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Online First version

HITTING TIMES OF HYPERBOLIC BESSEL PROCESSES

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Abstract. We investigate the first hitting time to a point of a hyperbolic Bessel process which is the generalization of the radial part of Brownian motion on a hyperbolic space. We give the Laplace transform of the hitting time and the probability that the hyperbolic Bessel process reaches a given point in some time. Moreover, the limiting behavior of the expectation of the hitting time is computed. These results are improvements of some preceding results.

1. Introduction. The probability distributions of the first time that a d-dimensional Brownian motion or an Ornstein–Uhlenbeck process arrives at some given point of the sphere have been determined explicitly (see [6, 7, 8]). Since the radial parts of these processes are represented by Brownian motion moving on $[0, \infty)$ with an appropriate drift, the general theory of one-dimensional diffusion processes can give the Laplace transforms of their hitting times in terms of suitable special functions. By calculating the inverse Laplace transforms we obtain the relevant distributions.

This article deals with hyperbolic Bessel processes. These are diffusion processes represented by one-dimensional Brownian motion with a suitable drift and their transition densities can be determined. More information is found in [1, 10, 16] and references therein. Just as a Bessel process is an extension of the radial part of Brownian motion on $\mathbb{R}^d$, a hyperbolic Bessel process is an extension of the radial part of Brownian motion on the Poincaré half-space $H^d(\mathbb{R})$, which is called d-dimensional hyperbolic Brownian motion.

The probability that d-dimensional hyperbolic Brownian motion reaches the boundary of a ball in some time is given in [3] when the starting point is outside the ball. Moreover, for $2 \leq d \leq 7$, the asymptotic behavior of the probability is discussed in [2] as the starting point tends to infinity. The limiting value of the logarithm of the hitting probability is linear in the radius of the ball and the same is expected in other cases.

2020 *Mathematics Subject Classification*: Primary 60J60; Secondary 60E10, 60G40.
Key words and phrases: hyperbolic Bessel process, first hitting time, associated Legendre functions.
Received 8 September 2023; revised 24 January 2024.
Published online 29 April 2024.

DOI: 10.4064/cm9222-2-2024 © Instytut Matematyczny PAN, 2024

The purpose of this paper is to improve these results. Namely, we give a formula for the hitting probability of a hyperbolic Bessel process for any starting point. Since the method in [3] is valid only when the starting point is close to the origin, we need to use a different method in other cases and the Laplace transform is useful for calculations. By using the derived representation we establish the exponential decay of the hitting probability, confirming the conjecture in [2].

In addition, we investigate the conditional expectation of the hitting time under the condition that the hitting time is finite. For the radial part of d-dimensional hyperbolic Brownian motion, the expectation of the exit time from a ball has been computed for $2 \le d \le 5$ (see [17]). Another purpose of this article is to find the limiting behavior of the conditional expectation as the arrival point or the starting point tends to infinity in the case of a hyperbolic Bessel process. We mention that the explicit form of the expectation can be deduced for $d = 6$.

We remark that hyperbolic Brownian motion is significant for mathematical finance, since it has a close connection with some exotic derivatives. In the framework of the Black–Scholes model several identities for the prices of some call options have been obtained (see [14]).

This paper is organized as follows. Section 2 provides the Laplace transform of the first hitting time of a hyperbolic Bessel process. Section 3 is devoted to giving a formula for the hitting probability and its limiting behavior. Sections 4 and 5 deal with the expectation or the conditional expectation of the first hitting time. We use the notation of special functions from [12].

2. Laplace transforms of hitting times. For $\alpha \in \mathbb{R}$ and $\theta > 0$ a hyperbolic Bessel process with index α and parameter θ, denoted by $\{X_t^{(\alpha,\theta)}\}_{t\ge 0}$, is a one-dimensional diffusion process on $[0,\infty)$ with the generator

$$\mathcal{L}_{\alpha,\theta} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\alpha + \frac{1}{2}\right)\theta\coth(\theta x)\frac{d}{dx}, \quad x > 0.$$

If $2\alpha + 2$ is a positive integer, the diffusion process $\{X_t^{(\alpha,1)}\}_{t\ge 0}$ can be regarded as the radial part of a Brownian motion on $(2\alpha+2)$-dimensional real hyperbolic space. Throughout this paper we consider the case where $X_0^{(\alpha,\theta)} = a$ for a given $a > 0$.

Let m and s be the speed measure and the scale function, respectively. Since

$$\mathcal{L}_{\alpha,\theta} = \frac{1}{2\sinh^{2\alpha+1}(\theta x)}\frac{d}{dx}\left(\sinh^{2\alpha+1}(\theta x)\frac{d}{dx}\right), \tag{2.1}$$

we can take

$$m(dx) = 2\sinh^{2\alpha+1}(\theta x)dx, \quad s'(x) = \sinh^{-2\alpha-1}(\theta x). \tag{2.2}$$

Hence the classification of boundary points, which are 0 and ∞, is the following. The endpoint ∞ is natural for all $\alpha \in \mathbb{R}$. The origin 0 is an entrance point if $\alpha \geq 0$ and an exit point if $\alpha \leq -1$. If $-1 < \alpha < 0$, then 0 is regular and in this paper we assume that 0 is instantaneously reflecting. We remark that the boundary points are classified independently of θ. The details are found in [9, 13] for example.

For $b > 0$ we define

$$\tau_{a,b}^{(\alpha,\theta)} = \inf\{t > 0;\ X_t^{(\alpha,\theta)} = b\}.$$

The Laplace transform of $\tau_{a,b}^{(\alpha,\theta)}$ can be obtained with the help of the general theory of one-dimensional diffusion processes. In order to give the Laplace transform of $\tau_{a,b}^{(\alpha,\theta)}$, we use the associated Legendre functions of the first and the second kinds, which are denoted by $\mathfrak{P}_\nu^\mu$ and $\mathfrak{Q}_\nu^\mu$ for $\mu, \nu \in \mathbb{R}$, respectively. It is known that $\mathfrak{P}_\nu^\mu$ and $e^{-i\pi\mu}\mathfrak{Q}_\nu^\mu$ are the independent fundamental solutions of the ordinary differential equation

$$(1-x^2)u''(x) - 2xu'(x) + \left[\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right]u(x) = 0, \quad x > 1.$$

For details, see [1, 12] for example. Moreover, for $\lambda \geq 0$ let

$$\alpha(\lambda) = \sqrt{\frac{2\lambda}{\theta} + \left(\alpha + \frac{1}{2}\right)^2} - \frac{1}{2}. \tag{2.3}$$

Now we are ready to provide the Laplace transform of $\tau_{a,b}^{(\alpha,\theta)}$.

THEOREM 2.1.

