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Exact controllability for nonlocal wave equations with nonlocal boundary conditions

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Abstract. We are interested in the controllability of nonlocal wave equations subject to nonlocal dynamical boundary conditions, where the nonlocality stems from integral terms on the bulk and on the boundary of the domain considered. First, we establish, in two geometric settings satisfying the geometric control condition (GCC), the internal observability of the corresponding local system using multipliers together with compactness-uniqueness results. Then, we prove that under analyticity assumptions on the kernels, the nonlocal system is also observable. Moreover, assuming the kernels are symmetric, the spectral properties of our system and a result on simultaneous observability allow us to show that, in a rectangular domain, the kernel on the bulk being analytic is enough for the system to be observable.

1. Introduction. Let Ω be a bounded open domain of $\mathbb{R}^n$, $n \geq 2$, with boundary $\Gamma = \Gamma^1 \cup \Gamma^2$ and let $T > 0$. We denote $\Omega_T = \Omega \times (0,T)$, $\Gamma_T^1 = \Gamma^1 \times (0,T)$, $\Gamma_T^2 = \Gamma^2 \times (0,T)$. Consider the following nonlocal hyperbolic problem:

$$\begin{cases} \partial_t^2 v - \Delta v + \int\limits_\Omega K_\Omega(x,y) v(y,t)\,dy = w_1 \quad \text{in } \Omega_T, \\ \partial_t^2 v_\Gamma + \partial_\nu v - \Delta_\Gamma v_\Gamma + \int\limits_{\Gamma^1} K_\Gamma(\xi,\zeta) v_\Gamma(\zeta,t)\,d\Gamma = w_2,\ v = v_\Gamma \quad \text{on } \Gamma_T^1, \\ v = 0 \quad \text{on } \Gamma_T^2, \\ (v(0), v_\Gamma(0)) = (v^0, v_\Gamma^0),\ (\partial_t v(0), \partial_t v_\Gamma(0)) = (v^1, v_\Gamma^1) \quad \text{in } \Omega \times \Gamma^1. \end{cases} \tag{1.1}$$

We investigate the issue of controllability of system (1.1) in two different geometric settings. Firstly, we analyse the case when Ω is a bounded domain with a smooth boundary $\Gamma = \Gamma^1 \cup \Gamma^2$ such that Γ^1, Γ^2 are nonempty, closed

2020 *Mathematics Subject Classification*: Primary 35L05; Secondary 93B05.
Key words and phrases: controllability, observability, nonlocal wave equation, Ventcel boundary condition.

Received 5 July 2023; revised 6 December 2023.
Published online 22 April 2024.

and $\Gamma^1 \cap \Gamma^2 = \emptyset$. Secondly, we consider the case when Ω is a rectangular domain in $\mathbb{R}^n$; here too we maintain the notation Γ^1 for Ventcel's part, which consists of one side of the boundary, and Γ^2 for Dirichlet's part, that is, the remaining sides of Γ.

The initial state of the system is given by v^0, v^0_Γ, v^1, v^1_Γ, and w_1, w_2 are the control functions that aim to steer, within finite time, system (1.1) toward a prescribed final state. We denote by ∂_ν the normal derivative on Γ, where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal vector to Γ, and by Δ_Γ the Laplace–Beltrami operator on Γ. Depending on the regularity of the solutions, v_Γ is not necessarily the trace of v on Γ^1.

System (1.1) describes the dynamics of a vibrating elastic body coated with a thin layer of high rigidity. Compared with classical boundary conditions, Ventcel's conditions allow for interaction between the boundary and the inside of the domain Ω by factoring into the total count of the energy the contribution of the boundary to the kinetic and potential energies of the system. We refer to [13] for a detailed derivation of dynamic boundary conditions for the wave equation, as well as their physical interpretations.

The community of researchers has paid considerable attention to different aspects of problems with these types of boundary conditions, from the existence of solutions, to asymptotic behavior, controllability and so on. We mention here the contributions in [12, 11, 2, 23, 19, 4, 5, 6], which are relevant to our work. The novelty here is the spatial integral terms introduced in both the wave equation and the Ventcel boundary condition. These nonlocal terms reflect the fact that the value of the solution at a certain point x may depend on its values at other points in space. They might also arise as a result of linearizing nonlinear systems, as was seen in [20], where a Burgers-type system with nonlocal viscosity was transformed, through linearization and change of functions, into a heat equation involving an integral term.

