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Invariant ideal axiom, beyond the countable sequential groups

by

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Abstract. We demonstrate applications of IIA to groups outside the class of countable sequential groups and answer several questions of D. Shakhmatov and others. A number of new open questions are stated as well.

1. Introduction. The study of phenomena resulting from marrying continuous and algebraic structures has a long history. In most cases, even a mild algebraic structure (say, that of a group) combined with an interesting topological property (e.g. compactness, or countable (π-)character) results in a space with a very rich topological structure (e.g. all compact groups are dyadic while first-countable groups are exactly the metrizable ones, see [3]). In some cases, the algebraic properties are affected, as well (as an example, every countably compact sequential semigroup with two-sides cancellations is in fact a topological group [35]).

Existence of (a rich supply of) convergent sequences in topological groups has been a subject of active research for several decades (see [1, 2, 7, 9, 16, 17, 18, 19, 21, 36] and also surveys [24, 25, 13]). The two early questions that guided the development of this field were asked by V. Malykhin and P. Nyikos in 1978 and 1980 respectively. Without getting into full details, let us point out that these questions involved topological groups in which convergent sequences determine the topology, i.e. Fréchet groups (in Malykhin's question) and sequential groups (in Nyikos' question).

The answers to both Malykhin's and Nyikos' problems were found to be independent of the usual axioms of set theory (see [5, 12, 28, 29]). Analyz-

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ing the proofs of these results, the present authors extracted a set-theoretic principle, called *the Invariant Ideal Axiom*, or IIA, and used it to obtain a full topological classification of countable sequential topological groups in [14]. In the same paper, the authors expressed their hope that the IIA is not merely a one-off technique for obtaining the classification mentioned above, but a tool providing a general (consistent) framework for investigating convergence in groups that are not necessarily sequential or countable.

In the present paper we continue the investigation started in [14] by considering some classes of uncountable sequential groups, as well as countable but not necessarily sequential groups. We introduce an extension of the class of sequential groups, the class of (strongly) groomed groups and use IIA to show that strongly groomed groups have almost the same structure as the class of sequential groups. An example showing that the 'almost' in the previous sentence cannot be dropped is given as well.

We also show that IIA implies the nonexistence of precompact (thus pseudocompact, as well) sequential groups that are not Fréchet, answering questions of D. Shakhmatov [24], and A. Arkhangel'skii and V. Tkachenko [3]. It is worth noting that this application of IIA requires an analysis of *subsequential* groups, so nearly full generality of IIA is essential.

Finally, we study embeddings into sequential topological groups and construct an example of a completely regular sequential space that cannot be embedded into a sequential topological group, answering questions of D. Shakhmatov [25] and A. Arkhangel'skii and V. Tkachenko [3]. We also show that in the model of IIA built in [14] every sequential topological group that is not Fréchet contains a closed copy of the sequential fan, $S(\omega)$, and build an example that demonstrates that some natural generalizations of IIA are false, answering a question asked by L. Zdomskiĭ.

2. Definitions and preliminary results. All spaces below are assumed to be Hausdorff unless stated otherwise. The following definitions are central to the study of convergence in topological spaces.

Definition 2.1. A space X is called *Fréchet* if for any $x \in \overline{A} \subseteq X$ there is a sequence $S \subseteq A$ such that $S \to x$.

An important subclass of Fréchet spaces is that of *first-countable* spaces, i.e. spaces that have a countable local base of open neighborhoods at every point.

Sequential spaces provide a nontrivial extension of the class of Fréchet spaces.

Definition 2.2. A space X is called *sequential* if for every $A \subseteq X$ such that $\overline{A} \neq A$ there is a sequence $C \subseteq A$ such that $C \to x \notin A$.

We will also need the following detailed description of the closure operator in sequential spaces.

DEFINITION 2.3 (Sequential closure). Let $A \subseteq X$. Define

$$[A]' = \{ x \in X : C \to x \text{ for some } C \subseteq A \}.$$

Put

$$[A]_0 = A, \qquad [A]_\alpha = \bigcup\{[[A]_\beta]' : \beta < \alpha\} \quad \text{for } 0 < \alpha \leq \omega_1.$$

It follows that X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subseteq X$.

We can now also define the *sequential order* as a complexity measure for the closure operator.

DEFINITION 2.4. Define $\mathfrak{so}(X) = \min\{\alpha \leq \omega_1 : [A]_\alpha = \overline{A}$ for every $A \subseteq X$. Similarly, for any $x \in X$ and any $A \subseteq X$ such that $x \in [A]_{\omega_1}$, define $\mathfrak{so}(x, A) = \min\{\alpha \leq \omega_1 : x \in [A]_\alpha\}$.

Various standard *test spaces* are defined below. They serve as quick examples that illustrate the various definitions introduced above.

Let $S_n = \omega^{\leq n}$ for $n \in \omega$ and $S_\omega = \bigcup_{n\in\omega} S_n$. Declare $U \subseteq S_n$, where $n \in \omega + 1$, to be open if and only if for every $s \in U$ the set $\{s^\frown k \in S_n : s^\frown k \notin U\}$ is finite. Now each S_n is sequential and $\mathfrak{so}(S_n) = n$ for $n < \omega$, whereas $\mathfrak{so}(S_\omega) = \omega_1$. S_2 is known also as *Arens' space*, while S_ω is referred to as the *Arkhangel'skii–Franklin space*.

The quotient $S(\omega) = S_2/\omega^{\leq 1}$ is called *the sequential fan*. Finally, define $D(\omega) = \omega \times \omega \cup \{(\omega, \omega)\} \subseteq (\omega + 1)^2$ in the natural product topology (this space is sometimes referred to as a *convergent sequence of discrete sets*).

We quickly note the following standard facts.

$S(\omega)$ is Fréchet but not first-countable (it has character $\mathfrak{d}$), while $S(\omega)^2$ as well as S_2 are sequential but not Fréchet. The space S_ω is homogeneous but not a topological group [22]. The space $D(\omega)$ is metrizable, while $S(\omega)\times D(\omega)$ is not sequential.

Some less immediate properties of test spaces will be listed next.

THEOREM 2.5 (P. Nyikos, [18]). *A topological group contains a copy of S_2 if and only if it contains a copy of $S(\omega)$.*

THEOREM 2.6 (Y. Tanaka, [34]). *Let X be a topological space in which every point is G_δ (for example, X is countable). Then X contains a copy of $S(\omega)$ (resp., S_2) if and only if X contains a closed copy of $S(\omega)$ (resp., S_2).*

Putting these facts together and recalling that S_2 is not Fréchet, one obtains the following characterization of Fréchet groups in terms of embeddings of test spaces.

COROLLARY 2.7. *A countable sequential topological group is Fréchet if and only if it does not contain a closed copy of $S(\omega)$ (equivalently S_2).*

We now introduce two important classes of topological spaces that will play a central role in the rest of the paper.

Definition 2.8. X is called a *k_ω-space* (resp. *c_ω-space*) if there exists a countable family $\mathcal{K}$ of compact (resp. countably compact) subspaces of X such that $A \subseteq X$ is closed if an only if $A \cap K$ is closed for every $K \in \mathcal{K}$.

Some authors use the notation $k_{\aleph_0}$ in place of k_ω, as well. A few basic properties of k_ω-spaces are listed below. Most of these results are folklore; see [8] for further details.

Lemma 2.9. *Countable k_ω-spaces are sequential. The class of k_ω-spaces is finitely productive, i.e. if X and Y are k_ω then so is $X \times Y$. Countable k_ω-spaces have definable* (*in fact $F_{\sigma\delta}$*) *topologies; in particular, all countable k_ω-spaces are* analytic (*see below for the definition*). *Every k_ω-space is a quotient image of a topological sum of countably many compact spaces.*

The class of c_ω-spaces is not equally well studied. A natural question about c_ω-spaces is how they behave with respect to products. Note that at most countable products (in fact, all Σ-products) of countably compact *sequential* spaces are countably compact and sequential [15]. We repeat the question from [14] here.

Question 1. Is the class of sequential c_ω-spaces closed under finite products?

To introduce the next property of k_ω-groups we need to recall the definition of the Cantor–Bendixson index.

Definition 2.10. Define $(Y)' = Y \setminus \{x \in Y : x \text{ is isolated in } Y\}$. Now put $(X)^0 = X$, $(X)^{\alpha+1} = ((X)^\alpha)'$, and $(X)^\alpha = \bigcap_{\beta<\alpha}(X)^\beta$ for limit α. A space X is called *scattered* if $(X)^\alpha = \varnothing$ for some α. The smallest such α is called the *scatteredness* (or the *Cantor–Bendixson*) index of X. Given $x \in X$, we write $\operatorname{scl}(x, X) = \alpha$ where α is the unique ordinal such that $x \in (X)^\alpha \setminus (X)^{\alpha+1}$.

It is well-known that every countable compact space is scattered (and is homeomorphic to a subspace of ω_1 in the standard order topology), which makes the following concept well-defined.

Definition 2.11. Define the *compact scatteredness rank* of a countable space X as the supremum of the scatteredness (Cantor–Bendixson) indices of compact subspaces of X.

The next result by E. Zelenyuk [36] is particularly useful.

Lemma 2.12 (E. Zelenyuk). *Countable k_ω-groups of the same compact scatteredness rank are homeomorphic.*

In particular, there are exactly ω_1 countable k_ω-group topologies (up to a homeomorphism).

If in the last property of Lemma 2.9 one replaces 'a topological sum of countably many compact spaces' with 'a separable metrizable space', the resulting space can be described using the following concept.

DEFINITION 2.13. A countable family $\mathcal{K}$ of subsets of a topological space X is called a *cs^*-network* if for any infinite convergent sequence $S \to x$ and any open $U \ni x$ there exists a $K \in \mathcal{K}$ such that $K \subseteq U$ and $K \cap S$ is infinite.

It can be shown that a sequential X is a quotient image of a separable metrizable space if and only if X has a countable cs^*-network. Among the many applications of Lemma 4.5 (see below) is the following, somewhat unexpected result.

THEOREM 2.14 (T. Banakh, L. Zdomskyĭ, [4]). *A countable sequential topological group is k_ω if and only if it has a countable cs^*-network.*

3. Questions of V. Malykhin and P. Nyikos and the Invariant Ideal Axiom. The following two questions have been especially influential in the study of convergence properties in groups.

QUESTION 2 (V. Malykhin, 1979). Does there exist a separable (equivalently, countable) Fréchet topological group that is not metrizable?

QUESTION 3 (P. Nyikos, 1980). Does there exist a (separable) sequential topological group that has an intermediate (i.e. not one of 0, 1, or ω_1) sequential order?

Relatively weak set-theoretic assumptions such as $\diamondsuit(2,=)$ [12] and $\mathfrak{p} = \mathfrak{b}$ [23] are known to produce examples that answer Malykhin's question in the affirmative.

The result establishing the independence of the answer to Question 2 from ZFC was proved in [12] by the first author and U. A. Ramos-García, where the following theorem was proved.

THEOREM 3.1. *There exists a model of* ZFC *in which all separable Fréchet topological groups are metrizable.*

Consistent examples of countable sequential groups with an intermediate sequential order were constructed in [27] and [28] using CH. Finally, in [29], the second author proved the result below, establishing the independence of the answer to P. Nyikos's question from the axioms of ZFC.

THEOREM 3.2. *There exists a model of* ZFC *in which the sequential order of every sequential group is in* $\{0, 1, \omega_1\}$.

Striving to unify the proofs of Theorems 3.1 and 3.2, the authors introduced a new set-theoretic axiom, called the *Invariant Ideal Axiom* or IIA

in [14]. The axiom allowed a complete topological characterization of countable sequential groups in a manner similar to that of Theorem 2.14. To reach such generality, the language of networks had to be replaced with that of ideals.

Our treatment of IIA here differs from the one in [14] in order to achieve greater generality and provide deeper insight into the behavior of general ideals in groups and other spaces.

Recall that an *ideal* is a family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$, closed under taking subsets and finite unions. Recall also that $\mathcal{I}^+ = \mathcal{P}(\mathbb{G}) \setminus \mathcal{I}$.

We now state the most general form of the *Ideal Axiom*. Let $\mathcal{P}$ be a class of countable topological spaces and $\mathcal{Q}$ be a class of ideals on the members of $\mathcal{P}$.

$\mathsf{IA}(\mathcal{P}, \mathcal{Q})$: *For every space $X \in \mathcal{P}$ and ideal $\mathcal{I} \subseteq \mathcal{P}(X)$ in $\mathcal{Q}$ one of the following holds for every $x \in X$:*

(1) *there is a countable $\mathcal{S} \subseteq \mathcal{I}$ such that for every infinite sequence C convergent to $x \in X$ there is an $I \in \mathcal{S}$ such that $C \cap I$ is infinite;*
(2) *there is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ such that for every open $U \subseteq X$ where $x \in U$, there is an $H \in \mathcal{H}$ such that $H \setminus U \in \mathcal{I}$.*

We will call the $\mathcal{S}$ from the first alternative a *sequence capturing* family, and the set $\mathcal{H}$ from the second alternative an *almost π-network*.

We will use $\cdot$ as a stand-in for 'an arbitrary ideal'. It is a trivial observation that $\mathsf{IA}(\{X\}, \cdot)$ holds for any space X without nontrivial convergent sequences. The following easy lemma provides useful examples of spaces rich in convergent sequences that have the same property.

