III.3. *Construction d'une commande avec retard pour le retour à l'origine*

THÉORÈME. *On suppose que le système*

$$Y(j+1) = {}^tA\,Y(j)+{}^tB\,Y(j-k) \tag{11}$$

est dégénéré en $2k$ *par rapport au vecteur* $X(0)$ *de* $\mathbf{R}^n$, *alors si l'on pose dans* (7)

$$\begin{aligned} U(j) &= 0, \quad 0 \leqslant j \leqslant k-1, \\ U(j) &= BX(j-k) \quad k \leqslant j \leqslant 2k, \end{aligned} \tag{12}$$

alors la solution de (7) *vérifie* $X(j) = 0$, $j \geqslant 2k$.

Démonstration. La matrice résolvante associée au système (11) est la transposée de la matrice résolvante K de (8); (11) est dégénéré donc

$${}^tX(0)\,{}^tK(j) = 0, \quad j \geqslant 2k,$$

c.-à.-d.

$$K(j)X(0) = 0, \quad j \geqslant 2k.$$

Or la solution de (7) (12) est donnée par

$$X(j+1) = K(j+1)X(0),$$

donc

$$X(j) = 0, \quad j \geqslant 2k. \ \blacksquare$$

On est donc ramené à la construction d'une matrice B telle que (11) soit dégénéré par rapport à X_0. La construction proposée par Popov se transpose sans difficulté au cas discret. On obtient alors une commande de la forme:

$$U(j) = 0, \quad 0 \leqslant j \leqslant k-1,$$

$$U(j) = (r, AX(j-k))_{R^n}A^kX_0 - (r, X(j-k))_{R^n}A^{k+1}X_0$$

où r est un vecteur de $\mathbf{R}^n$ vérifiant:

$$\begin{aligned} {}^tX_0 r &= 1, \\ {}^tX_0\,{}^tA^k r &= 0, \\ {}^tX_0\,{}^tA^{k+1} r &= 0. \end{aligned}$$

Bibliographie

[1] B. Asner and A. Halanay, *Algebraic theory of pointwise degenerate delay-differential systems*, J. Diff. Equations 14.2 (1973), pp. 293–306.

[2] P. Charrier, *Equations différentielles avec retard: applications de la dégénérescence à la théorie de la commande*, Thèse de spécialité, Université de Bordeaux I, 1972.

[3] —, and Y. Haugazeau, *On the degeneracy of linear time-invariant delay-differential systems*, J. Math. Anal. Appl. 52.1 (1975), pp. 42–55.

[4] V. M. Popov, *Delay-feedback time optimal linear time invariant control systems*, Proc. Confer. on Differ. Equations, Washington, 1971, Editor: L. Weiss.

[5] —, *Pointwise degeneracy of linear time invariant delay-differential equations*, J. Diff. Equations 11.3 (1972), pp. 541–561.

BANACH CENTER PUBLICATIONS
VOLUME 1

ESTIMATION OF FUNCTIONS OF A DEPENDENT VARIABLE

G. CHAVENT

Université Paris IX, Dauphine, France

1. Introduction

In many situations, systems are governed by ordinary (or partial) differential equations, which are nonlinear when some coefficients are modelled as functions of a dependent variable (as pressure, or temperature). In some cases, it is difficult to determine directly those dependences experimentally. But it is often possible to let the system evolve, and to record dynamic measurements from the system.

The aim of this paper is to show how to use those measurements in order to determine the unknown function of a dependent variable.

We do not suppose a priori any closed form of the formula for the unknown function, but only suppose it belongs to a given function space. In the numerical applications, the unknown functions are discretized and we determine them for a finite number of values of the dependent variable, which enables us to take into account many physical constraints on the unknown function (as upper and lower bounds, concavity or convexity etc.).

2. Theory

Let $\Omega \subset \mathbf{R}^n$ be a domain of boundary $\partial\Omega$ and let $(0, T) \subset \mathbf{R}$ be the time interval on which the system of interest evolves, and $Q = \Omega \times (0, T)$.

