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Nonlinear twists and moments of L-functions

by

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Dedicated to Henryk Iwaniec
with admiration and friendship

Abstract. Let F belong to the extended Selberg class $\mathcal{S}^\sharp$. We show how a suitable hypothesis on the analytic continuation of a certain nonlinear twist of F^2, namely the self-reciprocal twist, implies a sharp bound for the mean-square of $F(1/2+it)$.

1. Introduction. Let $\mathcal{S}$ and $\mathcal{S}^\sharp$ denote the Selberg and the extended Selberg class, respectively, and let $F \in \mathcal{S}^\sharp$ be of degree $d \geq 1$ and conductor q. We shall briefly recall the basic notation and results in Section 2. In our papers [1, 4–7] we studied the analytic properties of a class of nonlinear twists of F. Moreover, in these and other papers, notably in [8, 9], we refined and applied such properties to the study of the structure of the Selberg classes.

A significant role in the above research is played by the *standard twist* $F(s,\alpha)$, a special nonlinear twist of F defined for $\sigma > 1$ by

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n^{1/d}),$$

where $a(n)$ are the Dirichlet coefficients of F, $\alpha > 0$ and $e(x) = e^{2\pi i x}$. The analytic properties of $F(s,\alpha)$ are now rather well known (see Section 2 for some of them), and are crucial in several problems.

In this paper we introduce a more mysterious but equally important nonlinear twist, namely the *self-reciprocal twist* defined for $\sigma > 1$ by

$$F_{\mathrm{self}}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\kappa_F n^{2/d}), \qquad \kappa_F = \frac{1}{2} d q^{-1/d},$$

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and link it with a central problem in analytic number theory. Indeed, although at present the analytic properties of the self-reciprocal twist are essentially unknown, here we present an unexpected consequence of a seemingly mild hypothesis about its analytic continuation to the left of the line $\sigma = 1$. The name of this twist comes from the fact that the general transformation formula for nonlinear twists in [5] links the twists

$$F_\lambda(s,\alpha) = \sum_{n=1}^\infty \frac{a(n)}{n^s} e(-\alpha n^\lambda), \qquad \lambda > 1/d \text{ and } \alpha \neq 0,$$

to a certain combination of translates of so-called reciprocal (or dual) twists $\overline{F_{\lambda^*}}(s^*,\alpha^*)$. Here $\overline{F_{\lambda^*}}(s,\alpha)$ denotes the conjugate of $F_{\lambda^*}(s,\alpha)$ and

$$\lambda^* = \frac{\lambda}{d\lambda - 1}, \qquad \alpha^* = (d\lambda - 1)(q^\lambda \lambda^{d\lambda}\alpha)^{-1/(d\lambda-1)}$$

and, if $\theta_F = 0$,

$$s^* = \frac{s + \frac{d\lambda}{2} - 1}{d\lambda - 1},$$

where θ_F is the internal shift; see Section 2 for notation. Hence, when $\theta_F = 0$, we see that

$$(\lambda^*, \alpha^*, s^*) = (\lambda, \alpha, s) \quad \text{if and only if} \quad \lambda = 2/d \text{ and } \alpha = \kappa_F.$$

Therefore, in this case $\overline{F_{\lambda^*}}(s^*,\alpha^*) = \overline{F_{\text{self}}}(s)$, hence the name.

For powers of the Riemann zeta function, i.e. $F(s) = \zeta(s)^k$ with integer $k \geq 1$, we simply write

$$\zeta_k(s) = \sum_{n=1}^\infty \frac{d_{2k}(n)}{n^s} e(-kn^{1/k})$$

and our main result, Theorem 2 below, gives at once the following bound.

THEOREM 1. *Suppose that $\zeta_k(s)$ has holomorphic continuation to the half-plane $\sigma > \frac{1}{2} + \frac{1}{2k}$, with polynomial growth on every vertical strip inside this half-plane. Then for every $\varepsilon > 0$ we have*

$$\int_{-T}^{T} |\zeta(1/2+it)|^{2k}\, dt \ll T^{1+\varepsilon}.$$

We expect that $\zeta_k(s)$ has meromorphic continuation over $\mathbb{C}$, and has a pole of order k^2+1 at $s = \frac{1}{2} + \frac{1}{2k}$; cf. the Conjecture below.

Returning to the general case, we denote by $a_2(n) = a * a(n)$ the coefficients of F^2. Clearly, the degree and conductor of F^2 are $2d$ and q^2, respectively, hence

$$F^2_{\text{self}}(s) = \sum_{n=1}^\infty \frac{a_2(n)}{n^s} e(-\kappa_0 n^{1/d}), \qquad \kappa_0 := \kappa_{F^2} = dq^{-1/d},$$

is the self-reciprocal twist of F^2. For simplicity, in this paper we only consider L-functions $F \in \mathcal{S}^\sharp$ with integer degree d and $\theta_F = 0$. This is however the most interesting case, since the classical L-functions satisfy both conditions. Recalling that the conjugate function $\overline{F}$ has the same degree and conductor as F and conjugate coefficients, we assume the following hypothesis on the analytic continuation of the self-reciprocal twist.

δ-HYPOTHESIS. *Let $F \in \mathcal{S}^\sharp$ be of integer degree $d \geq 1$ with conductor q and $\theta_F = 0$. Moreover, let $\delta \geq 0$ be fixed. Then the self-reciprocal twists $F^2_{\mathrm{self}}(s)$ and $\overline{F^2}_{\mathrm{self}}(s)$ have holomorphic continuation to the half-plane $\sigma > \frac{1}{2} + \frac{1}{2d} + \delta$ with polynomial growth on every vertical strip contained in this half-plane.*

Interestingly, the point κ_0 is related to the spectrum of F; see Section 2. Moreover, the growth condition can be somewhat relaxed, still leading to the same conclusions in Theorem 2 below.

Turning to the applications of the self-reciprocal twist, for $T > 0$ sufficiently large we write

$$I_F(T) = \int_{-T}^{T} |F(1/2 + it)|^2 \, \mathrm{d}t$$

and prove the following result.

