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Congruences for modular forms and applications to crank functions

by

Hao Zhang and Helen W. J. Zhang (Changsha)

Abstract. Motivated by the work of Mahlburg, which refined the work of Ono, we find congruences for a large class of modular forms. Moreover, we generalize the generating function of the Andrews–Garvan–Dyson crank of partitions and establish several new infinite families of congruences. In this framework, we show that both the birank of an ordered pair of partitions introduced by Hammond and Lewis, and k-crank of k-colored partitions introduced by Fu and Tang, have the same properties as the partition function and crank.

1. Introduction. The objective of this paper is to generalize the work on partition congruences related to ranks and cranks. The previous literature focused on the partition functions; see for example [12, 14]. Here we go further by generalizing it to crank-type generating functions. This work was inspired by the key construction of Mahlburg [12] showing that the generating function of crank is deeply related to Klein forms and weakly holomorphic modular forms of half-integral weight on the congruence subgroup $\Gamma_1(l^j)$. The idea of Mahlburg's work follows originally from Ono [14]. This leads to arithmetic properties of the crank of partitions, which confirmed a conjecture of Ono [13].

Recall that a *partition* of a nonnegative integer n is any nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. A breakthrough of applying modular forms to partition functions is due to Ono [14]. He showed that for prime $\ell \geq 5$, there are infinitely many non-nested arithmetic progressions $An + B$ such that for every integer n we have

$$p(An + B) \equiv 0 \pmod{\ell}.$$

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Along this direction, the congruence properties for partition functions have been deeply investigated [1, 6, 8, 16, 18].

To give combinatorial interpretations of Ramanujan's well-known congruences, Dyson [5] conjectured the existence of a statistic named crank, and its definition was discovered by Andrews and Garvan [2]. The crank of a partition was defined as the largest part if the partition contains no ones, and otherwise as the number of parts larger than the number of ones minus the number of ones. For more details, we refer to [3, 2, 9].

Ono further conjectured that the crank functions enjoy all of the partition congruences. The work of Mahlburg [12, 13] provided an elegant answer to these questions. Let $M(m,n)$ denote the number of partitions of n with crank m. Then the generating function of $M(m,n)$ can be written as

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}, \tag{1.1}$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$. Mahlburg [12, 13] developed the theory of congruences for crank generating functions, which are shown to possess the same sort of arithmetic properties as partition functions. For instance, [12, 13] stated that if τ is a positive integer, then there are infinitely many non-nested arithmetic progressions $An+B$ such that

$$M(m,\ell^j,An+B) \equiv 0 \pmod{\ell^\tau},$$

simultaneously for every $0 \le m \le \ell^j - 1$.

Motivated by the work of Mahlburg, we define *crank-type generating function* as follows:

$$\begin{aligned} \mathfrak{F}_{r,d,t}(z,q) &= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M_{r,d,t}(m,n)z^m q^n \\ &:= \frac{(q^r;q^r)_\infty^d}{(q;q)_\infty^t (zq,q/z;q)_\infty}. \end{aligned} \tag{1.2}$$

Note that taking $r=t=1$ and $d=2$, (1.2) reduces to the ordinary crank generating function (1.1). When $r=d=t=1$, (1.2) becomes the generating function of the birank of an ordered pair of partitions given by Hammond and Lewis [10]. Taking $r=t=1$ and $d=k-1$ in (1.2), we obtain the generating function introduced by Bringmann and Dousse [4], whose combinatorial interpretation was given by Fu and Tang [7] as the k-crank of a k-colored partition.

Let N be a positive integer and set $\zeta = e^{2\pi i/N}$ in (1.2). For any residue class $m \pmod N$, elementary calculations give the generating function for the crank

$$\sum_{n=0}^{\infty} M_{r,d,t}(m,N,n)q^n := \frac{1}{N}\sum_{s=0}^{N-1}\mathfrak{F}_{r,d,t}(\zeta^s,q)\zeta^{-ms} = \frac{1}{N}\sum_{s=0}^{N-1}\zeta^{-ms}\left(\prod_{n=1}^{\infty}\frac{(1-q^{rn})^d}{(1-q^n)^t(1-\zeta^s q^n)(1-\zeta^{-s}q^n)}\right). \tag{1.3}$$

Our main result shows that the coefficients $M_{r,d,t}(m,N,n)$ enjoy the same arithmetic property as partition functions as well as crank.

