\dim V_3$. To this end, consider the affine cones $\widetilde{V}_1 = \mathbb{V}(\boldsymbol{u}^T A_i \boldsymbol{u})_{i=1,\dots,n_2} \subset \mathbb{A}_{\mathbb{C}}^{n_1}$ and $\widetilde{V}_3 = \mathbb{V}(\boldsymbol{x}^T A(\boldsymbol{z})) \subset \mathbb{A}_{\mathbb{C}}^{n_1} \times \mathbb{A}_{\mathbb{C}}^{n_1}$. Note in particular that $\widetilde{V}_1 \neq \emptyset$ even if $V_1 = \emptyset$.

Write $\widetilde{\Delta} \subset \mathbb{A}_{\mathbb{C}}^{n_1} \times \mathbb{A}_{\mathbb{C}}^{n_1}$ for the diagonal given by $\mathbb{V}(x_i = z_i)_i$. Then $\widetilde{V}_3 \cap \widetilde{\Delta} \cong \widetilde{V}_1 \neq \emptyset$. Thus, the affine dimension theorem [12, Proposition 7.1] yields

$$\dim \widetilde{V}_1 \geq \dim \widetilde{V}_3 - n_1.$$

Noting $\dim V_1 + 1 \geq \dim \widetilde{V}_1$ and $\dim \widetilde{V}_3 \geq \dim V_3 + 2$ now gives the desired result. We remind the reader at this point that this is compatible with the convention $\dim \emptyset = -1$. ∎

4. The auxiliary inequality. We remind the reader of the notation $e(x) = e^{2\pi i x}$. Starting with this section, we will often use the notation $\tilde{d} = d_1 + d_2 - 2$ throughout the paper. For $\boldsymbol{\alpha} \in [0,1]^R$ define

$$S(\boldsymbol{\alpha}, P_1, P_2) = S(\boldsymbol{\alpha}) := \sum_{\boldsymbol{x} \in P_1 \mathcal{B}_1} \sum_{\boldsymbol{y} \in P_2 \mathcal{B}_2} e(\boldsymbol{\alpha} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})),$$

where the sum ranges over $\boldsymbol{x} \in \mathbb{Z}^{n_1}$ such that $\boldsymbol{x}/P_1 \in \mathcal{B}_1$ and similarly for $\boldsymbol{y}$. Throughout this section we will assume $P_1 \geq P_2$. Note crucially that

$$N(P_1, P_2) = \int_{[0,1]^R} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}.$$

As noted in the introduction we can rewrite the forms as

$$F_i(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{j}} \sum_{\boldsymbol{k}} F_{\boldsymbol{j},\boldsymbol{k}}^{(i)} x_{j_1} \cdots x_{j_{d_1}} y_{k_1} \cdots y_{k_{d_2}},$$

and given $\boldsymbol{\alpha} \in \mathbb{R}^R$, as in [24], we consider the multilinear forms

$$\Gamma_{\boldsymbol{\alpha} \cdot \boldsymbol{F}}(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) := d_1! d_2! \sum_i \alpha_i \sum_{\boldsymbol{j}} \sum_{\boldsymbol{k}} F_{\boldsymbol{j},\boldsymbol{k}}^{(i)} x_{j_1}^{(1)} \cdots x_{j_{d_1}}^{(d_1)} y_{k_1}^{(1)} \cdots y_{k_{d_2}}^{(d_2)}.$$

Further we write $\widehat{\boldsymbol{x}} = (\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d_1-1)})$ and similarly for $\widehat{\boldsymbol{y}}$. For any real number λ we write $\|\lambda\| = \min_{k \in \mathbb{Z}} |\lambda - k|$. Now define $M_1(\boldsymbol{\alpha} \cdot \boldsymbol{F}; P_1, P_2, P^{-1})$ to be the number of integral $\widehat{\boldsymbol{x}} \in (-P_1, P_1)^{(d_1-1)n_1}$ and $\widetilde{\boldsymbol{y}} \in (-P_2, P_2)^{d_2 n_2}$ such that for all $\ell = 1, \dots, n_1$ we have

$$\|\Gamma_{\boldsymbol{\alpha} \cdot \boldsymbol{F}}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widetilde{\boldsymbol{y}})\| < P^{-1}.$$

Similarly, we define $M_2(\boldsymbol{\alpha} \cdot \boldsymbol{F}; P_1, P_2, P^{-1})$ to be the number of integral $\widetilde{\boldsymbol{x}} \in (-P_1, P_1)^{d_1 n_1}$ and $\widehat{\boldsymbol{y}} \in (-P_2, P_2)^{(d_2-1)n_2}$ such that for all $\ell = 1, \dots, n_2$

we have

$$\|\Gamma_{\boldsymbol{\alpha}\cdot\boldsymbol{F}}(\widetilde{\boldsymbol{x}},\widehat{\boldsymbol{y}},\boldsymbol{e}_\ell,)\| < P^{-1}.$$

For our purposes we will need a slight generalisation of [24, Lemma 2.1] that deals with a polynomial $G(\boldsymbol{x},\boldsymbol{y})$, which is not necessarily bihomogeneous. If $G(\boldsymbol{x},\boldsymbol{y})$ has bidegree (d_1,d_2) write

$$G(\boldsymbol{x},\boldsymbol{y}) = \sum_{\substack{0\le r\le d_1\\ 0\le l\le d_2}} G^{(r,l)}(\boldsymbol{x},\boldsymbol{y}),$$

where $G^{(r,l)}(\boldsymbol{x},\boldsymbol{y})$ is homogeneous of bidegree (r,l). Using notation as above we first show the following preliminary lemma, which is a version of Weyl's inequality for our context.

We remind the reader of the notation $\tilde{d} = d_1 + d_2 - 2$.

Lemma 4.1. *Let $\varepsilon > 0$. Let $G(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}[x_1,\dots,x_{n_1},y_1,\dots,y_{n_2}]$ be a polynomial of bidegree (d_1,d_2) with $d_1,d_2 \ge 1$. For the exponential sum*

$$S_G(P_1,P_2) = \sum_{\boldsymbol{x}\in P_1\mathcal{B}_1}\sum_{\boldsymbol{x}\in P_2\mathcal{B}_2} e(G(\boldsymbol{x},\boldsymbol{y}))$$

we have the following bound:

$$|S_G(P_1,P_2)|^{2^{\tilde{d}}} \ll P_1^{n_1(2^{\tilde{d}}-d_1+1)+\varepsilon} P_2^{n_2(2^{\tilde{d}}-d_2)} M_1(G^{(d_1,d_2)},P_1,P_2,P_1^{-1}).$$

Proof. The proof is quite involved but follows closely the proof of [24, Lemma 2.1], which in turn is based on ideas of Schmidt [27, Section 11] and Davenport [7, Section 3].

Our first goal is to apply a Weyl differencing process d_2-1 times to the $\boldsymbol{y}$ part of G and then d_1-1 times to the $\boldsymbol{x}$ part of the resulting polynomial. Clearly this is trivial if $d_2=1$ or $d_1=1$, respectively. Therefore assume for now that $d_2 \ge 2$. We start by applying the Cauchy–Schwarz inequality and the triangle inequality to find that

$$|S_G(P_1,P_2)|^{2^{d_2-1}} \ll P_1^{n_1(2^{d_2-1}-1)} \sum_{\boldsymbol{x}\in P_1\mathcal{B}_1} |S_{\boldsymbol{x}}(P_1,P_2)|^{2^{d_2-1}}, \tag{4.1}$$

where we define

$$S_{\boldsymbol{x}}(P_1,P_2) = \sum_{\boldsymbol{y}\in P_2\mathcal{B}_2} e(G(\boldsymbol{x},\boldsymbol{y})).$$

Now write $\mathcal{U} = P_2\mathcal{B}_2$, write $\mathcal{U}^D = \mathcal{U}-\mathcal{U}$ for the difference set and define

$$\mathcal{U}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(t)}) = \bigcap_{\varepsilon_1=0,1}\cdots\bigcap_{\varepsilon_t=0,1} (\mathcal{U}-\varepsilon_1\boldsymbol{y}^{(1)}-\cdots-\varepsilon_t\boldsymbol{y}^{(t)}).$$

Write $\mathcal{F}(\boldsymbol{y}) = G(\boldsymbol{x},\boldsymbol{y})$ and set

$$\mathcal{F}_d(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d)}) = \sum_{\varepsilon_1=0,1}\cdots\sum_{\varepsilon_d=0,1}(-1)^{\varepsilon_1+\cdots+\varepsilon_d}\mathcal{F}(\varepsilon_1\boldsymbol{y}^{(1)}+\cdots+\varepsilon_d\boldsymbol{y}^{(d)}).$$

Inequality (11.2) in [27] applied to our situation gives

$$|S_{\boldsymbol{x}}(P_1,P_2)|^{2^{d_2-1}} \ll |\mathcal{U}^D|^{2^{d_2-1}-d_2} \times \sum_{\boldsymbol{y}^{(1)}\in\mathcal{U}^D} \cdots \sum_{\boldsymbol{y}^{(d_2-2)}\in\mathcal{U}^D} \Big| \sum_{\boldsymbol{y}^{(d_2-1)}\in\mathcal{U}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-2)})} e(\mathcal{F}_{d_2-1}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)}))\Big|^2,$$

and we note that this did not require $\mathcal{F}(\boldsymbol{y})$ to be homogeneous in Schmidt's work. It is not hard to see that for $\boldsymbol{z},\boldsymbol{z}'\in\mathcal{U}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-2)})$ we have

$$\begin{aligned}\mathcal{F}_{d_2-1}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{z}) - \mathcal{F}_{d_2-1}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{z}') \\ = \mathcal{F}_{d_2}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)},\boldsymbol{y}^{(d_2)}) - \mathcal{F}_{d_2-1}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)}),\end{aligned}$$

where $\boldsymbol{y}^{(d_2-1)} = \boldsymbol{z}-\boldsymbol{z}' \in \mathcal{U}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-2)})^D$ and $\boldsymbol{y}^{(d_2)} = \boldsymbol{z}' \in \mathcal{U}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)})$. Thus we find

$$\begin{aligned}(4.2)\quad & |S_{\boldsymbol{x}}(P_1,P_2)|^{2^{d_2-1}} \\ & \ll |\mathcal{U}^D|^{2^{d_2-1}-d_2} \sum_{\boldsymbol{y}^{(1)}\in\mathcal{U}^D} \cdots \sum_{\boldsymbol{y}^{(d_2-2)}\in\mathcal{U}^D} \sum_{\boldsymbol{y}^{(d_2-1)}\in\mathcal{U}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-2)})^D} \\ & \times \sum_{\boldsymbol{y}^{(d_2)}\in\mathcal{U}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)})} e(\mathcal{F}_{d_2}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2)}) - \mathcal{F}_{d_2-1}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)})).\end{aligned}$$

We may write the polynomial $G(\boldsymbol{x},\boldsymbol{y})$ as

$$G(\boldsymbol{x},\boldsymbol{y}) = \sum_{\substack{0\le r\le d_1\\0\le l\le d_2}} \sum_{\boldsymbol{j}_r,\boldsymbol{k}_l} G^{(r,l)}_{\boldsymbol{j}_r,\boldsymbol{k}_l} \boldsymbol{x}_{\boldsymbol{j}_r}\boldsymbol{y}_{\boldsymbol{k}_l}$$

for some real $G^{(r,l)}_{\boldsymbol{j}_r,\boldsymbol{k}_l}$. Further write $\mathcal{F}(\boldsymbol{y}) = \mathcal{F}^{(0)}(\boldsymbol{y}) + \cdots + \mathcal{F}^{(d_2)}(\boldsymbol{y})$, where $\mathcal{F}^{(d)}(\boldsymbol{y})$ denotes the degree d homogeneous part of $\mathcal{F}(\boldsymbol{y})$. Lemma 11.4(A) in [27] states that $\mathcal{F}_{d_2}$ transpires to be the multilinear form associated to $\mathcal{F}^{(d_2)}(\boldsymbol{y})$. From this we see that

$$(4.3)\quad \mathcal{F}_{d_2} - \mathcal{F}_{d_2-1} = \sum_{\substack{0\le r\le d_1\\0\le l\le d_2}} \sum_{\boldsymbol{j}_r,\boldsymbol{k}_l} G^{(r,l)}_{\boldsymbol{j}_r,\boldsymbol{k}_l} x_{j_r(1)}\cdots x_{j_r(r)} h_{\boldsymbol{k}_l}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2)}),$$

where

$$h_{\boldsymbol{k}_{d_2}}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2)}) = d_2!\,y^{(1)}_{k_{d_2}(1)}\cdots y^{(d_2)}_{k_{d_2}(d_2)} + \tilde{h}_{\boldsymbol{k}_{d_2}}(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)})$$

for some polynomials $\tilde{h}_{\boldsymbol{k}_{d_2}}$ of degree d_2 that are independent of $\boldsymbol{y}^{(d_2)}$ and further $h_{\boldsymbol{k}_l}$ are polynomials of degree l that are always independent of $\boldsymbol{y}^{(d_2)}$ whenever $l\le d_2-1$. Write $\widetilde{\boldsymbol{y}} = (\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2)})$. Now set

$$S_{\widetilde{\boldsymbol{y}}} = \sum_{\boldsymbol{x}\in P_1\mathcal{B}_1} e\Big(\sum_{\substack{0\le r\le d_1\\0\le l\le d_2}} \sum_{\boldsymbol{j}_r,\boldsymbol{k}_l} G^{(r,l)}_{\boldsymbol{j}_r,\boldsymbol{k}_l} x_{j_r(1)}\cdots x_{j_r(r)} h_{\boldsymbol{k}_l}(\widetilde{\boldsymbol{y}})\Big).$$