THEOREM 2. *Let $F \in \mathcal{S}^\sharp$ be of integer degree $d \geq 1$ with $\theta_F = 0$, and suppose that the δ-Hypothesis holds true. Then for every $\varepsilon > 0$ we have*

$$I_F(T) \ll T^{1+\delta d+\varepsilon}.$$

Finally, some remarks are in order. First note that if the δ-Hypothesis holds with $\delta = 0$, then we have the optimal bound

$$I_F(T) \ll T^{1+\varepsilon}.$$

Thus, if the 0-Hypothesis holds true with F replaced by F^k for arbitrarily large integers k, then the Lindelöf Hypothesis holds for F. We have the following

CONJECTURE. *Let $F \in \mathcal{S}^\sharp$ be of integer degree $d \geq 1$ with $\theta_F = 0$, and satisfy the Ramanujan Conjecture. Then the self-reciprocal twist $F^2_{\mathrm{self}}(s)$ has meromorphic continuation to $\mathbb{C}$ with poles at most at the points*

$$s_k = \frac{1}{2} + \frac{1}{2d} - \frac{k}{d}, \qquad k \geq 0 \textit{ integer},$$

and polynomial growth on vertical strips. Moreover, the pole at s_0 has order $d^2 + 1$, while the other poles have order either $d^2 + 1$ or 0.

Note the analogy between the conjectural polar structure of the above self-reciprocal twist and the known polar structure of the standard twist reported in Section 2; note also the main difference in the polar orders.

We already pointed out that at present the range of δ for which the δ-Hypothesis holds, and a fortiori the above conjecture, are open problems. However, some partial results can be obtained in this direction, and we shall return to this problem. Trivially, the hypothesis holds with $\delta = \frac{1}{2} - \frac{1}{2d}$, thus giving

$$I_F(T) \ll T^{(d+1)/2+\varepsilon}$$

for every $F \in \mathcal{S}^\sharp$. For $d \geq 2$ this is weaker than the classical bound

$$I_F(T) \ll T^{\max(1,d/2)+\varepsilon},$$

following from the approximate functional equation coupled with the mean-value theorem for Dirichlet polynomials. Nevertheless, we believe that our approach is interesting as it opens up a new attack to the moment problem for L-functions. Moreover, our method shows that a finer heuristic conjecture concerning the polar structure of the self-reciprocal twist could be used to derive a precise asymptotics for $I_F(T)$. This would allow a comparison with other conjectures on the behaviour of $I_F(T)$ already appearing in the literature.

2. Notation. Throughout the paper we write $s = \sigma + it$, and $\overline{f}(s)$ for $\overline{f(\overline{s})}$. The extended Selberg class $\mathcal{S}^\sharp$ consists of non-identically-vanishing Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

absolutely convergent for $\sigma > 1$, such that $(s-1)^m F(s)$ is entire of finite order for some integer $m \geq 0$, and satisfying a functional equation of type

$$F(s)\gamma(s) = \omega\overline{\gamma}(1-s)\overline{F}(1-s),$$

where $|\omega| = 1$ and the γ-factor

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

has $Q > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re(\mu_j) \geq 0$. The Selberg class $\mathcal{S}$ is, roughly speaking, the subclass of $\mathcal{S}^\sharp$ of the functions with a general Euler product and satisfying the Ramanujan conjecture $a(n) \ll n^\varepsilon$. We refer to our survey paper [2] for further definitions, examples and the basic theory of the Selberg class.

The degree d, conductor q, root number ω_F and ξ-invariant ξ_F of $F \in \mathcal{S}^\sharp$ are defined by

$$d = 2\sum_{j=1}^r \lambda_j, \qquad q = (2\pi)^d Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j},$$
$$\omega_F = \omega \prod_{j=1}^r \lambda_j^{-2i\Im(\mu_j)}, \quad \xi_F = 2\sum_{j=1}^r (\mu_j - 1/2) := \eta_F + id\theta_F$$

with $\eta_F, \theta_F \in \mathbb{R}$; here θ_F is the internal shift of F.

We conclude this section by reporting some results on the meromorphic structure of the standard twist $F(s,\alpha)$; see [6]. The spectrum of $F \in \mathcal{S}^\sharp$ with $\theta_F = 0$ is defined as

$$\mathrm{Spec}(F) = \{\alpha > 0 : a(n_\alpha) \neq 0\}, \quad n_\alpha = qd^{-d}\alpha^d, \quad a(n_\alpha) = 0 \text{ if } n_\alpha \notin \mathbb{N}.$$

Then $F(s,\alpha)$ is entire if $\alpha \notin \mathrm{Spec}(F)$, while it is meromorphic over $\mathbb{C}$ if $\alpha \in \mathrm{Spec}(F)$. In the latter case, $F(s,\alpha)$ has at most simple poles at the points

$$s_k = \frac{1}{2} + \frac{1}{2d} - \frac{k}{d}, \qquad k = 0, 1, \dots,$$

with $\mathrm{res}_{s=s_0} F(s,\alpha) \neq 0$. Moreover, in all cases $F(s,\alpha)$ has polynomial growth on vertical strips.

3. Proof of Theorem 2. For $T > 0$ sufficiently large we also write

$$J_F(T) = \int_{-\infty}^{\infty} |F(1/2+it)|^2 e^{-(t/T)^2}\,\mathrm{d}t.$$

Clearly

$$I_F(T) \le eJ_F(T).$$

Moreover, since by convexity $F(1/2+it) \ll |t|^\xi$ for every $\xi > d/2$ as $|t| \to \infty$, in the opposite direction we have

$$J_F(T) \le I_F(T\sqrt{c\log T}) + O(1)$$

for every $c > 1 + d$. By partial integration we also obtain

$$J_F(T) = \frac{1}{T}\int_0^\infty I_F(t)\varphi(t/T)\,\mathrm{d}t,$$

where $\varphi(u) = 2ue^{-u^2}$. From the above relations it is easy to deduce that for any $A > 0$,

$$I_F(T) \ll T^A \iff J_F(T) \ll T^A,$$
$$I_F(T) \ll T\log^A T \iff J_F(T) \ll T\log^A T.$$

As a consequence, estimating $I_F(T)$ and $J_F(T)$ are essentially equivalent tasks; we shall deal with $J_F(T)$.

3.1. Set-up. In what follows, the implicit constants in the O- and $\ll$-symbols may always depend on F. The symbol c, with or without subscript, denotes complex constants depending on F, whose values will not necessarily be the same at each occurrence. Analogously, the symbol r, with or without suffix, denotes real constants with the same properties as c. Such constants c and r can be explicitly computed, but in this paper their value is not relevant.