THEOREM 1.1. *Let j,v be positive integers. Fix a prime number $\ell > \max\{5, dr^2\}$ such that $\left(\frac{g^2d-tr}{\ell}\right)$ are the same for all $g\,|\,r$. Then there are infinitely many non-nested arithmetic progressions $An+B$ such that*

$$M_{r,d,t}(m,\ell^j,An+B) \equiv 0 \pmod{\ell^v},$$

simultaneously for every $0 \le m \le \ell^j - 1$.

Theorem 1.1 provides the congruence of birank and k-crank. Recall that the *birank* $b(\pi)$ of an ordered pair of partitions $\pi = (\lambda(1),\lambda(2))$ (see [10]) is the number of parts in the first partition minus the number of parts in the second partition, that is, $b(\pi) = \#(\lambda(1)) - \#(\lambda(2))$. The number of ordered pairs of partitions of weight n having birank m will be written as $R(m,n)$, hence the generating function for $R(m,n)$ is

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} R(m,n)z^m q^n = \frac{1}{(zq,q/z;q)_\infty}.$$

COROLLARY 1.2. *Let $R(m,N,n)$ denote the number of ordered pairs of partitions of weight n with birank congruent to m modulo N. Suppose that $\ell \ge 5$ is prime and that τ and j are positive integers. Then there are infinitely many non-nested arithmetic progressions $An+b$ such that*

$$R(m,\ell^j,An+B) \equiv 0 \pmod{\ell^\tau},$$

simultaneously for every $0 \le m \le \ell^j - 1$.

The k-colored partition is a k-tuple of partitions $\lambda = (\lambda^{(1)},\ldots,\lambda^{(k)})$. For $k \ge 2$, Fu and Tang [7] defined the *k-crank* of a k-colored partition as follows:

$$k\text{-crank}(\lambda) = \ell(\lambda^{(1)}) - \ell(\lambda^{(2)}), \tag{1.4}$$

where $\ell(\lambda^{(i)})$ denotes the number of parts in $\lambda^{(i)}$. Let $M_k(m,n)$ denote the number of k-colored partitions of n with k-crank m. The generating function of $M_k(m,n)$ was derived by Bringmann and Dousse [4]:

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M_k(m,n)z^m q^n = \frac{(q;q)_\infty^{2-k}}{(zq;q)_\infty(z^{-1}q;q)_\infty}.$$

Corollary 1.3. *Let $M_k(m, N, n)$ denote the number of k-colored partitions of n with k-crank congruent to m modulo N. Suppose that $\ell \geq 5$ is prime and that τ and j are positive integers. Then there are infinitely many non-nested arithmetic progressions $An + b$ such that*

$$M_k(m, \ell^j, An + B) \equiv 0 \pmod{\ell^\tau},$$

simultaneously for every $0 \leq m \leq \ell^j - 1$.

To prove these results, we make use of the theory of modular forms, as developed by Serre. Ono was the first to show how the work of Serre and Shimura had deep implications for $p(n)$. Subsequent work on cranks and ranks reveals that these ideas apply as well to further partition generating functions. We further extend this method to the setting of the new refined cranks and ranks studies here. The notations and basic properties of modular forms and eta quotient will be introduced in Section 2, and we will prove the main result in Section 3.

2. Preliminaries. In this section, we present some basic properties of modular forms of half-integer weight which will be used in the following sections; see [15] for details.

Let $q = e^{2\pi i \tau}$ and

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)$$

with Nebentype character χ. Let ψ be a Dirichlet character modulo M. We define the twist of f by ψ to be

$$f(\tau) \otimes \psi = \sum_{n=0}^{\infty} \psi(n) a(n) q^n.$$

It is well-known that $f(\tau) \otimes \psi \in M_k(NM^2, \chi\psi^2)$. Fix a constant ϵ, and define

$$\widetilde{f_{\epsilon,\psi}}(\tau) = f(\tau) - \epsilon f(\tau) \otimes \psi. \tag{2.1}$$

In the following, when ϵ and the character ψ are specified, we write it as $\tilde{f}$.