LEMMA 3.3. *Let X be a countable k_ω-space or a first-countable space. Then $\mathsf{IA}(\{X\}, \cdot)$ holds.*

Proof. Suppose X is a countable k_ω-space. Let $\mathcal{K}$ be a countable family of compact subspaces of X such that $F \subseteq X$ is closed if and only if $F \cap K$ is closed for every $K \in \mathcal{K}$. Let $x \in X$ and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be any ideal.

Let $K \in \mathcal{K}$ be such that $x \in K$. Suppose for any $U \subseteq K$, $x \in U$, such that U is relatively open in K, there is no $I \in \mathcal{I}$ such that $U \subseteq I$. Then every such U is in $\mathcal{I}^+$. Since K is metrizable and thus first-countable, a countable base $\{U_n : n \in \omega\}$ of relatively open neighborhoods of x in K satisfies (2).

Otherwise, for every $K \in \mathcal{K}$, find an $I_K \in \mathcal{I}$ such that $\mathcal{I}_K$ contains a nonempty relatively open neighborhood of x in $K \cup \{x\}$.

It is well-known (see for example [8]) that for any infinite $S \to x$ there exists a $K \in \mathcal{K}$ such that $S \cap K$ is infinite. This implies that the countable family $\mathcal{S} = \{I_K : K \in \mathcal{K}\}$ satisfies (1).

The proof for the first-countable case is similar and is omitted. ∎

Thus $\mathsf{IA}(k_\omega \cup \chi_\omega, \cdot)$ holds where k_ω stands for the class of countable k_ω-spaces and χ_ω denotes the class of countable first-countable spaces.

In order to state a consistent version of $\mathsf{IA}(\mathcal{P}, \mathcal{Q})$ for a wide class of topological groups $\mathcal{P}$ relatively rich in convergent sequences, we need a few more definitions.

Given a topological group $\mathbb{G}$, an ideal $\mathcal{I} \subseteq P(\mathbb{G})$ is called *invariant* if both $g \cdot I = \{g \cdot h : h \in I\}$ and $I \cdot g = \{h \cdot g : h \in I\}$, as well as $I^{-1} = \{h^{-1} : h \in I\}$, are in $\mathcal{I}$ for every $I \in \mathcal{I}$ and $g \in \mathbb{G}$.

An ideal $\mathcal{I}$ on a set X is *ω-hitting* if for every countable family $\mathcal{Y}$ of infinite subsets of X there is an $I \in \mathcal{I}$ such that $Y \cap I$ is infinite for every $Y \in \mathcal{Y}$.

We call an ideal $\mathcal{I}$ *tame* if for every $Y \in \mathcal{I}^+$, every $f : Y \to \omega$, and every ω-hitting ideal $\mathcal{J}$ on ω there is a $J \in \mathcal{J}$ such that $f^{-1}[J] \notin \mathcal{I}$, i.e. if no ideal *Katětov-below* a restriction of $\mathcal{I}$ to a positive set is ω-hitting (an interested reader may want to consult [10] for further details about Katětov order and ω-hitting ideals).

We now present an extension of the class of sequential spaces that will be used in the statement of IIA.

Given a point x in a topological space (or a topological group) we denote by

$$\mathcal{I}_x = \{A \subseteq X : x \notin \overline{A}\}$$

the ideal dual to the filter of neighborhoods of x. Call a subset Y of a topological space X *entangled* if $\mathcal{I}_x \restriction Y$ is ω-hitting for every $x \in X$. We shall call a topological space X *groomed* if it does not contain a dense entangled set. In a later section we shall present a more topological description of the class of groomed spaces (see Lemma 4.10 below).

The class of groomed spaces includes all nondiscrete sequential (thus all Fréchet) spaces, as well as all *subsequential* spaces (i.e. subspaces of sequential spaces), as the next lemma shows (see [14]).

Lemma 3.4. *Every nondiscrete subsequential space is groomed.*

Finally, we call an ideal $\mathcal{I}$ of subsets of a topological group $\mathbb{G}$ *weakly closed* if for every set $A \subseteq \mathbb{G}$ and every sequence $C \subseteq \mathbb{G}$ convergent to $1_{\mathbb{G}}$,

$$A \in \mathcal{I} \quad \text{if and only if} \quad A \cup \{x : C \cdot x \subseteq^* A\} \in \mathcal{I}.$$

Let $\mathcal{G}$ be the class of countable groomed topological groups and $\mathcal{WC}$ be the class of all weakly closed tame invariant ideals. We will refer to $\mathsf{IA}(\mathcal{G}, \mathcal{WC})$ as simply IIA, in agreement with the terminology in [14].

The following consistency result was established in [14]. The definition of a strongly ω-hitting preserving notion of forcing can also be found in [14]. We only point out here that every countable poset is strongly ω-hitting preserving.

THEOREM 3.5. *The Invariant Ideal Axiom* IIA *together with Martin's Axiom* MA(*σ-centered strongly ω-hitting preserving*) *is consistent with* ZFC.

Let us now list a few ideals particularly suitable for the applications of IIA. Note that each one of these ideals is weakly closed and invariant (if the space is a topological group) by definition.

Let X be a topological space and τ be the topology on X. Throughout the paper, $\mathbf{csc}(\tau)$ denotes the ideal generated by closed scattered subsets of X, $\mathbf{nwd}(\tau)$ is the ideal of nowhere dense subsets of X, and $\mathbf{cpt}(\tau)$ stands for the ideal generated by all the compact subsets of X. When τ is clear from the context we shall also use the notation $\mathbf{csc}(X)$, $\mathbf{nwd}(X)$, and $\mathbf{cpt}(X)$.

The following topological classification of countable sequential groups was established in [14]. In this paper we prove a generalization of Theorem 3.6 applicable to countable groups that are not necessarily sequential (see Theorem 4.12 below).

THEOREM 3.6. *Assuming* IIA, *every countable sequential group is either* k_ω *or first-countable.*

Now Lemma 3.3 together with the theorem above imply the following corollary showing that for countable sequential groups one can drop all the restrictions on the ideal (including invariance), assuming IIA holds.

COROLLARY 3.7. *Let $\mathcal{S}_\omega$ be the class of all countable sequential groups. Then* IIA *implies* $\mathsf{IA}(\mathcal{S}_\omega, \cdot)$. *In particular,* $\mathsf{IA}(\mathcal{S}_\omega, \cdot)$ *is consistent.*

Since both classes of groups (k_ω and first-countable = metrizable) are rather well-behaved, Theorem 3.6 has a number of useful implications. For example, if IIA holds, there are exactly ω_1 nonhomeomorphic sequential group topologies on countable groups, those topologies are definable (analytic, in fact $F_{\sigma\delta}$), finite products of k_ω sequential non-Fréchet groups are sequential, etc. See [14] for a more exhaustive list.

4. Nonsequential groups and sequential coreflection. The following definition provides a natural way to 'adjust' a given topology in order to make it sequential.

DEFINITION 4.1. Let (X, τ) be a topological space. Define the *sequential coreflection* $[\tau]$ to be the finest topology on X that has the same set of convergent sequences as τ.

It is easy to check that the sequential coreflection is always defined and always sequential; moreover, $\mathfrak{so}(x, A)$ and $\mathfrak{so}(X)$ have identical values in τ and $[\tau]$ for any $x \in X$ and $A \subseteq X$. If the original topology was Hausdorff (or even Urysohn) then so will be its sequential coreflection. In general, however, the sequential coreflection of a regular space may not be regular.

If (G,τ) is a countable topological group, its sequential coreflection $(G,[\tau])$ may no longer be a topological group, although all translations remain continuous.

LEMMA 4.2. *Let (G,τ) be a topological group. If $[\tau]^2$ is sequential then $[\tau]$ is a group topology on G.*

Proof. Note that all the group operations are sequentially continuous in $[\tau]$. ∎

We now introduce an extension of the class of sequential spaces that may be thought of as located between the classes of groomed and sequential spaces.

DEFINITION 4.3. A space X is called *remotely sequential* if for every $A \subseteq X$ such that $\overline{A} \neq A$ there is an infinite sequence $C \subseteq A$ and an $x \in X$ such that $C \to x$.

Note that the only difference between the definition above and that of a sequential space (Definition 2.2) is that it is not required that $x \in X \setminus A$.

The following simple lemma follows immediately from the definition and illuminates the concept of a remotely sequential space.

LEMMA 4.4. *X is not remotely sequential if and only if there exists a $D \subseteq X$ such that $\overline{D} \neq D$ and $D \cap S$ is finite for every convergent sequence $S \subseteq X$.*

Thus non-remotely-sequential spaces contain a witness that exhibits the difference between the sequential closure and the topological closure in a dramatic way.

The following lemma was essentially proved in [4].

LEMMA 4.5 (T. Banakh, L. Zdomskyĭ). *Let $(\mathbb{G},\tau)$ be a topological group such that $(G,[\tau])$ contains closed copies of $D(\omega)$ and $S(\omega)$. Then (G,τ) contains a countable nondiscrete subspace that is almost disjoint from every convergent sequence. In particular, $\mathbb{G}$ is not remotely sequential.*

Call a space X *strongly groomed* if for every dense $D \subseteq X$ there exists an infinite convergent sequence $C \subseteq D$ (a justification for this choice of terminology will be given shortly below). Lemma 4.10 shows that every strongly groomed space is groomed.

LEMMA 4.6. *Every nondiscrete remotely sequential space is strongly groomed* (*and thus groomed*).

Proof. Let $\overline{D} = X$. Since X is not discrete, we may assume $\overline{D} \neq D$ by passing to a subset of D, if necessary. Now there exists an infinite sequence $C \subseteq D$ such that $C \to x$ for some $x \in X$, since X is remotely sequential. ∎

A quick example of a countable topological group that is groomed but not remotely sequential can be constructed using the following lemma.

Lemma 4.7. *Let G be a countable sequential non-Fréchet topological group, H be a subgroup of G such that H contains a copy of $S(\omega)$ and $\overline{H} \neq H$. Then H is groomed and is not remotely sequential.*

Proof. The proof is an easy application of Lemma 3.4 and Lemma 4.5. To show that H contains a closed copy of $D(\omega)$, find a $C \subseteq H$ such that $C \to g \notin H$ using $\overline{H} \neq H$. Let $C = \{c_i : i \in \omega\}$ and define $D = \{c_i \cdot c_j^{-1} : j > i\} \cup \{1_{\mathbb{G}}\}$. Note that every set $D^i = \{c_i \cdot c_j^{-1} : j > i\}$ is closed and discrete in H since $D^i \to c_i \cdot g^{-1} \notin H$. Let U be an open neighborhood of $1_{\mathbb{G}}$, and pick an open $V \subseteq G$ such that $g \in V$ and $V \cdot V^{-1} \subseteq U$. Let $k \in \omega$ be such that $c_i \in V$ for all $i > k$. Then $c_i \cdot c_j \in U$ for $i, j > k$, and thus $D^i \subseteq U$.

Now the argument above implies $D^i \to 1_{\mathbb{G}}$, which in turn means that D is a closed copy of $D(\omega)$ in H. ∎

The following definition serves as an important technical tool in several proofs below and also helps to provide a topological description of groomed spaces.

Definition 4.8. Let X be a topological space. Let $\mathcal{D}$ be a countable family of infinite closed discrete subspaces of X. We call $\mathcal{D}$ a (*strict*) *van Douwen network* (*vD-network* for short) at $x \in X$ if for every open $U \ni x$ there is a $D \in \mathcal{D}$ such that $D \cap U$ is infinite ($D \subseteq^* U$).

If $\mathcal{D}$ is a vD-network at x we will refer to the space $\bigcup \mathcal{D} \cup \{x\}$ as a *vD-subspace of* X and the point x as a *vD-point of* $\mathcal{D}$ *in* X. In case we need to mention the topology on X explicitly, we shall use the notation vD(τ)-point, etc.

A closed copy of $D(\omega)$ in X is a trivial example of a vD-subspace of X.

The following easy lemma was proved in [29] and may serve as an initial motivation for studying vD-points in sequential groups.

Lemma 4.9. *Let X be a countable space and $x \in X$ be a vD-point in X. Then $X \times S(\omega)$ is not sequential.*

The utility of the concept of a vD-network can also be seen in the following description of groomed spaces.

Lemma 4.10. *A regular countable topological space X is groomed if and only if for every dense $D \subseteq X$ there exists a point $x \in X$ such that there exists either an infinite sequence $S \subseteq D$ that converges to x or a strict vD-network $\mathcal{D} \subseteq \mathcal{P}(D)$ at x.*

Proof. Let X be a regular countable groomed space and $D \subseteq X$ be a dense subset of X. Since X is groomed, D is not entangled, that is, there exists a point $x \in X$ such that the ideal $\mathcal{I}_x \restriction D$ is not ω-hitting. Let $\mathcal{D} \in [D]^{\leq\omega}$ be a family of infinite subsets of D that witnesses that $\mathcal{I}_x \restriction D$ is not ω-hitting.

Let $C \in \mathcal{D}$. Consider the closure $\overline{C}$ and note that if $\overline{C}$ is a (countably) compact subspace of X, since X is countable, there exists a nontrivial convergent sequence $S \subseteq C$ such that $S \to x'$ for some $x' \in X$.