(1) *If* $0 < a < b$ *and* $\alpha > -1$, *then*

$$E[e^{-\lambda\tau_{a,b}^{(\alpha,\theta)}}] = \frac{\sinh^{-\alpha}(\theta a)\mathfrak{P}_{\alpha(\lambda)}^{-\alpha}(\cosh(\theta a))}{\sinh^{-\alpha}(\theta b)\mathfrak{P}_{\alpha(\lambda)}^{-\alpha}(\cosh(\theta b))} \quad \text{for } \lambda > 0. \tag{2.4}$$

(2) *If* $0 < a < b$ *and* $\alpha \leq -1$, *then*

$$E[e^{-\lambda\tau_{a,b}^{(\alpha,\theta)}}] = \frac{\sinh^{-\alpha}(\theta a)\mathfrak{P}_{\alpha(\lambda)}^{\alpha}(\cosh(\theta a))}{\sinh^{-\alpha}(\theta b)\mathfrak{P}_{\alpha(\lambda)}^{\alpha}(\cosh(\theta b))} \quad \text{for } \lambda > 0. \tag{2.5}$$

(3) *If* $0 < b < a$ *and* $\alpha \in \mathbb{R}$, *then*

$$E[e^{-\lambda\tau_{a,b}^{(\alpha,\theta)}}] = \frac{\sinh^{-\alpha}(\theta a)\mathfrak{Q}_{\alpha(\lambda)}^{-\alpha}(\cosh(\theta a))}{\sinh^{-\alpha}(\theta b)\mathfrak{Q}_{\alpha(\lambda)}^{-\alpha}(\cosh(\theta b))} \quad \text{for } \lambda > 0. \tag{2.6}$$

Before proving Theorem 2.1, we mention that $\mathfrak{B}_\nu^\mu(x)$ and $e^{-i\pi\mu}\mathfrak{D}_\nu^\mu(x)$ are real for $x>1$ and suitable $\mu,\nu\in\mathbb{R}$. Indeed, it is known that

$$\mathfrak{B}_\nu^\mu(x)=\frac{2^\mu(x^2-1)^{-\mu/2}}{\sqrt{\pi}\Gamma(1/2-\mu)}\int_0^\pi[x+(x^2-1)^{1/2}\cos y]^{\nu+\mu}\sin^{-2\mu}y\,dy \tag{2.7}$$

if $\mu<1/2$ (see [12, p. 184]), and

$$e^{-i\pi\mu}\mathfrak{D}_\nu^\mu(x)=\frac{(x^2-1)^{-\mu/2}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+1)}\times\int_0^\pi(x+\cos y)^{-\nu+\mu-1}\sin^{2\nu+1}y\,dy \tag{2.8}$$

if $\nu>-1$ and $\nu+\mu+1>0$ (see [12, p.186]).

For a proof of Theorem 2.1 we apply the general formula for the Laplace transform of the first hitting time of a diffusion process with generator $\mathcal{L}_{\alpha,\theta}$. According to the general theory of one-dimensional diffusions (see [9, p. 129]), if u is the solution of the differential equation

$$\mathcal{L}_{\alpha,\theta}u(x)=\lambda u(x),\quad x>0, \tag{2.9}$$

for $\lambda>0$ with appropriate conditions, then the Laplace transform of $\tau_{a,b}^{(\alpha,\theta)}$ is represented by $u(a)/u(b)$. In addition, for $0<a<b$, boundary conditions at 0 are required. If 0 is an entrance point or instantaneously reflecting regular, we need to solve (2.9) under the following conditions:

$$\lim_{x\downarrow 0}u(x)=1,\quad \lim_{x\downarrow 0}\frac{u'(x)}{s'(x)}=0. \tag{2.10}$$

If 0 is an exit point, the solution of (2.9) satisfying

$$\lim_{x\downarrow 0}u(x)=0,\quad \lim_{x\downarrow 0}\frac{u'(x)}{s'(x)}=1 \tag{2.11}$$

should be derived.

Standard calculations show that the functions $\sinh^{-\alpha}(\theta x)\mathfrak{B}_\nu^{\pm\alpha}(\cosh(\theta x))$ and $\sinh^{-\alpha}(\theta x)\mathfrak{D}_\nu^{\pm\alpha}(\cosh(\theta x))$ are solutions of (2.9).

We start with the case $0<a<b$ and $\alpha>-1$. Let u_1 be the function defined on $(0,\infty)$ by

$$u_1(x)=2^\alpha\Gamma(\alpha+1)\sinh^{-\alpha}(\theta x)\mathfrak{B}_{\alpha(\lambda)}^{-\alpha}(\cosh(\theta x)).$$

We can conclude that $E[\exp(-\lambda\tau_{a,b}^{(\alpha,\theta)})]=u_1(a)/u_1(b)$, which is equivalent to (2.4), if we succeed in proving that u_1 is the strictly increasing solution of (2.9) satisfying (2.10). It is sufficient to establish the following lemma.

LEMMA 2.2. *For $\mu,\nu\in\mathbb{R}$ define a function $f_{\nu,\mu}$ on $(1,\infty)$ by*

$$f_{\nu,\mu}(x)=(x^2-1)^{\mu/2}\mathfrak{B}_\nu^\mu(x). \tag{2.12}$$

If $\mu < 1$, then

$$\lim_{x \downarrow 1} f_{\nu,\mu}(x) = \frac{2^{\mu}}{\Gamma(1-\mu)}, \tag{2.13}$$

$$\lim_{x \downarrow 1} f'_{\nu,\mu}(x) = \frac{(\nu+\mu)(\nu-\mu+1)2^{\mu-1}}{\Gamma(2-\mu)}. \tag{2.14}$$

Moreover, if $\mu < 3/2$, $\nu + \mu > 0$ and $\nu - \mu + 1 > 0$, then

$$f'_{\nu,\mu}(x) > 0 \quad \text{for any } x > 1. \tag{2.15}$$

Proof. When $\mu < 1$ we have

$$f_{\nu,\mu}(x) = \frac{(x+1)^{\mu}}{\Gamma(1-\mu)} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right)$$

for $x > 1$, where ${}_2F_1$ is the hypergeometric function (see [12, p. 153]). This immediately gives (2.13).

It is known that

$$\frac{d\mathfrak{B}^{\mu}_{\nu}}{dx}(x) = \frac{\nu x}{x^2-1}\mathfrak{B}^{\mu}_{\nu}(x) - \frac{\nu+\mu}{x^2-1}\mathfrak{B}^{\mu}_{\nu-1}(x) \tag{2.16}$$

(see [12, p.165]), which yields

$$f'_{\nu,\mu}(x) = (x^2-1)^{\mu/2-1}(\nu+\mu)\{x\mathfrak{B}^{\mu}_{\nu}(x) - \mathfrak{B}^{\mu}_{\nu-1}(x)\}.$$

Since

$$x\mathfrak{B}^{\mu}_{\nu}(x) - \mathfrak{B}^{\mu}_{\nu-1}(x) = (\nu-\mu+1)(x^2-1)^{1/2}\mathfrak{B}^{\mu-1}_{\mu}(x)$$

(see [12, p.165]), we have

$$f'_{\nu,\mu}(x) = (\nu+\mu)(\nu-\mu+1)f_{\nu,\mu-1}(x). \tag{2.17}$$

Hence (2.14) follows from (2.13). Moreover, by (2.7), $f_{\nu,\mu-1}(x) > 0$ for any $x > 1$ and hence (2.17) yields (2.15). ■

We next consider the case of $0 < a < b$ and $\alpha \le -1$ and define a function u_2 on $(0,\infty)$ by

$$u_2(x) = \frac{\Gamma(1-\alpha)}{-2^{\alpha+1}\alpha\theta}\sinh^{-\alpha}(\theta x)\mathfrak{B}^{\alpha}_{\alpha(\lambda)}(\cosh(\theta x)).$$

It is obvious that u_2 is a solution of (2.9). For $\mu, \nu \in \mathbb{R}$ let $g_{\nu,\mu}$ be a function on $(0,\infty)$ defined by

$$g_{\nu,\mu}(x) = (\sinh^{-\mu} x)\mathfrak{B}^{\mu}_{\nu}(\cosh x).$$

Then

$$u_2(x) = \frac{\Gamma(1-\alpha)}{-2^{\alpha+1}\alpha\theta} g_{\alpha(\lambda),\alpha}(\theta x).$$

Since

$$g_{\nu,\mu}(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\Gamma(1/2-\mu)}\int_0^x (\cosh x - \cosh y)^{-\mu-1/2}\cosh\left[\left(\nu+\frac{1}{2}\right)y\right]dy$$

for $\mu < 1/2$ (see [12, p. 184]), $g_{\alpha(\lambda),\alpha}$ and u_2 are strictly increasing on $(0,\infty)$.