Now we swap the order of summation of $\sum_{\boldsymbol{x}}$ in (4.1) with the sums over $\boldsymbol{y}^{(i)}$ in (4.2). Using the Cauchy–Schwarz inequality and (4.3) we thus obtain

$$|S_G(P_1,P_2)|^{2^{\tilde{d}}} \ll P_1^{n_1(2^{\tilde{d}}-2^{d_1-1})} P_2^{n_2(2^{\tilde{d}}-d_2)} \sum_{\boldsymbol{y}^{(1)}} \cdots \sum_{\boldsymbol{y}^{(d_2)}} |S_{\widetilde{\boldsymbol{y}}}|^{2^{d_1-1}}.$$

The above still holds if $d_2 = 1$, which can be seen directly. Applying the same differencing process to $S_{\widetilde{\boldsymbol{y}}}$ gives
(4.4)

$$|S_G(P_1,P_2)|^{2^{\tilde{d}}} \ll P_1^{n_1(2^{\tilde{d}}-d_1)} P_2^{n_2(2^{\tilde{d}}-d_2)} \sum_{\boldsymbol{y}^{(1)}} \cdots \sum_{\boldsymbol{y}^{(d_2)}} \sum_{\boldsymbol{x}^{(1)}} \cdots \Big| \sum_{\boldsymbol{x}^{(d_1)}} e(\gamma(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}})) \Big|,$$

where

$$\gamma(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}) = \sum_{\substack{0\le r\le d_1\\ 0\le l\le d_2}} \sum_{\boldsymbol{j}_r,\boldsymbol{k}_l} G^{(r,l)}_{\boldsymbol{j}_r,\boldsymbol{k}_l} g_{\boldsymbol{j}_r}(\widetilde{\boldsymbol{x}}) h_{\boldsymbol{k}_l}(\widetilde{\boldsymbol{y}}),$$

and where similar to before we have

$$g_{\boldsymbol{j}_{d_1}}(\widetilde{\boldsymbol{x}}) = d_1! x^{(1)}_{j_{d_1}(1)} \cdots x^{(d_1)}_{j_{d_1}(d_1)} + \tilde{g}_{\boldsymbol{j}_{d_1}}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d_1-1)})$$

with $\tilde{g}_{\boldsymbol{j}_{d_1}}$ and $g_{\boldsymbol{j}_r}$ for $r < d_1$ not depending on $\boldsymbol{x}^{(d_1)}$. We note that (4.4) holds for all $d_1, d_2 \ge 1$ and all the summations $\sum_{\boldsymbol{x}^{(i)}}$ and $\sum_{\boldsymbol{y}^{(j)}}$ in (4.4) are over boxes contained in $[-P_1,P_1]^{n_1}$ and $[-P_2,P_2]^{n_2}$, respectively. Write $\widehat{\boldsymbol{x}} = (\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d_1-1)})$ and $\widehat{\boldsymbol{y}} = (\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(d_2-1)})$. We now wish to estimate the quantity

$$\sum(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}}) := \sum_{\boldsymbol{y}^{(d_2)}} \Big| \sum_{\boldsymbol{x}^{(d_1)}} e(\gamma(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}})) \Big|. \tag{4.5}$$

Viewing $\sum_{a<x\le b} e(\beta x)$ for $b-a \ge 1$ as a geometric series we recall the elementary estimate

$$\Big| \sum_{a<x\le b} e(\beta x) \Big| \ll \min\{b-a, \|\beta\|^{-1}\}.$$

This yields

$$\Big| \sum_{\boldsymbol{x}^{(d_1)}} e(\gamma(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}})) \Big| \ll \prod_{\ell=1}^{n_1} \min\{P_1, \|\widetilde{\gamma}(\widehat{\boldsymbol{x}},\boldsymbol{e}_\ell,\widetilde{\boldsymbol{y}})\|^{-1}\},$$

where $\boldsymbol{e}_\ell$ denotes the ℓth unit vector and where

$$\widetilde{\gamma}(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}) = d_1! \sum_{0\le l\le d_2} \sum_{\boldsymbol{j}_{d_1},\boldsymbol{k}_l} G^{(d_1,l)}_{\boldsymbol{j}_{d_1},\boldsymbol{k}_l} x^{(1)}_{j_{d_1}(1)} \cdots x^{(d_1)}_{j_{d_1}(d_1)} h_{\boldsymbol{k}_l}(\widetilde{\boldsymbol{y}}).$$

We now apply a standard argument in order to estimate this product, as in Davenport [8, Chapter 13]. For a real number z write $\{z\}$ for its fractional part. Let $\boldsymbol{r} = (r_1,\dots,r_{n_1}) \in \mathbb{Z}^{n_1}$ be such that $0 \le r_\ell < P_1$ for $\ell = 1,\dots,n_1$.

Define $\mathcal{A}(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \boldsymbol{r})$ to be the set of $\boldsymbol{y}^{(d_2)}$ in the sum in (4.5) such that

$$r_\ell P_1^{-1} \leq \{\widetilde{\gamma}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widehat{\boldsymbol{y}}, \boldsymbol{y}^{(d_2)})\} < (r_\ell + 1) P_1^{-1}$$

for all $\ell = 1, \ldots, n_1$ and write $A(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \boldsymbol{r})$ for its cardinality. We obtain the estimate

$$\sum(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}) \ll \sum_{\boldsymbol{r}} A(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \boldsymbol{r}) \prod_{\ell=1}^{n_1} \min\left\{P_1, \max\left\{\frac{P_1}{r_\ell}, \frac{P_1}{P_1 - r_\ell - 1}\right\}\right\},$$

where the sum $\sum_{\boldsymbol{r}}$ is over integral $\boldsymbol{r}$ with $0 \leq r_\ell < P_1$ for all $\ell = 1, \ldots, n_1$. Our next aim is to find a bound for $A(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \boldsymbol{r})$ that is independent of $\boldsymbol{r}$. Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \boldsymbol{r})$ then

$$\|\widetilde{\gamma}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widehat{\boldsymbol{y}}, \boldsymbol{u}) - \widetilde{\gamma}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widehat{\boldsymbol{y}}, \boldsymbol{v})\| < P_1^{-1}$$

for $\ell = 1, \ldots, n_1$. Similar to before we now define the multilinear forms

$$\varGamma_G(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) \coloneqq d_1! d_2! \sum_{\boldsymbol{j}_{d_1}, \boldsymbol{k}_{d_2}} G^{(d_1,d_2)}_{\boldsymbol{j}_{d_1}, \boldsymbol{k}_{d_2}} x^{(1)}_{j_{d_1}(1)} \cdots x^{(d_1)}_{j_{d_1}(d_1)} y^{(1)}_{k_{d_2}(1)} \cdots y^{(d_2)}_{k_{d_2}(d_2)},$$

which only depend on the (d_1, d_2)-degree part of G. For fixed $\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}$ let $N(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ be the number of $\boldsymbol{y} \in (-P_2, P_2)^{n_2}$ such that

$$\|\varGamma_G(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widehat{\boldsymbol{y}}, \boldsymbol{y})\| < P_1^{-1}$$

for all $\ell = 1, \ldots, n_1$. Observe now crucially

$$\widetilde{\gamma}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widehat{\boldsymbol{y}}, \boldsymbol{u}) - \widetilde{\gamma}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widehat{\boldsymbol{y}}, \boldsymbol{v}) = \varGamma_G(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widehat{\boldsymbol{y}}, \boldsymbol{u} - \boldsymbol{v}).$$

Thus we find $A(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \boldsymbol{r}) \leq N(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ for all $\boldsymbol{r}$ as specified above. Using this we get

$$\sum_{\boldsymbol{y}^{(d_2)}} \Big| \sum_{\boldsymbol{x}^{(d_1)}} e(\gamma(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}})) \Big| \ll N(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}) (P_1 \log P_1)^{n_1}.$$

Finally, summing over $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$ we obtain

$$|S_G(P_1, P_2)|^{2^{\tilde{d}}} \ll P_1^{n_1(2^{\tilde{d}} - d_1 + 1) + \varepsilon} P_2^{n_2(2^{\tilde{d}} - d_2)} M_1(G^{(d_1,d_2)}, P_1, P_2, P_1^{-1}). \ \blacksquare$$

Inspecting the proof of [24, Lemma 4.1] we find that for a polynomial $G(\boldsymbol{x}, \boldsymbol{y})$ as above given $\theta \in (0, 1]$ the following holds:

$$M_1(G^{(d_1,d_2)}, P_1, P_2, P_1^{-1}) \ll P_1^{n_1(d_1-1)} P_2^{n_2 d_2} P_2^{-\theta(n_1 d_1 + n_2 d_2)} \times \max_{i=1,2} \{P_2^{n_i \theta} M_i(G^{(d_1,d_2)}; P_2^\theta, P_2^\theta, P_1^{-d_1} P_2^{-d_2} P_2^{\theta(\tilde{d}+1)})\}.$$

Using this and Lemma 4.1 we deduce the next lemma.

Lemma 4.2. *Let $P_1, P_2 > 1$, $\theta \in (0, 1]$ and $\boldsymbol{\alpha} \in \mathbb{R}^R$. Write $S_G = S_G(P_1, P_2)$. Using the same notation as above for one of $i = 1$ or $i = 2$*

we have

$$|S_G|^{2^{\tilde{d}}} \ll_{d_i,n_i,\varepsilon} P_1^{n_1 2^{\tilde{d}}+\varepsilon} P_2^{n_2 2^{\tilde{d}}} P_2^{\theta n_i - \theta(n_1 d_1 + n_2 d_2)} \times M_i(G^{(d_1,d_2)}; P_2^{\theta}, P_2^{\theta}, P_1^{-d_1} P_2^{-d_2} P_2^{\theta(\tilde{d}+1)}).$$

Using the preceding lemma and adapting the proof of [22, Lemma 3.1] to our setting we can now show the following.

LEMMA 4.3. *Let* $\varepsilon > 0$, $\theta \in (0,1]$ *and* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$. *Then for* $i = 1$ *or* $i = 2$ *we have*

$$\min\left\{ \left| \frac{S(\boldsymbol{\alpha})}{P_1^{n_1+\varepsilon} P_2^{n_2}} \right|, \left| \frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1+\varepsilon} P_2^{n_2}} \right| \right\}^{2^{\tilde{d}+1}} \ll_{d_i,n_i,\varepsilon} \frac{M_i(\boldsymbol{\beta}\cdot\boldsymbol{F}; P_2^{\theta}, P_2^{\theta}, P_1^{-d_1} P_2^{-d_2} P_2^{\theta(\tilde{d}+1)})}{P_2^{\theta(n_1 d_1 + n_2 d_2) - \theta n_i}}. \tag{4.6}$$

Proof. Note first that for two real numbers $\lambda, \mu > 0$ we have

$$\min\{\lambda, \mu\} \le \sqrt{\lambda\mu}.$$

Therefore it suffices to show

$$\left| \frac{S(\boldsymbol{\alpha}) S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{2n_1+2\varepsilon} P_2^{2n_2}} \right|^{2^{\tilde{d}}} \ll_{d_i,n_i,\varepsilon} \frac{M_i(\boldsymbol{\beta}; P_2^{\theta}, P_2^{\theta}, P_1^{-d_1} P_2^{-d_2} P_2^{\theta(\tilde{d}+1)})}{P_2^{\theta(n_1 d_1 + n_2 d_2) - \theta n_i}}$$

for one of $i = 1$ or $i = 2$. Note first that

$$|S(\boldsymbol{\alpha}+\boldsymbol{\beta})\bar{S}(\boldsymbol{\alpha})| = \Bigg| \sum_{\substack{\boldsymbol{x}\in P_1\mathcal{B}_1 \\ \boldsymbol{y}\in P_2\mathcal{B}_2}} \sum_{\substack{\boldsymbol{x}+\boldsymbol{z}\in P_1\mathcal{B}_1 \\ \boldsymbol{y}+\boldsymbol{w}\in P_2\mathcal{B}_2}} e((\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{\alpha}\cdot\boldsymbol{F}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y}+\boldsymbol{w})) \Bigg|,$$

so by the triangle inequality we get

$$|S(\boldsymbol{\alpha}+\boldsymbol{\beta})\bar{S}(\boldsymbol{\alpha})| \le \sum_{\substack{\|\boldsymbol{z}\|_\infty \le P_1 \\ \|\boldsymbol{w}\|_\infty \le P_2}} \Bigg| \sum_{\substack{\boldsymbol{x}\in P_1\mathcal{B}_{\boldsymbol{z}} \\ \boldsymbol{y}\in P_2\mathcal{B}_{\boldsymbol{w}}}} e(\boldsymbol{\beta}\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) - g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})) \Bigg|,$$

where $g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})$ is of degree at most $d_1 + d_2 - 1$ in $(\boldsymbol{x},\boldsymbol{y})$ and we have some boxes $\mathcal{B}_{\boldsymbol{z}} \subset \mathcal{B}_1$ and $\mathcal{B}_{\boldsymbol{w}} \subset \mathcal{B}_2$. Applying Cauchy's inequality $\tilde{d}$ times we deduce

$$|S(\boldsymbol{\alpha}+\boldsymbol{\beta})\bar{S}(\boldsymbol{\alpha})|^{2^{\tilde{d}}} \le P_1^{n_1(2^{\tilde{d}}-1)} P_2^{n_2(2^{\tilde{d}}-1)} \sum_{\substack{\|\boldsymbol{z}\|_\infty \le P_1 \\ \|\boldsymbol{w}\|_\infty \le P_2}} \Bigg| \sum_{\substack{\boldsymbol{x}\in P_1\mathcal{B}_{\boldsymbol{z}} \\ \boldsymbol{y}\in P_2\mathcal{B}_{\boldsymbol{w}}}} e(\boldsymbol{\beta}\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) - g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})) \Bigg|^{2^{\tilde{d}}}.$$

If we write $G(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\beta}\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) - g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})$ then note that $G^{(d_1,d_2)} = \boldsymbol{\beta}\cdot\boldsymbol{F}$.