Let $y\colon Q \to \mathbf{R}$ be the *state* of the distributed system which is governed by an equation of the form

$$\psi(a(y), y) = f, \tag{1}$$

where $a\colon \mathbf{R} \to \mathbf{R}$ is the unknown nonlinearity (so $a(y)$ is a function of Q into $\mathbf{R}$), f is the known second member and ψ is a known mapping which relates the state y to the second member f and the unknown function a (ψ can be a heat equation with an initial and boundary condition for instance, as shown in Section 3.)

With a *given function* a we can associate a number $\mathcal{T}(a)$ (for instance by solving equation (1) formally and defining $\mathcal{T}(a)$ as the squared norm of the difference between the computed and the measured output).

Suppose now that:

(2) We are looking for a function a which is bounded with a bounded derivative on $\boldsymbol{R}$, i.e., $a \in C^1(\boldsymbol{R})$.

(3) We are given a set $\mathscr{A}_C \subset C^1(R)$ such that the criterion $\mathscr{T}$ can be calculated for every $a \in \mathscr{A}_C$ (that is, equation (1) has a unique solution in y for any $a \in \mathscr{A}_C$), and such that

$$\mathscr{T}\colon \mathscr{A}_C \to \boldsymbol{R} \text{ is Fréchet derivable on } \mathscr{A}_C.$$

The specification of the set $\mathscr{A}_C$ and the proof of the derivability of the functional $\mathscr{T}$ are to be made directly for each application.

Suppose finally that all the a priori information on the unknown function a results in the set $\mathscr{A}_{\text{ad}}$ of *admissible* functions (which is generally a closed subset of $\mathscr{A}_C$).

We can then formulate as usual the estimation problem of the function a as a minimization problem:

(4) *Find* $\hat{a} \in \mathscr{A}_{\text{ad}}$ *such that*

$$\mathscr{T}(\hat{a}) \leqslant \mathscr{T}(a) \quad \textit{for all } a \in \mathscr{A}_{\text{ad}}.$$

Under the preceding assumptions there is no result of existence and/or uniqueness of such a minimum; the answer to those questions, when possible, is to be found by the direct study of each application.

In this section we are only concerned with the effective minimization of the criterion. The classical available numerical methods require the computation of the gradient of $\mathscr{T}$: we will show here how this gradient can easily be determined.

THEOREM 1. *Under hypotheses* (2) *and* (3), *suppose that the derivative* $\mathscr{T}'(a)$ *of* $\mathscr{T}$ *is of the form*

$$\mathscr{T}'(a)\cdot\delta a = \int_Q \delta a\big(y(x,t)\big)\,\varphi(x,t)\,dx\,dt \tag{5}$$

with

$$y,\ \varphi \in L_1(Q). \tag{6}$$

Define, for every $\zeta \in \boldsymbol{R}$,

$$\begin{aligned} Q_\zeta &= \{(x,t)\in Q \mid y(x,t) \geqslant \zeta\},\\ S_\zeta &= \{(x,t)\in Q \mid y(x,t) = \zeta\} \end{aligned} \tag{7}$$

(defined up to a set of zero measure)

$$\gamma(\zeta) = \int_{Q_\zeta} \varphi(x,t)\,dx\,dt. \tag{8}$$

Then the function γ *is bounded on* $\boldsymbol{R}$, *and continuous except for the values of* ζ *for which* S_ζ *has non-zero measure*; *at such a point*, γ *has a discontinuity of the first kind.* γ *defines a distribution on* $\boldsymbol{R}$ *and we have for every* $\delta a \in C_1(\boldsymbol{R})$

$$\mathscr{T}'(a)\cdot\delta a = -\Big\langle \frac{d\gamma}{d\zeta},\,\delta a\Big\rangle. \tag{9}$$

Proof. For every $\zeta \in \boldsymbol{R}$ we have $|\gamma(\zeta)| \leqslant \|\varphi\|_{L^1(Q)}$ and we have proved that γ is bounded on $\boldsymbol{R}$. Denoting by Y the Heaviside function, and by y a representant of the class of functions y, we can rewrite formula (8) as:

$$\gamma(\zeta) = \int_Q Y\big(y(x,t)-\zeta\big)\varphi(x,t)\,dx\,dt$$

and the result on the continuity and discontinuity of γ follows from the Lebesgue convergence theorem and from the fact that the function Y is continuous, except at zero.