Recalling the definition of $J_F(T)$ and $\overline{F}$ we have

$$J_F(T) = \frac{1}{i} \int_{(1/2)} F(s)\overline{F}(1-s)e^{(\frac{s-1/2}{T})^2}\,\mathrm{d}s.$$

Then we shift the line of integration to $\sigma = 3/2$, taking into account the residue of the integrand at the possible pole of F at $s=1$. This is allowed by the polynomial growth of F and the exponential decay of $e^{((s-1/2)/T)^2}$. Estimating the residue we obtain

$$J_F(T) = \frac{1}{i} \int_{(3/2)} F(s)\overline{F}(1-s)e^{(\frac{s-1/2}{T})^2}\,\mathrm{d}s + O(1). \tag{3.1}$$

Recalling the functional equation of F, from (3.1) we deduce that

$$J_F(T) = \frac{1}{i\omega} \int_{(3/2)} F(s)^2 \frac{\gamma(s)}{\overline{\gamma}(1-s)} e^{(\frac{s-1/2}{T})^2}\,\mathrm{d}s + O(1).$$

Moreover, on the line $\sigma = 3/2$ we replace $F(s)^2$ by its Dirichlet series and switch summation and integration, thus getting

$$J_F(T) = \frac{1}{\omega}\sum_{n=1}^{\infty} \frac{a_2(n)}{n^{3/2}} \int_{-\infty}^{\infty} \frac{\gamma(3/2+it)}{\overline{\gamma}(-1/2-it)} n^{-it} e^{(\frac{1+it}{T})^2}\,\mathrm{d}t + O(1). \tag{3.2}$$

Recalling our assumption that $\theta_F = 0$, from the Stirling expansion in [3, equation (2.8)] we have, for fixed σ and large $|t|$,

$$\log\gamma(s) = As\log s + Bs + C\log s + \sum_{\nu=0}^{N} \frac{c_\nu}{s^\nu} + O(|s|^{-N-1}), \tag{3.3}$$

where $N \geq 0$ is a given integer and

$$A = d/2, \qquad B = (\log q - d\log(2\pi e))/2, \qquad C = \xi_F/2 \in \mathbb{R}, \qquad c_\nu \in \mathbb{C}. \tag{3.4}$$

Observing that

$$\overline{\gamma}(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \overline{\mu}_j),$$

we see that the values of A, B, C computed from the data of $\gamma(s)$ and $\overline{\gamma}(s)$ coincide. Hence from (3.3) we deduce that

$$\begin{aligned}(3.5)\qquad & \log\frac{\gamma(3/2+it)}{\overline{\gamma}(-1/2-it)} \\ &= A(3/2+it)\log(3/2+it)+B(3/2+it)+C\log(3/2+it) \\ &\quad + A(1/2+it)\log(-1/2-it)+B(1/2+it)-C\log(-1/2-it)+\varSigma_0(t),\end{aligned}$$

where $\varSigma_0(t)$ arises from the expansion of the terms in the sum over ν in (3.3) with $s = 3/2+it$ and $s = -1/2-it$. Recalling that the value of c_ν is not necessarily the same at each occurrence, $\varSigma_0(t)$ has the form

$$\varSigma_0(t) = \sum_{\nu=0}^{N}\frac{c_\nu}{t^\nu} + O(|t|^{-N-1}). \tag{3.6}$$

By further expansions we get

$$\begin{aligned}\log(3/2+it) &= \log|t| + i\frac{\pi}{2}\mathrm{sgn}(t) + \varSigma_1(t),\\ \log(-1/2-it) &= \log|t| - i\frac{\pi}{2}\mathrm{sgn}(t) + \varSigma_1(t).\end{aligned} \tag{3.7}$$

Here the form of $\varSigma_1(t)$ is similar to (3.6), but the summation starts with $\nu = 1$. Thus (3.4)–(3.7) give

$$\log\frac{\gamma(3/2+it)}{\overline{\gamma}(-1/2-it)} = d\log|t| + i(dt\log|t| + 2Bt + D\,\mathrm{sgn}(t)) + \varSigma_0(t),$$

where

$$D = \pi d/4 + \pi C \in \mathbb{R}. \tag{3.8}$$

Hence for large $|t|$, say $|t| > t_0$, we have

$$\frac{\gamma(3/2+it)}{\overline{\gamma}(-1/2-it)} = |t|^d e^{i\theta(t)}\varSigma_0(t), \tag{3.9}$$

where $\varSigma_0(t)$ is of the form (3.6) and

$$\theta(t) = dt\log|t| + 2Bt + D\,\mathrm{sgn}(t).$$

Recalling that $d \in \mathbb{N}$, choosing $N = d$ in (3.3), estimating trivially the part with $|t| \le t_0$ of the integral over $(-\infty, \infty)$ and writing

$$f_n(t) = \theta(t) - t\log n + 2t/T^2,$$

from (3.2), (3.6) and (3.9) we obtain

(3.10)
$$\begin{aligned} J_F(T) &= \frac{e^{1/T^2}}{\omega}\sum_{n=1}^\infty \frac{a_2(n)}{n^{3/2}}\int_{-\infty}^{\infty} e^{if_n(t)}|t|^d e^{-(t/T)^2}\Sigma_0(t)\,\mathrm{d}t + O(1) \\ &= e^{1/T^2}\sum_{\nu=0}^d c_\nu \sum_{n=1}^\infty \frac{a_2(n)}{n^{3/2}}\int_{-\infty}^{\infty} e^{if_n(t)}|t|^d t^{-\nu} e^{-(t/T)^2}\,\mathrm{d}t + O(\log T). \end{aligned}$$

Denoting the last integrand by $g_\nu(t)$, and observing that $f_n(-t) = -f_n(t)$ for $t\neq 0$, we have

$$\nu \text{ even} \implies g_\nu(-t) = \overline{g_\nu(t)} \quad\text{and}\quad \nu \text{ odd} \implies g_\nu(-t) = -\overline{g_\nu(t)}.$$

Hence, writing

$$J_\nu(n,T) = \int_0^\infty e^{iF_n(t)} t^{d-\nu} e^{-(t/T)^2}\,\mathrm{d}t \tag{3.11}$$

with

$$F_n(t) = dt\log t + (2B - \log n + 2/T^2)t \tag{3.12}$$

and B as in (3.4), from (3.10) we finally deduce that

$$\begin{aligned} J_F(T) &= e^{1/T^2}\sum_{\substack{\nu=0\\ \nu \text{ even}}}^d c_\nu \sum_{n=1}^\infty \frac{a_2(n)}{n^{3/2}}\Re(e^{iD}J_\nu(n,T)) \\ &\quad + e^{1/T^2}\sum_{\substack{\nu=1\\ \nu \text{ odd}}}^d c_\nu \sum_{n=1}^\infty \frac{a_2(n)}{n^{3/2}}\Im(e^{iD}J_\nu(n,T)) + O(\log T) \end{aligned} \tag{3.13}$$

with D as in (3.8).

3.2. Saddle point: preliminary reductions. Next we compute the saddle point of the exponential integrals $J_\nu(n,T)$ in (3.13) and then we use a suitable saddle point technique to extract their main contribution.