Using Lemma 4.2 we therefore obtain

$$|S(\boldsymbol{\alpha}+\boldsymbol{\beta})\bar{S}(\boldsymbol{\alpha})|^{2^{\tilde{d}}} \ll P_1^{2^{\tilde{d}+1}n_1+\varepsilon} P_2^{2^{\tilde{d}+1}n_2} P_2^{-\theta(n_1d_1+n_2d_2)+\theta n_i} \times M_i(\boldsymbol{\beta}\cdot\boldsymbol{F}, P_2^{\theta}, P_2^{\theta}, P_1^{-d_1}P_2^{-d_2}P_2^{\theta(\tilde{d}+1)}),$$

for one of $i=1$ or $i=2$, which readily delivers the result. ∎

As in the introduction, for $\boldsymbol{\beta}\in\mathbb{R}^R$ we define the auxiliary counting function $N_1^{\mathrm{aux}}(\boldsymbol{\beta};B)$ to be the number of integer vectors $\widehat{\boldsymbol{x}}\in(-B,B)^{(d_1-1)n_1}$ and $\widetilde{\boldsymbol{y}}\in(-B,B)^{d_2n_2}$ such that

$$|\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\widehat{\boldsymbol{x}},\boldsymbol{e}_\ell,\widetilde{\boldsymbol{y}})| < \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty B^{\tilde{d}}$$

for $\ell=1,\dots,n_1$ where $\|f\|_\infty := \frac{1}{d_1!d_2!}\max_{\boldsymbol{j},\boldsymbol{k}}\bigl|\frac{\partial^{d_1+d_2}f}{\partial x_{j_1}\cdots\partial x_{j_{d_1}}\partial y_{k_1}\cdots\partial y_{k_{d_2}}}\bigr|$. We also analogously define $N_2^{\mathrm{aux}}(\boldsymbol{\beta};B)$. We now formulate an analogue for [22, Proposition 3.1].

PROPOSITION 4.4. *Let $C_0\ge 1$ and $\mathscr{C}>0$ be such that for all $\boldsymbol{\beta}\in\mathbb{R}^R$ and $B>0$ we have, for $i=1,2$,*

$$N_i^{\mathrm{aux}}(\boldsymbol{\beta};B)\le C_0 B^{d_1n_1+d_2n_2-n_i-2^{\tilde{d}+1}\mathscr{C}}. \tag{4.7}$$

Assume further that the forms F_i are linearly independent, so that there exist $M>\mu>0$ such that

$$\mu\|\boldsymbol{\beta}\|_\infty \le \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty \le M\|\boldsymbol{\beta}\|_\infty. \tag{4.8}$$

Then there exists a constant $C>0$ depending on C_0, d_i, n_i, μ and M such that the following auxiliary inequality *holds:*

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|,\left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|\right\} \le C\max\{P_2^{-1}, P_1^{-d_1}P_2^{-d_2}\|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{(\tilde{d}+1)^{-1}}\}^{\mathscr{C}}$$

for all real numbers $P_1,P_2>1$.

Proof. The strategy of this proof will closely follow the proof of [22, Proposition 3.1]. By Lemma 4.3 we know that (4.6) holds for one of $i=1$ or $i=2$. Assume that there is some $\theta\in(0,1]$ such that for the same i we have

$$N_i^{\mathrm{aux}}(\boldsymbol{\beta};P_2^{\theta}) < M_i(\boldsymbol{\beta}\cdot\boldsymbol{F}, P_2^{\theta}, P_2^{\theta}, P_1^{-d_1}P_2^{-d_2}P_2^{\theta(\tilde{d}+1)}). \tag{4.9}$$

Going forward with the case $i=1$, noting that the case $i=2$ can be proven completely analogously, this means that there exists a (d_1-1)-tuple $\widehat{\boldsymbol{x}}$ and a d_2-tuple $\widetilde{\boldsymbol{y}}$ which is counted by $M_1(\boldsymbol{\beta}\cdot\boldsymbol{F}, P_2^{\theta}, P_2^{\theta}, P_1^{-d_1}P_2^{-d_2}P_2^{\theta(\tilde{d}+1)})$ but not by $N_1^{\mathrm{aux}}(\boldsymbol{\beta};P_2^{\theta})$. Therefore this pair of tuples satisfies

$$\|\widehat{\boldsymbol{x}}^{(i)}\|_\infty, \|\widetilde{\boldsymbol{y}}^{(j)}\|_\infty \le P_2^{\theta} \quad \text{for } i=1,\dots,d_1-1 \text{ and } j=1,\dots,d_2, \tag{4.10}$$

and

$$\|\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_\ell, \widetilde{\boldsymbol{y}})\| < P_1^{-d_1} P_2^{-d_2} P_2^{\theta(\tilde{d}+1)} \quad \text{for } \ell = 1, \dots, n_1, \tag{4.11}$$

since it is counted by $M_1(\boldsymbol{\beta}\cdot\boldsymbol{F}, P_2^\theta, P_2^\theta, P_1^{-d_1}P_2^{-d_2}P_2^{\theta(\tilde{d}+1)})$. On the other hand, since it is not counted by $N_1^{\mathrm{aux}}(\boldsymbol{\beta}; P_2^\theta)$ there exists $\ell_0 \in \{1, \dots, n_1\}$ such that

$$|\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_{\ell_0}, \widetilde{\boldsymbol{y}})| \geq \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty P_2^{\tilde{d}\theta}. \tag{4.12}$$

From (4.11) we deduce that for ℓ_0 we must have either

$$|\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_{\ell_0}, \widetilde{\boldsymbol{y}})| < P_1^{-d_1} P_2^{-d_2} P_2^{\theta(\tilde{d}+1)} \tag{4.13}$$

or

$$|\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_{\ell_0}, \widetilde{\boldsymbol{y}})| \geq 1/2. \tag{4.14}$$

If (4.13) holds then (4.12) implies

$$\|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty < \frac{P_1^{-d_1}P_2^{-d_2}P_2^{(\tilde{d}+1)\theta}}{P_2^{\tilde{d}\theta}} = P_2^\theta P_1^{-d_1} P_2^{-d_2}. \tag{4.15}$$

If on the other hand (4.14) holds, then (4.10) gives

$$1/2 \leq |\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\widehat{\boldsymbol{x}}, \boldsymbol{e}_{\ell_0}, \widetilde{\boldsymbol{y}})| \ll \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty P_2^{(\tilde{d}+1)\theta}. \tag{4.16}$$

Since either (4.15) or (4.16) holds, then via (4.8) we deduce

$$P_2^{-\theta} \ll_{\mu,M} \max\left\{P_1^{-d_1}P_2^{-d_2}\|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{(\tilde{d}+1)^{-1}}\right\}. \tag{4.17}$$

Since (4.6) holds for $i = 1$ and due to the assumption (4.7) we see that (4.9) holds if there exists some $C_1 > 0$ depending only on C_0, d_i, n_i and ε such that

$$P_2^{-\theta 2^{\tilde{d}+1}\mathscr{C}} \leq C_1 \min\left\{\left|\frac{S(\boldsymbol{\alpha})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|\right\}^{2^{\tilde{d}+1}}. \tag{4.18}$$

Now *define* θ such that we have equality above, i.e.

$$P_2^\theta = C_1^{\frac{1}{2^{\tilde{d}+1}\mathscr{C}}} \min\left\{\left|\frac{S(\boldsymbol{\alpha})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|\right\}^{-1/\mathscr{C}}. \tag{4.19}$$

If $\theta \in (0, 1]$ then (4.18) holds and so together with the assumption (4.7) as argued above this implies (4.17) holds, which gives the result in this case. But θ will always be positive; for if $\theta \leq 0$ then (4.19) implies

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|\right\} \geq C_1^{-1/2^{\tilde{d}+1}}.$$

However, note that clearly $|S(\boldsymbol{\alpha})| \leq (P_1+1)^{n_1}(P_2+1)^{n_2}$. Without loss of generality we may take P_i large enough, depending on ε, so that this clearly leads to a contradiction. Finally, if $\theta \geq 1$ then we find $P_2^{-\mathscr{C}\theta} \leq P_2^{-\mathscr{C}}$, and so from (4.19) we obtain

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1+\varepsilon}P_2^{n_2}}\right|\right\} \ll P_2^{-\mathscr{C}}.$$

This gives the result. ∎

5. The circle method. The aim of this section is to use the auxiliary inequality

$$P_1^{-\varepsilon}\min\left\{\left|\frac{S(\boldsymbol{\alpha})}{P_1^{n_1}P_2^{n_2}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1}P_2^{n_2}}\right|\right\} \leq C\max\left\{P_2^{-1}, P_1^{-d_1}P_2^{-d_2}\|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{(\tilde{d}+1)^{-1}}\right\}^{\mathscr{C}}, \tag{5.1}$$

where $C \geq 1$ and apply the circle method in order to deduce an estimate for $N(P_1, P_2)$. In this section we will use the notation $P = P_1^{d_1}P_2^{d_2}$. Write $b = \max\{1, \log P_1/\log P_2\}$ and $u = \max\{1, \log P_2/\log P_1\}$. If $P_1 \geq P_2$ then $b = \log P_1/\log P_2$ and thus $P_2^{bd_1+d_2} = P$. The main result will be the following.

Proposition 5.1. *Let $\mathscr{C} > (bd_1 + ud_2)R$, $C \geq 1$ and $\varepsilon > 0$ be such that the auxiliary inequality* (5.1) *holds for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$, all $P_1, P_2 > 1$ and all boxes $\mathcal{B}_i \subset [-1,1]^{n_i}$ with side lengths at most* 1 *and edges parallel to the coordinate axes. There exists some $\delta > 0$ depending on b, u, R, d_i and n_i such that*

$$N(P_1, P_2) = \sigma P_1^{n_1-d_1R}P_2^{n_2-d_2R} + O(P_1^{n_1-d_1R}P_2^{n_2-d_2R}P^{-\delta}).$$

The factor $\sigma = \mathfrak{I}\mathfrak{S}$ is the product of the singular integral $\mathfrak{I}$ and the singular series $\mathfrak{S}$, as defined in (5.11) *and* (5.8), *respectively.*

Note that this result holds for general bidegree, and therefore in the proof one may assume $P_1 \geq P_2$ throughout. For instance, if one wishes to show the above proposition for bidegree $(2,1)$, the result follows from the asymmetric results of bidegree $(2,1)$ and bidegree $(1,2)$.

5.1. The minor arcs. First we will show that the contributions from the minor arcs do not affect the main term. For this we will prove a lemma similar to [22, Lemma 2.1].

Lemma 5.2. *Let $r_1, r_2\colon (0,\infty) \to (0,\infty)$ be strictly decreasing and increasing bijections, respectively, and let $A > 0$ be a real number. Let $E_0 \subset \mathbb{R}^R$ be a hypercube of side lengths* 1 *whose edges are parallel to the coordinate axes. Let $E \subseteq E_0$ be a measurable set and let $\varphi\colon E \to [0,\infty)$ be a measurable function.*

Assume that for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ such that $\boldsymbol{\alpha}, \boldsymbol{\alpha}+\boldsymbol{\beta} \in E$ we have

$$\min\{\varphi(\boldsymbol{\alpha}), \varphi(\boldsymbol{\alpha}+\boldsymbol{\beta})\} \leq \max\{A, r_1^{-1}(\|\boldsymbol{\beta}\|_\infty), r_2^{-1}(\|\boldsymbol{\beta}\|_\infty)\}. \tag{5.2}$$

Then for all integers $k \le \ell$ such that $A < 2^k$ we get

$$\int_E \varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \ll_R 2^k + \sum_{i=k}^{\ell-1} 2^i \left(\frac{r_1(2^i)}{\min\{r_2(2^i),1\}}\right)^R + \left(\frac{r_1(2^\ell)}{\min\{r_2(2^\ell),1\}}\right)^R \sup_{\boldsymbol{\alpha}\in E} \varphi(\boldsymbol{\alpha}).$$

Note that if we take

$$\varphi(\boldsymbol{\alpha}) = C^{-1} P_1^{-n_1-\varepsilon} P_2^{-n_2} |S(\boldsymbol{\alpha})|,$$
$$r_1(t) = P_1^{-d_1} P_2^{-d_2} t^{-1/\mathscr{C}}, \quad r_2(t) = t^{(\tilde{d}+1)/\mathscr{C}}, \quad A = P_2^{-\mathscr{C}}$$

where C is the constant in (5.1), then the assumption (5.2) is just the auxiliary inequality (5.1).

Proof. The proof is very similar to the proof of [22, Lemma 2.1] so we shall be brief. Given $t \ge 0$ define the set

$$D(t) = \{\boldsymbol{\alpha} \in E \colon \varphi(\boldsymbol{\alpha}) \ge t\}.$$

If $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}+\boldsymbol{\beta}$ are both contained in $D(t)$ then by (5.2) one of the following must hold:

$$A \ge t, \quad \|\boldsymbol{\beta}\|_\infty \le r_1(t), \quad \text{or} \quad \|\boldsymbol{\beta}\|_\infty \ge r_2(t).$$

In particular, if $t > A$ then either $\|\boldsymbol{\beta}\|_\infty \le r_1(t)$ or $\|\boldsymbol{\beta}\|_\infty \ge r_2(t)$. Via the same considerations as in [22, the proof of Lemma 2.1] in this case we then find

$$\mu(D(t)) \ll_R \left(\frac{r_1(t)}{\min\{r_2(t),1\}}\right)^R.$$

Hence, if $2^k > A$ we obtain

$$\begin{aligned}\int_E \varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} &= \int_{E\setminus D(2^k)} \varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} + \sum_{i=k}^{\ell} \int_{D(2^i)\setminus D(2^{i+1})} \varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} + \int_{D(2^\ell)} \varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \\ &\ll_R 2^k + \sum_{i=k}^{\ell-1} 2^i \left(\frac{r_1(2^i)}{\min\{r_2(2^i),1\}}\right)^R + \left(\frac{r_1(2^\ell)}{\min\{r_2(2^\ell),1\}}\right)^R \sup_{\boldsymbol{\alpha}\in E} \varphi(\boldsymbol{\alpha}). \quad \blacksquare\end{aligned}$$

Recall the notation $P = P_1^{d_1} P_2^{d_2}$. From now on we will assume $P_1 \ge P_2$. Note that the assumption in Proposition 4.4 that $\mathscr{C} > R(bd_1 + ud_2)$ is equivalent to $\mathscr{C} > R(bd_1 + d_2)$ if $P_1 \ge P_2$.