With the notation above, let $p \nmid N$ be a prime number. Then the action of the half-integral Hecke operator T_{p^2} is defined by

$$\begin{aligned} f | T_{p^2} = \sum_{n \geq 0} \Bigg(a(p^2 n) &+ \chi(p) \left(\frac{(-1)^{k-1/2} n}{p} \right) p^{k-3/2} a(n) \\ &+ \chi(p^2) \left(\frac{(-1)^{k-1/2}}{p^2} \right) p^{2k-2} a\left(\frac{n}{p^2} \right) \Bigg) q^n. \end{aligned}$$

We need the following result which originally comes from Serre [17]. One can find the details of the proof in [14, pp. 300–301].

THEOREM 2.1. *Let $f_i(\tau) \in S_{k_i}(\Gamma_1(N_i))$ be half-integer weight cusp forms with algebraic integer coefficients where $k_i \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ for $i = 1, \dots, m$. Then for any $M \geq 1$, there exists a positive proportion of primes $p \equiv -1 \pmod{N_1 \cdots N_m M}$ such that*

$$f_i|T_{p^2} \equiv 0 \pmod{M} \quad \text{for every } i = 1, \dots, m.$$

Let $\vec{a} = (a_1, a_2) \in \mathbb{Q}^2$. We set $\tau' = a_1\tau + a_2$ and $q_{\tau'} = \mathbf{e}(\tau')$. The *Klein form* is given by

$$\mathfrak{k}_{\vec{a}}(\tau) = q_{\tau'}^{(a_1-1)/2} \frac{(q_{\tau'}; q)_\infty (q/q_{\tau'}; q)_\infty}{(q; q)_\infty^2}. \tag{2.2}$$

We recall some basic properties of Klein forms; one can find more details in [11].

PROPOSITION 2.2. *The Klein form $\mathfrak{k}_{\vec{a}}(\tau)$ satisfies the transformation formula, i.e. for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$\mathfrak{k}_{\vec{a}}|_\gamma(\tau) = \mathfrak{k}_{\vec{a}\cdot\gamma}(\tau).$$

If $(n_1, n_2) \in \mathbb{Z}^2$, then

$$\mathfrak{k}_{\vec{a}+(n_1,n_2)}(\tau) = (-1)^{n_1+n_2+n_1n_2} \mathbf{e}\left(\frac{a_1n_2 - a_2n_1}{2}\right) \mathfrak{k}_{\vec{a}}(\tau).$$

Proof. See [11, p. 260]. ∎

Finally, we introduce some basic properties of Dedekind's eta-function

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

The form $\prod_{1 \leq i \leq s} \eta^{r_i}(m_i\tau)$ is called an *eta quotient* where $r_i \in \mathbb{Z}$ and $m_i \in \mathbb{Z}_{\geq 1}$. The following classical result gives a sufficient condition for an eta quotient to be a modular function on a certain congruence subgroup.

THEOREM 2.3. *Let $f(\tau) = \prod_{m|N} \eta^{r_m}(mz)$ be an eta quotient with $k := \frac{1}{2}\sum_{m|N} r_m \in \mathbb{Z}$. If*

$$\sum_{m|N} mr_m \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{m|N} \frac{N}{m} r_m \equiv 0 \pmod{24},$$

then f is a modular function of weight k on the congruence subgroup $\Gamma_0(N)$ with character χ where $\chi(d) := \big(\frac{(-1)^k \prod_{m|N} m^{r_m}}{d}\big)$. Moreover, let a, c be two positive integers with $c \mid N$ and $\gcd(a, c) = 1$. Then the order of vanishing of

$f(\tau)$ at the cusp a/c is

$$\frac{N}{24}\sum_{m|N}\frac{\gcd(c,m)^2 r_m}{\gcd(c,N/c)cm}.$$

Proof. See [15, Theorems 1.64, 1.65]. ∎

As a corollary, we get the following result.