We aim to show that u_2 satisfies (2.11). Recall that $f_{\nu,\mu}$ is defined by (2.12) and so

$$g_{\nu,\mu}(x) = (\sinh^{-2\mu} x) f_{\nu,\mu}(\cosh x). \tag{2.18}$$

Applying (2.13) to (2.18), we easily obtain

$$\lim_{x\downarrow 0} g_{\alpha(\lambda),\alpha}(x) = 0,$$

which implies the first claim of (2.11) for u_2. Moreover, (2.18) gives

$$(\sinh^{2\mu+1} x) g'_{\nu,\mu}(x) = -2\mu(\cosh x) f_{\nu,\mu}(\cosh x) + (\sinh^2 x) f'_{\nu,\mu}(\cosh x).$$

In virtue of (2.14) we deduce that

$$\lim_{x\downarrow 0} (\sinh^{2\alpha+1} x) g'_{\alpha(\lambda),\alpha}(x) = -\frac{2^{\alpha+1}\alpha}{\Gamma(1-\alpha)}.$$

This shows that u_2 satisfies the second claim of (2.11) and hence (2.5) holds.

For the proof of (2.6) we need to find a decreasing solution of (2.9) that vanishes as $x \to \infty$. It is sufficient to establish the following lemma.

LEMMA 2.3. *Define a function $h_{\nu,\mu}$ on $(1,\infty)$ by*

$$h_{\nu,\mu}(x) = (x^2-1)^{-\mu/2} e^{i\pi\mu} \mathfrak{D}_\nu^{-\mu}(x).$$

For $\nu > -1$ and $\nu \pm \mu + 1 > 0$ we have the following properties of $h_{\nu,\mu}$:

$$h_{\nu,\mu}(x) \text{ is positive for any } x > 1, \tag{2.19}$$

$$h_{\nu,\mu} \text{ is decreasing on } (1,\infty), \tag{2.20}$$

$$h_{\nu,\mu}(x) \text{ converges to } 0 \text{ as } x \to \infty. \tag{2.21}$$

Proof. Formula (2.8) shows that $h_{\nu,\mu}$ takes values in $(0,\infty)$. In addition, (2.8) yields (2.20) since $x \mapsto (x+\cos y)^{-\nu-\mu-1}$ is decreasing for a fixed $y \in (0,\pi)$.

We have $x + \cos y > 1$ for $x > 2$ and $0 < y < \pi$ and thus

$$|(x+\cos y)^{-\nu-\mu-1} \sin^{2\nu+1} y| \le \sin^{2\nu+1} y.$$

Since $2\nu + 1 > -1$, the right hand side is integrable on $(0,\pi)$. Hence the dominated convergence theorem yields

$$\lim_{x\to\infty} \int_0^\pi (x+\cos y)^{-\nu-\mu-1} \sin^{2\nu+1} y \, dy = 0$$

and we can obtain (2.21) from (2.8). ∎

3. Hitting probabilities and their exponential decay. This section deals with the probability that $\tau_{a,b}^{(\alpha,\theta)} < \infty$. When $d = 2\alpha + 2$ for an integer $d \ge 2$, the explicit form of $P(\tau_{a,b}^{(\alpha,1)} < \infty)$ is provided in [3] for $0 < b < a$. However, the formula has a complicated form except for $d = 2, 3$. In virtue of the indefinite integral of $\sinh^{-n} x$ for an integer $n \geq 1$ (see [5, p. 113]), we can see that the formula in [3] is the same as

$$P(\tau_{a,b}^{(\alpha,1)} < \infty) = \frac{\int_a^\infty \sinh^{-d+1} x\,dx}{\int_b^\infty \sinh^{-d+1} x\,dx}.$$

Our result in this section improves the result in [3].

THEOREM 3.1.

(1) *If* $0 < a < b$ *and* $\alpha > -1$, *then*

$$P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = 1. \tag{3.1}$$

(2) *If* $0 < a < b$ *and* $\alpha \leq -1$, *then*

$$P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = \frac{\int_0^{\theta a} \sinh^{-2\alpha-1} x\,dx}{\int_0^{\theta b} \sinh^{-2\alpha-1} x\,dx}. \tag{3.2}$$

(3) *If* $0 < b < a$ *and* $\alpha > -1/2$, *then*

$$P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = \frac{\int_{\theta a}^\infty \sinh^{-2\alpha-1} x\,dx}{\int_{\theta b}^\infty \sinh^{-2\alpha-1} x\,dx}. \tag{3.3}$$

(4) *If* $0 < b < a$ *and* $\alpha \leq -1/2$, *then*

$$P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = 1. \tag{3.4}$$

We note that (3.2) and (3.3) are less than 1. Before giving a proof of Theorem 3.1, we shall give the asymptotic behavior of the probability that $\tau_{a,b}^{(\alpha,\theta)} < \infty$ for large a or b.

COROLLARY 3.2.

(1) *If* $0 < a < b$ *and* $\alpha \leq -1$, *then*

$$\lim_{b\to\infty} \frac{1}{b} \log P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = (2\alpha+1)\theta. \tag{3.5}$$

(2) *If* $0 < b < a$ *and* $\alpha > -1/2$, *then*

$$\lim_{a\to\infty} \frac{1}{a} \log P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = -(2\alpha+1)\theta. \tag{3.6}$$

Proof. Note that

$$\tfrac{1}{2}e^y(1-e^{-2y}) = \sinh y \leq \tfrac{1}{2}e^y \tag{3.7}$$

for $y > 0$. The standard calculation shows that (3.2) and (3.3) immediately imply (3.5) and (3.6), respectively. ∎

If $\theta = 1$ and $2\alpha + 2$ is an integer, then (3.6) is proved for $2 \leq 2\alpha + 2 \leq 7$ in [2]. It is expected in the same paper that (3.6) holds in the other cases. Corollary 3.2 confirms this.

The proof of Theorem 3.1 for $0 < b < a$ is easy and similar to the proof in [2]. The general theory yields

$$P(\tau_{a,b}^{(\alpha,\theta)} < \tau_{a,r}^{(\alpha,\theta)}) = \frac{s(r) - s(a)}{s(r) - s(b)} \tag{3.8}$$

for $r > a$ (see [9, p. 112]). By (2.2) we can take

$$s(x) = \int_b^x \sinh^{-2\alpha-1}(\theta y)\, dy.$$

Note that $\sinh x$ is asymptotically equal to $e^x/2$ as $x \to \infty$, which implies that $s(r)$ converges if $\alpha > -1/2$ and diverges if $\alpha \leq -1/2$ as $r \to \infty$. Hence (3.3) and (3.4) can be easily derived by letting $r \to \infty$ in (3.8).

We turn to the case $0 < a < b$. Recall that

$$P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = \lim_{\lambda \downarrow 0} E[e^{-\lambda \tau_{a,b}^{(\alpha,\theta)}}]. \tag{3.9}$$

Hence, in order to obtain (3.1) and (3.2), we should calculate the limiting values of the right hand sides of (2.4) and (2.5) as $\lambda \downarrow 0$.

First, we show the continuity of $\mathfrak{B}_\nu^\mu(\cosh x)$ with respect to ν. For $\mu \in \mathbb{R}$ and $x > 0$ define a function $\phi_{\mu,x}$ on $\mathbb{R}$ by

$$\phi_{\mu,x}(\nu) = \int_0^\pi (\cosh x + \sinh x \cos y)^{\nu+\mu} \sin^{-2\mu} y\, dy.$$

Formula (2.7) yields

$$\mathfrak{B}_\nu^\mu(\cosh x) = \frac{2^\mu \sin^{-\mu} x}{\sqrt{\pi}\Gamma(1/2 - \mu)} \phi_{\mu,x}(\nu) \tag{3.10}$$

for $\mu < 1/2$.