LEMMA 5.3. *Let $T\colon \mathbb{R}^R \to \mathbb{C}$ be a measurable function. With notation as in Lemma 5.2 assume that for all $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^R$ and for all $P_1 \ge P_2 > 1$, and $\mathscr{C} > 0$ we have*

(5.3)
$$\min\left\{\left|\frac{T(\boldsymbol{\alpha})}{P_1^{n_1}P_2^{n_2}}\right|, \left|\frac{T(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P_1^{n_1}P_2^{n_2}}\right|\right\} \le \max\left\{P_2^{-1}, P_1^{-d_1}P_2^{-d_2}\|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{(\tilde{d}+1)^{-1}}\right\}^{\mathscr{C}}.$$

Write $P = P_1^{d_1} P_2^{d_2}$ and assume that

$$\sup_{\boldsymbol{\alpha} \in E} |T(\boldsymbol{\alpha})| \leq P_1^{n_1} P_2^{n_2} P^{-\delta} \tag{5.4}$$

for some $\delta > 0$. If $\mathscr{C} > (d_1 + d_2)R$ then

$$\int_E \frac{T(\boldsymbol{\alpha})}{P_1^{n_1} P_2^{n_2}}\, d\boldsymbol{\alpha} \ll_{\mathscr{C}, d_i, R} P^{-R-\delta(1-(d_1+d_2)R/\mathscr{C})} + P^{-R-\delta(1-R/\mathscr{C})} + P_2^{-\mathscr{C}}.$$

Proof. The proof is a straightforward modification of [22, the proof of Lemma 2.2]. In our case it follows as an application of Lemma 5.2 by taking

$$\varphi(\boldsymbol{\alpha}) = \frac{|T(\boldsymbol{\alpha})|}{P_1^{n_1} P_2^{n_2}}, \quad r_1(t) = P_1^{-d_1} P_2^{-d_2} t^{-\frac{1}{\mathscr{C}}}, \quad r_2(t) = t^{\frac{\tilde{d}+1}{\mathscr{C}}}, \quad A = P_2^{-\mathscr{C}}. \ ■$$

We will finish this section by defining the major and minor arcs and showing that the minor arcs do not contribute to the main term. For $\Delta > 0$ we define the *major arcs* to be the set given by

$$\mathfrak{M}(\Delta) := \bigcup_{\substack{q \in \mathbb{N} \\ q \leq P^{\Delta}}} \bigcup_{\substack{0 \leq a_i \leq q \\ (a_1, \dots, a_R, q) = 1}} \{\boldsymbol{\alpha} \in [0,1]^R : 2\|q\boldsymbol{\alpha} - \boldsymbol{a}\|_\infty < P_1^{-d_1} P_2^{-d_2} P^{\Delta}\},$$

and the *minor arcs* to be the given by

$$\mathfrak{m}(\Delta) := [0,1]^R \setminus \mathfrak{M}(\Delta).$$

Write further

$$\delta_0 = \frac{\min_{i=1,2}\{n_1 + n_2 - \dim V_i^*\}}{(\tilde{d}+1)2^{\tilde{d}} R}.$$

Note that if the forms F_i are linearly independent, then V_i^* are proper subvarieties of $\mathbb{A}_{\mathbb{C}}^{n_1+n_2}$ so that $\dim V_i^* \leq n_1 + n_2 - 1$ whence $\delta_0 \geq \frac{1}{(\tilde{d}+1)2^{\tilde{d}} R}$. To see this for V_1^* note that requiring

$$\operatorname{rank} \left(\frac{\partial F_i}{\partial x_j} \right)_{i,j} < R$$

is equivalent to requiring all the $R \times R$ minors of $\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j}$ vanish. This defines a system of polynomials of degree $R(d_1 + d_2 - 1)$ in $(\boldsymbol{x}, \boldsymbol{y})$ which are not all zero unless there exists $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}$ such that

$$\sum_{i=1}^R \beta_i \left(\frac{\partial F_i}{\partial x_j} \right) = 0 \quad \text{for } j = 1, \dots, n_1$$

holds identically in $(\boldsymbol{x}, \boldsymbol{y})$. This is the same as saying that

$$\nabla_{\boldsymbol{x}} \Big(\sum_{i=1}^R \beta_i F_i \Big) = 0$$

holds identically. From this we find that $\sum_{i=1}^{R} \beta_i F_i$ must be a form entirely in the $\boldsymbol{y}$-variables. But this is a linear combination of homogeneous bidegree (d_1, d_2) forms with $d_1 \geq 1$ and thus we must in fact have $\sum_{i=1}^{R} \beta_i F_i = 0$ identically, contradicting linear independence. The argument works analogously for V_2^*.

The next lemma shows that the assumption (5.4) holds with $E = \mathfrak{m}(\varDelta)$ and $T(\boldsymbol{\alpha}) = C^{-1} P_1^{-\varepsilon} S(\boldsymbol{\alpha})$.

Lemma 5.4. *Let* $0 < \varDelta \leq R(\tilde{d}+1)(bd_1+d_2)^{-1}$ *and let* $\varepsilon > 0$. *Then we have the upper bound*

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}(\varDelta)} |S(\boldsymbol{\alpha})| \ll P_1^{n_1} P_2^{n_2} P^{-\varDelta\delta_0+\varepsilon}.$$

Proof. The result follows straightforward from [24, Lemma 4.3] by setting the parameter θ to be

$$\theta = \frac{\varDelta}{(\tilde{d}+1)R}.$$

If we have $0 < \varDelta \leq R(\tilde{d}+1)(bd_1+d_2)^{-1}$ this ensures that the assumption $0 < \theta \leq (bd_1+d_2)^{-1}$ in [24, Lemma 4.3] is satisfied. ∎

Before we state the next proposition, recall that we assume $P_1 \geq P_2$ throughout, as was mentioned at the beginning of this section.

Proposition 5.5. *Let* $\varepsilon > 0$ *and let* $0 < \varDelta \leq R(\tilde{d}+1)(bd_1+d_2)^{-1}$. *Under the assumptions of Proposition* 5.1 *we have*

$$\int_{\mathfrak{m}(\varDelta)} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll P_1^{n_1-d_1R} P_2^{n_2-d_2R} P^{-\varDelta\delta_0(1-(d_1+d_2)R/\mathscr{C})+\varepsilon}.$$

Proof. We apply Lemma 5.2 with

$$T(\boldsymbol{\alpha}) = C^{-1}P^{-\varepsilon}S(\boldsymbol{\alpha}), \quad E_0 = [0,1]^R, \quad E = \mathfrak{m}(\varDelta), \quad \text{and} \quad \delta = \varDelta\delta_0,$$

where $C > 0$ is some real number. With these choices (5.3) follows from the auxiliary inequality (5.1) since for any $\varepsilon > 0$ we have $P^{-\varepsilon} \leq P_1^{-\varepsilon}$. From Lemma 5.4 we have the bound

$$\sup_{\boldsymbol{\alpha} \in E} CT(\boldsymbol{\alpha}) \ll P_1^{n_1} P_2^{n_2} P^{-\delta}.$$

We may increase C if necessary so that we recover (5.4). Therefore the hypotheses of Lemma 5.3. Since we assume $\mathscr{C} > (bd_1+d_2)R$, we also note

$$P_2^{-\mathscr{C}} = P^{-R} P^{R-\mathscr{C}(bd_1+d_2)^{-1}} \ll_{\mathscr{C}} P^{-R-\tilde{\delta}}$$

for some $\tilde{\delta} > 0$. Therefore if we assume $\mathscr{C} > (bd_1+d_2)R$ then Lemma 5.3 gives

$$\int_{\mathfrak{m}(\varDelta)} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll P_1^{n_1-d_1R} P_2^{n_2-d_2R} P^{-\varDelta\delta_0(1-(d_1+d_2)R/\mathscr{C})+\varepsilon},$$

as desired. ∎

5.2. The major arcs. The aim of this section is to identify the main term via integrating the exponential sum $S(\boldsymbol{\alpha})$ over the major arcs, and analyse the singular integral and singular series appropriately. For $\boldsymbol{a} \in \mathbb{Z}^R$ and $q \in \mathbb{N}$ consider the complete exponential sum

$$S_{\boldsymbol{a},q} := q^{-n_1-n_2} \sum_{\boldsymbol{x},\boldsymbol{y}} e\left(\frac{\boldsymbol{a}}{q} \cdot \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})\right),$$

where the sum $\sum_{\boldsymbol{x},\boldsymbol{y}}$ runs through a complete set of residues modulo q. Further, for $P \geq 1$ and $\Delta > 0$ we define the truncated singular series

$$\mathfrak{S}(P) := \sum_{q \leq P^{\Delta}} \sum_{\boldsymbol{a}} S_{\boldsymbol{a},q},$$

where the sum $\sum_{\boldsymbol{a}}$ runs over $\boldsymbol{a} \in \mathbb{Z}^R$ such that the conditions $0 \leq a_i < q$ for $i = 1, \dots, R$ and $(a_1, \dots, a_R, q) = 1$ are satisfied. For $\boldsymbol{\gamma} \in \mathbb{R}^R$ we further define

$$S_{\infty}(\boldsymbol{\gamma}) := \int_{\mathcal{B}_1 \times \mathcal{B}_2} e(\boldsymbol{\gamma} \cdot \boldsymbol{F}(\boldsymbol{u},\boldsymbol{v}))\, d\boldsymbol{u}\, d\boldsymbol{v},$$

and we define the truncated singular integral for $P \geq 1$, $\Delta > 0$ as follows

$$\mathfrak{I}(P) := \int_{\|\boldsymbol{\gamma}\|_{\infty} \leq P^{\Delta}} S_{\infty}(\boldsymbol{\gamma})\, d\boldsymbol{\gamma}.$$

From now on we assume that our parameter $\Delta > 0$ satisfies

$$(bd_1 + d_2)^{-1} > \Delta(2R+3) + \delta \tag{5.5}$$

for some $\delta > 0$. Since $\mathscr{C} > R(bd_1 + d_2)$ we are always able to choose such Δ in terms of $\mathscr{C}$. Further as in [24] we now define some slightly modified major arcs $\mathfrak{M}'(\Delta)$ as

$$\mathfrak{M}'(\Delta) := \bigcup_{1 \leq q \leq P^{\Delta}} \bigcup_{\substack{0 \leq a_i < q \\ (a_1,\dots,a_R,q)=1}} \mathfrak{M}'_{\boldsymbol{a},q}(\Delta),$$

where $\mathfrak{M}'_{\boldsymbol{a},q}(\Delta) = \left\{\boldsymbol{\alpha} \in [0,1]^R \colon \left\|\boldsymbol{\alpha} - \frac{\boldsymbol{a}}{q}\right\|_{\infty} < P_1^{-d_1} P_2^{-d_2} P^{\Delta}\right\}$. The sets $\mathfrak{M}'_{\boldsymbol{a},q}$ are disjoint for our choice of Δ; for if there is some

$$\boldsymbol{\alpha} \in \mathfrak{M}'_{\boldsymbol{a},q}(\Delta) \cap \mathfrak{M}'_{\tilde{\boldsymbol{a}},\tilde{q}}(\Delta),$$

where $\mathfrak{M}'_{\tilde{\boldsymbol{a}},\tilde{q}}(\Delta) \neq \mathfrak{M}'_{\boldsymbol{a},q}(\Delta)$, then there is some $i \in \{1, \dots, R\}$ such that

$$P^{-2\Delta} \leq \frac{1}{q\tilde{q}} \leq \left|\frac{a_i}{q} - \frac{\tilde{a}_i}{\tilde{q}}\right| \leq 2P^{\Delta-1},$$

which is impossible for large P, since by (5.5) we have $3\Delta - 1 < 0$. Further we note that clearly $\mathfrak{M}'(\Delta) \supseteq \mathfrak{M}(\Delta)$ whence $\mathfrak{m}'(\Delta) \subseteq \mathfrak{m}(\Delta)$ and so the conclusions of Proposition 5.5 hold with $\mathfrak{m}(\Delta)$ replaced by $\mathfrak{m}'(\Delta)$.

By following the proof of [24, Lemma 5.3] it becomes transparent that

$$S(\boldsymbol{\alpha}) = P_1^{n_1} P_2^{n_2} S_{\boldsymbol{a},q} S_\infty(P\boldsymbol{\beta}) + O(qP_1^{n_1} P_2^{n_2-1}(1 + P\|\boldsymbol{\beta}\|_\infty)). \tag{5.6}$$

Using (5.6) in the same way as [22, (2.20)] was derived we find that

$$\int\limits_{\mathfrak{M}'(\Delta)} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = P_1^{n_1} P_2^{n_2} P^{-R} \mathfrak{S}(P)\mathfrak{J}(P) + O(P_1^{n_1} P_2^{n_2} P^{-R+\Delta(2R+3)-1/(bd_1+d_2)}). \tag{5.7}$$

Finally, using the auxiliary inequality (5.1) and the identity (5.6), the proofs of [22, Lemmas 2.5 and 2.6] go through mutatis mutandis.

Hence if $\mathscr{C} > (\tilde{d}+1)R$ and assuming that the forms $F_i(\boldsymbol{x}, \boldsymbol{y})$ are linearly independent then the *singular series*

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\boldsymbol{a} \bmod q} S_{\boldsymbol{a},q} \tag{5.8}$$

exists and converges absolutely, with

$$|\mathfrak{S}(P) - \mathfrak{S}| \ll_{C,\mathscr{C}} P^{-\Delta\delta_1} \tag{5.9}$$

for some $\delta_1 > 0$ depending only on $\mathscr{C}, d_i$ and R.