Thus assume that no $C \in \mathcal{D}$ is compact and for each $C \in \mathcal{D}$ pick a discrete and closed (in X) subset $C' \subseteq C$ (this is the only place where the regularity of X is used). Put $\mathcal{D}' = \{C' : C \in \mathcal{D}\}$ and observe that $\mathcal{D}'$ is the required vD-network at x. ∎

In view of the previous lemma, our next definition is natural. Call a topological space X *weakly groomed* if for every dense $D \subseteq X$ and every point $x \in X$ there either exists an infinite sequence $S \subseteq D$ such that $S \to x$ or a (not necessarily strict) vD-network $\mathcal{D} \subseteq \mathcal{P}(D)$ at x. It is natural to ask if the consistency of IIA can be extended to the wider class of weakly groomed countable groups. We have the following simple implications:

sequential $\Rightarrow$ remotely sequential $\Rightarrow$
strongly groomed $\Rightarrow$ groomed $\Rightarrow$ weakly groomed

We need the following technical definition to state Theorem 4.12 in full generality.

DEFINITION 4.11. A closed scattered subset P of a topological space (X, τ) is called *τ-weakly regular* if for each $p \in P$ there exists an open $O_p \ni p$ such that for every $q \in \overline{O_p \cap P} \setminus \{p\}$, $\mathrm{scl}(q, P) < \mathrm{scl}(p, P)$.

Note that, trivially, if P above is regular as a subspace of (X, τ), then it is τ-weakly regular. In all the applications below, $\tau = [\tau']$ for some topology τ'.

The rest of this section is dedicated to the proof of the following theorem.

THEOREM 4.12. (IIA) *Let $(\mathbb{G}, \tau)$ be a countable strongly groomed topological group in which every $[\tau]$-closed $[\tau]$-scattered subset is $[\tau]$-weakly regular. Then $(G, [\tau])$ is a k_ω-group or (G, τ) is metrizable.*

Before we proceed with the details of the proof, let us state some corollaries that illustrate possible applications of Theorem 4.12.

COROLLARY 4.13. (IIA) *Let $(\mathbb{G}, \tau)$ be a countable group. Then one of the following properties holds:*

(1) *$\mathbb{G}$ contains a dense subset that is almost disjoint from every convergent sequence in $\mathbb{G}$;*
(2) *$\mathbb{G}$ contains a subspace P that is closed and scattered in $[\tau]$ but is not regular in the topology inherited from $[\tau]$;*
(3) *$\mathbb{G}$ is metrizable;*
(4) *$(\mathbb{G}, [\tau])$ is a k_ω-group.*

The next corollary sheds some light on what prevents the sequential coreflection from being a topological group in a model of IIA.

Corollary 4.14. (IIA) *Let $(\mathbb{G},\tau)$ be a strongly groomed countable group. Then $(\mathbb{G},[\tau])$ is a topological group if and only if it is regular.*

The following corollary generalizes Corollary 3.7 to a wider class of countable groups.

Corollary 4.15. *Let $\mathcal{SG}_\omega$ be the class of all countable strongly groomed groups with regular sequential coreflection. Then* IIA *implies* $\mathsf{IA}(\mathcal{SG}_\omega,\cdot)$. *In particular,* $\mathsf{IA}(\mathcal{SG}_\omega,\cdot)$ *is consistent.*

Proof. Let $(\mathbb{G},\tau)\in\mathcal{SG}_\omega$ and $\mathcal{I}\subseteq\mathcal{P}(\mathbb{G})$ be any ideal. Theorem 4.12 implies that either τ is first countable or $(\mathbb{G},[\tau])$ is k_ω. In the former case $\mathsf{IA}(\{\mathbb{G}\},\cdot)$ holds by Lemma 3.3. In the latter, $\mathsf{IA}(\{(\mathbb{G},[\tau])\},\{\mathcal{I}\})$ holds similarly. Now alternative (1) of $\mathsf{IA}(\{(\mathbb{G},[\tau])\},\{\mathcal{I}\})$ implies alternative (1) of $\mathsf{IA}(\{(\mathbb{G},\tau)\},\{\mathcal{I}\})$, since every τ-convergent sequence is convergent in $[\tau]$. Similarly, alternative (2) of $\mathsf{IA}(\{(\mathbb{G},[\tau])\},\{\mathcal{I}\})$ implies alternative (2) of $\mathsf{IA}(\{(\mathbb{G},\tau)\},\{\mathcal{I}\})$ since $\tau\subseteq[\tau]$. ∎

It is not clear if the statement of Theorem 4.12 is optimal. Perhaps the most interesting question is whether the ($[\tau]$-weak) regularity in Theorem 4.12 may be dropped altogether (which would eliminate alternative (2)).

Example 4.22 shows that '$(\mathbb{G},[\tau])$ is k_ω' cannot, in general, be strengthened to '$(\mathbb{G},\tau)$ is k_ω' even if IIA is assumed.

The proof of Theorem 4.12 is split into several lemmata. Lemmata 4.16 and 4.18 generalize the corresponding propositions from [14].

Lemma 4.16. *Let $(\mathbb{G},\tau)$ be a countable strongly groomed group in which every $[\tau]$-closed, $[\tau]$-scattered subset is $[\tau]$-weakly regular. Suppose $\mathcal{P}\subseteq\mathbf{csc}([\tau])$ is a countable family such that for every $S\to 1_{\mathbb{G}}$ there exists a $P\in\mathcal{P}$ such that $|S\cap P|=\omega$. Let $\mathcal{D}\subseteq[\mathbb{G}]^\omega$ be a countable family of closed discrete subsets of $\mathbb{G}$. Then for every $g\in\mathbb{G}$ there exists an open $U\ni g$ such that $U\cap D$ is finite for every $D\in\mathcal{D}$.*

Proof. Let $\mathcal{D}=\{D_n:n\in\omega\}\subseteq\mathcal{P}(\mathbb{G})$ be a collection of closed discrete subsets of $\mathbb{G}$. The statement of the lemma is equivalent to claiming that there are no $\mathrm{vD}(\tau)$-points of $\mathcal{D}$ in $\mathbb{G}$. Suppose $g\in\mathbb{G}$ is a $\mathrm{vD}(\tau)$-point of $\mathcal{D}$. By translating each $D\in\mathcal{D}$ if necessary, we may assume that $g=1_{\mathbb{G}}$.

Let $\mathcal{P}=\{P_n:n\in\omega\}$ be a collection of $[\tau]$-closed, $[\tau]$-scattered subsets of $\mathbb{G}$ such that for any $S\to 1_{\mathbb{G}}$ there is a $P\in\mathcal{P}$ such that $|S\cap P|=\omega$. By requiring $\mathcal{P}$ to be closed under finite unions we may assume that such a P can be chosen so that $S\subseteq^* P$. Using the $[\tau]$-weak regularity of each $P\in\mathcal{P}$ we may also assume that $\mathrm{scl}(1_{\mathbb{G}},P)>\mathrm{scl}(p,P)$ for any $p\in P\setminus\{1_{\mathbb{G}}\}$ and any $P\in\mathcal{P}$. To simplify the notation, scatteredness $\mathrm{scl}(\cdot,\cdot)$ will always refer to $[\tau]$-scatteredness below. For the same reason, closure is assumed to be closure in τ unless stated otherwise.

Pick a family $\{O_n : n \in \omega\}$ of τ-open neighborhoods of $1_{\mathbb{G}}$ such that $\overline{O_{n+1}} \subseteq O_n$ and $\bigcap_{n\in\omega} O_n = \{1_{\mathbb{G}}\}$. Put $P = \bigcup_{n\in\omega} P_n \cap \overline{O_n}$. One may verify that P is $[\tau]$-closed, $[\tau]$-scattered, and for any $S \to 1_{\mathbb{G}}$, $S \subseteq^* P$. Note that the properties of $\mathcal{P}$ imply that $\mathrm{scl}(1_{\mathbb{G}}, P) = \mathrm{scl}(P) > \mathrm{scl}(p, P)$ for any $p \in P$ such that $p \neq 1_{\mathbb{G}}$. One may further require that $\alpha_P = \mathrm{scl}(1_{\mathbb{G}}, P)$ is the smallest ordinal among all $\alpha_{P'} = \mathrm{scl}(1_{\mathbb{G}}, P')$ such that P' has the properties above. Therefore

(3) P is a $[\tau]$-closed, $[\tau]$-scattered subset of $\mathbb{G}$ such that $S \subseteq^* P$ for every $S \to 1_{\mathbb{G}}$; moreover, $\alpha_P = \mathrm{scl}(P) = \mathrm{scl}(1_{\mathbb{G}}, P) > \mathrm{scl}(p, P)$ for any $p \in P \setminus \{1_{\mathbb{G}}\}$, and α_P is the smallest possible.

Thus, effectively, the countable family $\mathcal{P}$ has been replaced by a single element P. In what follows we occasionally construct scattered subsets P' to show that P may be assumed to have additional properties. Just as in the construction of P, we will use the notation P_n (or P'_n) for the pieces the subspace P' is 'assembled of'.

Let $\{g_i \in \mathbb{G} : i \in \omega\}$ be a 1-1 listing of $\mathbb{G}$. We will use the notation $G_n = \{g_i : i \leq n\}$ for brevity. By modifying D_n if necessary, we may assume that $D_n \cap G_n^{-1} = \varnothing$. For each $n \in \omega$ find τ-open $O_n \ni 1_{\mathbb{G}}$ such that $\overline{O_{n+1}} \subseteq O_n$, $\bigcap_{n\in\omega} O_n = \{1_{\mathbb{G}}\}$, and the following condition holds:

(4) $\overline{O_n} \cap D_n \cdot G_n = \varnothing$.

Such choice of O_n is possible since each D_i is closed and discrete.

Suppose there exists an $S \to 1_{\mathbb{G}}$ such that $(S \cdot p) \setminus P$ is infinite for every $p \in P \setminus \{1_{\mathbb{G}}\}$. We may then pick an infinite sequence $S' \subseteq S \cdot S$ so that $S' \to 1_{\mathbb{G}}$ and $S' \subseteq \mathbb{G} \setminus P$, contradicting (3). Thus for every sequence $S \to 1_{\mathbb{G}}$ there exists a $p \in P \setminus \{1_{\mathbb{G}}\}$ such that $S \cdot p \subseteq^* P$.

Let $P \setminus 1_{\mathbb{G}} = \{p_n : n \in \omega\}$ be a 1-1 listing of $P \setminus \{1_{\mathbb{G}}\}$. Using the $[\tau]$-weak regularity of P, for each $p_n \in P$ pick a $[\tau]$-open $U_n \ni p$ such that $\alpha_P > \mathrm{scl}(p_n, P) > \mathrm{scl}(q, P)$ for any $q \in \overline{U_n \cap P}^{[\tau]} \setminus \{p_n\}$. Put $P_n = \overline{O_n} \cap (\overline{U_n \cap P}^{[\tau]} \cdot p_n^{-1})$ and observe that $P' = \bigcup P_n$ is a $[\tau]$-closed, $[\tau]$-scattered subset of $\mathbb{G}$ that satisfies all the properties of P in (3). Indeed, if $S \to 1_{\mathbb{G}}$ then by the argument in the preceding paragraph, there is a $p_n \in P \setminus \{1_{\mathbb{G}}\}$ such that $S \cdot p_n \subseteq^* P$. Thus $S \cdot p_n \subseteq^* U_n \cap P$, therefore $S \subseteq^* P_n \subseteq P'$.

Let $p \in (D_n \cdot G_n) \cap P'$. Then, by the construction of P_n and O_n and (4), $p \in P' \setminus \overline{O_n}$, so $\mathrm{scl}(p, P') < \beta_n = \max_{i\leq n} \mathrm{scl}(p_i, P) < \alpha_P$.

To simplify notation, we will assume that P itself has the following additional property of P' we have just established:

(5) There exists a $\beta_n < \alpha_p$ such that $\mathrm{scl}(p, P) < \beta_n$ for any $p \in D_n \cdot G_n \cap P$.

Suppose $A \subseteq \mathbb{G}$ is such that there exists an ordinal β with the property $\mathrm{scl}(a, P) < \beta < \alpha_P$ for every $a \in A \cap P$. We will show that there exists a sequence $S \to 1_{\mathbb{G}}$ such that $S \setminus (P \cdot F^{-1})$ is infinite for every $F \in [A]^{<\omega}$.

Indeed, suppose no such S exists and let $A = \{a_n : n \in \omega\}$ list all the points in A. First, note that it is enough to establish the existence of such S for any $F \subseteq A$, since trivially $|S \cap P \cdot a| < \omega$ for any $S \to 1_{\mathbb{G}}$ whenever $a \notin P$ due to the closedness of P. For each $n \in \omega$ use the $[\tau]$-weak regularity of P to find a $[\tau]$-open neighborhood $U_n \ni a_n$ such that $\mathrm{scl}(q, P) < \beta' < \beta$ whenever $a_n \in P$ and $q \in \overline{U_n \cap P}^{[\tau]} \setminus \{a_n\}$. Put $P'_n = \overline{U_n \cap P}^{[\tau]} \cdot a_n^{-1}$ if $a_n \in P$, and $P'_n = \varnothing$ otherwise. Note that $\mathcal{P}' = \{P'_n : n \in \omega\}$ is a collection of $[\tau]$-closed, $[\tau]$-scattered subsets of $\mathbb{G}$ with $\mathrm{scl}(P_n) < \beta$ for every $n \in \omega$, and for every $S \to 1_{\mathbb{G}}$ there exists an $F \in [\omega]^{<\omega}$ such that $S \subseteq^* \bigcup_{n \in F} P'_n$.