LEMMA 3.3. *Let $\mu < 1/2$ and $x > 0$. Then $\phi_{\mu,x}$ is a continuously differentiable function on $\mathbb{R}$ and*

$$\phi'_{\mu,x}(\nu) = \int_0^\pi (\cosh x + \sinh x \cos y)^{\nu+\mu} \sin^{-2\mu} y \log(\cosh x + \sinh x \cos y)\, dy \tag{3.11}$$

for $\nu \in \mathbb{R}$. In particular, the function $\nu \mapsto \mathfrak{B}_\mu^\nu(\cosh x)$ is continuous on $\mathbb{R}$.

Proof. Let $x > 0$ and $0 < y < \pi$. For simplicity we define

$$\xi(x, y) = \cosh x + \sinh x \cos y$$

and so

$$\phi_{\mu,x}(\nu) = \int_0^\pi \xi(x, y)^{\nu+\mu} \sin^{-2\mu} y\, dy.$$

Note that

$$e^{-x} \le \xi(x,y) \le e^{x}. \tag{3.12}$$

Let $\nu_0 \in \mathbb{R}$ be arbitrary. We shall prove that $\phi_{\mu,x}$ is differentiable at $\nu = \nu_0$ and $\phi'_{\mu,x}$ is continuous at $\nu = \nu_0$. Since $\sin^{-2\mu} y$ is integrable with respect to y, it is sufficient to show that $|\xi(x,y)^{\nu+\mu}\log\xi(x,y)|$ can be dominated by a constant which depends only on x, ν_0 and μ.

Note that $|\log\xi(x,y)| \le x$ by (3.12). We now estimate $\xi(x,y)^{\nu+\mu}$. If $\nu_0+\mu>0$, it follows from (3.12) that

$$\xi(x,y)^{\nu+\mu} \le e^{(\nu_0+\mu+1)x}$$

for $-\mu<\nu<\nu_0+1$. If $\nu_0+\mu=0$, we have

$$\xi(x,y)^{\nu+\mu} \le \max\{e^{(\nu+\mu)x}, e^{-(\nu+\mu)x}\} \le e^{|\mu|x/2}$$

for $\nu_0-|\mu|/2<\nu<\nu_0+|\mu|/2$. Finally, if $\nu_0+\mu<0$, then

$$\xi(x,y)^{\nu+\mu} \le e^{-(\nu_0+\mu-1)x}$$

for $\nu_0-1<\nu<-\mu$. This finishes the proof of the lemma. ∎

When $\mu<3/2$, the well-known formula

$$\mathfrak{B}^{\mu}_{\nu}(x) = -2(\mu-1)x(x^2-1)^{\mu/2}\mathfrak{B}^{\mu-1}_{\nu}(x) + (\nu-\mu+2)(\nu+\mu-1)\mathfrak{B}^{\mu-2}_{\nu}(x)$$

for $x>1$ permits application of Lemma 3.3. Hence we deduce that

$$\lim_{\delta\to 0}\mathfrak{B}^{\mu}_{\nu+\delta}(\cosh x) = \mathfrak{B}^{\mu}_{\nu}(\cosh x) \tag{3.13}$$

for $\mu<3/2$ and $x>0$. Recall that $\alpha(\lambda)\ge\alpha(0)$ and

$$\lim_{\lambda\downarrow 0}\alpha(\lambda) = \alpha(0).$$

For $x>0$ let

$$\rho^{\pm}_{\alpha}(x) = \sinh^{-\alpha}(\theta x)\mathfrak{B}^{\pm\alpha}_{\alpha(0)}(\cosh(\theta x)).$$

We deduce from (3.9) and (3.13) that

$$P(\tau^{(\alpha,\theta)}_{a,b}<\infty) = \frac{\rho^{-}_{\alpha}(a)}{\rho^{-}_{\alpha}(b)} \quad \text{for } 0<a<b \text{ and } \alpha>-1$$

and

$$P(\tau^{(\alpha,\theta)}_{a,b}<\infty) = \frac{\rho^{+}_{\alpha}(a)}{\rho^{+}_{\alpha}(b)} \quad \text{for } 0<a<b \text{ and } \alpha\le-1. \tag{3.14}$$

Hence it is sufficient to evaluate $\rho^{\pm}_{\alpha}(x)$.

It is obvious that

$$\alpha(0) = \begin{cases} \alpha & \text{if } \alpha>-1/2, \\ -\alpha-1 & \text{if } \alpha\le-1/2. \end{cases} \tag{3.15}$$

If $\alpha > -1/2$, we have

$$\rho_\alpha^-(x) = \sinh^{-\alpha}(\theta x)\mathfrak{B}_\alpha^{-\alpha}(\cosh(\theta x)). \tag{3.16}$$

The formula

$$\mathfrak{B}_\mu^{-\mu}(x) = \frac{(x^2-1)^{\mu/2}}{2^\mu \Gamma(\mu+1)} \quad \text{for } \mu > -1 \text{ and } x > 1 \tag{3.17}$$

(see [12, p.172]) yields

$$\rho_\alpha^-(x) = \frac{1}{2^\alpha \Gamma(\alpha+1)}.$$

This gives $\rho_\alpha^-(a)/\rho_\alpha^-(b) = 1$.

If $-1 < \alpha \leq -1/2$, it follows from (3.15) that

$$\rho_\alpha^-(x) = \sinh^{-\alpha}(\theta x)\mathfrak{B}_{-\alpha-1}^{-\alpha}(\cosh(\theta x)),$$

which is the same as the right hand side of (3.16) since $\mathfrak{B}_\nu^\mu(x) = \mathfrak{B}_{-\nu-1}^\mu(x)$ for $x > 1$ (see [12, p. 164]). Hence $\rho_\alpha^-(a)/\rho_\alpha^-(b) = 1$ for $-1 < \alpha \leq -1/2$.

If $\alpha \leq -1$, we need to consider

$$\rho_\alpha^+(x) = \sinh^{-\alpha}(\theta x)\mathfrak{B}_\alpha^\alpha(\cosh(\theta x)). \tag{3.18}$$

When $\nu + \mu < 0$ and $x > 1$, the Whipple formula states that

$$\Gamma(-\nu-\mu)\mathfrak{B}_\nu^\mu(x) = \sqrt{2/\pi}\, e^{i\pi(\nu+1/2)}(x^2-1)^{-1/4}\mathfrak{D}_{-\mu-1/2}^{-\nu-1/2}[x(x^2-1)^{-1/2}]$$

(see [12, p. 164]). This shows that (3.18) is equal to

$$\sqrt{\frac{2}{\pi}}\,\frac{1}{\Gamma(-2\alpha)}\sinh^{-\alpha-1/2}(\theta x)e^{i\pi(\alpha+1/2)}\mathfrak{D}_{-\alpha-1/2}^{-\alpha-1/2}(\coth(\theta x)) \tag{3.19}$$

and thus we have to evaluate $e^{-i\pi\mu}\mathfrak{D}_\mu^\mu(x)$ for $x > 1$.