In order to prove (9) divide $\boldsymbol{R}$ into intervals of length h and set

$$y_i = ih, \qquad i \in Z.$$

Then formula (5) becomes

$$\mathscr{T}'(a)\cdot\delta a = \sum_{i=-\infty}^{+\infty} \int_{Q_{y_i}\setminus Q_{y_{i+1}}} \delta a(y)\varphi(x,t)\,dx\,dt$$

or

$$\mathscr{T}'(a)\cdot\delta a = \sum_{i=-\infty}^{+\infty} \delta a(y_i)\big(\gamma(y_i)-\gamma(y_{i+1})\big)+\mathscr{R},$$

where

$$\mathscr{R} = \sum_{i=-\infty}^{+\infty} \int_{Q_{y_i}\setminus Q_{y_{i+1}}} (\delta a)'\big(y_i+\theta(y-y_i)\big)(y-y_i)\varphi\,dx\,dt.$$

But we have for $\mathscr{R}$ the estimation

$$|\mathscr{R}| \leqslant \sum_{i=-\infty}^{+\infty} h\cdot\|\delta a\| \int_{Q_{y_i}\setminus Q_{y_{i+1}}} |\varphi|\,dx\,dt = h\|\delta a\|\cdot\|\varphi\|_{L^1(Q)},$$

which proves that $\mathscr{R} \to 0$ with $h \to 0$ and thus

$$\mathscr{T}'(a)\,\delta a = \lim_{h\to 0}\sum_{i=-\infty}^{+\infty} \delta a(y_i)\big(\gamma(y_i)-\gamma(y_{i+1})\big). \tag{10}$$

This can be rewritten as

$$\mathscr{T}'(a)\,\delta a = \lim_{h\to 0}\sum_{i=-\infty}^{+\infty} \frac{\delta a(y_i)-\delta a(y_{i-1})}{h}\,\gamma(y_i)h = \int_R (\delta a)'(y)\gamma(y)\,dy$$

which is (9).

The distribution $d\gamma/d\zeta$ is the "gradient" of $\mathscr{T}$ with respect to the function a; it includes, in general, Dirac functions at the values ζ for which S_ζ has non-zero measure. Thus we will be able, in the numerical application, to minimize $\mathcal{T}$ by a gradient method operating on the function a itself (and not on parameters appearing in some a priori-chosen closed form formula for a).

3. Application to the nonlinear heat equation

Let us now specify equation (1) more precisely. We consider a system governed by a nonlinear heat equation:

$$\frac{\partial y}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a(y)\frac{\partial y}{\partial x_i}\right)=f(x,t) \text{ in } \Omega,$$

$$\frac{\partial y}{\partial n}=0 \text{ on } \partial\Omega x(0,T), \tag{11}$$

$$y(x,0)=y_0(x) \text{ in } \Omega,$$

where f, y_0 are known functions, and the function $y\to a(y)$ is to be adjusted.

For this purpose, let us measure (for instance, at each time t) the mean value of the solution $y(x,t)$ in sub-regions $\Omega_1,\Omega_2,\ldots,\Omega_K$ of Ω, which results in K function $z_1(t),\ldots,z_K(t)$, $t\in(0,T)$, all known.

Our identification problem (4) is now to find a function a such that

$$0<\bar{\alpha}\leqslant a(\zeta)\leqslant \bar{M} \quad \text{for} \quad \zeta\in R\ (\bar{\alpha}, \bar{M}, \text{given})$$

which minimizes

$$\mathcal{T}(a)=\sum_{i=1}^{K}\int_0^T\frac{1}{|\Omega_i|}\left[\iint_{\Omega_i}(y(x_i,t)-z_i(t))^2\,dx\right]dt. \tag{12}$$

We now have to put this problem into a rigorous functional frame in order to prove that the assumptions of Theorem 1 are satisfied.