Similarly if $\mathscr{C} - \varepsilon' > R$ then for all $P > 1$ we have

$$|\mathfrak{J}(P) - \mathfrak{J}| \ll_{\mathscr{C},C,\varepsilon'} P^{-\Delta(\mathscr{C}-\varepsilon'-R)}, \tag{5.10}$$

where $\mathfrak{J}$ is the *singular integral*

$$\mathfrak{J} = \int\limits_{\boldsymbol{\gamma}\in\mathbb{R}^R} S_\infty(\boldsymbol{\gamma})\, d\boldsymbol{\gamma}. \tag{5.11}$$

In particular, we see that $\mathfrak{J}$ exists and converges absolutely.

Before we finish the proof of the main result we state two different expressions for the singular series and the singular integral that will be useful later on. If $\mathscr{C} > R(d_1 + d_2)$ then $\mathfrak{J}$ and $\mathfrak{S}$ converge absolutely, as was shown in the previous two lemmas. Therefore, as in [3, §7], by regarding the bihomogeneous forms under investigation simply as homogeneous forms we may express the singular series as an absolutely convergent product

$$\mathfrak{S} = \prod_p \mathfrak{S}_p, \tag{5.12}$$

where

$$\mathfrak{S}_p = \lim_{k\to\infty} \frac{1}{p^{k(n_1+n_2-R)}} \#\{(\boldsymbol{u}, \boldsymbol{v}) \in \{1, \ldots, p^k\}^{n_1+n_2} : F_i(\boldsymbol{u}, \boldsymbol{v}) \equiv 0 \ (\mathrm{mod}\ p),\ i = 1, \ldots, R\}.$$

Lemma 2.6 in [22] further shows that we can write the singular integral as

$$\mathfrak{I} = \lim_{P\to\infty} \frac{1}{P^{n_1+n_2-(d_1+d_2)R}} \mu\big\{(\boldsymbol{t}_1,\boldsymbol{t}_2)/P \in \mathcal{B}_1 \times \mathcal{B}_2 \colon |F_i(\boldsymbol{t}_1,\boldsymbol{t}_2)| \le 1/2,\quad i = 1,\dots,R\big\}, \tag{5.13}$$

where $\mu(\cdot)$ denotes the Lebesgue measure. We may therefore interpret the quantities $\mathfrak{I}$ and $\mathfrak{S}_p$ as the real and p-adic *densities*, respectively, of the system of equations $F_1(\boldsymbol{x},\boldsymbol{y}) = \cdots = F_R(\boldsymbol{x},\boldsymbol{y}) = 0$.

5.3. Proofs of Proposition 5.1 and Theorem 2.1

Proof of Proposition 5.1. From Proposition 5.5 and the estimates (5.7), (5.9) and (5.10), for any $\varepsilon > 0$ we find that

$$\frac{N(P_1,P_2)}{P_1^{n_1}P_2^{n_2}P^{-R}} - \mathfrak{S}\mathfrak{I} \ll P^{-\Delta\delta_1} + P^{-\Delta\delta_0(1-(d_1+d_2)R/\mathscr{C})+\varepsilon} + P^{(2R+3)\Delta-1/(bd_1+d_2)} + P^{-\Delta(\mathscr{C}-\varepsilon'-R)}$$

for some $\delta_1 > 0$ and some $1 > \varepsilon' > 0$. Recall we assumed $\mathscr{C} > (bd_1+d_2)R$, and assuming the forms F_i are linearly independent we also have $\delta_0 \ge 1/(\tilde{d}+1)2^{\tilde{d}}R$. Therefore choosing suitably small $\Delta > 0$ there exists some $\delta > 0$ such that

$$\frac{N(P_1,P_2)}{P_1^{n_1}P_2^{n_2}P^{-R}} - \mathfrak{S}\mathfrak{I} \ll P^{-\delta}$$

as desired. Finally, since we assume that the equations F_i define a complete intersection, it is a standard fact to see that $\mathfrak{S}$ is positive if there exists a non-singular p-adic zero for all primes P, and similarly $\mathfrak{I}$ is positive if there exists a non-singular real zero within $\mathcal{B}_1 \times \mathcal{B}_2$. A detailed argument of this fact using a version of Hensel's Lemma for $\mathfrak{S}$ and the implicit function theorem for $\mathfrak{I}$ can be found for example in [22, §4]. ∎

Proof of Theorem 2.1. Assume the estimate in (2.1) holds for some constant $C_0 > 0$. From Proposition 4.4 it thus follows that the auxiliary inequality (5.1) holds with a constant $C > 0$ depending on C_0, d_i, n_i, μ and M, where all of these quantities follow the same notation as in Section 4. Therefore the assumptions of Proposition 5.1 are satisfied whence we can apply it to obtain the desired conclusions. ∎

6. Systems of bilinear forms. In this section we assume $d_1 = d_2 = 1$. We can thus write our system as

$$F_i(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{y}^T A_i \boldsymbol{x},$$

where A_i are $n_2 \times n_1$-dimensional matrices with integer entries. For $\boldsymbol{\beta} \in \mathbb{R}^R$ we now have

$$\boldsymbol{\beta} \cdot \boldsymbol{F} = \boldsymbol{y}^T A_{\boldsymbol{\beta}} \boldsymbol{x},$$

where $A_{\boldsymbol{\beta}} = \sum_i \beta_i A_i$. Recall that we put

$$\sigma_{\mathbb{R}}^{(1)} = \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}} \dim \ker(A_{\boldsymbol{\beta}}) \quad \text{and} \quad \sigma_{\mathbb{R}}^{(2)} = \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}} \dim \ker(A_{\boldsymbol{\beta}}^T).$$

Since the row rank of a matrix is equal to its column rank we can also define

$$\rho_{\mathbb{R}} := \min_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}} \operatorname{rank}(A_{\boldsymbol{\beta}}) = \min_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}} \operatorname{rank}(A_{\boldsymbol{\beta}}^T).$$

Due to the rank-nullity theorem the conditions

$$n_i - \sigma_{\mathbb{R}}^{(i)} > (2b+2)R$$

for $i = 1, 2$ are equivalent to

$$\rho_{\mathbb{R}} > (2b+2)R.$$

LEMMA 6.1. *Assume that* $\mathbb{V}(F_1, \dots, F_R) \subset \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ *is a smooth complete intersection. Let* $b \geq 1$ *be a real number. Assume further*

$$\min\{n_1, n_2\} > (2b+2)R \quad \textit{and} \quad n_1 + n_2 > (4b+5)R. \tag{6.1}$$

Then

$$n_i - \sigma_{\mathbb{R}}^{(i)} > (2b+2)R \tag{6.2}$$

for $i = 1, 2$.

Proof. Without loss of generality assume $n_1 \geq n_2$. Pick $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}$ such that $\operatorname{rank}(A_{\boldsymbol{\beta}}) = \rho_{\mathbb{R}}$. In particular then

$$\dim \ker(A_{\boldsymbol{\beta}}) = \sigma_{\mathbb{R}}^{(1)} \quad \text{and} \quad \dim \ker(A_{\boldsymbol{\beta}}^T) = \sigma_{\mathbb{R}}^{(2)}.$$

We proceed in distinguishing two cases. Firstly, if $\sigma_{\mathbb{R}}^{(2)} = 0$ then (6.2) follows for $i = 2$ by the assumption (6.1). Further by comparing row rank and column rank of $A_{\boldsymbol{\beta}}$ in this case we must then have $\sigma_{\mathbb{R}}^{(1)} \leq n_1 - n_2$, and therefore

$$n_1 - \sigma_{\mathbb{R}}^{(1)} \geq n_2 > (2b+2)R,$$

so (6.2) follows for $i = 1$.

Now we turn to the case $\sigma_{\mathbb{R}}^{(2)} > 0$. Then also $\sigma_{\mathbb{R}}^{(1)} > 0$. The singular locus of the variety $\mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \subset \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ is given by

$$\operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) = \mathbb{V}(\boldsymbol{y}^T A_{\boldsymbol{\beta}}) \cap \mathbb{V}(A_{\boldsymbol{\beta}} \boldsymbol{x}).$$

Therefore

$$\dim \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) = \sigma_{\mathbb{R}}^{(1)} + \sigma_{\mathbb{R}}^{(2)} - 2.$$

Since we assumed $\mathbb{V}(\boldsymbol{F})$ to be a smooth complete intersection we can apply Lemma 3.3 to get $\dim \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \leq R-2$. Therefore we find

$$\sigma_{\mathbb{R}}^{(1)} + \sigma_{\mathbb{R}}^{(2)} \leq R.$$

From our previous remarks we know that showing (6.2) is equivalent to showing $\rho_{\mathbb{R}} > (2b+2)R$. But now

$$\rho_{\mathbb{R}} = \frac{1}{2}(n_1 + n_2 - \sigma_{\mathbb{R}}^{(1)} - \sigma_{\mathbb{R}}^{(2)}) \geq \frac{1}{2}(n_1 + n_2 - R) > (2b+2)R,$$

where the last inequality followed from the assumption (6.1). Therefore (6.2) follows as desired. ∎

Proof of Theorem 1.1. Recall the notation $b = \frac{\log P_1}{\log P_2}$. By virtue of Theorem 2.1 it suffices to show that assuming

$$n_i - \sigma_{\mathbb{R}}^{(i)} > (2b+2)R$$

for $i = 1, 2$ implies (2.1). We will show (2.1) for $i = 1$, the other case follows analogously. Let $\mathscr{C} = \frac{n_2 - \sigma_{\mathbb{R}}^{(2)}}{2}$; we note that $\mathscr{C} > (bd_1 + d_2)R = (b+1)R$ precisely when $n_2 - \sigma_{\mathbb{R}}^{(2)} > (2b+2)R$. Therefore it suffices to show that

$$N_1^{\mathrm{aux}}(\boldsymbol{\beta}, B) \ll B^{\sigma_{\mathbb{R}}^{(2)}} \tag{6.3}$$

for all $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}$ with the implied constant not depending on $\boldsymbol{\beta}$. In our case we have

$$\boldsymbol{\Gamma}(\boldsymbol{u}) = \boldsymbol{u}^T A(\boldsymbol{\beta}),$$

where $\boldsymbol{u} \in \mathbb{Z}^{n_2}$. Therefore $N_1^{\mathrm{aux}}(\boldsymbol{\beta}, B)$ counts vectors $\boldsymbol{u} \in \mathbb{Z}^{n_2}$ such that

$$\|\boldsymbol{u}\|_\infty \leq B \quad \text{and} \quad \|\boldsymbol{u}^T A(\boldsymbol{\beta})\|_\infty \leq \|A(\boldsymbol{\beta})\|_\infty = \|\boldsymbol{\beta} \cdot \boldsymbol{F}\|_\infty.$$

Precisely the same argument that leads to [22, (4.3)] now yields (6.3). ∎

7. Systems of forms of bidegree $(2,1)$**.** We consider a system $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})$ of homogeneous equations of bidegree $(2,1)$, where $\boldsymbol{x} = (x_1, \ldots, x_{n_1})$ and $\boldsymbol{y} = (y_1, \ldots, y_{n_2})$. We will first assume $n_1 = n_2 = n$, say, and then deduce Theorem 1.2. Therefore the initial main goal is to establish the following.

Proposition 7.1. *Let $F_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, F_R(\boldsymbol{x}, \boldsymbol{y})$ be bihomogeneous forms of bidegree $(2,1)$ such that the biprojective variety $\mathbb{V}(F_1, \ldots, F_R) \subset \mathbb{P}_{\mathbb{Q}}^{n-1} \times \mathbb{P}_{\mathbb{Q}}^{n-1}$ is a complete intersection. Write $b = \max\{\log P_1 / \log P_2, 1\}$ and $u = \max\{\log P_2 / \log P_1, 1\}$. Assume that*

$$n - s_{\mathbb{R}}^{(i)} > (8b + 4u)R \tag{7.1}$$

for $i = 1, 2$, where $s_{\mathbb{R}}^{(i)}$ are as defined in (1.7) *and* (1.8)*. Then there exists some $\delta > 0$ depending at most on $\boldsymbol{F}$, R, n, b and u such that*

$$N(P_1, P_2) = \sigma P_1^{n-2R} P_2^{n-R} + O(P_1^{n-2R} P_2^{n-R} \min\{P_1, P_2\}^{-\delta})$$

where $\sigma > 0$ if the system $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{0}$ has a smooth p-adic zero for all primes p and a smooth real zero in $\mathcal{B}_1 \times \mathcal{B}_2$.

If we assume that $\mathbb{V}(F_1,\dots,F_R) \subset \mathbb{P}_{\mathbb{Q}}^{n-1} \times \mathbb{P}_{\mathbb{Q}}^{n-1}$ is smooth, then the same conclusions hold if we assume

$$n > (16b + 8u + 1)R$$

instead of (7.1).