Lemma 3.4. *If $\mu > -1/2$, then*

$$e^{-i\pi\mu}\mathfrak{D}_\mu^\mu(x) = 2^\mu \Gamma(\mu+1)(x^2-1)^{\mu/2}\int_x^\infty \frac{dy}{(y^2-1)^{\mu+1}} \quad \textit{for } x > 1. \tag{3.20}$$

Proof. For $x > 1$ let $\sigma_\mu(x) = e^{-i\pi\mu}\mathfrak{D}_\mu^\mu(x)$. Since

$$\mathfrak{B}_\mu^{-\mu}(x)\sigma_\mu'(x) - \sigma_\mu(x)\frac{d\mathfrak{B}_\mu^{-\mu}}{dx}(x) = \frac{1}{1-x^2},$$

(see [12, p. 165]), by (3.17) we have

$$\sigma_\mu'(x) - \frac{\mu x}{x^2-1}\sigma_\mu(x) = -\frac{2^\mu \Gamma(\mu+1)}{(x^2-1)^{\mu/2+1}}$$

for any $x > 1$. Hence

$$(r^2-1)^{-\mu/2}\sigma_\mu(r) - (x^2-1)^{-\mu/2}\sigma_\mu(x) = -2^\mu \Gamma(\mu+1)\int_x^r \frac{dy}{(y^2-1)^{\mu+1}} \tag{3.21}$$

for $1 < x < r$. Formula (2.8) gives

$$(r^2-1)^{-\mu/2}\sigma_\mu(r) = \frac{\Gamma(2\mu+1)}{2^{\mu+1}\Gamma(\mu+1)}\,\frac{1}{(r^2-1)^\mu}\int_0^\pi \frac{\sin^{2\mu+1} y}{r+\cos y}\,dy,$$

which is bounded by

$$\frac{\Gamma(2\mu+1)}{2^{\mu+1}\Gamma(\mu+1)}\,\frac{\pi}{(r^2-1)^\mu(r-1)},$$

since $\mu > -1/2$. We let $r \to \infty$ in (3.21) and deduce (3.20). ∎

It follows from $\alpha \le -1$ that $-\alpha - 1/2 \ge 1/2$. This shows that (3.20) can be applied and so

$$e^{i\pi(\alpha+1/2)}\mathfrak{D}^{-\alpha-1/2}_{-\alpha-1/2}(\coth(\theta x)) = \frac{\Gamma(1/2-\alpha)}{2^{\alpha+1/2}}(\coth^2(\theta x)-1)^{-(\alpha+1/2)/2}\kappa_\alpha(x),$$

where

$$\kappa_\alpha(x) = \int_{\coth(\theta x)}^\infty \frac{d\xi}{(\xi^2-1)^{-\alpha+1/2}}.$$

Hence we obtain (3.19) and also

$$\rho^+_\alpha(x) = \frac{\Gamma(1/2-\alpha)}{\sqrt{\pi}2^\alpha\Gamma(-2\alpha)}\kappa_\alpha(x),$$

which implies that the right hand side of (3.14) coincides with $\kappa_\alpha(a)/\kappa_\alpha(b)$. In order to complete the proof of (3.2) we shall calculate $\kappa_\alpha(x)$ for $x > 0$. Changing the variable via $\xi = \coth y$, we deduce from a simple calculation that

$$\kappa_\alpha(x) = \int_0^{\theta x} \sinh^{-2\alpha-1} y\,dy.$$

Hence we deduce (3.2) and the proof of Theorem 3.1 is complete.

4. Expectations of the hitting times for $a < b$. When $0 < a < b$, the explicit form of the expectation of $\tau^{(\alpha,\theta)}_{a,b}$ is provided in [17] for $\alpha = 0, 1/2, 1, 3/2$. Our purpose in this section is to improve those results. Throughout we consider only the case $0 < a < b$.

For $\alpha > -1$ the main tool in calculating $E[\tau^{(\alpha,\theta)}_{a,b}]$ is the Dynkin formula (see [9, p. 99]). For $x > 0$ let

$$f(x) = 2\int_0^x \sinh^{-2\alpha-1}(\theta\xi)\int_0^\xi \sinh^{2\alpha+1}(\theta\eta)\,d\eta\,d\xi + 1.$$

A simple calculation shows that $\mathcal{L}_{\alpha,\theta} f(x) = 1$ and that $f(x)$ and $f'(x)/s'(x)$ converge to 1 and 0 as $x \downarrow 0$, respectively. The argument in [9, p. 99] implies that the Dynkin formula ensures the existence of the expectation of $\tau_{a,b}^{(\alpha,\theta)}$. Applying the Dynkin formula again, we get

$$E[\tau_{a,b}^{(\alpha,\theta)}] = \frac{2}{\theta^2} \int_{\theta a}^{\theta b} \sinh^{-2\alpha-1} \xi \int_0^{\xi} \sinh^{2\alpha+1} \eta \, d\eta \, d\xi \tag{4.1}$$

for $0 < a < b$ and $\alpha > -1$. The limiting behavior of $E[\tau_{a,b}^{(\alpha,\theta)}]$ for large b can be trivially derived from (3.7) and (4.1); the detailed calculations are omitted.

PROPOSITION 4.1. *We have, as $b \to \infty$,*

$$E[\tau_{a,b}^{(\alpha,\theta)}] = \begin{cases} \frac{2b}{(2\alpha+1)\theta}(1+o[1]) & \text{if } \alpha > -1/2, \\ b^2(1+o[1]) & \text{if } \alpha = -1/2, \\ \frac{2e^{|2\alpha+1|\theta b}}{(2\alpha+1)^2\theta^2}(1+o[1]) & \text{if } -1 < \alpha < -1/2. \end{cases}$$

Moreover, the explicit form of $E[\tau_{a,b}^{(\alpha,\theta)}]$ for small α can be deduced from indefinite integrals of rational functions of $\cosh x$ and $\sinh x$ which are described in [5]. The following results are straightforward consequences of (4.1).

PROPOSITION 4.2. *We have*

$$E[\tau_{a,b}^{(0,\theta)}] = \frac{2}{\theta^2} \log \frac{\cosh(\theta b)+1}{\cosh(\theta a)+1},$$

$$E[\tau_{a,b}^{(1/2,\theta)}] = \frac{b\coth(\theta b) - a\coth(\theta a)}{\theta},$$

$$E[\tau_{a,b}^{(1,\theta)}] = \frac{2}{3\theta^2}\left[\log \frac{\cosh(\theta b)+1}{\cosh(\theta a)+1} - \frac{1}{\cosh(\theta b)+1} + \frac{1}{\cosh(\theta a)+1}\right],$$

$$E[\tau_{a,b}^{(3/2,\theta)}] = \frac{1}{4\theta^2}\left[\frac{1+\theta b\{\sinh(2\theta b) - \coth(\theta b)\}}{\sinh^2(\theta b)} - \frac{1+\theta a\{\sinh(2\theta a) - \coth(\theta a)\}}{\sinh^2(\theta a)}\right],$$

$$E[\tau_{a,b}^{(2,\theta)}] = \frac{2}{15\theta^2}\left[3\log \frac{\cosh(\theta b)+1}{\cosh(\theta a)+1} - \frac{4+3\cosh(\theta b)}{(\cosh(\theta b)+1)^2} + \frac{4+3\cosh(\theta a)}{(\cosh(\theta a)+1)^2}\right].$$

The first four formulas have been obtained in [17] for $\theta = 1$ by solving the boundary value problem

$$\mathcal{L}_{\alpha,1}u(x) = -1, \quad u(b) = 0, \quad u(0) \text{ is finite}$$

(see [4]). Our method is different.