Since $F_n'(t) = d\log t - d\log(2\pi(n/q)^{1/d}) + 2/T^2$, see (3.12), we have

$$F_n'(t) = 0 \quad\text{if and only if}\quad t = t_n := 2\pi(n/q)^{1/d}e^{-2/(dT^2)}. \tag{3.14}$$

Accordingly, we shift the integration over $(0,\infty)$ in $J_\nu(n,T)$ in the following way. For a fixed $0<\theta<\pi/4$ to be chosen later on, we consider the three lines in the complex z-plane

$$\ell_A:\ z = \rho e^{-i\theta}, \quad \ell_B:\ z=\rho e^{i\theta}, \quad \ell_n:\ z = t_n + \rho e^{i\pi/4} \quad \text{with } \rho\in\mathbb{R},$$

and define the two points

$$z_A = \ell_A\cap\ell_n \quad\text{and}\quad z_B = \ell_B\cap\ell_n$$

lying in the half-plane $\Re(z) > 0$. The functions $F_n(z)$ in (3.12) and $z^{d-\nu}$, $e^{-(z/T)^2}$ in (3.11) are holomorphic for $\Re(z) > 0$. Moreover, writing $z = \rho e^{i\phi}$, we have

$$e^{iF_n(z)} z^{d-\nu} e^{-(z/T)^2} \\ \ll e^{-\rho(d\log\rho\sin\phi + d\phi\cos\phi + (2B - \log n + 2/T^2)\sin\phi)} \rho^{d-\nu} e^{-(\rho/T)^2\cos(2\phi)},$$

thus the integrand in (3.11) has exponential decay as $\rho \to \infty$ uniformly for $0 \le \phi \le \theta$, i.e. in the sector between the half-lines $z = \rho$ and ℓ_B with $\rho > 0$, thanks to $0 < \theta < \pi/4$. Hence by Cauchy's theorem we can shift the path of integration, thus getting

$$J_\nu(n,T) = \Big(\int_0^{z_A} + \int_{z_A}^{z_B} + \int_{z_B}^{\infty e^{i\theta}} \Big) e^{iF_n(z)} z^{d-\nu} e^{-(z/T)^2}\, \mathrm{d}z \\ = J_\nu^{(1)}(n,T) + J_\nu^{(2)}(n,T) + J_\nu^{(3)}(n,T), \tag{3.15}$$

say, where the paths of integration are along ℓ_A, ℓ_n and ℓ_B, respectively.

Before treating the integrals in (3.15) we compute z_A and z_B. Let h, h' be the distances of z_A, z_B from the real axis, respectively. Then

$$\tan\theta = \frac{h}{t_n - h}, \qquad h = \frac{\tan\theta}{1+\tan\theta} t_n,$$
$$|z_A| = \frac{h}{\sin\theta} = \frac{\tan\theta}{1+\tan\theta}\frac{1}{\sin\theta} t_n = \frac{t_n}{\sin\theta + \cos\theta}$$

and hence

$$z_A = \frac{e^{-i\theta}}{\sin\theta + \cos\theta} t_n. \tag{3.16}$$

Arguing in a similar way we obtain

$$z_B = \frac{e^{i\theta}}{\cos\theta - \sin\theta} t_n. \tag{3.17}$$

In view of the definition of ℓ_A and recalling that $0 \le \nu \le d$, we have

$$J_\nu^{(1)}(n,T) \ll \int_0^{|z_A|} e^{-\Im F_n(\rho e^{-i\theta})} (1+\rho^d)\, \mathrm{d}\rho.$$

Moreover, from (3.4), (3.12) and (3.14) we get

$$\Im F_n(\rho e^{-i\theta}) = -d\rho\log\rho\sin\theta - d\rho\theta\cos\theta - \rho(2B - \log n + 2/T^2)\sin\theta \\ = -d\rho\sin\theta\log\Big(\frac{\rho(q/n)^{1/d} e^{2/(dT^2)} e^{\theta\cot\theta}}{2\pi e}\Big) \\ = d\rho\sin\theta\log\Big(\frac{e t_n}{\rho e^{\theta\cot\theta}}\Big). \tag{3.18}$$

But, thanks to (3.16), for $0 \le \rho \le |z_A|$ we obtain
$$\frac{et_n}{\rho e^{\theta\cot\theta}} \ge e\frac{\sin\theta+\cos\theta}{e^{\theta\cot\theta}} = 1+\theta+O(\theta^2)$$
as $\theta \to 0^+$, therefore
$$\Im F_n(\rho e^{-i\theta}) \ge d\rho\theta^2/2$$
for $0 \le \rho \le |z_A|$ and $0 < \theta < \theta_0$, where $\theta_0 > 0$ is sufficiently small. As a consequence we have
$$J_\nu^{(1)}(n,T) \ll \int_0^{|z_A|} e^{-d\rho\theta^2/4}(1+\rho^d)\,\mathrm{d}\rho \ll 1 \tag{3.19}$$
uniformly in n,ν and T, with any fixed $0<\theta<\theta_0$ and a sufficiently small $\theta_0>0$.

The treatment of $J_\nu^{(3)}(n,T)$ is similar. Thanks to (3.14), (3.17) and (3.18), for $\rho \ge |z_B|$ we have
$$\Im F_n(\rho e^{i\theta}) = d\rho\sin\theta\log\left(\frac{\rho e^{\theta\cot\theta}}{et_n}\right) \ge d\rho\sin\theta\log\left(\frac{e^{\theta\cot\theta}}{e(\cos\theta-\sin\theta)}\right),$$
and as $\theta \to 0^+$,
$$\frac{e^{\theta\cot\theta}}{e(\cos\theta-\sin\theta)} = 1+\theta+O(\theta^2).$$
Hence, as before, we deduce that
$$J_\nu^{(3)}(n,T) \ll \int_{|z_B|}^{\infty} e^{-d\rho\theta^2/4}\rho^d\,\mathrm{d}\rho \ll 1,$$
thus from (3.15) and (3.19) also that
$$J_\nu(n,T) = J_\nu^{(2)}(n,T) + O(1), \tag{3.20}$$
with the same uniformity and conditions stated after (3.19).