Let V and H be two Hilbert spaces such that

$$H_0^1(\Omega)\subset V\subset H^1(\Omega), \qquad H=L^2(\Omega) \tag{13}$$

and denote respectively by $\|\cdot\|$, $((\,,\,))$, $|\cdot|$, $(\,,\,)$, $\|\,\|_*$, $((\,,\,))_*$ the norm and scalar product in V, H, V'. Remember that $((u,v))_*=((D^{-1}u,D^{-1}v))$ for $u,v\in V'$, where D is the canonical isomorphism from V onto V' defined by: $\forall v\in V$, $Dv\in V'$ is defined by $(Dv,u)_{V',V}=((v,u))$ $\forall u\in V$. We will identify V with a part of H, and H with its dual H':

$$\underbrace{V'\subset H\subset V}_{D}, \tag{14}$$

and we consider the following non-linear equation:

$$\left(\frac{dy}{dt}(t),v\right)+a(y(t),v)=(f(t),v), \quad \forall v\in V, \text{ a.e. on } (0,T), \tag{15}$$

$$y(0)=y_0,$$

where

$$f\in L^2(0,T;H)=L^2(Q), \qquad y_0\in L^2(\Omega)=H \tag{16}$$

and

$$a(u,v)=\sum_{i=1}^{n}\int_\Omega a(u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx, \qquad \forall u,v\in V. \tag{17}$$

It is easy to see that, when $V=H^1(\Omega)$, a solution of (15) is a weak solution of (11). We now give the results concerning the existence and uniqueness of the solution of (15), its dependence on the function a, and the existence of an optimal estimate $\hat{a}$. The proofs are rather technical and will not be reproduced here. They can be found in [3].

(i) *Existence, uniqueness of the solution of* (14):

THEOREM 2. *Under assumptions* (13)–(17), *equation* (15) *has a unique solution such that*

$$y\in L^2(0,T;V)\cap L^\infty(0,T;H), \qquad \frac{dy}{dt}\in L^2(0,T;V'). \tag{18}$$

Let us suppose additionally that

$$f\in L^\infty(Q), \qquad y_0\in L^\infty(\Omega). \tag{19}$$

Then y satisfies (18) *and*

$$y\in L^\infty(Q). \tag{20}$$

(ii) *Dependence of y with respect to a*:

THEOREM 3. *Let*

$$\mathcal{A}=C^1(R) \text{ as in } (2),$$
$$\mathcal{A}_C=\{a\in\mathcal{A}|\ \exists\alpha>0 \text{ such that } a(\zeta)\geqslant\alpha>0 \text{ a.e. on } \boldsymbol{R}\}, \tag{2.1}$$

and suppose $\bar{a}\in\mathcal{A}_C$ given and such that the corresponding solution y of (15) *satisfies*

$$\frac{\partial y}{\partial x_i}\in L^\infty(Q), \qquad i=1,2,\ldots,n. \tag{22}$$

Then, under hypotheses (13)–(17) *and* (19) *the mapping $a\to y$ from $\mathcal{A}_C$ into $L^p(Q)$ defined by* (15) *is strongly continuous at $\bar{a}$ for $p\geqslant 2$.*

Moreover, if $n\leqslant 3$, one has:

$$\|\Delta y\|_{L^p(Q)}\leqslant\begin{cases} C\cdot\|\delta_a\| & \text{for } \quad 2\leqslant p\leqslant\frac{10}{3},\\ C\cdot\|\delta_a\|^{10/3p} & \text{for } \quad \frac{10}{3}\leqslant p<+\infty,\end{cases} \tag{23}$$

$(C = constant)$.