Repeating the construction used to build P out of P_n at the beginning of this argument, we may construct a $[\tau]$-closed, $[\tau]$-scattered $P' \subseteq \mathbb{G}$ such that $\mathrm{scl}(P') \leq \beta < \alpha_P$ and $S \subseteq^* P'$ for every $S \to 1_{\mathbb{G}}$, contradicting the minimality of α_P in (3).

Using (5) and the argument above, for each $n \in \omega$ construct an $S_n \to 1_{\mathbb{G}}$ such that $S_n \setminus P \cdot G_n^{-1} \cdot d^{-1}$ is infinite for every $d \in D_n$. Let $D_n = \{d_i : i \in \omega\}$ and $S_n = \{s_i : i \in \omega\}$ be 1-1 listings of D_n and S_n. For each $i \in \omega$ pick an $m(i) > n$ so that $m(i)$ is strictly increasing, $s_{m(i)} \cdot d_i \cdot G_n \subseteq \mathbb{G} \setminus P$ for every $i \in \omega$ and put $e_i^n = s_{m(i)} \cdot d_i$. Put $B_n = \{e_i^n : i \in \omega\}$, $B = \bigcup_{n \in \omega} B_n$ and note that each B_n is a closed and discrete subspace of $\mathbb{G}$.

Now, $1_{\mathbb{G}} \in \overline{B}$. Indeed, let U be any open neighborhood of $1_{\mathbb{G}}$. Find an open $V \ni 1_{\mathbb{G}}$ such that $V \cdot V \subseteq U$. Then $V \cap D_n$ is infinite for some $n \in \omega$, since $1_{\mathbb{G}}$ is a vD-point of $\mathcal{D}$. Also, $S_n \subseteq^* V$. Thus for some large enough $i \in \omega$, both $d_i \in V$ and $s_{m(i)} \in V$, showing that $e_i^n \in U$.

Let $g \in \mathbb{G}$ and $F \in [\mathbb{G}]^{<\omega}$. Find $k \in \omega$ such that $g \cdot F^{-1} \subseteq G_k$. Let $b \in B_n$ for some $n > k$. Then $b = e_i^n = s_{m(i)} \cdot d_i$ for some $n > k$, $i \in \omega$, $s_{m(i)} \in S_n$, and $d_i \in D_n$. Hence $b \cdot g \cdot F^{-1} = s_{m(i)} \cdot d_i \cdot g \cdot F^{-1} \subseteq s_{m(i)} \cdot d_i \cdot G_n \subseteq \mathbb{G} \setminus P$. We have established:

(6) For any $g \in \mathbb{G}$ and any finite $F \subseteq \mathbb{G}$ the intersection $B \cdot g \cap P \cdot F$ is a closed discrete subset of $\mathbb{G}$. In particular, $g \in \overline{B \cdot g \setminus P \cdot F}$.

Put $C_n = B \cdot g_n \setminus P \cdot G_n$ for $n \in \omega$, and $C = \bigcup_{n \in \omega} C_n$. Then by (6), $g_n \in \overline{C_n}$ for every $n \in \omega$ so C is dense in $\mathbb{G}$.

Suppose $S \subseteq C$ is an infinite sequence such that $S \to g$ for some $g \in \mathbb{G}$. By (3), $S \subseteq^* P \cdot g \subseteq P \cdot G_n$ for some $n \in \omega$. If $S \cap C_k$ is infinite for some $k \in \omega$ then, passing to a subsequence if necessary, we may assume that $S \subseteq B \cdot g_k \cap P \cdot G_n$, contradicting $B \cdot g_k \cap P \cdot G_n$ being closed and discrete by (6).

Otherwise, there exists an $s \in S \cap C_{n'}$ for some $n' > n$; but $C_{n'} \cap P \cdot G_n \subseteq C_{n'} \cap P \cdot G_{n'} = \varnothing$ by the choice of C_i, a contradiction. Therefore C is a dense subset of $\mathbb{G}$, almost disjoint from every convergent sequence in $\mathbb{G}$, contradicting the assumption that $\mathbb{G}$ is strongly groomed. ∎

While the sequential coreflection of a topological group may not be a topological group itself, the following weak version of joint continuity holds. We only need the lemma below, while a more general result can be shown that uses k-spaces instead of sequential ones.

Lemma 4.17. *Let $\mathbb{G}$ be a countable group, U be a $[\tau]$-open neighborhood of $1_{\mathbb{G}}$, and $S \to 1_{\mathbb{G}}$ be a sequence such that $S \subseteq U$. Then there exists a $[\tau]$-open $O \ni 1_{\mathbb{G}}$ such that $O \cdot S \subseteq U$.*

Proof. Define $O = \{g \in U : S \cdot g \subseteq U\}$. Let $F = \mathbb{G} \setminus O$ and suppose F is not $[\tau]$-closed. Then there exists an infinite sequence $C \subseteq F$ such that $C \subseteq U$ and $C \to x$ for some $x \in O$. Since $S \cdot C$ is compact in τ, the set $S \cdot C \setminus U$ has an accumulation point $c \in S \cdot x$. Then $c \notin U$, contradicting the choice of x. Thus $O \ni 1_{\mathbb{G}}$ is the desired neighborhood. ∎

Lemma 4.18. *Let $(\mathbb{G}, \tau)$ be a countable strongly groomed group. Suppose $\mathcal{P} \subseteq \mathbf{nwd}([\tau])$ is a countable family such that for every $S \to 1_{\mathbb{G}}$ there exists a $P \in \mathcal{P}$ such that $|S \cap P| = \omega$. Then $\mathbb{G}$ has no countable π-network at $1_{\mathbb{G}}$ that consists of subspaces whose closures are not $[\tau]$-locally compact.*

Proof. Let $\mathcal{D} = \{D_n : n \in \omega\}$ be a π-network (in τ) at $1_{\mathbb{G}}$ such that each $D_n \in \mathcal{D}$ is $[\tau]$-dense in itself. Since τ is regular, we may assume that each D_n is closed in $[\tau]$. By translating each element of $\mathcal{D}$ if necessary, we may assume that $1_{\mathbb{G}} \in D$ is a point at which D is not locally compact for every $D \in \mathcal{D}$.

Fix an open $O_n \ni 1_{\mathbb{G}}$ so that $\overline{O_{n+1}} \subseteq O_n$ and $\bigcap_{n\in\omega} O_n = \{1_{\mathbb{G}}\}$.

Let $\mathcal{P} = \{P_n : n \in \omega\} \subseteq \mathbf{nwd}([\tau])$ be such that for every $S \to 1_{\mathbb{G}}$ there exists a $P \in \mathcal{P}$ such that $|S \cap P| = \omega$. Just as in the proof of Lemma 4.16 we may construct a $P \in \mathbf{nwd}([\tau])$ such that for every $S \to 1_{\mathbb{G}}$, $S \subseteq^* P$. By taking the $[\tau]$-closure of P if necessary, we may assume that P is $[\tau]$-closed. Additionally, given any $N \in \mathbf{nwd}([\tau])$ we may assume that $N \subseteq P$ by replacing P with $P \cup N$ in what follows.

Let $g \in \mathbb{G}$. Define $d(g) = \mathfrak{so}(g, \mathbb{G} \setminus P)$. Let $\alpha_P = d(1_{\mathbb{G}})$. Note that $\alpha_P > 1$ by the choice of P. We now prove the following claim by induction on α.

(7) Let $p \in P$ and $d(p) = \alpha$ for some $\alpha < \omega_1$. There exists a $T \subseteq \mathbb{G} \setminus P$ and a neighborhood assignment $W : T \to \tau$ such that

  (a) $p \in [T]_\alpha$, and if $p' \in \overline{T}$ then $d(p') = \mathfrak{so}(p', T)$;
  (b) $g \in W(g) \setminus P$ for every $g \in T$ and the $W(g)$ are disjoint; if $s_i \in W(g_i) \setminus P$ is such that $s_i \to g$ for some $g \in \mathbb{G}$ and all g_i are distinct then $g_i \to g$.

Let $d(p) = \alpha$ for some $p \in P$. If $\alpha = 1$, there exists an infinite sequence of $g_i \in \mathbb{G} \setminus P$ such that $g_i \to p$. Thinning out the sequence and reindexing if necessary, pick disjoint open $W(g_i) \ni g_i$ so that $W(g_i) \subseteq O_i \cdot p$. Put $T = \{g_i : i \in \omega\}$. Properties (7a) and (7b) are easy to check.

Thus we may assume $\alpha > 1$. Let $p^n \to p$ and $\alpha_n < \alpha$ be such that $p^n \in O_n \cdot p$ and $p^n \in [\mathbb{G} \setminus P]_{\alpha_n}$ for every $n \in \omega$. Since $d(p^n) \leq \alpha_n < \alpha$, by the induction hypothesis there exist $T_n \subseteq \mathbb{G} \setminus P$ and $W_n : T_n \to \tau$ that satisfy (7). Pick a sequence of τ-open disjoint $V_n \ni p^n$ such that $V_n \subseteq O_n \cdot p$ after thinning out and reindexing if necessary. By passing to subsets and possibly reindexing again, we may assume that the $\overline{T_n}$ are disjoint, $T_n \subseteq O_n \cdot p \cap V_n$, and $W_n(g) \subseteq O_n \cdot p \cap V_n$ for every $n \in \omega$ and $g \in T_n$. Let $T = \bigcup_{n \in \omega} T_n$ and define $W : T \to \tau$ by $W(g) = W_n(g)$ whenever $g \in T_n$.

By the choice of T_n and O_n, $\overline{T} = \{p\} \cup \bigcup_{n \in \omega} \overline{T_n}$. If $p' \in \overline{T_n}$ then $d(p') = \mathfrak{so}(p', T_n) = \mathfrak{so}(p', T)$ by the inductive hypothesis and the choice of T_n. Since $d(p) = \sup_n \alpha_n + 1$ and $d(p^n) \leq \alpha_n$, we find that $d(p) = \mathfrak{so}(p, T)$.

Let $s_i \in W(g_i) \setminus P$ for some $g_i \in T$ be such that $s_i \to g$. By thinning out and reindexing we may assume that either $g_i \in T_n$ for some fixed $n \in \omega$ or $g_i \in T_{n(i)} \subseteq O_{n(i)} \cdot p$ for some strictly increasing $n(i)$. In the first case $g_i \to g \in P$ by the choice of T_n. Otherwise, $s_i \in W_{n(i)}(g_i) \subseteq O_{n(i)} \cdot p$ so $s_i \to p$ by the choice of O_n, contradicting $s_i \in \mathbb{G} \setminus P$ and $d(p) = \alpha > 1$.

Pick T and W that satisfy (7) for $p = 1_{\mathbb{G}}$ and let $T = \{t_n : n \in \omega\}$ be a 1-1 enumeration of the points of T. Pick $[\tau]$-open $U_n \subseteq (W(t_n) \setminus P) \cdot t_n^{-1}$ so that $U_{n+1} \subseteq U_n \subseteq O_n$, and $1_{\mathbb{G}} \in \overline{\{t_k : U_k \cdot t_k \subseteq O_n\}}$ for every $n \in \omega$. To see that such a choice of U_n is possible, note that $1_{\mathbb{G}} \in \overline{T \cap O_n}^{[\tau]}$ for every $n \in \omega$. Observe that $(W(t_n) \setminus P) \cdot t_n^{-1}$ is a $[\tau]$-open neighborhood of $1_{\mathbb{G}}$ and recursively pick U_i so that (1) $U_i \subseteq (W(t_i) \setminus P) \cdot t_i^{-1}$, (2) $U_{i+1} \subseteq U_i \subseteq O_i$, and (3) $U_i \subseteq O_n \cdot t_i^{-1}$ whenever $t_i \in O_n$. To see that (3) is possible, note that for each t_i there are only finitely many $n \in \omega$ such that $t_i \in O_n$. Now $T \cap O_n \subseteq \{t_k : U_k \cdot t_k \subseteq O_n\}$, implying the required property of U_i.

Note that $\bigcap_{n \in \omega} U_n = \{1_{\mathbb{G}}\}$, and $U_n \cdot t_n \subseteq \mathbb{G} \setminus P$ for each n. Let $k \in \omega$ and show that

(8) there exists a $[\tau]$-closed $[\tau]$-discrete subset $E_k \subseteq D_k$ such that $E_k = \{e^k_n : n \in \omega\}$ and $e^k_n \in U_n$ for every $n \in \omega$, or there is a $[\tau]$-closed copy of $D(\omega)$ in $(\mathbb{G}, [\tau])$;

If no such E_k exists, the $U_n \cap D_k$ form a countable base of $[\tau]$-open neighborhoods of $1_{\mathbb{G}}$ in D_k. Now $\overline{U_n}^{\tau}$ form a nested countable local network at $1_{\mathbb{G}}$ (in both τ and $[\tau]$). Note that no $\overline{U_n}^{\tau}$ is τ-compact. Otherwise, since τ and $[\tau]$ share the same compact subspaces, D_k would be locally compact at $1_{\mathbb{G}}$. Selecting an infinite closed discrete subset in each $\overline{U_n}^{\tau}$ results in a closed (in both τ and $[\tau]$) copy of $D(\omega)$ (as a subspace of $(\mathbb{G}, \tau)$). Since $D(\omega)$ is first-countable, it remains a $D(\omega)$ in $[\tau]$.