We conclude this subsection by rewriting $J_\nu^{(2)}(n,T)$ in a more convenient form. To this end we write $z \in [z_A, z_B]$ as
$$z = t_n(1+ue^{i\pi/4}) \quad \text{with } -A(\theta) \le u \le B(\theta) \tag{3.21}$$
and compute $A(\theta)$ and $B(\theta)$. From (3.16) we see that $A(\theta)$ satisfies
$$\frac{\cos\theta - i\sin\theta}{\sin\theta+\cos\theta} = 1 - A(\theta)e^{i\pi/4},$$
hence
$$A(\theta) = (1+i)\frac{\sin\theta}{\sin\theta+\cos\theta}e^{-i\pi/4} = \sqrt{2}\,\frac{\sin\theta}{\sin\theta+\cos\theta} = \frac{\sqrt{2}}{1+\cot\theta}. \tag{3.22}$$
Similarly, from (3.17) we get
$$B(\theta) = (1+i)\frac{\sin\theta}{\cos\theta-\sin\theta}e^{-i\pi/4} = \frac{\sqrt{2}}{\cot\theta-1}. \tag{3.23}$$

Therefore, in view of (3.15) and (3.21), $J_\nu^{(2)}(n,T)$ can be rewritten as

$$(3.24)\qquad J_\nu^{(2)}(n,T)$$
$$= t_n e^{i\pi/4}\int_{-A(\theta)}^{B(\theta)} e^{iF_n(t_n(1+ue^{i\pi/4}))} t_n^{d-\nu}(1+ue^{i\pi/4})^{d-\nu} e^{-(t_n/T)^2(1+ue^{i\pi/4})^2}\,\mathrm{d}u$$
$$= t_n^{d+1-\nu} e^{i\pi/4} e^{iF_n(t_n)}\int_{-A(\theta)}^{B(\theta)} e^{i(F_n(t_n(1+ue^{i\pi/4}))-F_n(t_n))} \times (1+ue^{i\pi/4})^{d-\nu} e^{-(t_n/T)^2(1+ue^{i\pi/4})^2}\,\mathrm{d}u,$$

where $A(\theta)$ and $B(\theta)$ are given by (3.22) and (3.23), respectively. Moreover, θ is fixed with $0<\theta<\theta_0$ and $\theta_0>0$ sufficiently small. Therefore, in what follows we assume that $|u|$ is sufficiently small as well, since $A(\theta), B(\theta)\to 0$ as $\theta\to 0^+$.

3.3. Saddle point: further reductions. From (3.4), (3.12) and (3.14) we see that

$$(3.25)\qquad F_n(z) = dz\log\Big(\frac{z}{et_n}\Big),$$

hence

$$F_n\big(t_n(1+ue^{i\pi/4})\big) - F_n(t_n) = dt_n\big((1+ue^{i\pi/4})\log(1+ue^{i\pi/4}) - ue^{i\pi/4}\big).$$

But for $|w|\le 1/2$ we have

$$(1+w)\log(1+w)-w = (1+w)\sum_{m=1}^\infty \frac{(-1)^{m+1}}{m}w^m - w = \sum_{m=2}^\infty \frac{(-1)^m}{m(m-1)}w^m,$$

therefore for any integer $M\ge 2$ we deduce, with the obvious meaning of the real constants r_m, that

$$(3.26)\qquad F_n(t_n(1+ue^{i\pi/4}))-F_n(t_n) = dt_n\sum_{m=2}^M r_m e^{im\pi/4}u^m + O\Big(\frac{t_n|u|^{M+1}}{M}\Big).$$

As a consequence, from (3.24) we obtain

$$(3.27)\qquad J_\nu^{(2)}(n,T) = t_n^{d+1-\nu} e^{iF_n(t_n)} e^{-(t_n/T)^2}\sum_{\mu=0}^{d-\nu}\binom{d-\nu}{\mu} e^{i(\mu+1)\pi/4}$$
$$\times\int_{-A(\theta)}^{B(\theta)} e^{-\frac12 dt_nu^2} e^{idt_n\sum_{m=3}^M r_m e^{im\pi/4}u^m} e^{O\left(\frac{t_n|u|^{M+1}}{M}\right)} u^\mu e^{-(t_n/T)^2(2e^{i\pi/4}u+iu^2)}\,\mathrm{d}u$$
$$= t_n^{d+1-\nu} e^{iF_n(t_n)} e^{-(t_n/T)^2}\sum_{\mu=0}^{d-\nu}\binom{d-\nu}{\mu} e^{i(\mu+1)\pi/4}\widetilde{\mathfrak{J}}_\mu(n,T),$$

say.

By the substitution

$$\sqrt{dt_n/2}\,u = \xi \tag{3.28}$$

we have

(3.29)
$$\begin{aligned}\widetilde{\mathfrak{J}}_\mu(n,T) &= \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} \int\limits_{-\sqrt{dt_n/2}\,A(\theta)}^{\sqrt{dt_n/2}\,B(\theta)} e^{-\xi^2}\xi^\mu e^{idt_n\sum_{m=3}^{M} r_m(\frac{2}{dt_n})^{m/2}e^{im\pi/4}\xi^m} \\ &\qquad \times e^{O(\frac{t_n}{M}(2\xi^2/(dt_n))^{(M+1)/2})} e^{-(t_n/T)^2(2e^{i\pi/4}\sqrt{\frac{2}{dt_n}}\xi + i\frac{2}{dt_n}\xi^2)}\,\mathrm{d}\xi \\ &= \left(\frac{2}{dt_n}\right)^{(\mu+1)/2}\Big(\int\limits_{-\sqrt{dt_n/2}\,A(\theta)}^{\sqrt{dt_n/2}\,B(\theta)} e^{-\xi^2}\xi^\mu e^{\Sigma(\xi,n)}e^{-g(\xi,n,T)}\,\mathrm{d}\xi + R_\mu(n,T)\Big) \\ &= \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} (\mathfrak{J}_\mu(n,T) + R_\mu(n,T)).\end{aligned}$$

Here $R_\mu(n,T)$ is the error term arising from the replacement of $e^{O\left(\frac{t_n}{M}(2\xi^2/(dt_n))^{(M+1)/2}\right)}$ by 1 inside the first integral, and

$$\Sigma(\xi,n) = idt_n\sum_{m=3}^{M} r_m\left(\frac{2}{dt_n}\right)^{m/2} e^{im\pi/4}\xi^m, \tag{3.30}$$

$$g(\xi,n,T) = (t_n/T)^2\left(2e^{i\pi/4}\sqrt{\frac{2}{dt_n}}\,\xi + i\frac{2}{dt_n}\xi^2\right), \tag{3.31}$$

$$\mathfrak{J}_\mu(n,T) = \int\limits_{-\sqrt{dt_n/2}\,A(\theta)}^{\sqrt{dt_n/2}\,B(\theta)} e^{-\xi^2+\Sigma(\xi,n)-g(\xi,n,T)}\xi^\mu\,\mathrm{d}\xi. \tag{3.32}$$