In order to get differentiability results for the mapping $a \to y$, we differentiate formally system (11). This leads us to the study of the following linear system:

$$(24)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(C(x,t)u) = h \text{ in } Q, \\ \dfrac{\partial u}{\partial n}(C(x,t)u) = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \quad \text{on } \Omega. \end{cases}$$

We shall show the existence and unicity of a weak solution of this system by taking V' as Pivot-space([1]), but without identifying V' with V; consider the linear equation

$$(25)\quad \begin{aligned} &\frac{d}{dt}((u,\omega))_* + C(t,u,\omega) = (L(t),\omega), \\ &u(0) = u_0, \end{aligned} \qquad \forall \omega \in H \text{ a.e. on } (0,T),$$

where

$$(26)\quad L \in L^2(0,T;H), \quad u_0 \in V',$$

and

$$(27)\quad C(t;u,v) = \int_\Omega C(x,t)u(x)\big(v(x) - D^{-1}v(x)\big)\,dx, \quad \forall u, v \in H,$$

where $C(x,t)$ is a function satisfying

$$(28)\quad \exists \alpha, M \in \mathbf{R} \quad \text{such that } 0 < \alpha \leqslant C(x,t) \leqslant M \text{ a.e. on } Q.$$

It is easy to show that for $V = H^1(\Omega)$ and $(L(t),\omega) = ((h(t),\omega))_*$ a solution of (25) is a weak solution of (24).

THEOREM 4. *Under assumptions* (26)–(28), *equation* (25) *has a unique solution* u *satisfying*

$$(29)\quad u \in L^2(Q), \quad \frac{dD^{-1}u}{dt} \in L^2(Q)$$

and depending continuously on the right-hand side.

It is now possible to give the differentiability theorem for the mapping $a \to y$:

THEOREM 5. *Under the assumptions of Theorem* 3, *for* $n \leqslant 3$ *the mapping* $a \to y$ *defined by* (15) *is differentiable as a transformation from* $\mathscr{A}_C$ *into* $L^2(Q)$. *Its derivative is the linear mapping which maps* $\delta a \in \mathscr{A}$ *onto* $\delta y \in L^2(Q)$ *solution of* (see Theorem 4):

$$(30)\quad \begin{aligned} &\frac{d}{dt}((\delta y,\omega))_* + \int_\Omega a\big(y(x,t)\big)\,\delta y(\omega - D^{-1}\omega)\,dx \\ &\qquad = -\int_\Omega \delta\beta(y)(\omega - D^{-1}\omega)\,dx, \quad \forall \omega \in \mathscr{A}, \\ &\delta y(0) = 0, \end{aligned}$$

([1]) Then, for a.e. $t \in (0,T)$, $u(t)$ will belong to V'.

where

$$\delta\beta(\zeta) = \int_0^\zeta \delta a(\xi)\,d\xi.$$

When $V = H^1(\Omega)$, δy *is a weak solution of*

$$(31)\quad \frac{\partial \delta y}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2}{x_i^2}\big(a(y(x,t))\,\delta y\big) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\delta a(y(x,t))\frac{\partial y}{\partial x_i}\right) \text{ a.e. on } Q,$$

$$\frac{\partial}{\partial n}\big(a(y(x,t))\,\delta y\big) = 0 \text{ on } \partial\Omega \times (0,T),$$

$$\delta y(x,0) = 0.$$

(iii) *The inverse problem*:

We now want to identify the function $\zeta \to a(\zeta)$ from measurements over the solution y, as defined above:

$$(32)\quad z_i \in L^2(0,T), \quad i = 1, 2, \ldots, K.$$

From (32) and Theorem 2 it follows that the cost function $\mathscr{T}$ is well defined by (12).

It is possible to give here an existence theorem for a minimum of the function $\mathscr{T}$:

THEOREM 6. *Let us set "locally"*

$$(33)\quad a \in L^\infty(R), \quad \mathscr{A}_{ad} = \{a \in L^\infty(R) \mid 0 < \bar{\alpha} \leqslant a(\zeta) \leqslant \overline{M} \text{ a.e. on } R\} \quad (\bar{\alpha}, M \text{ given}).$$

Then under assumptions (13)–(17), (19), (32), (33), *there exists at least one function* $a \in \mathscr{A}_{ad}$ *which minimizes* $\mathscr{T}$ *over* $\mathscr{A}_{ad}$.