Now consider the following cases.

CASE 1. It is possible to pick an $E_k \subseteq D_k$ as in (8) for every $k \in \omega$.

Consider the set $D^k = \{d^n : d^n = e^k_n \cdot t_n, U_n \cdot t_n \subseteq O_k, n \in \omega\}$. Now $D^k \subseteq O_k \setminus P$ for every $k \in \omega$ and $d^n \in W(t_n)$ for every $n \in \omega$ by $e^k_n \in U_n$ and the choice of U_n. Suppose $d^{n(i)} \to d$ for some $d \in \mathbb{G}$. By (7b), $t_{n(i)} \to d$ so $e^k_{n(i)} = d^{n(i)} \cdot t^{-1}_{n(i)} \to 1_{\mathbb{G}}$, contradicting the choice of e^k_n. Thus each $D^k \subseteq O_k$ is closed and discrete in $(\mathbb{G}, [\tau])$.

Suppose $S \to g$ for some $g \in \mathbb{G}$ is an infinite sequence such that $S \subseteq \bigcup_{k\in\omega} D^k$. Since each D^k is closed discrete in $[\tau]$, we may assume that $S = \{s_n : n \in \omega\}$, where $s_n \in D^{k(n)}$ for some strictly increasing $k(n)$. Then $s_n \in O_{k(n)}$ and $S \to 1_{\mathbb{G}}$, contradicting $S \subseteq \mathbb{G} \setminus P$ and the choice of P. Thus $\bigcup_{k\in\omega} D^k$ is almost disjoint from every convergent sequence in $\mathbb{G}$.

Let $U \ni 1_{\mathbb{G}}$ be open (for the rest of the paragraph we work in τ) and find an open $V \ni 1_{\mathbb{G}}$ such that $V \cdot V \subseteq U$. Let $k \in \omega$ be such that $D_k \subseteq V$, and let t_n be such that $U_n \cdot t_n \subseteq O_k$ and $t_n \in V$. Then $d^n = e^k_n \cdot t_n \in D^k \cap V \cdot V$. Thus $1_{\mathbb{G}} \in \overline{\bigcup_{k\in\omega} D^k}$. Note that $\overline{\bigcup_{k\in\omega} D^k} \in \mathbf{nwd}([\tau])$.

CASE 2. There exists a $[\tau]$-closed copy of $D(\omega)$ in $(\mathbb{G}, [\tau])$.

Since $P \in \mathbf{nwd}([\tau])$ and there is no $S \subseteq \mathbb{G}\setminus P$ such that $S \to 1_{\mathbb{G}}$, $(\mathbb{G}, [\tau])$ is not Fréchet. Since $\mathbb{G}$ is countable, there exists a $[\tau]$-closed copy of $S(\omega)$ in $\mathbb{G}$. Let $\bigcup_{i,j\in\omega} S^j_i$ be a $[\tau]$-closed copy of $S(\omega)$ in $\mathbb{G}$ such that each $S^j_i \to 1_{\mathbb{G}}$, the S^j_is are disjoint, and any $\bigcup_{k\in\omega} S^{j(k)}_{i(k)}$ is a $[\tau]$-closed copy of $S(\omega)$ provided the set $\{(i(k), j(k)) : k \in \omega\}$ is infinite. Similarly, let $\bigcup_{i,j\in\omega} D^j_i$ be such that each D^j_i is closed and discrete, the D^j_is are disjoint, and $\bigcup_{k\in\omega} D^{j(k)}_{i(k)} \cup \{1_{\mathbb{G}}\}$ is a closed copy of $D(\omega)$ provided the set $\{(i(k), j(k)) : k \in \omega\}$ is infinite.

Let $D^n_i = \{d^n_i(j) : j \in \omega\}$ and $S^n_i = \{s^n_i(j) : j \in \omega\}$ be 1-1 listings of D^n_i and S^n_i. Using Lemma 4.17 and trimming D^n_i and S^n_i if necessary, we may assume that $B^n_i = \{s^n_i(j) \cdot d^n_i(j) : j \in \omega\} \subseteq U_n$ for every $i \in \omega$. Put $B^n = \bigcup_{i\in\omega} B^n_i \setminus \{1_{\mathbb{G}}\} \subseteq U_n$. A standard argument shows that $1_{\mathbb{G}} \in \overline{B^n}$. Let $B = \bigcup_{n\in\omega} B^n \cdot t_n$ and observe that $B \subseteq \mathbb{G} \setminus P$ and $1_{\mathbb{G}} \in \overline{B}$.

Suppose there exists an infinite $C \subseteq B$ such that $C \to x$ for some $x \in \mathbb{G}$. First assume we can pick a strictly increasing $n(i) \in \omega$ such that $c_i \in C \cap B^{n(i)} \cdot t_{n(i)}$ for each $i \in \omega$. Then $c_i \in U_{n(i)} \cdot t_{n(i)} \subseteq W_{n(i)} \setminus P$, so by (7b) $t_{n(i)} \to x$. Now $c_i = s^{n(i)}_{j(i)}(k_i) \cdot d^{n(i)}_{j(i)}(k_i) \cdot t_{n(i)}$, where $d^{n(i)}_{j(i)}(k_i) \to 1_{\mathbb{G}}$, $c_i \to x$ and $t_{n(i)} \to x$. This implies $s^{n(i)}_{j(i)}(k_i) \to 1_{\mathbb{G}}$, contradicting the choice of S^n_i.

Thus we may assume that $C \subseteq B^n \cdot t_n$ for some $n \in \omega$. If there exists a strictly increasing $i(k)$, $k \in \omega$, such that $c_k \in B^n_{i(k)} \cdot t_n$ for infinitely many $c_k \in C$ then $c_k = s^n_{i(k)}(m_k) \cdot d^n_{i(k)}(m_k) \cdot t_n$, where $c_k \to x$ and $d^n_{i(k)}(m_k) \to 1_{\mathbb{G}}$, implying $s^n_{i(k)}(m_k) \to x \cdot t_n^{-1}$, contradicting the choice of S^j_i. Otherwise, $c_k \in B^n_i \cdot t_n$ for some $i, n \in \omega$ and infinitely many $c_k \in C$. An argument similar to the one above results in a contradiction.

We have therefore proved the following claim (the argument above deals with the case of $p = 1_{\mathbb{G}}$ only; the general case follows by translating N if necessary):

(9) Let $N \in \mathbf{nwd}([\tau])$ and $p \in \mathbb{G}$. There exists a set $B \subseteq \mathbb{G} \setminus N$ such that $p \in \overline{B}$, $B \in \mathbf{nwd}([\tau])$, and B is almost disjoint from every convergent sequence in τ.

Now let $\mathbb{G} = \{g_i : i \in \omega\}$ list all the points of $\mathbb{G}$. Using (9), recursively build $B_i \in \mathbf{nwd}([\tau])$ such that $g_i \in \overline{B_i}$, B_i is almost disjoint from every convergent sequence in τ and $B_n \subseteq \mathbb{G} \setminus \bigcup_{i<n}(P \cdot g_i \cup B_i)$ where $P \in \mathbf{nwd}([\tau])$ is a sequential neighborhood of $1_{\mathbb{G}}$ constructed at the beginning of the proof.

Put $B = \bigcup_{i\in\omega} B_i$. Then B is dense in $\mathbb{G}$ by construction. Suppose $S \subseteq B$ is an infinite sequence with $S \to g_i$ for some $i \in \omega$. By trimming S if necessary, we may assume $S \subseteq P \cdot g_i$. Since $B_j \cap S$ is finite for every $j \in \omega$ there is an $s \in S \cap P \cdot g_i$ such that $s \in B_n$ for some $n > i$, contradicting $s \in B_n \subseteq \mathbb{G} \setminus \bigcup_{i<n}(P \cdot g_i \cup B_i)$. Thus B is almost disjoint from every convergent sequence in τ, contradicting the condition that $(\mathbb{G}, \tau)$ is strongly groomed. ∎

Lemma 4.19. *Let (G, τ) be a nonmetrizable countable topological group such that τ has a countable π-network $\mathcal{U}$ at $1_{\mathbb{G}}$ such that each $U \in \mathcal{U}$ is $[\tau]$-dense in some open subset of $(G, [\tau])$. Then $(G, [\tau])$ is a first-countable topological group and there exists a subset $D \subseteq G$, dense in τ and almost disjoint from every convergent sequence in (G, τ). In particular, (G, τ) is not strongly groomed.*

Proof. Since (G, τ) is regular, we may assume that each $U \in \mathcal{U}$ is closed in τ. By taking the interior of each $U \in \mathcal{U}$ in $[\tau]$ we may construct a countable π-network for (G, τ) consisting of open subsets of $(G, [\tau])$. To simplify the notation we will assume that $\mathcal{U}$ is such a network.

By considering the appropriate translations we may additionally assume that $1_{\mathbb{G}} \in U$ for every $U \in \mathcal{U}$.

Let $S \subseteq^* U$ for every $U \in \mathcal{U}$. Since $\mathcal{U}$ is a π-network at $1_{\mathbb{G}}$, it follows that $S \to 1_{\mathbb{G}}$ in τ, so $S \to 1_{\mathbb{G}}$ in $[\tau]$. Since $[\tau]$ is sequential and homogeneous, $(G, [\tau])$ is first-countable. By Lemma 4.2, $(G, [\tau])$ is a topological group.

Let $\{U_n : n \in \omega\}$ be a countable base of open neighborhoods at $1_{\mathbb{G}}$ for $(G, [\tau])$ such that $U_{n+1} \subseteq U_n$ for every $n \in \omega$. Since (G, τ) is not metrizable, we may assume that each U_n is nowhere dense in τ. Since G is countable, we may further assume that each U_n is clopen in $[\tau]$. Recursively build an open partition $\{V_i : i \in \omega\}$ of $(G, [\tau])$ so that $V_i \subseteq U_{n(i)} \cdot g_i$ for some strictly increasing $n(i) \in \omega$ and $g_i \in V_i$.

Let $U \subseteq G$ be open in τ. Find a $V \ni 1_{\mathbb{G}}$ open in τ such that $V^{-1} \cdot V \cdot x \subseteq U$ for some $x \in U$. Let $m \in \omega$ be such that $U_{n(i)} \subseteq V$ for all $n(i) > m$. Since each V_i is nowhere dense in τ, there exists an $i \in \omega$ such that $n(i) > m$ and $g \in V \cdot x \cap V_i$ for some $g \in G$. Then $g_i \in U_{n(i)}^{-1} \cdot g \subseteq V^{-1} \cdot V \cdot x \subseteq U$. Thus the set $D = \{g_i : i \in \omega\}$ is dense in (G, τ). Since V_i form an open partition of $(G, [\tau])$ and $g_i \in V_i$ for each $i \in \omega$, D is closed and discrete in $[\tau]$. ∎

The proof of the following lemma is identical to that of [14, Lemma 26] (although the statement in [14] uses the group topology instead of $[\tau]$, the basic argument stays the same).

LEMMA 4.20. *Let* $(\mathbb{G}, \tau)$ *be a countable topological group. Then each of* $\mathbf{nwd}([\tau])$, $\mathbf{csc}([\tau])$, *and* $\mathbf{cpt}([\tau])$ *is a weakly closed, tame, invariant ideal.*

The next lemma finishes the proof of Theorem 4.12.

LEMMA 4.21. *Let* $\mathbb{G}$ *be a countable, strongly groomed nonmetrizable group such that* $(\mathbb{G}, [\tau])$ *is not* k_ω. *Then one of the* $\mathbf{nwd}([\tau])$, $\mathbf{csc}([\tau])$, *and* $\mathbf{cpt}([\tau])$ *is a tame, weakly closed invariant ideal that satisfies neither* (1) *nor* (2) *of the* IIA.

Proof. Suppose the contrary. Since $\mathbb{G}$ is not k_ω, $\mathbf{cpt}([\tau])$ does not have a countable sequence capturing subfamily. Thus alternative (2) of IIA, applied to $\mathbf{cpt}([\tau])$, must hold and there exists a countable almost π-network consisting of $\mathbf{cpt}([\tau])^+$-subsets. Thinning each of the subsets if necessary we may assume that $\mathbb{G}$ has a π-network $\mathcal{D}$ consisting of infinite closed discrete subspaces.

Applying Lemma 4.16 shows that alternative (1) of IIA, applied to $\mathbf{csc}([\tau])$, fails and therefore there exists a countable almost π-network of subsets in $\mathbf{csc}([\tau])^+$. By thinning out we may assume that $\mathbb{G}$ has a countable π-network consisting of subsets $[\tau]$-dense in themselves. Note that no such set can have a locally $[\tau]$-compact closure.