For $r = 1,\dots,R$ we can write each form $F_r(\boldsymbol{x},\boldsymbol{y})$ as

$$F_r(\boldsymbol{x},\boldsymbol{y}) = \sum_{i,j,k} F_{ijk}^{(r)} x_i x_j y_k,$$

where the coefficients $F_{ijk}^{(r)}$ are symmetric in i and j. In particular, for any $r = 1,\dots,R$ we have an $n \times n$ matrix given by $H_r(\boldsymbol{y}) = (\sum_k F_{ijk}^{(r)} y_k)_{ij}$ whose entries are linear homogeneous polynomials in $\boldsymbol{y}$. We may thus also write each equation in the form

$$F_r(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^T H_r(\boldsymbol{y})\boldsymbol{x}.$$

The strategy of the proof of Proposition 7.1 is the same as in the bilinear case, but this time more technical arguments are required. We need to obtain a good upper bound for the counting functions $N_i^{\mathrm{aux}}(\boldsymbol{\beta}; B)$ so that we can apply Theorem 2.1. For $\boldsymbol{\beta} \in \mathbb{R}^R$ we consider $\boldsymbol{\beta} \cdot \boldsymbol{F}$, which we can rewrite in our case as

$$\boldsymbol{\beta} \cdot \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}$$

where $H_{\boldsymbol{\beta}}(\boldsymbol{y}) = \sum_{i=1}^R \beta_i H_i(\boldsymbol{y})$ is a symmetric $n \times n$ matrix whose entries are linear and homogeneous in $\boldsymbol{y}$. The associated multilinear form $\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)},\boldsymbol{y})$ is thus given by

$$\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)},\boldsymbol{y}) = 2(\boldsymbol{x}^{(1)})^T H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}^{(2)}.$$

Recall $N_1^{\mathrm{aux}}(\boldsymbol{\beta}, B)$ counts integral tuples $\boldsymbol{x},\boldsymbol{y} \in \mathbb{Z}^n$ satisfying $\|\boldsymbol{x}\|_\infty, \|\boldsymbol{y}\|_\infty \le B$ and

$$\|(\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{y}),\dots,\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\boldsymbol{x},\boldsymbol{e}_n,\boldsymbol{y}))^T\|_\infty = 2\|H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}\|_\infty \le \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty B.$$

Now $N_2^{\mathrm{aux}}(\boldsymbol{\beta}, B)$ counts integral tuples $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$ with $\|\boldsymbol{x}^{(1)}\|_\infty, \|\boldsymbol{x}^{(2)}\|_\infty \le B$ and

$$\|(\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)},\boldsymbol{e}_1),\dots,\Gamma_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)},\boldsymbol{e}_n))^T\|_\infty \le \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty B.$$

We may rewrite this as

$$\|\boldsymbol{x}^{(1)} H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x}^{(2)}\| \le \|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty B$$

for $\ell = 1,\dots,n$. As in the proof of Theorem 1.1 using Propositions 4.4 and 5.1 we find that for the proof of Theorem 7.1 it is enough to show that there exists a positive constant C_0 such that for all $B \geq 1$ and all $\boldsymbol{\beta} \in \mathbb{R}^r \setminus \{0\}$ we have

$$N_i^{\mathrm{aux}}(\boldsymbol{\beta}; B) \leq C_0 B^{2n-4\mathscr{C}}$$

for $i = 1, 2$, where $\mathscr{C} > (2b+u)R$. The remainder of this section establishes these upper bounds.

7.1. The first auxiliary counting function. This is the easier case and the problem of finding a suitable upper bound for $N_1^{\mathrm{aux}}(\boldsymbol{\beta}; B)$ is essentially handled in [23]. Note that in [23] there is the additional symmetry $H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x} = H_{\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{y}$ present, however the proof of [23, Corollary 5.2] only uses the fact that $H_{\boldsymbol{\beta}}(\boldsymbol{y})$ is a symmetric matrix. In fact, we recover the next lemma as a special case of Corollary 7.5.

LEMMA 7.2 ([23, Corollary 5.2]). *Let $H_{\boldsymbol{\beta}}(\boldsymbol{y})$ and $N_1^{\mathrm{aux}}(\boldsymbol{\beta}; B)$ be as above. Let $B, C \geq 1$, let $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}$ and let $\sigma \in \{0, \dots, n-1\}$. Then either*

$$N_1^{\mathrm{aux}}(\boldsymbol{\beta}; B) \ll_{C,n} B^{n+\sigma}(\log B)^n,$$

or there exist non-trivial linear subspaces $U, V \subseteq \mathbb{R}^n$ with $\dim U + \dim V = n + \sigma + 1$ such that for all $\boldsymbol{v} \in V$ and $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U$ we have

$$\frac{|\boldsymbol{u}_1^T H_{\boldsymbol{\beta}}(\boldsymbol{v})\boldsymbol{u}_2|}{\|\boldsymbol{\beta} \cdot \boldsymbol{F}\|_\infty} \ll_n C^{-1} \|\boldsymbol{u}_1\|_\infty \|\boldsymbol{v}\|_\infty \|\boldsymbol{u}_2\|_\infty.$$

Recall the quantity

$$s_{\mathbb{R}}^{(1)} := 1 + \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}} \dim \mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2},$$

where we regard $\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} \subset \mathbb{P}_{\mathbb{C}}^{n_1-1}$ as a projective variety. Note that for this definition we do not necessarily require $n_1 = n_2$. In the same way Proposition 1.4 in [23] is proven we therefore obtain

$$N_1^{\mathrm{aux}}(\boldsymbol{\beta}; B) \ll_\varepsilon B^{n+s_{\mathbb{R}}^{(1)}+\varepsilon} \tag{7.2}$$

for all $B \geq 1$, $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}$ and $\varepsilon > 0$.

Now that we have found an upper bound in terms of the geometry of $\mathbb{V}(\boldsymbol{F})$ the next lemma shows that if $\boldsymbol{F}$ defines a non-singular variety then $s_{\mathbb{R}}^{(1)}$ is not too large. For the next lemma we will not assume $n_1 = n_2$, as we will require it later in the slightly more general context when this assumption is not necessarily satisfied.

LEMMA 7.3. *Let $s_{\mathbb{R}}^{(1)}$ be defined as above and assume that $\boldsymbol{F}$ is a system of bihomogeneous equations of bidegree $(2,1)$ that defines a smooth complete intersection $\mathbb{V}(\boldsymbol{F}) \subset \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$. Then*

$$s_{\mathbb{R}}^{(1)} \leq \max\{0, R + n_1 - n_2\}.$$

Proof. Consider $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}$ such that $\dim \mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} = s_{\mathbb{R}}^{(1)} - 1$. When $\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} = \emptyset$, the statement in the lemma is trivially true. Hence we may assume that this is not the case. The singular locus of $\mathbb{V}(\boldsymbol{\beta}\cdot\boldsymbol{F}) \subseteq \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ is given by

$$\operatorname{Sing}\mathbb{V}(\boldsymbol{\beta}\cdot\boldsymbol{F}) = (\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}) \cap \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}).$$

From Lemma 3.3 we obtain

$$\dim \operatorname{Sing}\mathbb{V}(\boldsymbol{\beta}\cdot\boldsymbol{F}) \le R-2.$$

Further, since $\mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})$ is a system of n_1 bilinear equations, Lemma 3.1 gives

$$\dim \operatorname{Sing}\mathbb{V}(\boldsymbol{\beta}\cdot\boldsymbol{F}) \ge s_{\mathbb{R}}^{(1)} - 1 + n_2 - 1 - n_1.$$

Combining the previous two inequalities yields

$$s_{\mathbb{R}}^{(1)} \le R + n_1 - n_2,$$

as desired. ∎

We remark here that the proof of Lemma 7.3 shows that if $\mathbb{V}(\boldsymbol{F})$ defines a smooth complete intersection and if $s_{\mathbb{R}}^{(1)} > 0$ then $n_2 < n_1 + R$.

7.2. The second auxiliary counting function. Define $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(1)})$ to be the $n \times n$ matrix with the rows given by $(\boldsymbol{x}^{(1)})^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)/\|\boldsymbol{\beta}\cdot\boldsymbol{F}\|_\infty$ for $\ell = 1,\dots,n$. Using this notation $N_2^{\mathrm{aux}}(\boldsymbol{\beta}, B)$ counts the number of integer tuples $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$ such that $\|\boldsymbol{x}^{(1)}\|_\infty, \|\boldsymbol{x}^{(2)}\|_\infty \le B$ and

$$\|\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(1)})\boldsymbol{x}^{(2)}\|_\infty \le B$$

is satisfied. The entries of $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(1)})$ are homogeneous linear polynomials in $\boldsymbol{x}^{(1)}$ whose coefficients do not exceed absolute value 1.

Let A be a real $m \times n$ matrix. Then A^TA is a symmetric and positive definite $n \times n$ matrix, with eigenvalues $\lambda_1^2, \dots, \lambda_n^2$. The non-negative real numbers $\{\lambda_i\}$ are the *singular values* of A.

Notation. Given a matrix $M = (m_{ij})$ we define $\|M\|_\infty := \max_{i,j}|m_{ij}|$. For simplicity we will from now on write $\boldsymbol{x}$ instead of $\boldsymbol{x}^{(1)}$ and $\boldsymbol{y}$ instead of $\boldsymbol{x}^{(2)}$. For $\boldsymbol{x} \in \mathbb{R}^n$ let $\lambda_{\boldsymbol{\beta},1}(\boldsymbol{x}), \dots, \lambda_{\boldsymbol{\beta},n}(\boldsymbol{x})$ denote the singular values of the real $n\times n$ matrix $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})$ in descending order, counted with multiplicity. Note that $\lambda_{\boldsymbol{\beta},i}(\boldsymbol{x})$ are real and non-negative. Also note

$$\lambda_{\boldsymbol{\beta},1}^2(\boldsymbol{x}) \le n\|\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})^T\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})\|_\infty \le n^2\|\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})\|_\infty^2 \le n^4\|\boldsymbol{x}\|_\infty^2.$$

Taking square roots we find the following useful estimates:

$$\lambda_{\boldsymbol{\beta},1}(\boldsymbol{x}) \le n\|\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})\|_\infty \le n^2\|\boldsymbol{x}\|_\infty.$$

Let $i \in \{1, \ldots, n\}$ and write $\boldsymbol{D}^{(\boldsymbol{\beta},i)}(\boldsymbol{x})$ for the vector with $\binom{n}{i}^2$ entries being the $i\times i$ minors of $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})$. Note that the entries are homogeneous polynomials in $\boldsymbol{x}$ of degree i.

Finally write $J_{\boldsymbol{D}^{(\boldsymbol{\beta},i)}}(\boldsymbol{x})$ for the Jacobian matrix of $\boldsymbol{D}^{(\boldsymbol{\beta},i)}(\boldsymbol{x})$. That is, $J_{\boldsymbol{D}^{(\boldsymbol{\beta},i)}}(\boldsymbol{x})$ is the $\binom{n}{i}^2 \times n$ matrix given by

$$(J_{\boldsymbol{D}^{(\boldsymbol{\beta},i)}}(\boldsymbol{x}))_{jk} = \frac{\partial D_j^{(\boldsymbol{\beta},i)}}{\partial x_k}.$$

We begin by showing a generalisation of [23, Lemma 5.1], where we need to account for the fact that $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})$ is not necessarily a symmetric matrix.

Lemma 7.4. *Let $b \in \{1, \ldots, n-1\}$ and $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ be such that $\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}^{(0)}) \neq 0$. Then there exist subspaces $Y_1, Y_2 \subseteq \mathbb{R}^n$ with $\dim Y_1 = \dim Y_2 = n-b$ such that for all $\boldsymbol{Y}_1 \in Y_1$, $\boldsymbol{Y}_2 \in Y_2$ and $\boldsymbol{t} \in \mathbb{R}^n$ we have*

(7.3)
$$\boldsymbol{Y}_1^T \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{t})\boldsymbol{Y}_2 \ll_n \left(\frac{\|J_{\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{t}\|_\infty}{\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}^{(0)})\|_\infty} + \frac{\lambda_{\boldsymbol{\beta},b+1}(\boldsymbol{x}^{(0)}) \cdot \|\boldsymbol{t}\|_\infty}{\lambda_{\boldsymbol{\beta},b}(\boldsymbol{x}^{(0)})} \right) \|\boldsymbol{Y}_1\|_\infty \|\boldsymbol{Y}_2\|_\infty$$

where the implied constant only depends on n but is otherwise independent of $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{t})$. If $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{t})$ is symmetric then we may take $Y_1 = Y_2$.

Proof. Given $\boldsymbol{x} \in \mathbb{R}^n$ define $\boldsymbol{y}_1^{(1)}(\boldsymbol{x}), \ldots, \boldsymbol{y}_1^{(n-b)}(\boldsymbol{x})$ in the following way. The jth entries are given by

$$(y_1^{(i)}(\boldsymbol{x}))_j = \begin{cases} (-1)^{n-b} \det \big((\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})_{k\ell})_{\substack{k=n-b+1,\ldots,n \\ \ell=n-b+1,\ldots,n}} \big) & \text{if } j = i, \\ (-1)^j \det \big((\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})_{k\ell})_{\substack{k=i,n-b+1,\ldots,n;\, k\neq j \\ \ell=n-b+1,\ldots,n}} \big) & \text{if } j > n-b, \\ 0 & \text{otherwise,} \end{cases} \tag{7.4}$$

where $k = i, n-b+1, \ldots, n$; $k \neq j$ means that we let the index k run over the values $i, n-b+1, \ldots, n$ with $k = j$ omitted. Similarly we define $\boldsymbol{y}_2^{(1)}(\boldsymbol{x}), \ldots, \boldsymbol{y}_2^{(n-b)}(\boldsymbol{x})$ by

$$(y_2^{(i)}(\boldsymbol{x}))_j = \begin{cases} (-1)^{n-b} \det \big((\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})_{k\ell})_{\substack{k=n-b+1,\ldots,n \\ \ell=n-b+1,\ldots,n}} \big) & \text{if } j = i, \\ (-1)^j \det \big((\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})_{k\ell})_{\substack{k=n-b+1,\ldots,n \\ \ell=i,n-b+1,\ldots,n;\, \ell\neq j}} \big) & \text{if } j > n-b, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})$ is symmetric then $\boldsymbol{y}_1^{(i)} = \boldsymbol{y}_2^{(i)}$ for all $i = 1, \ldots, n-b$. Using the Laplace expansion of a determinant along columns and rows we thus obtain

$$(\boldsymbol{y}_1^{(i)}(\boldsymbol{x})^T \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}))_j = \begin{cases} (-1)^{n-b} \det \big((\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})_{k\ell})_{\substack{k=i,n-b+1,\ldots,n \\ \ell=j,n-b+1,\ldots,n}} \big) & \text{if } j \leq n-b, \\ 0 & \text{otherwise,} \end{cases}$$

and

(7.5)
$$(\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{y}_2^{(i)}(\boldsymbol{x}))_j = \begin{cases} (-1)^{n-b}\det\left((\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})_{k\ell})_{\substack{k=j,n-b+1,\dots,n\\ \ell=i,n-b+1,\dots,n}}\right) & \text{if } j\le n-b, \\ 0 & \text{otherwise,}\end{cases}$$

respectively. It follows from (7.4)–(7.5) that there exist matrices $L_1^{(i)}$, $L_2^{(i)}$, $M_1^{(i)}$ and $M_2^{(i)}$ for $i=1,\dots,n-b$ with entries only in $\{0,\pm1\}$ such that

$$\boldsymbol{y}_1^{(i)}(\boldsymbol{x}) = L_1^{(i)}\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}), \tag{7.6}$$

$$\boldsymbol{y}_2^{(i)}(\boldsymbol{x}) = L_2^{(i)}\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}), \tag{7.7}$$

$$(\boldsymbol{y}_1^{(i)}(\boldsymbol{x}))^T\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}) = [M_1^{(i)}\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}(\boldsymbol{x})]^T, \tag{7.8}$$

$$\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})\boldsymbol{y}_2^{(i)}(\boldsymbol{x}) = M_2^{(i)}\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}(\boldsymbol{x}). \tag{7.9}$$