Next we estimate the contribution to $J_\nu^{(2)}(n,T)$ of the error terms $R_\mu(n,T)$ in (3.29). Recalling (3.28), the value of r_m in (3.26) and the fact that $|u|$ is assumed to be sufficiently small (see after (3.24)), from (3.30) and (3.31) we have that

$$\Sigma(\xi,n) \ll \frac{|\xi|^3}{\sqrt{t_n}} \le \xi^2/2 \quad \text{and} \quad |g(\xi,n,T)| \le \frac{(t_n/T)^2}{2} \tag{3.33}$$

for $-\sqrt{dt_n/2}\,A(\theta) \le \xi \le \sqrt{dt_n/2}\,B(\theta)$. Then we choose $M = M(n)$ so large that

$$\frac{t_n}{M2^M} < c\,t_n^{-d-1/2}, \tag{3.34}$$

with a sufficiently small constant $c > 0$. Hence, recalling that $A(\theta), B(\theta)$ are sufficiently small and $\mu \geq 0$, from (3.29), (3.33) and (3.34) we obtain

$$\left(\frac{2}{dt_n}\right)^{(\mu+1)/2} R_\mu(n,T) \ll \frac{1}{t_n^{d+1}} e^{(t_n/T)^2/2} \int_{-\infty}^{\infty} e^{-\xi^2/2}\,\mathrm{d}\xi \ll \frac{1}{t_n^{d+1}} e^{(t_n/T)^2/2}. \tag{3.35}$$

Since $F_n(t_n) \in \mathbb{R}$, by (3.27), (3.29) and (3.35) the contribution to $J_\nu^{(2)}(n,T)$ of the error terms $R_\mu(n,T)$ is

$$O(1) \tag{3.36}$$

uniformly in ν, n and T, provided $\theta_0 > 0$ is sufficiently small and M satisfies (3.34). Therefore, from (3.20), (3.27), (3.29) and (3.36) we deduce that

$$\begin{aligned} J_\nu(n,T) &= e^{iF_n(t_n)} t_n^{d+1-\nu} e^{-(t_n/T)^2} \sum_{\mu=0}^{d-\nu} \binom{d-\nu}{\mu} e^{i(\mu+1)\pi/4} \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} \mathfrak{J}_\mu(n,T) \\ &\quad + O(1) \end{aligned} \tag{3.37}$$

with the same uniformity and conditions after (3.36), where $\mathfrak{J}_\mu(n,T)$ is defined by (3.32).

3.4. Saddle point: computing the main terms. Now we study the integrals $\mathfrak{J}_\mu(n,T)$. We may assume that n is sufficiently large, say $n \geq n_0$, since for $n < n_0$ we have $J_\nu(n,T) = O(1)$ and their contribution to $J_F(T)$ amounts to $O(1)$. We first show that the range of integration in $\mathfrak{J}_\mu(n,T)$ can be replaced, up to a negligible quantity, by $|\xi| \leq c(\theta)\log t_n$, where $c(\theta) > 0$ is such that

$$\min(\sqrt{kt_n/2}\,A(\theta), \sqrt{kt_n/2}\,B(\theta)) > c(\theta)\log t_n$$

for every $n \geq n_0$. To this end we write

$$I_\mu(n,T) = \int_{-c(\theta)\log t_n}^{c(\theta)\log t_n} e^{-\xi^2+\Sigma(\xi,n)-g(\xi,n,T)} \xi^\mu\,\mathrm{d}\xi, \tag{3.38}$$

and arguing as in (3.35), from (3.33) we find that

$$\mathfrak{J}_\mu(n,T) - I_\mu(n,T) \ll \Big(\int_{-\infty}^{-c(\theta)\log t_n} + \int_{c(\theta)\log t_n}^{\infty}\Big) e^{-\xi^2/2+(t_n/T)^2/2} |\xi|^\mu\,\mathrm{d}\xi \ll t_n^{-A} e^{(t_n/T)^2/2} \tag{3.39}$$

for every $A > 0$. Therefore, arguing as for (3.36), from (3.37) and (3.39) we get

$$J_\nu(n,T) = e^{iF_n(t_n)} e^{-(t_n/T)^2} \sum_{\mu=0}^{d-\nu} \alpha_{\nu,\mu}\, t_n^{d+1-\nu-(\mu+1)/2} I_\mu(n,T) + O(1) \tag{3.40}$$

with the same uniformity and conditions after (3.36), where $I_\mu(n,T)$ is defined by (3.38) and

$$\alpha_{\nu,\mu} = \binom{d-\nu}{\mu} \left(\frac{2}{d}\right)^{(\mu+1)/2} e^{i(\mu+1)\pi/4}. \tag{3.41}$$

Next we estimate the contribution of the integrals $I_\mu(n,T)$ to $J_\nu(n,T)$ for the values $n \geq n_0$ such that $t_n > T \log t_n$. In this case, again thanks to (3.33), we have

$$I_\mu(n,T) \ll \int_{-c(\theta)\log t_n}^{c(\theta)\log t_n} e^{(t_n/T)^2/2} |\xi|^\mu \, d\xi \ll e^{(t_n/T)^2/2} \log^{\mu+1} t_n.$$

Hence such a contribution is

$$O(1), \tag{3.42}$$

once more with the same uniformity and conditions after (3.36).

Finally, we compute the contribution of $I_\mu(n,T)$ to $J_\nu(n,T)$ for $n \geq n_0$ with $t_n \leq T \log t_n$. From (3.30), (3.31) and the first estimate in (3.33) we see that for $|\xi| \leq c(\theta) \log t_n$,

$$|\Sigma(\xi,n)|, |g(\xi,n,T)| \leq \frac{\log^3 t_n}{\sqrt{t_n}},$$

provided $c(\theta)$ is sufficiently small. Hence, given an arbitrarily large constant $A > 0$, there exist integers $Q = Q(A) > 0$ and $0 \leq p,k,\ell \leq Q$, and coefficients $\beta_{p,k,\ell} \in \mathbb{C}$, such that for $|\xi| \leq c(\theta) \log t_n$ we have

$$e^{\Sigma(\xi,n) - g(\xi,n,T)} = \sum_{0\leq p\leq Q} \sum_{0\leq k\leq Q} \sum_{0\leq \ell\leq Q} \beta_{p,k,\ell} \frac{\xi^p}{t_n^{k/2}} \left(\frac{t_n}{T}\right)^{2\ell} + O(t_n^{-A-1}).$$

Note that $\beta_{0,0,0} = 1$, while the other coefficients $\beta_{p,k,\ell}$ are computable from the expressions in (3.30) and (3.31). Note also that, although M in (3.30) now depends on n, the integer Q is independent of n. Indeed, the contribution of the terms in (3.30) with m sufficiently large in terms of A is directly absorbed into the error term $O(t_n^{-A-1})$. As a consequence,

$$I_\mu(n,T) \tag{3.43}$$
$$= \sum_{0\le p\le Q}\sum_{0\le k\le Q}\sum_{0\le \ell\le Q} \beta_{p,k,\ell}\frac{1}{t_n^{k/2}}\left(\frac{t_n}{T}\right)^{2\ell}\int\limits_{-c(\theta)\log t_n}^{c(\theta)\log t_n} e^{-\xi^2}\xi^{\mu+p}\,\mathrm{d}\xi + O(t_n^{-A-1}).$$