The remainder of this section is devoted to the case $\alpha \leq -1$. Since (3.2) gives $P(\tau_{a,b}^{(\alpha,\theta)} < \infty) < 1$, we should consider the expectation of $\tau_{a,b}^{(\alpha,\theta)}$ under the condition $\tau_{a,b}^{(\alpha,\theta)} < \infty$. Formula (3.10) gives

$$\mathfrak{B}^{\alpha}_{\alpha(\lambda)}(\cosh(\theta x)) = \frac{2^{\alpha}\sinh^{-\alpha}(\theta x)}{\sqrt{\pi}\Gamma(1/2-\alpha)}\phi_{\alpha,\theta x}(\alpha(\lambda)) \tag{4.2}$$

for $\lambda \geq 0$. This shows that (2.5) and (3.14) are equivalent to

$$E[e^{-\lambda\tau_{a,b}^{(\alpha,\theta)}}] = \frac{\sinh^{-2\alpha}(\theta a)\phi_{\alpha,\theta a}(\alpha(\lambda))}{\sinh^{-2\alpha}(\theta b)\phi_{\alpha,\theta b}(\alpha(\lambda))} \tag{4.3}$$

and

$$P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = \frac{\sinh^{-2\alpha}(\theta a)\phi_{\alpha,\theta a}(\alpha(0))}{\sinh^{-2\alpha}(\theta b)\phi_{\alpha,\theta b}(\alpha(0))}, \tag{4.4}$$

respectively. Hence

$$E[e^{-\lambda\tau_{a,b}^{(\alpha,\theta)}} \,|\, \tau_{a,b}^{(\alpha,\theta)} < \infty] = \frac{\phi_{\alpha,\theta a}(\alpha(\lambda))}{\phi_{\alpha,\theta a}(\alpha(0))}\,\frac{\phi_{\alpha,\theta b}(\alpha(0))}{\phi_{\alpha,\theta b}(\alpha(\lambda))}$$

for any $\lambda > 0$. Since

$$E[\tau_{a,b}^{(\alpha,\theta)} \,|\, \tau_{a,b}^{(\alpha,\theta)} < \infty] = -\lim_{\lambda\downarrow 0}\frac{d}{d\lambda}E[e^{-\lambda\tau_{a,b}^{(\alpha,\theta)}} \,|\, \tau_{a,b}^{(\alpha,\theta)} < \infty], \tag{4.5}$$

we deduce from Lemma 3.3 that

$$E[\tau_{a,b}^{(\alpha,\theta)} \,|\, \tau_{a,b}^{(\alpha,\theta)} < \infty] = \alpha'(0)\left[\frac{\phi'_{\alpha,\theta b}(\alpha(0))}{\phi_{\alpha,\theta b}(\alpha(0))} - \frac{\phi'_{\alpha,\theta a}(\alpha(0))}{\phi_{\alpha,\theta a}(\alpha(0))}\right]. \tag{4.6}$$

We can compute the right hand side of (4.6) with the help of Lemma 3.3, and the following theorem is the result for $\alpha \leq -1$.

THEOREM 4.3. *For $0 < a < b$ and $\alpha \leq -1$ we have*

$$\begin{aligned} E[\tau_{a,b}^{(\alpha,\theta)} \,|\, \tau_{a,b}^{(\alpha,\theta)} < \infty] \\ = \frac{2}{|2\alpha+1|}\left[b + \frac{1}{\theta}\left(\frac{p_\alpha}{q_\alpha} - \log 2 - \frac{\phi'_{\alpha,\theta a}(\alpha(0))}{\phi_{\alpha,\theta a}(\alpha(0))}\right) + o[1]\right] \end{aligned} \tag{4.7}$$

as $b \to \infty$, where

$$p_\alpha = \int_0^\pi (1-\cos y)\sin^{-2\alpha-2} y \log(1+\cos y)\,dy,$$

$$q_\alpha = \int_0^\pi (1-\cos y)\sin^{-2\alpha-2} y\,dy.$$

Proof. Recall that $\alpha(0) = -\alpha-1 \geq 0$. We can apply (3.11) to $\phi'_{\alpha,\theta b}(\alpha(0))$ to get

$$\begin{aligned} \phi'_{\alpha,\theta b}(\alpha(0)) &= \log(\cosh(\theta b))\phi_{\alpha,\theta b}(\alpha(0)) \\ &+ \frac{1}{\cosh(\theta b)}\int_0^\pi \frac{\log(1+\tanh(\theta b)\cos y)}{1+\tanh(\theta b)\cos y}\sin^{-2\alpha} y\,dy. \end{aligned} \tag{4.8}$$

We consider the asymptotic behavior for large b of the last integral, which is equal to

$$\int_0^\pi \frac{1-\tanh(\theta b)\cos y}{1-\tanh^2(\theta b)\cos^2 y}\sin^{-2\alpha} y \log(1+\tanh(\theta b)\cos y)\, dy. \tag{4.9}$$

Since $0 < \tanh x < 1$ for $x > 0$, it follows that

$$|1-\tanh(\theta b)\cos y| \le 2, \qquad \left|\frac{\sin^2 y}{1-\tanh^2(\theta b)\cos^2 y}\right| \le 1. \tag{4.10}$$

Moreover, $|\sin^{-2\alpha-2} y| \le 1$ because $2\alpha+2 \le 0$. Hence the absolute value of the integrand of (4.9) is bounded by

$$2|\log(1+\tanh(\theta b)\cos y)| \le \begin{cases} 2\log 2 & \text{if } 0 < y \le \pi/2, \\ 4|\log(1-y/\pi)| & \text{if } \pi/2 < y < \pi, \end{cases} \tag{4.11}$$

which can be easily obtained from

$$0 \le \log(1+\tanh(\theta b)\cos y) \le \log 2$$

for $0 < y \le \pi/2$, and

$$\frac{(\pi-y)^2}{\pi^2} \le 1+\cos y \le 1+\tanh(\theta b)\cos y \le 1$$

for $\pi/2 < y < \pi$. Since the right hand side of (4.11) is integrable on $(0,\pi)$, the dominated convergence theorem shows that we can interchange the limit in b and the integral in y in (4.9). Thus (4.9) converges to p_α as $b \to \infty$. Moreover, similarly to (4.9), by (4.10) we obtain

$$\phi_{\alpha,\theta b}(\alpha(0)) = \frac{1}{\cosh(\theta b)}\int_0^\pi \frac{1-\tanh(\theta b)\cos y}{1-\tanh^2(\theta b)\cos^2 y}\sin^{-2\alpha} y\, dy = \frac{q_\alpha + o[1]}{\cosh(\theta b)}$$

as $b \to \infty$. Hence (4.8) gives

$$\frac{\phi'_{\alpha,\theta b}(\alpha(0))}{\phi_{\alpha,\theta b}(\alpha(0))} = \theta b + \frac{p_\alpha}{q_\alpha} - \log 2 + o[1].$$

Since $\alpha'(0) = 2/|2\alpha+1|\theta$, we deduce (4.4) from (4.6). ∎

5. Expectations of the hitting times for $b < a$. For $0 < b < a$ we shall use (2.6) to compute the expectation of $\tau_{a,b}^{(\alpha,\theta)}$. If $\alpha = -1/2$, by (2.6) we have

$$E[e^{-\lambda\tau_{a,b}^{(-1/2,\theta)}}] = e^{-(a-b)\sqrt{2\lambda\theta}},$$

since

$$e^{-i\pi/2}\mathfrak{D}_\nu^{1/2}(x) = \sqrt{\frac{\pi}{2}}(x^2-1)^{-1/4}[x+(x^2-1)^{1/2}]^{-\nu-1/2}$$

(see [12, p. 72]). Hence the density of $\tau_{a,b}^{(-1/2,\theta)}$ is represented by

$$\frac{(a-b)\sqrt{\theta}}{\sqrt{2\pi t^3}}e^{-(a-b)^2\theta/2t}$$

for $t>0$ (see [15, p. 258]). In virtue of (3.4) we have

$$E[\tau_{a,b}^{(-1/2,\theta)}] = E[\tau_{a,b}^{(-1/2,\theta)} \mid \tau_{a,b}^{(-1/2,\theta)} < \infty] = \infty.$$

We concentrate on the case $\alpha \neq -1/2$. We define a function $\psi_{\mu,x}$ on $\mathbb{R}$ by

$$\psi_{\mu,x}(\nu) = \int_0^\pi (\cosh x + \cos y)^{-\nu-\mu-1} \sin^{2\nu+1} y\, dy \tag{5.1}$$

for $\mu \in \mathbb{R}$ and $x>0$. The purpose of this section is to show the following theorem.