Given $\boldsymbol{t}\in\mathbb{R}^n$ we write $\partial_{\boldsymbol{t}}$ for the directional derivative given by $\sum t_i\frac{\partial}{\partial x_i}$. Applying $\partial_{\boldsymbol{t}}$ to both sides of (7.9) we obtain

$$[\partial_{\boldsymbol{t}}\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})]\boldsymbol{y}_2^{(i)}(\boldsymbol{x}) + \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})[\partial_{\boldsymbol{t}}\boldsymbol{y}_2^{(i)}(\boldsymbol{x})] = M_2^{(i)}[\partial_{\boldsymbol{t}}\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}(\boldsymbol{x})]. \tag{7.10}$$

Now note

$$\partial_{\boldsymbol{t}}\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}(\boldsymbol{x}) = J_{\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}}(\boldsymbol{x})\boldsymbol{t} \quad \text{and} \quad \partial_{\boldsymbol{t}}\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}) = \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{t}). \tag{7.11}$$

Substituting (7.11) and (7.7) into (7.10) yields

$$\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{t})\boldsymbol{y}_2^{(i)}(\boldsymbol{x}) = M_2^{(i)}J_{\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}}(\boldsymbol{x})\boldsymbol{t} - \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})L_2^{(i)}\partial_{\boldsymbol{t}}\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}).$$

If we premultiply this by $\boldsymbol{y}_1^{(j)}(\boldsymbol{x})^T$ and use (7.8) then we obtain

$$\begin{aligned}\boldsymbol{y}_1^{(j)}(\boldsymbol{x})^T\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{t})\boldsymbol{y}_2^{(i)}(\boldsymbol{x}) &= \boldsymbol{y}_1^{(j)}(\boldsymbol{x})^T M_2^{(i)}J_{\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}}(\boldsymbol{x})\boldsymbol{t} \\ &\quad - [M_1^{(j)}\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}(\boldsymbol{x})]^T[L_2^{(i)}\partial_{\boldsymbol{t}}\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x})].\end{aligned} \tag{7.12}$$

Lemma 3.2(i) in [23] yields the bounds

$$\frac{\|\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}(\boldsymbol{x})\|_\infty}{\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x})\|_\infty} \ll_n \lambda_{\boldsymbol{\beta},b+1}(\boldsymbol{x}), \tag{7.13}$$

and

$$\frac{\|\partial_{\boldsymbol{t}}\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x})\|_\infty}{\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x})\|_\infty} \ll_n \frac{\|\boldsymbol{t}\|_\infty}{\lambda_{\boldsymbol{\beta},b}(\boldsymbol{x})}. \tag{7.14}$$

Now we specify $\boldsymbol{x}=\boldsymbol{x}^{(0)}$ so by assumption we have $\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}^{(0)})\|_\infty > 0$. Thus define

$$\boldsymbol{Y}_k^{(i)} = \frac{\boldsymbol{y}_k^{(i)}(\boldsymbol{x}^{(0)})}{\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}^{(0)})\|_\infty} \qquad \text{for } i=1,\dots,n-b \text{ and } k=1,2. \tag{7.15}$$

Dividing (7.12) by $1/\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}^{(0)})\|_\infty^2$ and using (7.15) as well as the bounds (7.13) and (7.14) gives

$$|\boldsymbol{Y}_1^{(j)}\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{t})\boldsymbol{Y}_2^{(i)}| \ll_n \frac{\|J_{\boldsymbol{D}^{(\boldsymbol{\beta},b+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{t}\|_\infty}{\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}^{(0)})\|_\infty} + \frac{\lambda_{\boldsymbol{\beta},b+1}(\boldsymbol{x}^{(0)})\|\boldsymbol{t}\|_\infty}{\lambda_{\boldsymbol{\beta},b}(\boldsymbol{x}^{(0)})}.$$

We now claim that we can take the subspaces $Y_k \subseteq \mathbb{R}^n$ to be defined as the span of $\boldsymbol{Y}_k^{(1)},\dots,\boldsymbol{Y}_k^{(n-b)}$ for $k=1,2$ respectively, so that the lemma holds. For this we need to show that (7.3) holds, and also that $\dim Y_1 = \dim Y_2 = n-b$. Therefore it suffices to show the following claim: Given $\boldsymbol{\gamma}\in\mathbb{R}^{n-b}$ if we take $\boldsymbol{Y}_k = \sum \gamma_i Y_k^{(i)}$ then $\|\boldsymbol{\gamma}\|_\infty \ll_n \|\boldsymbol{Y}_k\|_\infty$, for $k=1,2$ respectively.

Assume that the $b\times b$ minor of $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(0)})$ of largest absolute value lies in the bottom right corner of $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(0)})$. In other words, we assume

$$\|\boldsymbol{D}^{(\boldsymbol{\beta},b)}(\boldsymbol{x}^{(0)})\|_\infty = \Big|\det\big((\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(0)})_{k\ell})_{\substack{k=n-b+1,\dots,n\\ \ell=n-b+1,\dots,n}}\big)\Big|. \tag{7.16}$$

After permuting the rows and columns of $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(0)})$ the identity (7.16) will always be true. The vectors $\boldsymbol{Y}_k^{(i)}$ depend on minors of $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(0)})$. Thus we can apply the same permutations to $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(0)})$ that ensure that (7.16) holds to the definition of these vectors. From this we see that we can always reduce the general case to the case where (7.16) holds.

Now for $k=1,2$ we define matrices

$$Q_k = (\boldsymbol{Y}_k^{(1)}|\cdots|\boldsymbol{Y}_k^{(n-b)}|\boldsymbol{e}_{n-b+1}|\cdots|\boldsymbol{e}_n).$$

By the definition of $\boldsymbol{Y}_k^{(i)}$ we see that Q_k must be of the form

$$Q_k = \begin{pmatrix} I_{n-b} & 0 \\ \widetilde{Q}_k & I_b \end{pmatrix}$$

for some matrix $\widetilde{Q}_k$. In particular, we find $\det Q_k = 1$ and so $\|Q_k^{-1}\|_\infty \ll_n 1$. Given $\boldsymbol{Y}_k = \sum \gamma_i Y_k^{(i)}$ we thus find

$$\|\boldsymbol{\gamma}\|_\infty = \|Q_k^{-1}\boldsymbol{Y}_k\|_\infty \ll_n \|\boldsymbol{Y}_k\|_\infty,$$

and so the lemma follows. ■

The next corollary can be deduced from Lemma 7.4 just like [23, Corollary 5.2] was proven. In [23] several other results are used that we have not stated here. The results in question are [23, Lemmas 2.2, 3.1, 3.2, 4.1, and Corollary 2.2]. These hold in our situation upon replacing the word *eigenvalue* in [23] by the word *singular value*. Otherwise the proofs remain unchanged and thus we did not find it necessary to repeat the details here.

COROLLARY 7.5. *Let $B, C \geq 1$ and let $\sigma \in \{0, \ldots, n-1\}$. Then one of the following alternatives is true. Either we have the bound*

$$N_2^{\mathrm{aux}}(\boldsymbol{\beta}, B) \ll_{C,n} B^{n+\sigma}(\log B)^n,$$

or there exist subspaces $X, Y_1, Y_2 \subseteq \mathbb{R}^n$ with $\dim X + \dim Y_1 = \dim X + \dim Y_2 = n + \sigma + 1$, *such that*

$$|\boldsymbol{Y}_1^T \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{X})\boldsymbol{Y}_2| \ll_n C^{-1}\|\boldsymbol{Y}_1\|_\infty \|\boldsymbol{X}\|_\infty \|\boldsymbol{Y}_2\|_\infty$$

for all $\boldsymbol{X} \in X, \boldsymbol{Y}_1 \in Y_1, \boldsymbol{Y}_2 \in Y_2$. If $\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x})$ is symmetric then we may take $Y_1 = Y_2$.

Recall the definition of the quantity

$$s_{\mathbb{R}}^{(2)} := \left\lfloor \frac{\max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}} \dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})}{2} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ denotes the largest integer m such that $m \leq x$. Although we have been assuming $n_1 = n_2$ throughout, the definition of this quantity remains valid if $n_1 \neq n_2$. Note that we have $\mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) \subsetneq \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ for all $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}$. For if not, then the matrix $H_{\boldsymbol{\beta}}(\boldsymbol{y})$ is identically zero for some $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}$ contradicting the fact that $\mathbb{V}(\boldsymbol{F})$ is a complete intersection. In particular this yields $s_{\mathbb{R}}^{(2)} \leq \frac{n_1+n_2}{2} - 1$.

Before we prove the main result of this section we require another small lemma.

LEMMA 7.6. *Let $\boldsymbol{\beta} \in \mathbb{R} \setminus \{0\}$. The system of equations*

$$\boldsymbol{y}^T \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x} = 0, \quad \ell = 1, \ldots, n,$$

and the equation $H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x} = \boldsymbol{0}$ define the same variety in $\mathbb{P}_{\mathbb{C}}^{n-1} \times \mathbb{P}_{\mathbb{C}}^{n-1}$.

Proof. Recall that by definition we have

$$\widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{z}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_1) \\ \vdots \\ \boldsymbol{z}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_n) \end{pmatrix}.$$

For $\ell \in \{1, \ldots, n\}$ we get

$$\boldsymbol{y}^T \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x} = \boldsymbol{y}^T \begin{pmatrix} \boldsymbol{e}_\ell^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_1)\boldsymbol{x} \\ \vdots \\ \boldsymbol{e}_\ell^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_n)\boldsymbol{x} \end{pmatrix} = \sum_{i=1}^n y_i \boldsymbol{e}_\ell^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_i)\boldsymbol{x} = \boldsymbol{e}_\ell^T H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x},$$

where the last line follows since the entries of $H_{\boldsymbol{\beta}}(\boldsymbol{y})$ are linear homogeneous in $\boldsymbol{y}$. The result is now immediate. ∎

Proposition 7.7. *Let* $s_{\mathbb{R}}^{(2)}$ *be defined as above and let* $B \geq 1$. *Then for all* $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}$ *we have*

$$N_2^{\mathrm{aux}}(\boldsymbol{\beta}, B) \ll_n B^{n+s_{\mathbb{R}}^{(2)}} (\log B)^n.$$

Proof. Suppose for a contradiction the result were false. Then for each positive integer N there exists some $\boldsymbol{\beta}_N$ such that

$$N_2^{\mathrm{aux}}(\boldsymbol{\beta}_N, B) \geq N B^{n+s_{\mathbb{R}}^{(2)}} (\log B)^n.$$

From Corollary 7.5 it follows that there are linear subspaces $X^{(N)}, Y_1^{(N)}, Y_2^{(N)} \subset \mathbb{R}^n$ with

$$\dim X^{(N)} + \dim Y_i^{(N)} = n + s_{\mathbb{R}}^{(2)} + 1, \qquad i = 1, 2,$$

such that for all $\boldsymbol{X} \in X^{(N)}$, $\boldsymbol{Y}_i \in Y_i^{(N)}$ we get

$$|\boldsymbol{Y}_1^T \widetilde{H}_{\boldsymbol{\beta}_N}(\boldsymbol{X}) \boldsymbol{Y}_2| \leq N^{-1} \|\boldsymbol{Y}_1\|_\infty \|\boldsymbol{X}\|_\infty \|\boldsymbol{Y}_2\|_\infty.$$

Note that $\widetilde{H}_{\boldsymbol{\beta}_N}(\boldsymbol{\beta})$ is unchanged when $\boldsymbol{\beta}_N$ is multiplied by a constant. Thus we may assume $\|\boldsymbol{\beta}_N\|_\infty = 1$ and consider a converging subsequence of $\boldsymbol{\beta}_{N_r}$ converging to $\boldsymbol{\beta}$, say, as $N \to \infty$. This delivers subspaces $X, Y_1, Y_2 \subset \mathbb{R}^n$ with $\dim X + \dim Y_i = n + s_{\mathbb{R}}^{(2)} + 1$ for $i = 1, 2$ such that

$$\boldsymbol{Y}_1^T \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{X}) \boldsymbol{Y}_2 = 0 \quad \text{for all } \boldsymbol{X} \in X,\ \boldsymbol{Y}_1 \in Y_1,\ \boldsymbol{Y}_2 \in Y_2.$$

There exists some $b \in \{0, \ldots, n - s_{\mathbb{R}}^{(2)} - 1\}$ such that $\dim X = n - b$ and $\dim Y_i = s_{\mathbb{R}}^{(2)} + b + 1$. Now let $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ be a basis for $\mathbb{R}^n$ such that $\boldsymbol{x}^{(b+1)}, \ldots, \boldsymbol{x}^{(n)}$ is a basis for X. Write $[Y_i] \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ for the linear subspace of $\mathbb{P}_{\mathbb{C}}^{n-1}$ associated to Y_i for $i = 1, 2$.