But for an arbitrarily large constant $B>0$ we have

$$\int\limits_{-c(\theta)\log t_n}^{c(\theta)\log t_n} e^{-\xi^2}\xi^{\mu+p}\,\mathrm{d}\xi = \int\limits_{-\infty}^{\infty} e^{-\xi^2}\xi^{\mu+p}\,\mathrm{d}\xi + O(t_n^{-B})$$
$$= \frac{1}{2}\Gamma\left(\frac{\mu+p+1}{2}\right)(1+(-1)^{\mu+p}) + O(t_n^{-B}).$$

Hence, by a suitable choice of $B=B(A)$ and after summation over p, (3.43) becomes

$$I_\mu(n,T) = \sum_{0\le k\le Q}\sum_{0\le \ell\le Q}\gamma_{k,\ell,\mu}\frac{1}{t_n^{k/2}}\left(\frac{t_n}{T}\right)^{2\ell} + O(t_n^{-A}) \tag{3.44}$$

for $n\ge n_0$ with $t_n\le T\log t_n$, where

$$\gamma_{k,\ell,\mu} = \frac{1}{2}\sum_{0\le p\le Q}\beta_{p,k,\ell}\Gamma\left(\frac{\mu+p+1}{2}\right)(1+(-1)^{\mu+p}). \tag{3.45}$$

Now we fix a sufficiently small $\theta>0$ such that all the above estimates hold, and a sufficiently large A. Hence from the remark at the beginning of this subsection and from equations (3.40), (3.42), (3.44) we deduce, for the values of n such that $n<n_0$ or $t_n>T\log t_n$, that

$$J_\nu(n,T) = O(1). \tag{3.46}$$

Moreover, for the values of n with $n\ge n_0$ and $t_n\le T\log t_n$ we have

$$J_\nu(n,T) = e^{iF_n(t_n)}e^{-(t_n/T)^2}\sum_{\mu=0}^{d-\nu}\sum_{k=0}^{Q}\sum_{\ell=0}^{Q}\delta_{\nu,\mu,k,\ell}t_n^{d+1/2-\nu-(\mu+k)/2}\left(\frac{t_n}{T}\right)^{2\ell} + O(1) \tag{3.47}$$

uniformly in ν, n and T. Here

$$\delta_{\nu,\mu,k,\ell} = \alpha_{\nu,\mu}\gamma_{k,\ell,\mu}, \tag{3.48}$$

where $\alpha_{\nu,\mu}$ is given by (3.41) and $\gamma_{k,\ell,\mu}$ is as in (3.45). Note that $\delta_{\nu,\mu,k,\ell}$ are complex numbers due to the powers of $e^{i\pi/4}$ involved in $\alpha_{\nu,\mu}$ and $\beta_{p,k,\ell}$.

3.5. Entering the self-reciprocal twist and completion of the proof. From (3.13), (3.46), (3.47) and a simple estimate for the terms with

$n < n_0$ we obtain

$$J_F(T) = e^{1/T^2} \sum_{\substack{\nu=0 \\ \nu \text{ even}}}^{d} c_\nu \sum_{\mu=0}^{d-\nu} \sum_{k=0}^{Q} \sum_{\ell=0}^{Q} \widetilde{S}_{\mathrm{Re}}(T) + e^{1/T^2} \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{d} c_\nu \sum_{\mu=0}^{d-\nu} \sum_{k=0}^{Q} \sum_{\ell=0}^{Q} \widetilde{S}_{\mathrm{Im}}(T) + O(\log T), \tag{3.49}$$

where the coefficients c_ν are as in (3.13),

$$\widetilde{S}_{\mathrm{Re}}(T) = \sum_{t_n \le T \log t_n} \frac{a_2(n)}{n^{3/2}} \Re(\delta_{\nu,\mu,k,\ell} e^{iD} e^{iF_n(t_n)}) t_n^{d-\nu+\frac{1-\mu-k}{2}} \phi_{2\ell}(t_n/T), \tag{3.50}$$

$$\phi_{2\ell}(\xi) = e^{-\xi^2} \xi^{2\ell},$$

the coefficients $\delta_{\nu,\mu,k,\ell} \in \mathbb{C}$ are given by (3.48), and D is as in (3.8). Moreover, $\widetilde{S}_{\mathrm{Im}}(T)$ is similar to $\widetilde{S}_{\mathrm{Re}}(T)$, but with the real part replaced by the imaginary part.

Recalling the value of t_n in (3.14) we have

$$t_n^{d-\nu+\frac{1-k-\mu}{2}} = (2\pi)^{d-\nu+\frac{1-k-\mu}{2}} \left(\frac{n}{q}\right)^{\frac{d-\nu}{d}+\frac{1-k-\mu}{2d}} \left(\sum_{h=0}^{d} \frac{r_h}{T^{2h}} + O\Big(\frac{1}{T^{2d+2}}\Big)\right),$$

where $r_h \in \mathbb{R}$ are easily computed and $r_0 = 1$. Moreover, by (3.14) and (3.25) we also get

$$F_n(t_n) = -dt_n = -2\pi d \Big(\frac{n}{q}\Big)^{1/d} \left(\sum_{h=0}^{d} \frac{r_h}{T^{2h}} + O\Big(\frac{1}{T^{2d+2}}\Big)\right)$$

with certain coefficients r_h. Therefore, since $t_n \le T \log t_n$ implies that $n^{1/d} \ll T \log T$, by a further expansion of the exponential we obtain

$$e^{iF_n(t_n)} t_n^{d-\nu+\frac{1-k-\mu}{2}} = (2\pi)^{d-\nu+\frac{1-k-\mu}{2}} \left(\frac{n}{q}\right)^{\frac{d-\nu}{d}+\frac{1-k-\mu}{2d}} e(-\kappa_0 n^{1/d}) \times \left(\sum_{h=0}^{d} \sum_{j=0}^{h} \eta_{h,j} \frac{n^{j/d}}{T^{2h}} + O(T^{-d}) \right) \tag{3.51}$$

uniformly in ν, μ and k, where $\eta_{h,j} \in \mathbb{C}$ can be computed from the above expressions and

$$\kappa_0 = dq^{-1/d}. \tag{3.52}$$

Plugging (3.51) into (3.50) and then completing to ∞ the resulting sum over n, thanks to the decay of the function $\phi_{2\ell}(t_n/T)$ we obtain

$$\widetilde{S}_{\mathrm{Re}}(T) = \sum_{h=0}^{d} \sum_{j=0}^{h} \frac{1}{T^{2h}} S_{\mathrm{Re}}(T) + O(1), \tag{3.53}$$

where

$$S_{\mathrm{Re}}(T) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^{\frac{1}{2}-\frac{1}{2d}+\frac{\nu-j}{d}+\frac{k+\mu}{2d}}} \Re(\omega_{\nu,\mu,k,\ell,h,j} e(-\kappa_0 n^{1/d}))\phi_{2\ell}(t_n/T), \tag{3.54}$$