Theorem 5.1. *For $0<b<a$ and $\alpha \neq -1/2$ we have*

$$\begin{aligned} E[\tau_{a,b}^{(\alpha,\theta)} \mid \tau_{a,b}^{(\alpha,\theta)} < \infty] \\ = \frac{2}{|2\alpha+1|}\left[a - \frac{1}{\theta}\left(\frac{\bar{p}_\alpha}{\bar{q}_\alpha} + \log 2 - \frac{\psi'_{\alpha,\theta b}(\alpha(0))}{\psi_{\alpha,\theta b}(\alpha(0))}\right) + o[1]\right] \end{aligned} \tag{5.2}$$

as $a \to \infty$, where

$$\bar{p}_\alpha = 2\int_0^\pi \sin^{-2\alpha-1} y \log(\sin y)\, dy, \quad \bar{q}_\alpha = \int_0^\pi \sin^{-2\alpha-1} y\, dy.$$

To prove this theorem we need to establish the differentiability of $\psi_{\mu,x}$ on a suitable interval.

Lemma 5.2. *Let $\mu \in \mathbb{R}$ and $x>0$. Put $\mu_0 = \max\{\mu-1,-1\}$. Then $\psi_{\mu,x}$ is continuously differentiable on (μ_0,∞) and*

$$\begin{aligned} \psi'_{\mu,x}(\nu) = &-\int_0^\pi (\cosh x + \cos y)^{-\nu-\mu-1} \sin^{2\nu+1} y \log(\cosh x + \cos y)\, dy \\ &+ 2\int_0^\pi (\cosh x + \cos y)^{-\nu-\mu-1} \sin^{2\nu+1} y \log(\sin y)\, dy. \end{aligned} \tag{5.3}$$

The proof is postponed to the end of this section; we first prove Theorem 5.1. Note that (2.3) gives

$$\alpha(\lambda) \geq -1/2, \quad \alpha(\lambda) \geq \alpha \quad \text{for } \lambda \geq 0. \tag{5.4}$$

In particular, $\alpha(\lambda) > \max\{\alpha-1,-1\}$. Formula (2.8) yields

$$e^{i\pi\alpha} \sinh^{-\alpha} x\, \mathfrak{D}_{\alpha(\lambda)}^{-\alpha}(\cosh x) = \frac{\Gamma(\alpha(\lambda)-\alpha+1)}{2^{\alpha(\lambda)+1}\Gamma(\alpha(\lambda)+1)}\psi_{\alpha,x}(\alpha(\lambda)) \tag{5.5}$$

for any $x > 0$ and $\lambda > 0$. Hence the combination of (2.6) and (5.5) yields

$$E[e^{-\lambda\tau_{a,b}^{(\alpha,\theta)}}] = \frac{\psi_{\alpha,\theta a}(\alpha(\lambda))}{\psi_{\alpha,\theta b}(\alpha(\lambda))}. \tag{5.6}$$

Since Lemma 5.2 implies that $\psi_{\alpha,x}$ is continuous, by (3.9) we have

$$P(\tau_{a,b}^{(\alpha,\theta)} < \infty) = \frac{\psi_{\alpha,\theta a}(\alpha(0))}{\psi_{\alpha,\theta b}(\alpha(0))}. \tag{5.7}$$

Similarly to (4.6) we can apply (5.3) to deduce by (4.5) and (5.7) that

$$E[\tau_{a,b}^{(\alpha,\theta)} \mid \tau_{a,b}^{(\alpha,\theta)} < \infty] = \frac{2}{|2\alpha+1|\theta}\left[\frac{\psi'_{\alpha,\theta b}(\alpha(0))}{\psi_{\alpha,\theta b}(\alpha(0))} - \frac{\psi'_{\alpha,\theta a}(\alpha(0))}{\psi_{\alpha,\theta a}(\alpha(0))}\right] \tag{5.8}$$

if $\alpha \neq -1/2$. When $\alpha < -1/2$, the left hand side of (5.8) coincides with $E[\tau_{a,b}^{(\alpha,\theta)}]$ since (3.4) holds.

We first show (5.2) for $\alpha > -1/2$, so $\alpha(0) = \alpha$ in this case. It follows that

$$\psi_{\alpha,\theta a}(\alpha(0)) = \int_0^\pi (\cosh(\theta a) + \cos y)^{-2\alpha-1} \sin^{2\alpha+1} y \, dy. \tag{5.9}$$

Moreover, since $\alpha(0) > \max\{\alpha - 1, -1\}$, we infer by (5.3) that

$$\begin{aligned} \psi'_{\alpha,\theta a}(\alpha(0)) &= -\log(\cosh(\theta a))\psi_{\alpha,\theta a}(\alpha(0)) \\ &\quad - \int_0^\pi (\cosh(\theta a) + \cos y)^{-2\alpha-1} \sin^{2\alpha+1} y \log\left(1 + \frac{\cos y}{\cosh(\theta a)}\right) dy \\ &\quad + 2\int_0^\pi (\cosh(\theta a) + \cos y)^{-2\alpha-1} \sin^{2\alpha+1} y \log(\sin y) \, dy. \end{aligned} \tag{5.10}$$

Since $|\log(1+\xi)| \leq 2|\xi|$ for $|\xi| < 1/2$, the second term of (5.10) is $\psi_{\alpha,\theta a}(\alpha(0)) \times O[1/\cosh(\theta a)]$. Note that

$$\frac{1}{2} < 1 + \frac{\cos y}{\cosh(\theta a)} < \frac{3}{2}$$

for $a > 0$ with $\cosh(\theta a) > 2$. The dominated convergence theorem shows that (5.9) and the third term of (5.10) are

$$\cosh^{-2\alpha-1}(\theta a)(\bar{q}_\alpha + o[1]) \quad \text{and} \quad \cosh^{-2\alpha-1}(\theta a)(\bar{p}_\alpha + o[1])$$

as $a \to \infty$, respectively. Therefore

$$\frac{\psi'_{\alpha,\theta a}(\alpha(0))}{\psi_{\alpha,\theta a}(\alpha(0))} = -\theta a + \frac{\bar{p}_\alpha}{\bar{q}_\alpha} + \log 2 + o[1]$$

as $a \to \infty$. Combining this with (5.8), we deduce (5.2) for $\alpha > -1/2$.

We next consider the case $\alpha < -1/2$ and so $\alpha(0) = -\alpha - 1$. Hence it follows from (5.1) and (5.3) that

$$\psi_{\alpha,\theta a}(\alpha(0)) = \int_0^\pi \sin^{-2\alpha-1} y\, dy = \bar{q}_\alpha$$

and that

$$\begin{aligned}\psi'_{\alpha,\theta a}(\alpha(0)) = &- \log(\cosh(\theta a))\psi_{\alpha,\theta a}(\alpha(0)) \\ &- \int_0^\pi \sin^{-2\alpha-1} y \log\left(1 + \frac{\cos y}{\cosh(\theta a)}\right) dy \\ &+ 2\int_0^\pi \sin^{-2\alpha-1} y \log(\sin y)\, dy.\end{aligned}$$

Note that the third term of the right hand side is equal to $\bar{p}_\alpha$. Similarly to the case $\alpha > -1/2$ we conclude that

$$\frac{\psi'_{\alpha,\theta a}(\alpha(0))}{\psi_{\alpha,\theta a}(\alpha(0))} = -\log(\cosh(\theta a)) + \frac{\bar{p}_\alpha}{\bar{q}_\alpha} + O\left[\frac{1}{\cosh(\theta a)}\right],$$

which yields (5.2) for $\alpha < -1/2$.