LEMMA 2.4. *Let $\ell \geq 5$ be a prime number. Then*

$$\frac{\eta^\ell(\ell\tau)}{\eta(\tau)} \in M_{(\ell-1)/2}\Big(\Gamma_0(\ell), \Big(\frac{*}{\ell}\Big)\Big). \tag{2.3}$$

For integer $j \geq 2$, we have

$$E_j(\tau) := \frac{\eta^{\ell^j}(\tau)}{\eta(\ell^j\tau)} \in M_{(\ell^j-1)/2}(\Gamma_0(\ell^j), \chi_j),$$

where $\chi_j(d) = \big(\frac{(-1)^{(\ell^j-1)/2}\ell^j}{d}\big)$, which is the same as in Theorem 2.3 with $r_1 = \ell^j$ and $r_{\ell^j} = -1$. Moreover, $E_j(\tau)$ vanishes at every cusp a/c with ℓ^j not dividing c.

Proof. For the first part, we just need to note that when d is an odd prime, the law of quadratic reciprocity gives

$$\Big(\frac{(-1)^{(\ell-1)/2}\ell^\ell}{d}\Big) = \Big(\frac{(-1)^{(\ell-1)/2}}{d}\Big)\Big(\frac{\ell}{d}\Big) = \Big(\frac{d}{\ell}\Big).$$

When $d = 2$, the above equality also holds since both sides are equal to $(-1)^{(\ell^2-1)/8}$. Then $\frac{\eta^\ell(\ell\tau)}{\eta(\tau)} \in M_{(\ell-1)/2}\big(\Gamma_0(\ell), \big(\frac{*}{\ell}\big)\big)$ directly by Theorem 2.3. As for the vanishing property of $E_j(\tau)$, we note that for any $c = \ell^b c'$ with $\gcd(\ell, c') = 1$, the cusp a/c is $\Gamma_0(\ell^j)$-equivalent to a_0/ℓ^b for some a_0. So when $0 \leq b < j$, by Theorem 2.3, the order of vanishing of $E_j(\tau)$ at a_0/ℓ^b is greater than 0. ∎

3. Proof of Theorem 1.1. The following lemmas play central roles in the proof of Theorem 1.1.

LEMMA 3.1. *Let $e_\ell = \sum_{n=1}^{\ell-1}\big(\frac{n}{\ell}\big)e^{2\pi in/\ell}$ be the Gauss sum. Then the twisted modular form has the following expression:*

$$f(z)\otimes\Big(\frac{*}{\ell}\Big) = \frac{e_\ell}{\ell}\sum_{n=1}^{\ell-1}\Big(\frac{n}{\ell}\Big)f(\tau)\Big|\begin{pmatrix}1 & -n/\ell\\ 0 & 1\end{pmatrix}. \tag{3.1}$$

Proof. The right hand side of (3.1) can be written as

$$\frac{e_\ell}{\ell}\sum_{n=1}^{\ell-1}\sum_{m\geq 1} a(m)\Big(\frac{n}{\ell}\Big)\mathbf{e}\Big(2\pi i m\Big(\tau-\frac{n}{\ell}\Big)\Big)$$
$$= \frac{e_\ell}{\ell}\sum_{m\geq 1} e_\ell a(m)\Big(\frac{-m}{\ell}\Big)q^m = \sum_{m\geq 1} a(m)\Big(\frac{m}{\ell}\Big)q^m,$$

which completes the proof. ∎

We recall that the generating series of $M_{r,d,t}(m,N,n)$ is given by

$$\text{(3.2)}\qquad \sum_{n=0}^{\infty} M_{r,d,t}(m,N,n)q^n = \frac{1}{N}\sum_{s=0}^{N-1}\mathfrak{F}_{r,d,t}(\zeta^s,q)\zeta^{-ms}$$
$$= \frac{1}{N}\sum_{s=0}^{N-1}\zeta^{-ms}\Big(\prod_{n=1}^{\infty}\frac{(1-q^{rn})^d}{(1-q^n)^t(1-\zeta^s q^n)(1-\zeta^{-s}q^n)}\Big).$$