Remark. In many specific applications, the above set $\mathscr{A}_{ad}$ of admissible non-linearity does not satisfy the assumptions required in Theorem 6, because it contains very irregular functions. For instance, in the numerical applications we have also used the following sets $\mathscr{A}$ and $\mathscr{A}_{ad}$:

$$(34)\quad \begin{aligned} \mathscr{A} &= \{a \in L^\infty(\mathbf{R}) \mid a'' \in L^\infty(\mathbf{R})\}, \\ \mathscr{A}_{ad} &= \{a \in \mathscr{A} \mid 0 < \bar{\alpha} \leqslant a(y) \leqslant \overline{M},\ \gamma_m(y) \leqslant a''(y) \leqslant \gamma_M(y) \text{ a.e. on } \mathbf{R}\} \end{aligned}$$

when $\bar{\alpha}$, $\overline{M} \in \mathbf{R}$ and the functions γ_m, γ_M are given. It is easy to show the existence of a minimum on this new admissible set.

It is now possible to prove, as a corollary to Theorem 5, that the functional $\mathscr{T}$ defined in (12) satisfies the assumptions (3) of the theory, i.e., that it is differentiable.

THEOREM 7. *Under assumptions* (13)–(17), (19), (21), (22), *the functional* $\mathscr{T}$ *defined by* (12) *is differentiable as a mapping from* $\mathscr{A}_C$ *into* $\mathbf{R}$ *for* $n \leqslant 3$, *and its derivative* $\mathscr{T}'(a)$ *is given by*

$$\mathcal{T}'(a)\,\delta a = \int_Q \delta a\big(y(x,t)\big)\sum_{i=1}^{n}\frac{\partial y}{\partial x_i}\frac{\partial p}{\partial x_i}\,dx\,dt, \qquad \forall a \in C^1(\boldsymbol{R}), \tag{35}$$

where y is the unique solution of (15) *and p is the adjoint state given by*

$$\begin{cases} p \in L^2(0,T;V), \quad \dfrac{dp}{dt} \in L^2(Q) \quad \textit{and} \quad p(t) = D^{-1}\varrho(t), \quad \textit{a.e. on } (0,T), \\ \varrho \in L^2(Q) \ \textit{is the unique solution of} \\ \quad -\dfrac{d}{dt}\big((\varrho(t),\omega)\big)_* + \displaystyle\int_\Omega a\big(y(x,t)\big)(\varrho - D^{-1}\varrho)\omega\,dx \\ \quad = -2\displaystyle\sum_{l=1}^{K}\int_{\Omega_l}\frac{1}{|\Omega_l|}\left(\frac{1}{|\Omega_l|}\int_{\Omega_l} y\,dx - z_l(t)\right)\omega\,dx \quad \forall\omega \in H, \ \textit{a.e. on } (0,T). \end{cases} \tag{36}$$

If $V = H^1(\Omega)$, p is the strong solution in $L^2(0,T;H^2(\Omega))$ of:

$$\begin{aligned} &-\frac{\partial p}{\partial t} - a\big(y(x,t)\big)\sum_{i=1}^{n}\frac{\partial^2 p}{\partial x_i^2} = -2\sum_{l=1}^{K}\frac{\chi_l}{|\Omega_l|}\left(\frac{1}{|\Omega_l|}\int_{\Omega_l} y(x,t)\,dx - z_l(t)\right) \quad \text{in } Q, \\ &\frac{\partial p}{\partial n} = 0 \quad \text{on } \partial\Omega x(0,T), \\ &p(x,T) = 0 \ \text{ on } \Omega. \end{aligned} \tag{37}$$

So all the conditions of Theorem 1 are satisfied, which enables us to calculate the "gradient" $d\gamma/d\zeta$ of $\mathcal{T}$ with respect to a. We can remark that in the case of the heat equation the function γ is always continuous:

For a value of ζ such that S_ζ has a non-zero measure, it happens that

$$\varphi(x,t) = \sum_{i=1}^{n}\frac{\partial y}{\partial x_i}(x,t)\frac{\partial p}{\partial x_i}(x,t)$$

is equal to zero on S_ζ, so that γ is also continuous for those values: the distribution $d\gamma/d\zeta$ does not include Dirac functions.