κ_0 is given by (3.52) and

$$\omega_{\nu,\mu,k,\ell,h,j} = e^{iD}\left(\frac{(2\pi)^d}{q}\right)^{\frac{d-\nu}{d}+\frac{1-k-\mu}{2d}} \delta_{\nu,\mu,k,\ell}\eta_{h,j}.$$

Clearly, a completely analogous expression holds for $\widetilde{S}_{\mathrm{Im}}(T)$, with the imaginary part in place of the real part. Therefore, inserting (3.53) into (3.49) we finally obtain

$$\begin{aligned} J_F(T) = {} & e^{1/T^2} \sum_{\substack{\nu=0\\ \nu \text{ even}}}^{d} \sum_{\mu=0}^{d-\nu} \sum_{k=0}^{Q} \sum_{\ell=0}^{Q} \sum_{h=0}^{d} \sum_{j=0}^{h} \frac{c_\nu}{T^{2h}} S_{\mathrm{Re}}(T) \\ & + e^{1/T^2} \sum_{\substack{\nu=1\\ \nu \text{ odd}}}^{d} \sum_{\mu=0}^{d-\nu} \sum_{k=0}^{Q} \sum_{\ell=0}^{Q} \sum_{h=0}^{d} \sum_{j=0}^{h} \frac{c_\nu}{T^{2h}} S_{\mathrm{Im}}(T) + O(\log T). \end{aligned} \tag{3.55}$$

Now we recall that $a_2(n)$ are the coefficients of $F(s)^2$, whose degree is $2d$. Thus the above quantities $S_{\mathrm{Re}}(T)$ and $S_{\mathrm{Im}}(T)$, and hence also $J_F(T)$ thanks to (3.55), are closely related to the self-reciprocal twists $F^2_{\mathrm{self}}(s)$ and $\overline{F^2}_{\mathrm{self}}(s)$. More precisely, for $\sigma > 1$ and $\alpha \neq 0$ we write

$$F^2_{\cos}(s) := \frac{1}{2}(F^2_{\mathrm{self}}(s) + \overline{\overline{F^2}_{\mathrm{self}}(s)}) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s} \cos(-2\pi\kappa_0 n^{1/d}), \tag{3.56}$$

$$F^2_{\sin}(s) := \frac{1}{2i}(F^2_{\mathrm{self}}(s,\alpha) - \overline{\overline{F^2}_{\mathrm{self}}(s)}) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s} \sin(-2\pi\kappa_0 n^{1/d}). \tag{3.57}$$

Hence, writing for simplicity

$$a = a_{\nu,\mu,k,\ell,h,j} := \Re(\omega_{\nu,\mu,k,\ell,h,j}), \qquad b = b_{\nu,\mu,k,\ell,h,j} := \Im(\omega_{\nu,\mu,k,\ell,h,j})$$

and

$$\sigma_0 := \frac{1}{2} - \frac{1}{2d} + \frac{\nu-j}{d} + \frac{k+\mu}{2d} = \frac{1}{2} - \frac{1}{2d} - \frac{j}{d} + f, \tag{3.58}$$

say, in view of (3.54), (3.56), (3.57) the quantities $S_{\mathrm{Re}}(T)$ and $S_{\mathrm{Im}}(T)$ are closely related to

$$aF^2_{\cos}(\sigma_0) - bF^2_{\sin}(\sigma_0) \quad \text{and} \quad bF^2_{\cos}(\sigma_0) + aF^2_{\sin}(\sigma_0),$$

respectively. But, thanks to the δ-Hypothesis, the functions $F_{\cos}(s)$ and $F_{\sin}(s)$ have holomorphic continuation to the half-plane $\sigma > 1/2+1/(2d)+\delta$, with polynomial growth on vertical strips. Thus we may apply a Mellin transform technique to get bounds for $S_{\mathrm{Re}}(T)$ and $S_{\mathrm{Im}}(T)$.

To this end we first compute the Mellin transform of $\phi_{2\ell}(\xi)$, namely

$$\widetilde{\phi}_{2\ell}(s) = \int_0^\infty \phi_{2\ell}(\xi)\xi^{s-1}\,\mathrm{d}\xi = \frac{1}{2}\Gamma\Big(\frac{s+2\ell}{2}\Big), \tag{3.59}$$

and let

$$Y = \frac{q^{1/d}}{2\pi}Te^{2/(dT^2)} = \frac{q^{1/d}}{2\pi}T\Big(1+O\Big(\frac{1}{T^2}\Big)\Big), \tag{3.60}$$

so that by (3.14) we have

$$\phi_{2\ell}(t_n/T) = \phi_{2\ell}(n^{1/d}/Y). \tag{3.61}$$

Thus from (3.59), (3.61) and the inverse Mellin transform we obtain

$$S_{\mathrm{Re}}(T) = \frac{1}{2\pi i}\int_{(c)} (aF_{\cos}^2(s/d+\sigma_0) - bF_{\sin}^2(s/d+\sigma_0))\frac{1}{2}\Gamma\Big(\frac{s+2\ell}{2}\Big)Y^s\,\mathrm{d}s \tag{3.62}$$

with a sufficiently large constant $c>0$, and similarly for $S_{\mathrm{Im}}(T)$. Let $\varepsilon>0$ be arbitrarily small. Recalling the value of σ_0 in (3.58), thanks to our hypothesis and the decay of the Γ function we can shift the integration in (3.62) to the line $\sigma = 1+\delta d+j-df+\varepsilon$. Indeed, on this line we have

$$\Re(s/d+\sigma_0) = \frac{1}{2}+\frac{1}{2d}+\delta+\frac{\varepsilon}{d},$$

and hence in view of (3.56), (3.57), (3.60) and (3.62) we get

$$S_{\mathrm{Re}}(T), S_{\mathrm{Im}}(T) \ll Y^{1+\delta d+j-df+\varepsilon} \ll T^{1+\delta d+j-df+\varepsilon}. \tag{3.63}$$

Finally, from (3.55) and (3.58) we see that the worst case in (3.63) happens when $\nu\le 1$ and $\mu=k=h=j=0$, so $f=0$ as well, thus

$$J_F(T) \ll T^{1+\delta d+\varepsilon},$$

and Theorem 2 follows.

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