Applying Lemma 4.18 shows that alternative (1) of IIA, applied to $\mathbf{nwd}([\tau])$, fails, so by alternative (2) there exists a countable π-network of subsets of $\mathbb{G}$ whose $[\tau]$-closures have nonempty $[\tau]$-interiors. Now Lemma 4.19 implies that $\mathbb{G}$ is not strongly groomed, a contradiction. ∎

The next example shows that remotely sequential does not imply sequential even if IIA holds and the sequential coreflection is k_ω. Before describing the construction, we introduce some useful notation. Given a countable topological group G and a countable family of compact subspaces $\mathcal{K}$, one may construct the finest group topology τ on G in which each $K \in \mathcal{K}$ retains its topology inherited from G. It is easy to show that the topology is well-defined and is k_ω (see e.g. [31, Lemma 4]). We write $k_\omega(\mathcal{K})$ to denote such τ. One useful property of $k_\omega(\mathcal{K})$ is that any subset $C \subset G$, compact in $k_\omega(\mathcal{K})$, is a subset of a finite sum of translations of elements of $\mathcal{K}$, or, if $\mathcal{K}$ covers G, 'translations of' above may be omitted. If G is abelian and $B \subseteq G$, we write $\sum^n B = B + \cdots + B$ where the sum consists of n copies of B.

EXAMPLE 4.22. (MA(countable)) There exists a remotely sequential non-sequential Hausdorff group topology with a k_ω-sequential coreflection on a countable boolean group.

Proof. Let $\mathbb{B}$ be a countable infinite boolean group and τ_1 be a first-countable Hausdorff group topology on $\mathbb{B}$ such that $S \to 0$ in τ_1 for some algebraic basis $S \subseteq \mathbb{B}$ of $\mathbb{B}$ (for example, one can take a *free boolean group* over a convergent sequence to obtain such $\mathbb{B}$ and S). Let $S = \{v_0, v_1, \ldots\}$, $m \in \omega$ and define $\mathbb{B}_0(m) = \{0\}$, $\mathbb{B}_1(m) = \{v_i : i \geq m\} \cup \{0\}$, $\mathbb{B}_{n+1}(m) = \mathbb{B}_n(m) + \mathbb{B}_1(m)$ (i.e. $\mathbb{B}_n(m)$ is a sum of n copies of the m-tail of $S \cup \{0\}$). For brevity, we shall write $\mathbb{B}_n = \mathbb{B}_n(0)$. Note that every $\mathbb{B}_n(m)$ is a compact subspace of $(\mathbb{B}, \tau_1)$. Put $\tau_0 = k_\omega(\{\mathbb{B}_1\})$ and note that every $K \subseteq \mathbb{B}$ compact in τ_0 is a subset of some $\mathbb{B}_n$.

Given an $a \in \mathbb{B}$ one can uniquely (up to the order of the summands) write $a = v_{i(1)} + v_{i(2)} + \cdots + v_{i(s)}$ for some distinct $v_{i(j)} \in S$. Put $|a| = s$ and $\|a\| = \max\{i(j) : j \leq s\}$. Note that $|b| \leq n$ for every $b \in \mathbb{B}_n$.

Pick a clopen neighborhood $U_0 \in \tau_0$ of 0 and an infinite subset $D \subseteq \mathbb{B} \setminus U_0$ such that D is closed discrete in τ_0 and $D \to 0$ in τ_1. Let $D = \{d_i : i \in \omega\}$ and pick a sequence $m_0, m_1, \ldots$ such that $K^i = d_i + \mathbb{B}_i(m_i) \subseteq \mathbb{B} \setminus U_0$, $K^i \cap K^j = \varnothing$ for $i \neq j$, and $K_0 = \bigcup\{d_i + \mathbb{B}_i(m_i) : i \in \omega\} \cup \{0\}$ is compact in τ_1. Note that this choice of K^i and the compactness of K_0 imply that for any sequence $b_i \in K^{q(i)}$, $b_i \to 0$ in τ_1 as long as $q(i)$ is strictly increasing.

Let $\{L_\lambda : \lambda < \mathfrak{c}\}$ list all closed discrete subspaces of $(\mathbb{B}, \tau_0)$. For convenience we shall assume that L_0 is a closed discrete subspace of $(\mathbb{B}, \tau_1)$ as well.

For each $\lambda < \mathfrak{c}$ construct, by recursion on λ, topologies $\tau^\lambda \subseteq \tau_0$ on $\mathbb{B}$, families $\tau_0^\lambda \subseteq \tau^\lambda$, and compact subspaces $H_\lambda \subseteq K_0$ of $(\mathbb{B}, \tau_1)$ such that

(10) $\tau^\alpha \subseteq \tau^\lambda$ for $\alpha < \lambda$;
(11) τ_0^λ is a local base of open neighborhoods of 0 for τ^λ and $|\tau_0^\lambda| < \mathfrak{c}$;
(12) whenever $U \in \tau^\lambda$, $k \in \omega$, and $o \in [\lambda+1]^{<\omega}$ there exist $b \in \mathbb{B}$ and $m, n \in \omega$ such that $b + \mathbb{B}_n(m) \subseteq U \cap \bigcap\{H_\beta : \beta \in o\}$ and $n > k$;
(13) L_λ is a closed discrete subspace of $(\mathbb{B}, \tau^\lambda)$.

Let $\tau^0 = \tau_1$, $H_0 = K_0$, and let τ_0^0 be a countable local base of neighborhoods of 0 in τ^0. Properties (10)–(13) follow for $\lambda = 0$.

Let $\lambda < \mathfrak{c}$ and suppose τ^α, τ_0^α, and H_α have been constructed to satisfy (10)–(13) for all $\alpha < \lambda$. Put $\tau^{<\lambda} = \bigcup\{\tau^\alpha : \alpha < \lambda\}$. Then $\tau^{<\lambda} \subseteq \tau_0$ is a

Hausdorff group topology on $\mathbb{B}$. Let $\tau_0^{<\lambda}$ be a local base of open neighborhoods of 0 for $\tau^{<\lambda}$ such that $|\tau_0^{<\lambda}| < \mathfrak{c}$.

Let P be the set of all compact F in $(\mathbb{B}, \tau_0)$ with the following properties:

(14) $F = \bigcup \{b_i + \mathbb{B}_{q(i)}(p(i)) : i \le s\}$ for some $s \in \omega$, where the sequence of $q(i)$ is strictly increasing; put $F(i) = b_i + \mathbb{B}_{q(i)}(p(i))$;

(15) $F(i) \subseteq K^{r(i)}$ for each i, where the sequence $r(i)$ is strictly increasing;

(16) $(b_i + \sum^{q(i)}(\{b_j : j < i\} \cup \{0\}) + \mathbb{B}_i) \cap L_\lambda = \varnothing$ for every $i \le s$.

Now P is countable. Given $F_0, F_1 \in P$, define $F_0 \le F_1$ if and only if $F_1 \subseteq F_0$ and $F_0 \cap K^i \cap F_1 \ne \varnothing$ implies $F_0 \cap K^i = F_1 \cap K^i$.

Let $U \in \tau_0^{<\lambda}$, $o \in [\lambda]^{<\omega}$, and $k \in \omega$. Define $\mathcal{D}(U, o, k)$ to be the set of all $F \in P$ such that $b + \mathbb{B}_n(m) \subseteq F \cap U \cap \bigcap \{H_\beta : \beta \in o\}$ for some $n \ge k$ and $m \in \omega$.

Let $F_1 \in P$, $U \in \tau_0^{<\lambda}$, $o \in [\lambda]^{<\omega}$, and $k \in \omega$. Define $q(s+1) = \max\{\{q(i) : i \le s\} \cup \{k\}\} + 1$ where $q(i)$ and s are as in the definition of F in (14). Now $\sum^{q(s+1)}(\{b_i : i \le s\} \cup \{0\}) + \mathbb{B}_{s+1} \subseteq \mathbb{B}_t$ for some $t \in \omega$. Using (12) find $b \in \mathbb{B}$, $q \in \omega$, and $p \in \omega$ so that $b + \mathbb{B}_q(p) \subseteq K^{r(s+1)} \cap U \cap \bigcap \{H_\beta : \beta \in o\}$ for some $r(s+1) > \max\{r(i) : i \le s\}$ and $q > q(s+1) + t + 1$.

Since L_λ is closed and discrete in τ_0, the intersection $L_\lambda \cap \mathbb{B}_{|b|+2t+1}$ is finite. Hence there is a $p' \in \omega$ such that $p' > \max\{\|g\| : g \in (L_\lambda \cap \mathbb{B}_{|b|+2t+1}) \cup \{b\}\}$. Put $p'' = \max\{p, p'\} + 1$, $v = v_{p''} + v_{p''+1} + \cdots + v_{p''+t}$, and $b_{s+1} = b + v$. Define $F_0 = F_1 \cup b_{s+1} + \mathbb{B}_{q(s+1)}(p)$ and note that $F_0(s+1) = b_{s+1} + \mathbb{B}_{q(s+1)}(p) \subseteq b + \mathbb{B}_q(p) \subseteq K^{r(s+1)}$ by the choice of b_{s+1} and $q(s+1)$.

Now (14) and (15) are satisfied by the choice of F_0. Let $a \in b_{s+1} + \sum^{q(s+1)}(\{b_i : i \le s\} \cup \{0\}) + \mathbb{B}_{s+1}$. Then $a = b + v + v'$, where $v' \in \mathbb{B}_t$. Thus $a \in \mathbb{B}_{|b|+2t+1}$. By the choice of p'', $\|b\| < p''$. Since $|v| = t + 1 > |v'|$, we have $\|v + v'\| \ge p'' > p' > \|b\|$, which implies $\|a\| > p'$ and $a \notin L_\lambda$. This shows (16), so $F_0 \in P$. Now $F_0 \le F_1$ and $F_0 \in \mathcal{D}(U, o, k)$ by the construction. Thus each $\mathcal{D}(U, o, k)$ is dense in $(P, \le)$.

Using MA(countable) pick a directed $G \subseteq P$ such that $G \cap \mathcal{D}(U, o, k) \ne \varnothing$ for every $U \in \tau_0^{<\lambda}$, $o \in [\lambda]^{<\omega}$, and $k \in \omega$. Put $H'_\lambda = \bigcup G \cup \{0\}$. Properties (14) and (15) imply that $H'_\lambda = \bigcup \{H'_\lambda(i) : i \in \omega\} \cup \{0\}$, where $H'_\lambda(i) = b_i + \mathbb{B}_{q(i)}(p(i)) \subseteq K^{r(i)}$ and the sequences $q(i)$ and $r(i)$ are strictly increasing. Observe that $b_i \in K^{q(i)}$, so by the remark immediately following the definition of K_0 we have $T = \{b_i : i \in \omega\} \to 0$ in τ_1. Note that the choice of b_i involves no ambiguity using simple scatteredness arguments. Thus $T \cup \{0\}$ is a compact subspace of $(\mathbb{B}, \tau_1)$.

Put $\tau' = k_\omega(\{\mathbb{B}_1, T \cup \{0\}\})$. Let $k, m \in \omega$ be arbitrary and let $a \in K = \sum^k (T \cup \{0\}) + \mathbb{B}_m$. Write $a = b_{i(1)} + \cdots + b_{i(k')} + b$, where $i(1) > i(j)$ and $i(j)$ are distinct for any $1 < j \le k' \le k$, and $b \in \mathbb{B}_m$. If $i(1) > m$ and $q(i(1)) > k$ then $a \notin L_\lambda$ by (16). Therefore $K \cap L_\lambda \subseteq \sum^k \{b_i : q(i) \le k \text{ or } i \le m\} + \mathbb{B}_m$

and is thus finite, since L_λ is closed and discrete in τ_0 and the sum on the right hand side is a compact subspace of $(\mathbb{B},\tau_0)$.

Let $\tau'' \subseteq \tau' \subseteq \tau_0$ be a first-countable Hausdorff group topology on $\mathbb{B}$ such that L_λ is a closed discrete subspace of $(\mathbb{B},\tau'')$. Let τ_0'' be a countable base of neighborhoods of τ'' at 0. Then $T \to 0$ in τ'', so we can pick a sequence $p'(i) \geq p(i)$ such that $b_i + \mathbb{B}_{q(i)}(p'(i)) \subseteq U$ for all but finitely many $i \in \omega$ whenever $U \in \tau_0''$. Put $H_\lambda = \bigcup\{b_i + \mathbb{B}_{q(i)}(p'(i)) : i \in \omega\}$ and let τ^λ be the topology generated by the subbase $\tau'' \cup \tau^{<\lambda}$. As both τ'' and $\tau^{<\lambda}$ are Hausdorff group topologies on $\mathbb{B}$, so is τ^λ. The sets $U \cap V$ where $U \in \tau_0^{<\lambda}$ and $V \in \tau_0''$ form a local base τ_0^λ of open neighborhoods of 0 in τ^λ. Thus $|\tau_0^\lambda| = |\tau_0^{<\lambda}| < \mathfrak{c}$. This establishes (10)–(11) and (13).

Given any finite $o \in [\lambda]^{<\omega}$, $W \in \tau_0^\lambda$, and $k \in \omega$, write $W = U \cap V$ where $U \in \tau_0^{<\lambda}$ and $V \in \tau_0''$, and find $F \in G$ such that $F \in \mathcal{D}(U, o, k)$. Then $F \subseteq H'_\lambda$ and therefore there exist $b \in \mathbb{B}$ and $n, m \in \omega$ such that $b + \mathbb{B}_n(m) \subseteq U \cap \{H_\beta : \beta \in o \cup \{\lambda\}\}$ and $n > k$. We may also arrange (by making k larger if necessary and using scatteredness arguments) for $b = b_i$ where i is sufficiently large so that $b + \mathbb{B}_n(m) \subseteq V$. It follows that $b + \mathbb{B}_n(m) \subseteq W \cap \{H_\beta : \beta \in o \cup \{\lambda\}\}$, which shows (12) is satisfied.