4. Numerical results

From a numerical point of view, it is relatively easy to compute the distribution $d\gamma/d\zeta$; the main operation consists in an integration on the time-space domain contained between two level surfaces of the solution $y(x,t)$.

The direct and adjoint equations (11) and (37) are solved by classical finite differences.

We have considered a one-dimensional heat equation ($n = 1$):

$\Omega = (0,1)$ discretized into twenty intervals,

$(0,T) = (0,1)$ discretized into forty intervals.

There were five measurements points

$$.1,\ .3,\ .5,\ .7,\ .9,$$

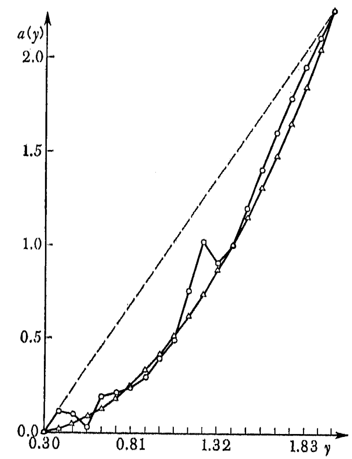

Fig. 1. Estimation of $a(y)$ as a "free" function (no regularity constraints)
--- initial value; △ exact function; ∘ identified function

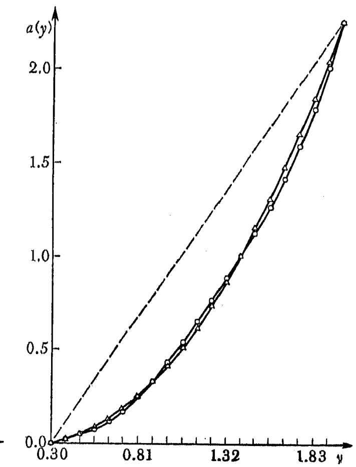

Fig. 2. Estimation of $a(y)$ with constraints on the second derivative $|a''(y)| \leqslant 5$
--- initial value; △ exact function; ∘ identified function

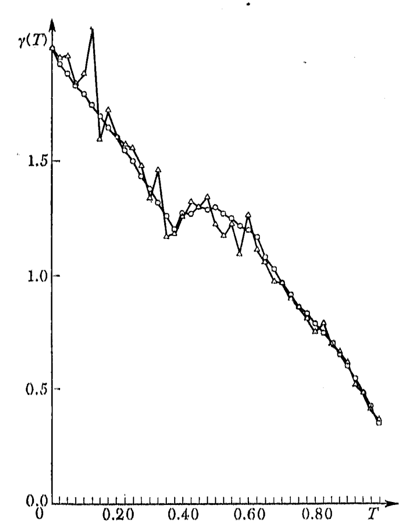

Fig. 3. The data of the noisy case: recorded and computed pressure at the unique observation point (the pressure that should have been observed in absence of noise is practically confounded with the computed pressure)
△ observed pressure; ∘ computed pressure

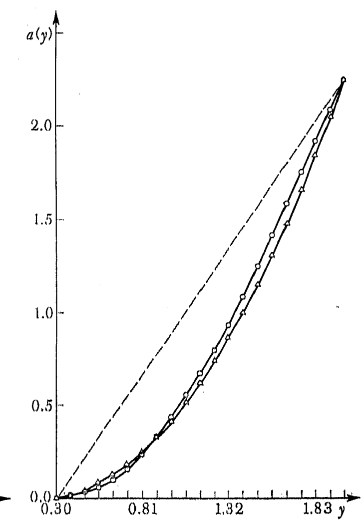

Fig. 4. Estimation of $a(y)$ in the presence of noise, with constraints on the second derivative: $0 \leqslant a''(y) \leqslant 5$
--- initial value; △ exact function; ∘ identified function

We simulated the measurements $z_i(t)$ by use of the "true" function

$$a_0(y) = .21-.28y+.7y^2.$$

The lower and upper bounds of the solution y were chosen as $\gamma_m = .3$, $\gamma_M = 2$, the interval (γ_m, γ_M) was divided into 20 intervals of length Δ, and the function $a(y)$ was represented on this interval by a continuous piecewise linear function.