Substituting the definition of the Klein form (2.2) into (3.2), we consider the series

$$g_m(\tau) := \frac{1}{2\pi i}\sum_{s=1}^{N-1}\frac{\omega_s\zeta^{-ms}}{\mathfrak{k}_{(0,s/N)}(\tau)}\frac{\eta^{t\ell}(\ell\tau)\eta^{d\ell^v}(r\tau)}{\eta^t(\tau)} + \frac{\eta^{t\ell}(\ell\tau)\eta^{d\ell^v}(r\tau)}{\eta^t(\tau)}$$
$$=: G_m(\tau) + P(\tau)$$

where $\omega_s = \zeta^{s/2}(1-\zeta^{-s})$.

LEMMA 3.2. *Let r,d,t,ℓ,v be defined as before. Then*

$$\text{(3.3)}\qquad \frac{\widetilde{G_m}(24\tau)}{\eta^{t\ell}(24\ell\tau)\eta^{d\ell^v}(24r\tau)}E_{j+1}(24\tau)^{\ell^v} \in S_{\lambda+1/2}(\Gamma_0(576r\ell^{j+1}),\chi)$$

for some integer λ, where $\widetilde{G_m}$ is defined as in (2.1).

Proof. By [15, Lemma 2.4 and Theorem 1.65], we can show that the order of $E_{j+1}(\tau)$ at the cusp a/c with ℓ^{j+1} not dividing c is at least $\frac{\ell^2-1}{24}$. On the other hand, we have

$$\mathrm{ord}_{a/c}\,\eta^{t\ell}(\ell\tau)\eta^{d\ell^v}(r\tau) = \frac{t\ell^3+dr^2\ell^v}{24} < \frac{\ell^2-1}{24}\ell^v.$$

Hence it is enough to show that $\frac{\widetilde{G_m}(\tau)}{\eta^{k\ell}(\ell\tau)\eta^{d\ell^v}(r\tau)}$ vanishes at each cusp a/c with $\ell^{j+1}\mid c$. Choose integers b,d such that $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(\ell^{j+1})$.

Now we compute the order of G_m at the cusp a/c. To do that, let $g = \gcd(c,r)$, $a_1 = ra/g$, $c_1 = c/g$. It is easy to see $\gcd(a_1,c_1) = 1$, so we can choose integers b_1,d_1 such that $B = \left(\begin{smallmatrix} a_1 & b_1\\ c_1 & d_1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\Delta(r\tau)|_{12}A = (c\tau+d)^{12}\Delta\left(\frac{ar\tau+br}{c\tau+d}\right) = (c\tau+d)^{12}\Delta\left(B\left(\frac{g\tau+h}{r/g}\right)\right)$$
$$= \Delta\left(\frac{g\tau+h}{r/g}\right),$$

where $h = rbd_1 - b_1 d$. On the other hand, by Lemma 2.4, we see that $\frac{\eta^\ell(\ell\tau)}{\eta(\tau)}$ belongs to $M_{(\ell-1)/2}\big(\Gamma_0(\ell), \left(\frac{*}{\ell}\right)\big)$. By Proposition 2.2, we have

$$\mathfrak{k}_{(0,s/N)}|A(\tau) = \mathbf{e}\left(\frac{N(cs+ds-\overline{ds})+cds^2}{2N^2}\right)\mathfrak{k}_{(0,\overline{ds}/N)}(\tau),$$

where $\overline{ds}$ is the unique integer between 0 and $N-1$ such that $N \mid ds - \overline{ds}$. Hence combining the above, we get

$$G_m(\tau)|A = \frac{1}{2\pi i}\sum_{s=1}^{N-1}\frac{w_s\zeta^{-ms}}{e_N(c,d,s)\mathfrak{k}_{(0,\overline{ds}/N)}(\tau)}\left(\frac{d}{\ell}\right)^t\frac{\eta^{t\ell}(\ell\tau)}{\eta^t(\tau)}\Delta^{(d\ell^v+d)/24}\left(\frac{g\tau+h}{r/g}\right),$$