Define $\tau = \bigcup_{\lambda<\mathfrak{c}} \tau^\lambda$. Then τ is a Hausdorff group topology on $\mathbb{B}$. Since $\tau \subseteq \tau_0$, every compact subspace of τ_0 is also compact in τ. Property (13) implies that every closed discrete subspace $L = L_\gamma$ of $(\mathbb{B},\tau_0)$ for some $\gamma < \mathfrak{c}$ is also closed and discrete in τ. This in turn implies that every sequence converging in τ is also converging in τ_0, so the sequential coreflection of $(\mathbb{B},\tau)$ is $(\mathbb{B},\tau_0)$ and τ is remotely sequential.

Let $0 \in U \in \tau$. Then (12) shows that $0 \in \overline{K_0 \setminus \{0\}} \subseteq \mathbb{B} \setminus U_0$. Thus $\tau \neq \tau_0$, so τ is not sequential. ∎

Remarks 4.23. The two topologies constructed above (τ_0 and τ) share the same closed discrete subspaces while there exists a locally compact subspace ($K_0 \setminus \{0\}$) that is closed in τ_0 and has an accumulation point in τ. A similar construction may be used to build a topology τ^+ that has the same closed locally compact subspaces as τ_0, yet $\tau^+ \neq \tau_0$. Any such τ^+ will also be remotely sequential. We omit the proofs.

The topology above cannot be made *linear* (i.e. have a neighborhood base at 0 consisting of subgroups). The proof is omitted.

To finish this section, we consider the following definition that generalizes the concept of a k_ω-space.

Definition 4.24. Let γ be an infinite cardinal. Let X be a topological space. Call X a *k_γ-space* (resp. a *c_γ-space*) if there exists a family $\mathcal{K}$ of compact (resp. countably compact) subspaces of X such that $|\mathcal{K}| \leq \gamma$ and $U \subseteq X$ is open in X if and only if $U \cap K$ is relatively open in K for every $K \in \mathcal{K}$.

The next theorem shows that strongly groomed groups have sequential coreflections that are either k_ω or determined by a large number of compact subspaces.

THEOREM 4.25. (IIA) *Let G be a countable topological group whose sequential coreflection is a k_γ-space for $\gamma < \mathfrak{b}$. Then either the sequential coreflection of G is k_ω or there exists a dense subset of G almost disjoint from every convergent sequence.*

Proof. Suppose there is no dense subset of $\mathbb{G}$ almost disjoint from every convergent sequence, i.e. $\mathbb{G}$ is strongly groomed. Then $\mathbb{G}$ is groomed by Lemma 4.10, so IIA applies.

Let τ be the topology of G and let $\mathcal{K}$ be the family of compact subspaces of G of size at most $\gamma < \mathfrak{b}$. Apply IIA to $\mathbf{cpt}(G)$. Alternative (1) implies that $[\tau]$ is k_ω. Otherwise, (2) implies the existence of a sequence of $D_i \subseteq G$ closed and discrete in $[\tau]$ such that each intersection $D_i \cap K$ is finite where $K \in \mathcal{K}$ and for every open $U \ni 0$ there exists a D_i such that $D_i \setminus U$ is finite.

Let $D_i = \{d_i^j : j \in \omega\}$. Assume D_i are disjoint and $\mathcal{K}$ is closed under finite unions. For each $K \in \mathcal{K}$ define a function $f_K : \omega \to \omega$ by $f_K(i) = \max\{j : d_i^j \in K\} \cup \{0\}$. Since $|\mathcal{K}| < \mathfrak{b}$, there exists an $f : \omega \to \omega$ such that $f \geq^* f_K$ for every $K \in \mathcal{K}$. Define $D'_i = D_i \setminus \{d_i^j : j \leq f(i)\}$. It is a simple observation that $D = \bigcup_{i\in\omega} D'_i$ has the property that $\overline{D} \ni 0$ and every intersection $D \cap K$ is finite where $K \in \mathcal{K}$. In particular D is almost disjoint from every convergent sequence in G.

Let $D = \{c_j : j \in \omega\}$ and $G = \{g_i : i \in \omega\}$. Put $C'_i = Dg_i$. Recursively pick $C_i \subseteq C'_i$ such that $g_i \in \overline{\bigcup_{n\leq i} C_n \setminus \{g_i\}}$ and C_i are disjoint. At step k if $g_k \in \overline{\bigcup_{n<k} C_n \setminus \{g_k\}}$ put $C_k = \varnothing$. Otherwise put $C_k = C'_k \setminus \overline{\bigcup_{n<k} C_n \setminus \{g_k\}}$. Note that $g \in \overline{C_i \setminus F}$ for some $i \in \omega$ and any finite F whenever $g \in G$. For simplicity assume every $C_i \neq \varnothing$ (this assumption can be eliminated by reindexing C_i) and let $C_i = \{c_i^j : j \in \omega\}$

Define $g_K : \omega \to \omega$ by $g_K(i) = \max\{j : c_i^j \in K\}$ for every $K \in \mathcal{K}$. As before, let $g : \omega \to \omega$ be such that $g \geq^* g_K$ for every $K \in \mathcal{K}$. Put $B = \bigcup\{c_i^j : j > g(i)\}$. Then B is dense in G by the property of C_i mentioned in the previous paragraph. The choice of B implies that every intersection $B \cap K$ with $K \in \mathcal{K}$ is finite. Since every convergent sequence is contained in some $K \in \mathcal{K}$, B is almost disjoint from every sequence convergent in τ. ∎

5. Embeddings in sequential groups. The following example answers [24, Question 4.3(v)] as well as [3, Problem 7.1.5].

EXAMPLE 5.1. There exist (1) a separable completely regular space that cannot be embedded into a sequential topological group and (2) a separable completely regular first-countable space that cannot be embedded into a sequentially compact Hausdorff space.

Proof. Let $A \subseteq [\omega]^\omega$, $A = \{a_\lambda : \lambda < \mathfrak{c}\}$ be a MAD family of size $\mathfrak{c}$. Let also $\mathcal{C} = \{C_\lambda : \lambda < \mathfrak{c}\}$ list all the $C \in ([\omega]^\omega)^\omega$. For brevity, put $A_i = A \times \{i\}$, $i \in \{1, 2\}$, and recursively define a topology on $X = \omega \times \omega \cup A_1 \cup A_2$. Let $\lambda < \mathfrak{c}$. Pick disjoint $D_1^\lambda, D_2^\lambda \subseteq \omega$ so that $D_1^\lambda \cup D_2^\lambda = \omega$ and $D_i^\lambda \cap C_\lambda(j)$ is infinite for $i \in \{1, 2\}$ and any $j \in \omega$. Leave every point in $\omega \times \omega \subseteq X$ isolated and define the basic neighborhoods of $a \in A_i$ by $U_n(a) = \{a\} \cup (A_i \setminus n) \times D_i^\lambda$ where $a = a_\lambda \times \{i\}$ and $n \in \omega$.

Thus defined, X becomes 2-scattered, Hausdorff, first-countable, and zero-dimensional (hence completely regular). Let $X \subseteq Y$, where Y is such that for every $n \in \omega$ there exists an infinite $C(n) \subseteq \omega$ such that $\{n\} \times C(n) \to y_n \in Y$. Then $C \in \mathcal{C}$ so $C = C_\lambda$ for some $\lambda < \mathfrak{c}$. Let $n \in \omega$; then $U_n(a_\lambda \times \{i\}) \cap (\{n+1\} \times C(n+1) = \{n+1\} \times (D_i^\lambda \cap C_\lambda(n+1)) \to y_{n+1}$ where $i \in \{1, 2\}$. This shows that Y is not Urysohn. A more detailed analysis would show that if Y is sequentially compact then Y cannot be Hausdorff.

Define Z to be the topological sum of X constructed above and the sequential fan $\mathbb{S}(\omega)$. Suppose $Z \subseteq G$ for some sequential topological group G. By [34], G contains a closed copy of $\mathbb{S}(\omega)$. The argument in the preceding paragraph implies that for all but finitely many $n \in \omega$, the sets $\{n\} \times \omega \subseteq X \subseteq G$ are closed and discrete in G. Therefore for some $n \in \omega$, every subspace $\{a_\lambda \times \{i\}\} \cup ((\omega \setminus n) \times D_i^\lambda$ is a closed copy of $D(\omega)$ in G. This contradicts the assumption that G is sequential (see Lemma 4.5). ∎

REMARKS 5.2. The space above is not normal so the existence of normal (or better still, countable) examples of spaces not embeddable in sequential topological groups remains an open question. In a model of IIA *most* countable Fréchet spaces with a single nonisolated point (even with a countable cs^*-network) cannot be embedded into *countable* sequential groups. Indeed, in such models there are only ω_1 different sequential group topologies on countable groups and thus at most $\mathfrak{c}$ different topologies embeddable into such groups. At the same time there are $2^\mathfrak{c}$ nonhomeomorphic countable Fréchet spaces with a countable cs^*-network that have a single nonisolated point (see e.g. [30, Example 2]). The next lemma gives a more specific example.

LEMMA 5.3. (IIA) *Let X be a countable, Fréchet, α_4, and non-first-countable space. Then X cannot be embedded into a countable sequential topological group.*

Sketch. Only non-Fréchet groups are of interest. Use the k_ω property to recursively build a copy of $\mathbb{S}(\omega)$ in X (the recursion does not stop because X is not first-countable). ∎

We now show that $S(\omega)$ distinguishes between Fréchet and non-Fréchet groups in the class of general (not necessarily countable) sequential groups in the model of IIA built in [14] (this is referred to as $\mathsf{IIA}+$ below). Whether IIA by itself implies the conclusion of Theorem 5.5 is still an open question.

LEMMA 5.4. ($\mathsf{IIA}+$) *Let $H \in V[\mathbb{G}]$ be a separable sequential group that is not Fréchet. Then there exists a closed normal subgroup $K \subseteq H$ such that $\psi\chi(H/K) \leq \omega_1$ and H/K contains a closed copy of $\mathbb{S}(\omega)$.*

Proof. Let $H \in V[\mathbb{G}]$ be a separable sequential group and $K \subseteq H$ be a closed normal subgroup of H such that $\psi\chi(H/K) \leq \omega_1$. Suppose H/K does not contain a closed copy of $\mathbb{S}(\omega)$. If H/K is Fréchet then it is metrizable, since H is separable. Otherwise, there exist disjoint convergent sequences $H/K \supseteq S_n \to 1 \in H/K$, $n \in \omega$, such that no 'diagonal' sequence in $\bigcup S_n$ converges to 1.

Since $\psi\chi(H/K) \leq \omega_1 < \mathfrak{b}$, one may find a cofinite $S'_n \subseteq S_n$ such that any 'diagonal' convergent sequence in $\bigcup S'_n$ must converge to 1. Since no 'diagonal' sequence in $\bigcup S_n$ converges to $1 \in H/K$ and H/K is sequential, it follows that $\bigcup S'_n$ is a closed copy of $\mathbb{S}(\omega)$ in H/K, contradicting the choice of K.

Therefore we may assume that whenever a closed normal subgroup $K \subseteq H$ is such that $\psi\chi(H/K) \leq \omega_1$, the group H/K is metrizable (i.e. H is ω_1-collapsible in the terminology of [29]).

Let $X \subseteq H$ be a countable dense subgroup of H large enough to witness that H is not Fréchet. Using standard arguments, find an $\alpha < \omega_2$ such that X, H, and $G = V[\mathbb{G}_\alpha] \cap H$ satisfy the conditions of [29, Lemma 18], as well as the additional property that if $\psi\chi(H) > \omega$, then $\psi\chi(G) > \omega$ as well.

Just as in the argument following the proof of Lemma 20 in [29], we conclude that G is hereditarily Lindelöf and thus has pseudocharacter ω. Thus $\psi\chi(H) = \omega$ and we may use an argument similar to the one above to find a closed copy of $\mathbb{S}(\omega)$ in H. ∎

THEOREM 5.5. ($\mathsf{IIA}+$) *Every separable nonmetrizable sequential group contains a closed copy of $\mathbb{S}(\omega)$.*

Proof. Using Lemma 5.4 find a closed normal subgroup $K \subseteq H$ such that H/K contains a closed copy $\bigcup S_n$ of $\mathbb{S}(\omega)$ where $S_n \to 1$ are disjoint convergent sequences. Let $p : H \to H/K$ be the appropriate quotient map. Since $S_n \to 1$ in H/K, there exists an infinite $T_n \subseteq p^{-1}(S_n)$ such that $T_n \to s_n \in K$ and $p|_{T_n}$ is 1-1. It follows that $\bigcup T_n \cdot s_n^{-1}$ is a closed copy of $\mathbb{S}(\omega)$ in H. ∎

COROLLARY 5.6. ($\mathsf{IIA}+$) *Every countably compact subspace of a separable sequential nonmetrizable group is nowhere dense.*

Corollary 5.7. (IIA+) *The character of every separable sequential group is either* $\mathfrak{c} > \omega_1$ *or* ω.

Recall that a group G is called *sequentially complete* if for any H such that G is a dense subgroup of H, and any $S \subseteq G$, if $S \to h$ in H then $h \in G$.