In order to complete the proof of Theorem 5.1 we need to establish Lemma 5.2. Let $x > 0$ and $\mu \in \mathbb{R}$. For $\nu, y \in \mathbb{R}$ we put

$$\zeta_{\mu,x}(y,\nu) = (\cosh x + \cos y)^{-\nu-\mu-1}.$$

The following uniform estimate of $\zeta_{\mu,x}(y,\nu)$ for y and ν is useful.

Lemma 5.3. *Let $x > 0$ and $\nu_0 \in (\mu_0, \infty)$. There exist positive constants $L(\mu, x, \nu_0)$ and $\delta(\mu, \nu_0)$ such that*

$$\zeta_{\mu,x}(y,\nu) \le L(\mu, x, \nu_0) \tag{5.11}$$

for any $y \in \mathbb{R}$ and $\nu > \mu_0$ satisfying $|\nu - \nu_0| < \delta(\mu, \nu_0)$.

Proof. A simple calculation shows that

$$(\cosh x + \cos y)^{-\gamma} \le L_0(x,\gamma) \tag{5.12}$$

for any $y \in \mathbb{R}$, where

$$L_0(x,\gamma) = \begin{cases} \left(1 - \dfrac{1}{\cosh x}\right)^{-\gamma} & \text{if } \gamma \ge 0, \\ (\cosh x + 1)^{-\gamma} & \text{if } \gamma < 0. \end{cases}$$

It is obvious that the function $\gamma \mapsto L_0(x,\gamma)$ is increasing on $[0,\infty)$ and is decreasing on $(-\infty, 0]$. Moreover, the definition of μ_0 implies that $\nu_0 > \mu - 1$ if $\mu \ge 0$ and $\nu_0 > -1$ if $\mu < 0$.

We first consider the case $\mu \geq 0$. When $\mu - 1 < \nu < 2\nu_0 - \mu + 1$, it follows that $0 \leq 2\mu < \nu + \mu + 1 < 2\nu_0 + 2$. Hence (5.12) gives

$$\zeta_{\mu,x}(y,\nu) \leq L_0(x, 2\nu_0 + 2)$$

for any $y \in \mathbb{R}$. This implies that we can choose

$$L(\mu, x, \nu_0) = L_0(x, 2\nu_0 + 2), \qquad \delta(\mu, \nu_0) = \nu_0 - \mu + 1.$$

We next show (5.11) for $\mu < 0$ by considering the following three cases:

$$\text{(i) } \nu_0 + \mu + 1 > 0, \quad \text{(ii) } \nu_0 + \mu + 1 = 0, \quad \text{(iii) } \nu_0 + \mu + 1 < 0.$$

Case (i). If $-\mu - 1 < \nu < 2\nu_0 + \mu + 1$, then $0 < \nu + \mu + 1 < 2(\nu_0 + \mu + 1)$, which implies that

$$\zeta_{\mu,x}(y,\nu) \leq L_0(x, 2(\nu_0 + \mu + 1))$$

for any $y \in \mathbb{R}$. Hence (5.11) holds with

$$L(\mu, x, \nu_0) = L_0(x, 2(\nu_0 + \mu + 1)), \qquad \delta(\mu, \nu_0) = \nu_0 + \mu + 1.$$

Case (ii). If $-1 < \nu < \nu_0$, then $\mu < \nu + \mu + 1 < \nu_0 + \mu + 1 = 0$. This yields

$$\zeta_{\mu,x}(y,\nu) \leq L_0(x, \mu).$$

In addition, if $\nu_0 \leq \nu < \nu_0 + 1$, then $0 \leq \nu + \mu + 1 < \nu_0 + \mu + 2 = 1$, which gives

$$\zeta_{\mu,x}(y,\nu) \leq L_0(x, 1).$$

We put

$$L(\mu, x, \nu_0) = \max\{L_0(x,\mu), L_0(x,1)\}, \qquad \delta(\mu, \nu_0) = \min\{\nu_0 + 1, 1\}$$

and obtain (5.11).

Case (iii). When $-1 < \nu < \nu_0$, as $\mu < \nu + \mu + 1 < \nu_0 + \mu + 1 < 0$ we have

$$\zeta_{\mu,x}(y,\nu) \leq L_0(x, \mu).$$

Moreover, if $\nu_0 \leq \nu < -\mu - 1$, it follows that $\nu_0 + \mu + 1 \leq \nu + \mu + 1 < 0$ and thus

$$\zeta_{\mu,x}(y,\nu) \leq L_0(x, \nu_0 + \mu + 1).$$

Hence taking

$$\begin{cases} L(\mu, x, \nu_0) = \max\{L_0(x,\mu), L_0(x, \nu_0 + \mu + 1)\}, \\ \delta(\mu, \nu_0) = \min\{\nu_0 + 1, -(\nu_0 + \mu + 1)\}, \end{cases}$$

we get (5.11).

This finishes the proof of the lemma. ■

We are now ready to prove Lemma 5.2. Let $\nu_0 \in (\mu_0, \infty)$. It is sufficient to see that $\psi_{\mu,x}$ is differentiable at $\nu = \nu_0$ and that its first derivative is continuous at $\nu = \nu_0$.

For $x > 0$ let

$$l(x) = \max\left\{\log(\cosh x + 1), \log\frac{1}{\cosh x - 1}\right\}.$$

Note that

$$|\log(\cosh x + \cos y)| \leq l(x),$$

because $0 < \cosh x - 1 \leq \cosh x + \cos y \leq \cosh x + 1$.

For $y \in (0, \pi)$ and $\nu \in \mathbb{R}$ satisfying $|\nu - \nu_0| < \delta(\mu, \nu_0)$, we can deduce from Lemma 5.3 that

$$\left|\zeta_{\mu,x}(y,\nu)\sin^{2\nu+1} y \log(\cosh x + \cos y)\right| \leq L(\mu, x, \nu_0) l(x) \sin^{2\nu+1} y.$$

If $|\nu - \nu_0| < (\nu_0 - \mu_0)/2$, we have $2\nu + 1 > \nu_0 + \mu_0 + 1 > 2\mu_0 + 1 \geq -1$. Hence

$$\left|\zeta_{\mu,x}(y,\nu)\sin^{2\nu+1} y \log(\cosh x + \cos y)\right| \leq L(\mu, x, \nu_0) l(x) \sin^{\nu_0+\mu+1} y \tag{5.13}$$

for $y \in (0, \pi)$. The right hand side is an integrable function of y on $(0, \pi)$. Moreover, if $\nu \in \mathbb{R}$ satisfies $|\nu - \nu_0| < \min\{\delta(\mu, \nu_0), (\nu_0 - \mu_0)/2\}$, then

$$\left|\zeta_{\mu,x}(y,\nu)\sin^{2\nu+1} y \log(\sin y)\right| \leq L(\mu, x, \nu_0) \sin^{\nu_0+\mu+1} y |\log(\sin y)|, \tag{5.14}$$

which is integrable on $(0, \pi)$.

Therefore $\psi_{\mu,x}$ is differentiable at $\nu = \nu_0$ and (5.3) holds for $\nu = \nu_0$. In addition, (5.13) and (5.14) also yield the continuity of $\psi'_{\mu,x}$ at $\nu = \nu_0$. The proof of Lemma 5.2 is complete.

Acknowledgements. This work is partially supported by the Grant-in-Aid for Scientific Research (C) No. 20K03634 of the Japan Society for the Promotion of Science (JSPS). The authors are grateful to Professor Jun-ichi Inoguchi for his useful comments and advice.

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