where $e_N(c,d,s) = \mathbf{e}\big(\frac{N(cs+ds-\overline{ds})+cds^2}{2N^2}\big)$. Next we consider $G_m \otimes \left(\frac{*}{\ell}\right)$. For an integer $0 \le u \le \ell-1$, we choose u' such that $u' \equiv d^2u \pmod{\ell}$. Then we have

$$\begin{pmatrix}1 & -u/\ell\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a-cu/\ell & b-cuu'/\ell^2+(au'-du)/\ell\\ c & d+cu'/\ell\end{pmatrix}\begin{pmatrix}1 & -u'/\ell\\ 0 & 1\end{pmatrix}.$$

With the help of Lemma 3.1 we can show that

$$G_m \otimes \left(\frac{*}{\ell}\right)|A$$
$$= \frac{e_\ell}{2\pi i\ell}\sum_{s=1}^{N-1}\sum_{u=1}^{\ell}\frac{w_s\zeta^{-ms}}{e_N(c,d',s)'\mathfrak{k}_{(0,\overline{d's}/N)}(\tau)}\left(\frac{d'^t u}{\ell}\right)\frac{\eta^{t\ell}(\ell\tau)}{\eta^t(\tau)}\Delta^{(d\ell^v+d)/24}\left(\frac{g\tau+h_s}{r/g}\right),$$

where $d' = d + cu'/\ell$ and h_s are integers. We compare the first nonzero coefficient of $G_m(\tau)|A$ and $G_m \otimes \left(\frac{*}{\ell}\right)|A$. In fact, if we assume that

$$G_m(\tau)|A = a_n q^n + a_{n+1}q^{n+1} + \cdots,$$

where $n = t\delta_\ell + \frac{g^2(d\ell^v+d)}{24r}$, then since r divides $d\ell^v + d$, the expansion of $G_m \otimes \left(\frac{*}{\ell}\right)|A$ is

$$a_n\left(\frac{n}{\ell}\right)q^n + a'_{n+1}q^{n+1} + \cdots.$$

By the assumption on ℓ, this implies that

$$\mathrm{ord}_{a/c}\,\widetilde{G_m}(\tau) \ge t\delta_\ell + \frac{g^2(d\ell^v+d)}{24r} + 1.$$

By comparing the order, we see that $\frac{\widetilde{G_m}(\tau)}{\eta^{k\ell}(\ell\tau)\eta^{d\ell^v}(r\tau)}$ also vanishes at the cusp a/c when $\ell^{j+1} \mid c$. ∎

The proof of the following lemma is similar to that of Lemma 3.2, so we omit it.

LEMMA 3.3. *Let v be an integer large enough. Then*

$$\frac{\widetilde{P}(24\tau)}{\eta^{t\ell}(24\ell\tau)\eta^{d\ell^v}(24r\tau)}E_{j+1}(24\tau)^{\ell^v} \in S_{\lambda'+1/2}(\Gamma_0(576\ell^{\max\{3,j+1\}}),\chi) \tag{3.4}$$

for some integer λ'.

LEMMA 3.4. *Let $f(q)$ be a formal power series such that $f(q) \equiv \sum_{n\geq 0} a_n q^{n\ell} \pmod{\ell^v}$. If the series $\sum b_n q^n$ and $\sum b'_n q^n$ coincide on the subsequence $\{\ell m+d \mid m\in\mathbb{Z}\}$, then the series $\sum b_n q^n f(q)$ and $\sum b'_n q^n f(q)$ also coincide on the subsequence $\{\ell m+d \mid m\in\mathbb{Z}\}$ modulo ℓ^v.*

Proof. By the assumption, ℓ^v divides the coefficient of q^n of $f(q)$ when $\ell\nmid n$. So in the expression of $\sum b_n q^n f(q)$, the coefficient of $q^{\ell m+d}$ is congruent to $\sum b_{\ell n+d}a_{m-n} \pmod{\ell^v}$. This implies that $\sum b_n q^n f(q)$ and $\sum b'_n q^n f(q)$ coincide on the subsequence $\{\ell m+d \mid m\in\mathbb{Z}\}$ modulo ℓ^v. ∎

The following theorem gives the precise congruence relations for $M_{r,d,t}$, which implies Theorem 1.1.