Corollary 5.8. (IIA+) *A sequential non-Fréchet group is sequentially complete.*

Proof. Let H be a topological group and $G \subseteq H$ is a sequential subgroup of H that is not Fréchet such that there exists an $S \to a \in H \setminus G$ with $S \subseteq G$. Find a separable subgroup $G' \subseteq G$ that is sequential and not Fréchet and $S \subseteq G'$. Then G' contains closed copies of $\mathbb{S}(\omega)$ and $\mathbb{D}(\omega) \subseteq S \cdot S^{-1}$ which contradicts Lemma 4.5. ∎

Corollary 5.9 (IIA+). *A sequential subgroup of a sequential group is either Fréchet or closed.*

We now turn to the question of strengthening the statement of IIA.

We use [14, Example 29] to show that the alternative (1) of IIA cannot be strengthened to require that the countable family $\mathcal{S}$ satisfy the additional property that for any sequence $C \to x$ and any open neighborhood $U \ni x$ there exists an element $S \in \mathcal{S}$ such that $S \subseteq U$ and $S \cap C$ is infinite. A family with this property is called a *cs*-network at* x (see for example [4]). The authors of [4] proved that any countable sequential group with a countable *cs**-network is k_ω (see Lemma 2.14 above). Thus such a stronger version of the IIA would have enabled much shorter proofs for most of the results in this paper. The question about a possible extension of IIA along these lines was asked by L. Zdomskyĭ during a talk given by the first author.

After establishing some simple preliminary results, we show that such an extension of IIA is false. More specifically, we shall show below that the subgroup K of $\mathbb{G}$ generated by D in [14, Example 29] does not have a countable *cs**-network at $0_{\mathbb{G}}$, while the restriction to K of the ideal generated by the countably compact subspaces of $\mathbb{G}$ satisfies alternative (1) of IIA. Additionally, such a $\mathbb{G}$ can be constructed in a model of IIA, while K is subsequential, and therefore groomed.

Lemma 5.10. *Let R be countably compact and Fréchet, and $Q \subseteq R$ be a dense subspace of R. If Q has a countable cs*-network at $x \in Q$ then R is first-countable at x.*

Proof. Let $\mathcal{K} \subseteq 2^Q$ be a countable *cs**-network for Q at $x \in Q$ closed under finite unions. Put $\overline{\mathcal{K}} = \{\overline{P} : P \in \mathcal{K}\}$ where the closure is taken in R. To show that $\overline{\mathcal{K}}$ is a countable *cs**-network at x in R consider $S \to x$ with $S \subseteq R$, and an open $U \ni x$. Since R is Fréchet, we may pick $S_i \to x_i$ where $S = \langle x_i : i \in \omega \rangle$ and $S_i \subseteq Q$. If there exists a finite $\mathcal{K}_U \subseteq \{P \in \overline{\mathcal{K}} : P \subseteq U\}$

such that $(\bigcup \mathcal{K}_U) \cap S_i$ is infinite for infinitely many $i \in \omega$ then $(\bigcup \mathcal{K}_U) \cap S$ is infinite and we are done.

Suppose such a finite subfamily does not exist and let $\{P \in \overline{\mathcal{K}} : P \subseteq U\} = \{P_i : i \in \omega\}$. Recursively choose strictly increasing indices $i_k \in \omega$ such that $\bigcup\{P_i : i \leq k\} \cap S_{i_k} = F_k$ is finite. Using the Fréchet property again, find a sequence $T \subseteq \bigcup_{k\in\omega}(S_{i_k} \setminus F_k)$ such that $T \to x$. Now $T \subseteq Q$ and $P \cap T$ is finite for every $P \in \mathcal{K}$ such that $P \subseteq U$, contradicting the choice of $\mathcal{K}$ together with the regularity of R.

Now Proposition 7(1) of [4] finishes the proof. ∎

LEMMA 5.11. *Let $\mathbb{G}$ be a topological group whose topology is determined by a countable family $\mathcal{C}$ of countably compact sequential subspaces. If $\mathbb{G}$ has a nonmetrizable countably compact subspace then there exists a $C \in \mathcal{C}$ that has an uncountable pseudocharacter.*

Proof. Let $\mathcal{C} = \{C_i : i \in \omega\}$. Since $\mathbb{G}$ has a nonmetrizable countably compact subspace, $\mathbb{G}$ must also have a countably compact subspace K of uncountable pseudocharacter. To show this we will modify the proof of [3, Lemma 3.3.22]. Let C be a countably compact nonmetrizable subspace of $\mathbb{G}$. Note that C is sequential (since $\mathbb{G}$ must be) and thus $C \times C$ is also sequential and countably compact [15]. Let $K = C \cdot C^{-1}$. Since K is a continuous image of $C \times C$ and a subspace of $\mathbb{G}$, it is also sequential and countably compact. Now if $1_{\mathbb{G}} \in K$ has a countable pseudocharacter in K it follows, just as in [3, proof of Lemma 3.3.22], that C has a G_δ diagonal and is thus compact and metrizable by the classical result of [6], contradicting the choice of C.

Note that the sequentiality of $\mathbb{G}$ implies that every $C \in \mathcal{C}$ is closed in $\mathbb{G}$. Since the topology of $\mathbb{G}$ is determined by $\mathcal{C}$, there exists a finite $\mathcal{C}' \subseteq \mathcal{C}$ such that $K \subseteq \bigcup \mathcal{C}'$, so one of $C \in \mathcal{C}'$ has an uncountable pseudocharacter. ∎

The proof of the next lemma uses standard techniques and is omitted.

LEMMA 5.12. *Let B be a continuous image of C, where both C and B are sequential and countably compact. If C is Fréchet then so is B.*

Before we proceed to construct the example promised at the beginning of this section, we shall briefly review the construction of [14, Example 29] for the reader's benefit.

The construction uses a space $X = D \cup \omega_1$ with the following properties: (1) D is a dense countable set of isolated points, (2) the subspace ω_1 has the natural topology, and (3) X is first-countable, locally compact and countably compact. Such spaces were called $\gamma\mathbb{N}$ in [20] where their existence was shown to follow from $\mathbf{t} = \omega_1$, which holds in any model of IIA (this is because IIA implies that all countable Fréchet groups are metrizable, which is false under $\mathbf{t} > \omega_1$ [11]).

The group $\mathbb{G}$ is constructed as a free boolean topological group over X. This means that (4) the points of X form an algebraic basis for $\mathbb{G}$ (thus the elements of $\mathbb{G}$ may be thought of as finite subsets of X with the symmetric difference as the algebraic operation), (5) the topology of X as a subspace of $\mathbb{G}$ is the original topology of X, and (6) the topology of $\mathbb{G}$ is the finest group topology with these properties. Theorem 2 of [33] states that such objects exist for any space X.

The argument in [14, Example 29] shows that $\mathbb{G}$ is sequential and that the topology of $\mathbb{G}$ is determined by the countable family of subspaces $\{\sum^n X : n \in \omega\}$ (i.e. $F \subseteq \mathbb{G}$ is closed if and only each $F \cap \{\sum^n X : n \in \omega\}$ is closed). Since X is first-countable, these subspaces are sequential (because $\mathbb{G}$ is and because they are closed in $\mathbb{G}$ due to the next property), as well as countably compact (as continuous images of finite products of sequential countably compact spaces). This shows that $\mathbb{G}$ is c_ω.

Let K be the subgroup of $\mathbb{G}$ generated by $D \subseteq X$. Further, let $\mathcal{I}$ be the ideal generated by the countably compact subspaces of $\mathbb{G}$ and put $\mathcal{J} = \mathcal{I}|_K$. Then $\mathcal{J}$ is a tame invariant ideal that satisfies alternative (1) (but not (2)!) of IIA.

Suppose K has a countable cs^*-network $\mathcal{K}$. The topology of $\mathbb{G}$ is determined by the family $\{\sum^n X : n \in \omega\}$. Since X^n is first-countable, and each $\sum^n X$ is sequential, Lemma 5.12 implies that each $\sum^n X$ is Fréchet. Since X is not metrizable, Lemma 5.11 implies that one of $\sum^n X$ is not first-countable. Since X algebraically generates $\mathbb{G}$ we may assume (by picking a larger n if necessary) that $0_{\mathbb{G}}$ has uncountable character in $\sum^n X$. Now $\mathcal{K}|_{\sum^n D}$ is a countable cs^*-network in $\sum^n D \ni 0_{\mathbb{G}}$ which is dense in $\sum^n X$. Lemma 5.10 gives the desired contradiction.

QUESTION 4. Does there exist, in ZFC, a countable group $(\mathbb{G}, \tau)$ such that $[\tau]$ is not regular? Does such a group exist in a model of IIA?

QUESTION 5. Are the ideals $\mathbf{nwd}(\tau)$ and $\mathbf{csc}(\tau)$ tame for a countable topological group $\mathbb{G}$?

The final result of this section answers [25, Question 7.4] by showing that IIA implies the nonexistence of sequential precompact groups that are not Fréchet. For countable sequential groups in the Cohen model this was established in [32]. It was shown in [26] that countably compact sequential non-Fréchet groups exist under $\diamondsuit$, which together with the result below implies that the existence of precompact sequential non-Fréchet groups is independent of the usual axioms of ZFC.

LEMMA 5.13. *Let G be a precompact sequential group and $H \subseteq G$ be a dense subgroup. Then for any countable $\mathcal{I} \subseteq \mathbf{nwd}(G)$ there exists a nontrivial* *sequence* $\langle y_i : i \in \omega\rangle \subseteq H$ *converging to* $1_{\mathbb{G}}$ *such that* $\langle y_i : i \in \omega\rangle \cap N$ *is finite for every* $N \in \mathcal{I}$.

Proof. Let $H \subseteq G \subseteq X$ be such that H is a dense subgroup of a sequential group G, which is a dense subgroup of a compact group X. Let $\mathcal{I} = \{N_i : i \in \omega\} \subseteq \mathbf{nwd}(X)$. Recursively pick $x_i \in H$ and open (in X) neighborhoods O_i of $\mathbf{1}_X$ so that

(17) $x_{i+1} \in O_i$ and $O_{i+1} \subseteq O_i$;
(18) $x_i \cdot O_i^{-1} \cdot (O_i \cdot O_i^{-1})^{-1} \subseteq X \setminus \bigcup_{j\le i} N_j$.

Since X is compact, the set $\{x_i : i \in \omega\}$ has an accumulation point $x \in X$. Let O be an arbitrary open neighborhood of $\mathbf{1}_X$ in X. Pick an open $U \ni x$ so that $U \cdot U^{-1} \subseteq O$. Since x is an accumulation point of $\{x_i : i \in \omega\}$, there are $i > j$ such that $x_i, x_j \in U$. Thus $x_j \cdot x_i^{-1} \in O$, which shows that $\mathbf{1}_X = 1_{\mathbb{G}}$ is an accumulation point of $S = \{x_j \cdot x_i^{-1} : i > j,\, i, j \in \omega\} \subseteq H$. Since G is sequential, there exists a nontrivial convergent sequence $T \subseteq S$.

Consider the following two cases. First, suppose T can be picked so that $T = \langle x_j \cdot x_{n(i)}^{-1} : i \in \omega\rangle$ for some fixed $j \in \omega$ and an increasing $n(i)$. Then the sequence $\langle x_{n(i)} : i \in \omega\rangle$ converges. Now the sequence $\langle x_{n(i)} \cdot x_{n(i+1)}^{-1} : i \in \omega\rangle$ converges to $1_{\mathbb{G}}$. By (18) and (17), $x_{n(i+1)} \in O_{n(i)}$ and $y_i = x_{n(i)} \cdot x_{n(i+1)}^{-1} \in x_{n(i)} \cdot O_{n(i)}^{-1} \cdot (O_{n(i)} \cdot O_{n(i)}^{-1})^{-1} \subseteq X \setminus \bigcup_{j\le n(i)} N_j$. Since $n(i)$ is increasing, $\langle y_i : i \in \omega\rangle \cap N_i$ is finite for every $i \in \omega$.

If no such T exists, we may assume, after thinning T if necessary, that $T = \langle x_{n(i)} \cdot x_{m(i)}^{-1} : i \in \omega\rangle$ where $n(i)$ is strictly increasing and $m(i) > n(i)$. Put $y_i = x_{n(i)} \cdot x_{m(i)}^{-1} \cdot (x_{n(i+1)} \cdot x_{m(i+1)}^{-1})^{-1}$ and note that $y_i \to 1_{\mathbb{G}}$ and $y_i \in x_{n(i)} \cdot O_{n(i)}^{-1} \cdot (O_{n(i)} \cdot O_{n(i)}^{-1})^{-1} \subseteq X \setminus \bigcup_{j\le n(i)} N_j$. Thus $\langle y_i : i \in \omega\rangle \cap N_j$ is finite for every $j \in \omega$. ∎

COROLLARY 5.14. (IIA) *Every separable precompact sequential group is metrizable.*

Proof. Let G be precompact sequential and let $H \subseteq G$ be a dense countable subgroup. Since H is subsequential, H is groomed. Now H has no isolated points and G is sequential, so the ideal $\mathbf{nwd}(H)$ is tame by [29, Lemma 19] and [14, Lemma 10] ([29, Lemma 19] is stated in terms of preservation of the ω-hitting property under some forcing extensions and [14, Lemma 10] states that this is equivalent to tameness). Now Lemma 5.13 implies that alternative (1) of the IIA does not hold, so H has a countable π-base. Thus, H is metrizable and so is G. ∎

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