To recover the function $a(y)$, we used the standard gradient method (steepest descent with projection for the case of $\mathscr{A}_{ad}$ as in (33), Franck and Wolf algorithm for the case of $\mathscr{A}_{ad}$ as in (34)).

Our numerical results are shown in figures 1 through 4.

Detailed numerical comparisons are to be found in [3].

5. Conclusion

We have given a method of computing the gradient of a functional depending on a function of the state variable and applied it to the nonlinear heat-equation.

Numerical results have been given, which show the feasibility of the method.

References

[1] G. Chavent, *Analyse fonctionnelle et identification de coefficients repartis dans les équations aux dérivées partielles paraboliques*, Thèse, Paris, 1971.

[2] —, M. Dupuy et P. Lemonnier, *History matching by use of optimal Control Theory*, SPE 4627, Las Vegas, 1973.

[3] —, P. Lemonnier, *Identification de la non-linéarité d'une équation parabolique quasilinéaire*, Applied Mathematics and Optimization, Vol. 1 No. 2, 1974.

[4] T. L. Lions, *Quelques méthodes de resolution de problèmes aux limites non-linéaires*, Dunod-Gauthier Villars, Paris 1969.

BANACH CENTER PUBLICATIONS
VOLUME 1

A NOTE ON THE POISSON DISORDER PROBLEM

M. H. A. DAVIS

Department of Computing and Control, Imperial College, London SW7 2BZ, Great Britain

1. Introduction

The problem can be stated roughly as follows. We observe a Poisson process N, whose rate changes from λ_0 to λ_1 (positive constants) at a certain time T. T is a random variable which is zero with probability π, or, given that $T \neq 0$, exponentially distributed with parameter λ. We want to tell when T occurred, from the observations of $\{N_t\}$. Thus the problem is to choose a stopping time τ of $\mathfrak{F}_t = \{N_s,\ s \leqslant t\}$ so as to minimize the expected value of some cost function depending on the difference between τ and T. Two forms of cost function are considered here; they are

$$s_\tau^1(\omega) = d(T-\tau)I_{(\tau<T)}+c(\tau-T)I_{(\tau\geqslant T)}, \tag{1.1}$$

$$s_\tau^2(\omega) = I_{(\tau<T-\varepsilon)}+c(\tau-T)I_{(\tau\geqslant T)}, \tag{1.2}$$

where ε, c, d are positive constants. It will turn out that these are special cases of a "standard problem" (see § 4). A third natural form of cost function, the "hit or miss" cost

$$s_\tau^3(\omega) = 1-I_{(T-\varepsilon\leqslant\tau\leqslant T+\varepsilon)}$$

is not standard and presents a more difficult problem.

The Wiener process version of this problem (where the observation is $N_t = \lambda(t-T)I_{(t\geqslant T)}+W_t$, $\{W_t\}$ a Wiener process) was studied by Shiryayev [5]. Shiryayev's methods were applied to the Poisson case the cost function s^2 with $\varepsilon = 0$ by Galchuk and Rozovsky [2] who with a rather complicated proof solved the problem in case $\lambda+c \geqslant \lambda_1 > \lambda_0$. Here we show that this result (Theorem 2 below) is a very simple consequence of the martingale or innovations approach to point process filtering developed in [4]. Furthermore, the solution is in fact valid for $\lambda+c \geqslant \lambda_1-\lambda_0 \geqslant 0$ and we can also obtain solutions for other cost functions such as (1.1) and (1.2) which can be rewritten in standard form.

In § 2 we state the recursive filtering result of [4], which is applied in § 3 to derive a stochastic differential equation satisfied by the process $\pi_t = P[t \geqslant T|\ \mathfrak{F}_t]$. In § 4 the standard problem is formulated and solved under certain conditions on the coefficients. When these conditions are not met things are more complicated