THEOREM 3.5. *Let r,d,t,j,v be positive integers and $\ell > dr^2$ be a prime number such that $\left(\frac{g^2d-tr}{\ell}\right)$ are the same for all $g\,|\,r$. Then there exists a positive proportion of primes $p\equiv -1 \pmod{24\ell}$ such that*

$$M_{r,d,t}\left(m,\ell^j,\frac{np^3+t-dr}{24}\right)\equiv 0 \pmod{\ell^v} \tag{3.5}$$

for all $0\leq m<\ell^j$ and $n\equiv t-dr-24\beta \pmod{24\ell}$ with $p\nmid n$ where β is a certain integer between 0 and ℓ.

Proof. We first assume that $v\geq \max\{5,t\}$ satisfies $\ell^v\equiv -1 \pmod{24r}$. We set

$$\epsilon=\left(\frac{24(dr-t)}{\ell}\right),\qquad \alpha=\frac{t(\ell^2-1)+dr(\ell^v+1)}{24}$$

and $0\leq\beta<\ell$ such that $\left(\frac{\alpha+\beta}{\ell}\right)=0$ or $-\epsilon$. Recall the definition of $\widetilde{g_m}(\tau)$ in (2.1), it is easy to see that $g_m(\tau)$ coincides with $\widetilde{g_m}(\tau)$ on the subsequent $\{\ell n'+\alpha+\beta\}$. Moreover, we have

$$\eta^{d\ell^v}(24r\tau)=q^{dr\ell^v}\prod_{n\geq 1}(1-q^{24rn})^{d\ell^v}\equiv q^{dr\ell^v}\prod_{n\geq 1}(1-q^{24rd\ell^v n}) \pmod{\ell^v}.$$

Hence by Lemma 3.4, we see that the subsequence of $\frac{\widetilde{g_m}(24\tau)}{\eta^{t\ell}(24\ell\tau)\eta^{d\ell^v}(24r\tau)}$ with the indices $\{\ell n'+\alpha+\beta\}$ coincides with $\frac{g_m(24\tau)}{\eta^{t\ell}(24\ell\tau)\eta^{d\ell^v}(24r\tau)}$ on these indices, which is exactly

$$\sum_{n'\equiv\alpha+\beta\,(\mathrm{mod}\,\ell)} \ell^j M_{r,d,t}(m,\ell^j,n'-\alpha)q^{24n'-t\ell^2-dr\ell^v} \pmod{\ell^v}.$$

By changing the variable, we get

$$\sum_{n\equiv 24\beta+dr-t\,(\mathrm{mod}\,24\ell)} \ell^j M_{r,d,t}\left(m,\ell^j,\frac{n+t-dr}{24}\right)q^n.$$

Combining Lemmas 3.2 and 3.3, we see that there exist modular forms f_m and f such that

$$\sum_{n\equiv 24\beta+dr-t\,(\mathrm{mod}\,24\ell)} \ell^j M_{r,d,t}\left(m,\ell^j,\frac{n+t-dr}{24}\right)q^n \equiv f_m(\tau)+f(\tau) \pmod{\ell^v}.$$

Moreover, by applying Theorem 2.1, we can find a positive proportion of primes $p\equiv -1 \pmod{24\ell}$ such that

$$f_m|T_{p^2}\equiv f|T_{p^2}\equiv 0 \pmod{\ell^v}.$$

This implies that

$$\begin{aligned}
&\ell^j M_{r,d,t}\left(m,\ell^j,\frac{p^2n+t-dr}{24}\right)\\
&\quad+\ell^j\chi(p)\left(\frac{(-1)^\lambda n}{p}\right)p^{\lambda-1}M_{r,d,t}\left(m,\ell^j,\frac{n+t-dr}{24}\right)\\
&\quad+\ell^j\chi(p^2)\left(\frac{(-1)^\lambda}{p^2}\right)p^{2\lambda-1}M_{r,d,t}\left(m,\ell^j,\frac{n/p^2+t-dr}{24}\right)\equiv 0 \pmod{\ell^v}.
\end{aligned}$$