Define the biprojective variety $W \subset [Y_1] \times [Y_2]$ in the variables $(\boldsymbol{y}_1, \boldsymbol{y}_2)$:

$$W = \mathbb{V}(\boldsymbol{y}_1 \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(i)}) \boldsymbol{y}_2)_{i=1,\ldots,b}.$$

Since the non-trivial equations defining W have bidegree $(1, 1)$ we can apply Corollary 3.2 to find that

$$\dim W \geq \dim [Y_1] \times [Y_2] - b = 2 s_{\mathbb{R}}^{(2)} + b. \tag{7.17}$$

Given $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in W$ we have in particular $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in [Y_1] \times [Y_2]$ and so

$$\boldsymbol{y}_1 \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{x}^{(i)}) \boldsymbol{y}_2 = 0 \quad \text{for } i = b + 1, \ldots, n,$$

and hence $\boldsymbol{y}_1 \widetilde{H}_{\boldsymbol{\beta}}(\boldsymbol{z}) \boldsymbol{y}_2 = 0$ for all $\boldsymbol{z} \in \mathbb{R}^n$. From Lemma 7.6 we thus see $H_{\boldsymbol{\beta}}(\boldsymbol{y}_1) \boldsymbol{y}_2 = 0$ for all $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in W$. Hence in particular

$$\dim W \leq \dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y}) \boldsymbol{x}) \leq 2 s_{\mathbb{R}}^{(2)} - 1,$$

where we regard $\mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y}) \boldsymbol{x})$ as a variety in $\mathbb{P}_{\mathbb{C}}^{n-1} \times \mathbb{P}_{\mathbb{C}}^{n-1}$ in the variables $(\boldsymbol{x}, \boldsymbol{y})$. This together with (7.17) implies $b \leq -1$, which is clearly a contradiction. ∎

In the next lemma we show that $s_{\mathbb{R}}^{(2)}$ is small if $\mathbb{V}(\boldsymbol{F})$ defines a smooth complete intersection. For this we no longer assume $n_1 = n_2$.

LEMMA 7.8. *Let $s_{\mathbb{R}}^{(2)}$ be defined as above. If $\mathbb{V}(\boldsymbol{F})$ is a smooth complete intersection in $\mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ then*

$$\frac{n_2 - 1}{2} \leq s_{\mathbb{R}}^{(2)} \leq \frac{n_2 + R}{2}. \tag{7.18}$$

Proof. Let $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}$ be such that

$$s_{\mathbb{R}}^{(2)} = \left\lfloor \frac{\dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})}{2} \right\rfloor + 1.$$

Note that then

$$2s_{\mathbb{R}}^{(2)} - 2 \leq \dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) \leq 2s_{\mathbb{R}}^{(2)} - 1. \tag{7.19}$$

The variety $\mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) \subset \mathbb{P}_{\mathbb{C}}^{n_1-1} \times \mathbb{P}_{\mathbb{C}}^{n_2-1}$ is defined by n_1 bilinear polynomials. Using Corollary 3.2 we thus find that

$$\dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) \geq n_2 - 2$$

so the lower bound in (7.18) follows. We proceed by considering two cases.

CASE 1: $\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} = \emptyset$. Note that this can only happen if $n_2 \geq n_1$. We can thus apply Lemma 3.4 with $V_1 = \mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2}$, $V_2 = \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})$ and $A_i = H_{\boldsymbol{\beta}}(\boldsymbol{e}_i)$ to find that

$$\dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) \leq n_2 - 1 + \dim \mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} = n_2 - 2.$$

From this and (7.19) the upper bound in (7.18) follows for this case.

CASE 2: $\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} \neq \emptyset$. By assumption there exists $\boldsymbol{x} \in \mathbb{C}^{n_1} \setminus \{\mathbf{0}\}$ such that

$$\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x} = 0 \quad \text{for all } \ell = 1, \dots, n_2.$$

We claim that there exists $\boldsymbol{y} \in \mathbb{C}^{n_2} \setminus \{\mathbf{0}\}$ such that $H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x} = \mathbf{0}$. For this define the vectors

$$\boldsymbol{u}_\ell = H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x}, \quad \ell = 1, \dots, n_2.$$

Note that $\boldsymbol{x} \in \langle \boldsymbol{u}_1, \dots, \boldsymbol{u}_{n_2} \rangle^\perp$ so these vectors must be linearly dependent. Thus there exist $y_1, \dots, y_{n_2} \in \mathbb{C}$ not all zero such that

$$H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x} = \sum_{\ell=1}^{n_2} y_\ell H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x} = \mathbf{0},$$

where the first equality followed since the entries of $H_{\boldsymbol{\beta}}(\boldsymbol{y})$ are linear homogeneous in $\boldsymbol{y}$. The claim follows. In particular it follows from this that

$$(\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} \times \mathbb{P}^{n_2-1}) \cap \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) \neq \emptyset.$$

Using Lemma 3.1 and (7.19) we therefore find that

$$\begin{aligned}(7.20)\qquad \dim[(\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} \times \mathbb{P}^{n_2-1}) \cap \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x})]&\\ \geq \dim \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}) - n_2 \geq 2s_{\mathbb{R}}^{(2)} - n_2 - 2.&\end{aligned}$$

Recall $\boldsymbol{\beta} \cdot \boldsymbol{F} = \boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}$ so that

$$\operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) = (\mathbb{V}(\boldsymbol{x}^T H_{\boldsymbol{\beta}}(\boldsymbol{e}_\ell)\boldsymbol{x})_{\ell=1,\dots,n_2} \times \mathbb{P}^{n_2-1}) \cap \mathbb{V}(H_{\boldsymbol{\beta}}(\boldsymbol{y})\boldsymbol{x}).$$

Under our assumptions we can apply Lemma 3.3 to find $\dim \operatorname{Sing} \mathbb{V}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \leq R - 2$. The result follows from this and (7.20). ∎

Proof of Theorem 7.1. Applying Theorem 2.1 it suffices to show

$$N_i^{\mathrm{aux}}(\boldsymbol{\beta}; B) \leq C_0 B^{2n-4\mathscr{C}}, \tag{7.21}$$

for all $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{0\}$ and $i = 1, 2$, where $\mathscr{C} > (2b + u)R$. Let

$$s = \max\{s_{\mathbb{R}}^{(1)}, s_{\mathbb{R}}^{(2)}\},$$

where $s_{\mathbb{R}}^{(1)}$ and $s_{\mathbb{R}}^{(2)}$ are defined as in (1.7) and (1.8), respectively. From (7.2) and Proposition 7.7 for any $\varepsilon > 0$ we get

$$N_i^{\mathrm{aux}}(\boldsymbol{\beta}; B) \ll_\varepsilon B^{n+s+\varepsilon},$$

with the implied constant not depending on $\boldsymbol{\beta}$. Choose $\varepsilon = \frac{n-s-(8b+4u)R}{2}$, which is a positive real number by our assumption (7.1). Taking

$$\mathscr{C} = \frac{n - s - \varepsilon}{4},$$

we see that from the assumption $n - s_{\mathbb{R}}^{(i)} > (8b + 4u)R$ for $i = 1, 2$ we must have $\mathscr{C} > (2b + u)R$ for this choice. Therefore (7.21) holds and the first part of the theorem follows upon applying Theorem 2.1.

For the second part recall we assume $n > (16b + 8u + 1)R$ and that the forms $F_i(\boldsymbol{x}, \boldsymbol{y})$ define a smooth complete intersection in $\mathbb{P}_{\mathbb{C}}^{n-1} \times \mathbb{P}_{\mathbb{C}}^{n-1}$. By Lemma 7.3 in this case we obtain

$$s_{\mathbb{R}}^{(1)} \leq R,$$

and from Lemma 7.8 we get

$$s_{\mathbb{R}}^{(2)} \leq \frac{n + R}{2}.$$

Therefore it is easily seen that $n > (16b + 8u + 1)R$ implies that

$$n - s_{\mathbb{R}}^{(i)} > (8b + 4u)R$$

for $i = 1, 2$, which is what we wanted to show. ∎

7.3. Proof of Theorem 1.2. If $n_1 = n_2$ then the result follows immediately from Proposition 7.1. We have two cases to consider and although their strategies are very similar they are not entirely symmetric. Therefore it is necessary to consider them individually.

CASE 1: $n_1 > n_2$. We consider a new system of equations $\widetilde{F}_i(\boldsymbol{x}, \tilde{\boldsymbol{y}})$ in the variables $\boldsymbol{x} = (x_1, \dots, x_{n_1})$ and $\tilde{\boldsymbol{y}} = (y_1, \dots, y_{n_2}, y_{n_2+1}, \dots, y_{n_1})$ where the forms $\widetilde{F}_i(\boldsymbol{x}, \tilde{\boldsymbol{y}})$ satisfy

$$\widetilde{F}_i(\boldsymbol{x}, \tilde{\boldsymbol{y}}) = F(\boldsymbol{x}, \boldsymbol{y}),$$

where $\boldsymbol{y} = (y_1, \dots, y_{n_2})$. Write $\widetilde{N}(P_1, P_2)$ for the counting function associated to the system $\widetilde{\boldsymbol{F}} = \boldsymbol{0}$ and the boxes $\mathcal{B}_1 \times (\mathcal{B}_2 \times [0,1]^{n_1 - n_2})$. Note, in particular, that if we replace F by $\widetilde{F}$ in (5.13) and (5.12) then the expressions for the singular series and the singular integral remain unchanged. Further denote by $\tilde{s}_{\mathbb{R}}^{(i)}$ the quantities defined in (1.7) and (1.8) but with F replaced by $\widetilde{F}$. Note that we have $\tilde{s}_{\mathbb{R}}^{(1)} = s_{\mathbb{R}}^{(1)}$ and $\tilde{s}_{\mathbb{R}}^{(2)} \leq s_{\mathbb{R}}^{(2)} + \frac{n_1 - n_2}{2}$. Therefore the assumptions (1.9) imply

$$n_1 - \tilde{s}_{\mathbb{R}}^{(i)} > (8b + 4u)R$$

for $i = 1, 2$. Hence we may apply Proposition 7.1 in order to obtain

$$\widetilde{N}(P_1, P_2) = \mathfrak{I}\mathfrak{S} P_1^{n_1 - 2R} P_2^{n_1 - R} + O(P_1^{n_1 - 2R} P_2^{n_1 - R} \min\{P_1, P_2\}^{-\delta}),$$

for some $\delta > 0$. Finally it is easy to see that

$$\begin{aligned} \widetilde{N}(P_1, P_2) &= N(P_1, P_2) \#\{\boldsymbol{t} \in \mathbb{Z}^{n_1 - n_2} \cap [0, P_2]^{n_1 - n_2}\} \\ &= N(P_1, P_2)(P_2^{n_1 - n_2} + O(P_2^{n_1 - n_2 - 1})), \end{aligned}$$

and so (1.10) follows.

CASE 2: $n_2 > n_1$. We deal with this very similarly to the first case; we define a new system of forms $\widetilde{F}_i(\tilde{\boldsymbol{x}}, \boldsymbol{y})$ in the variables $\tilde{\boldsymbol{x}} = (x_1, \dots, x_{n_2})$ and $\boldsymbol{y} = (y_1, \dots, y_{n_2})$ such that

$$\widetilde{F}_i(\boldsymbol{x}, \tilde{\boldsymbol{y}}) = F_i(\boldsymbol{x}, \boldsymbol{y}).$$

As before we define a new counting function $\widetilde{N}(P_1, P_2)$ with respect to the new product of boxes $(\mathcal{B}_1 \times [0,1]^{n_2 - n_1}) \times \mathcal{B}_2$, and we define $\tilde{s}_{\mathbb{R}}^{(i)}$ similarly to the previous case. Note that $\tilde{s}_{\mathbb{R}}^{(1)} = s_{\mathbb{R}}^{(1)} + n_2 - n_1$ and $\tilde{s}_{\mathbb{R}}^{(2)} \leq s_{\mathbb{R}}^{(2)} + \frac{n_2 - n_1}{2}$ so that (1.9) gives

$$n_2 - \tilde{s}_{\mathbb{R}}^{(i)} > (8b + 4u)R,$$

for $i = 1, 2$. Therefore Proposition 7.1 applies and we deduce again that (1.10) holds as desired.

Finally, we turn to the case when $\mathbb{V}(\boldsymbol{F})$ defines a smooth complete intersection. Note first that by Lemma 7.8 we have

$$s_{\mathbb{R}}^{(2)} \leq \frac{n_2 + R}{2},$$

and therefore the condition

$$\frac{n_1 + n_2}{2} - s_{\mathbb{R}}^{(2)} > (8b + 4u)R$$

is satisfied if we assume $n_1 > (16b + 8u + 1)R$. Further, by Lemma 7.3 we have

$$s_{\mathbb{R}}^{(1)} \leq \max\{0, n_1 + R - n_2\},$$

and so we may replace the condition $n_1 - s_{\mathbb{R}}^{(1)} > (8b + 4u)R$ by

$$n_1 - \max\{0, n_1 + R - n_2\} > (8b + 4u)R.$$

If $n_2 \geq n_1 + R$ then this reduces to assuming $n_1 > (8b + 4u + 1)R$, which follows immediately since we have assumed $n_1 > (16b + 8u + 1)R$. If $n_2 \leq n_1 + R$ on the other hand, then this is equivalent to assuming

$$n_2 > (8b + 4u + 1)R.$$

In any case, the assumptions (1.11) imply the assumptions (1.9) as desired.

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