Finally, replacing n by pn' with $p\nmid n'$, we get

$$\ell^j M_{r,d,t}\left(m,\ell^j,\frac{p^3n'+t-dr}{24}\right)\equiv 0 \pmod{\ell^v},$$

where $n'\equiv t-dr-24\beta \pmod{24\ell}$. This gives

$$M_{r,d,t}\left(m,\ell^j,\frac{p^3n'+t-dr}{24}\right)\equiv 0 \pmod{\ell^{v-j}}.$$

But we note that there are infinitely many v that satisfy $\ell^v\equiv -1 \pmod{24r}$, so the congruence relation (3.5) holds for all $v\in\mathbb{N}$. ∎

We end this article with two questions arising from this project. A natural question is: can one give a combinatorial interpretation of $M_{r,d,t}(m,n)$? Moreover, can one find any other kind of congruences for $M_{r,d,t}(m,n)$?

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References

[1] S. Ahlgren and K. Ono, *Congruence properties for the partition function*, Proc. Nat. Acad. Sci. USA 98 (2001), 12882–12884.

[2] G. E. Andrews and F. G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. 18 (1988), 167–171.

[3] G. E. Andrews and K. Ono, *Ramanujan's congruences and Dyson's crank*, Proc. Nat. Acad. Sci. USA 102 (2005), 15277.

[4] K. Bringmann and J. Dousse, *On Dyson's crank conjecture and the uniform asymptotic behavior of certain inverse theta functions*, Trans. Amer. Math. Soc. 368 (2016), 3141–3155.

[5] F. J. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) 8 (1944), 10–15.

[6] A. Folsom, *Mock modular forms and d-distinct partitions*, Adv. Math. 254 (2014), 682–705.

[7] S. Fu and D. Tang, *On a generalized crank for k-colored partitions*, J. Number Theory 184 (2018), 485–497.

[8] S. A. Garthwaite and M. Jameson, *Incongruences for modular forms and applications to partition functions*, Adv. Math. 376 (2021), art. 107448, 17 pp.

[9] F. G. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11*, Trans. Amer. Math. Soc. 305 (1988), 47–77.

[10] P. Hammond and R. Lewis, *Congruences in ordered pairs of partitions*, Int. J. Math. Math. Sci. 2004, 2509–2512.

[11] S. Lang, *Elliptic Functions*, Springer, New York, 1987.

[12] K. Mahlburg, *Partition congruences and the Andrews–Garvan–Dyson crank*, Proc. Nat. Acad. Sci. USA 102 (2005), 15373–15376.

[13] K. Mahlburg, *Congruences for the coefficients of modular forms and applications to number theory*, Ph.D. thesis, Univ. of Wisconsin, Madison, 2006.

[14] K. Ono, *Distribution of the partition function modulo m*, Ann. of Math. (2) 151 (2000), 293–307.

[15] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, CBMS Reg. Conf. Ser. Math. 102, Amer. Math. Soc., 2003.

[16] C.-S. Radu, *A proof of Subbarao's conjecture*, J. Reine Angew. Math. 672 (2012), 161–175.

[17] J.-P. Serre, *Divisibilité de certaines fonctions arithmétiques*, Enseign. Math. 22 (1976), 227–260.

[18] S. Treneer, *Congruences for the coefficients of weakly holomorphic modular forms*, Proc. London Math. Soc. (3) 93 (2006), 304–324.

Hao Zhang, Helen W. J. Zhang
School of Mathematics
Hunan University
Changsha 410082, P.R. China
and
Hunan Provincial Key Laboratory of
Intelligent Information Processing and Applied Mathematics
Changsha 410082, P.R. China
E-mail: zhanghaomath@hnu.edu.cn
helenzhang@hnu.edu.cn