System (1.1) is said to be exact-controllable in time $T > 0$ if we can find control functions (w_1, w_2) that will drive the solution from the initial state $(v^0, v^0_\Gamma, v^1, v^1_\Gamma)$ to the equilibrium

$$\begin{aligned} &(v(T), \partial_t v(T)) = (0,0) && \text{in } \Omega, \\ &(v_\Gamma(T), \partial_t v_\Gamma(T)) = (0,0) && \text{on } \Gamma^1. \end{aligned} \tag{1.2}$$

Notice that we need to act on the system (1.1) using two internal controls. The control w_1 serves to kill the vibrations inside the domain Ω, while the control w_2 is for the vibrations of Ventcel's boundary Γ^1; see the final remarks in [12].

To our knowledge, [10] is among the first works dealing with the control of similar nonlocal systems, in which the authors have established, under some analyticity assumptions on the kernel, the null controllability of linear nonlocal heat and wave equations with Dirichlet boundary condition. In the same direction, papers [7, 3, 18, 20] focus mostly on the nonlocal heat equation. It is important to note that to get controllability results, one needs additional requirements on the kernels besides just being bounded. In fact, unique continuation, crucial for controllability results, fails when no further conditions are imposed on the kernels. This is evidenced by a counterexample proposed by P. Gerard and presented in detail in [3]. To fix this, some solutions were advanced in the literature.

In [10], the authors considered an analytic kernel. The analyticity condition allows them to use linear perturbation techniques to obtain the observability inequality for the wave equation. There is also [20], where the one-dimensional nonlocal heat equation is proven to be controllable using spectral analysis techniques, under the assumption that the kernel is separable. In [3], the kernel is time-dependent, bounded, and assumed to have exponential decay at the extrema of the time interval. The authors were able to show the null controllability of the heat equation through Carleman estimates. All the works cited have the common feature of analyzing systems subject to Dirichlet boundary conditions. As for nonlinear PDEs, we have the recent papers [9, 14, 16] devoted to the analysis of their controllability properties.

In this work, we assume that the kernels K_Ω, K_Γ belong to $L^2(\Omega\times\Omega)$, $L^2(\Gamma^1\times\Gamma^1)$ respectively, and satisfy the analyticity assumptions

$$\begin{aligned} &y\mapsto \int_\Omega K_\Omega(x,y)f(x)\,dx \text{ is analytic for all } f\in L^2(\Omega),\\ &y\mapsto \int_{\Gamma^1} K_\Gamma(x,y)g(x)\,d\Gamma(x) \text{ is analytic for all } g\in L^2(\Gamma^1). \end{aligned} \tag{1.3}$$

This analyticity condition holds for instance for convolution kernels $K_\Omega(x,y)=K_\Omega(x-y)$, $K_\Gamma(x,y)=K_\Gamma(x-y)$ and for separable kernels $K_\Omega(x,y)=K_{\Omega,1}(x)K_{\Omega,2}(y)$, $K_\Gamma(x,y)=K_{\Gamma,1}(x)K_{\Gamma,2}(y)$ if the kernel is analytic with respect to the variable y and bounded with respect to x. We emphasize that no connection is assumed between K_Ω and K_Γ.

As is standard by now, we do not tackle the controllability problem directly. We exploit, in the framework of the Hilbert Uniqueness Method [17], its equivalence to the observability problem for the adjoint system

$$\begin{cases} \partial_t^2 u-\Delta u+\int_\Omega K_\Omega(x,y)u(x,t)\,dx=0 \quad \text{in } \Omega_T,\\ \partial_t^2 u_\Gamma+\partial_\nu u-\Delta_\Gamma u_\Gamma+\int_{\Gamma^1} K_\Gamma(\xi,\zeta)u_\Gamma(\xi,t)\,d\Gamma=0,\ u=u_\Gamma \quad \text{on } \Gamma^1_T,\\ u=0 \quad \text{on } \Gamma^2_T,\\ (u(T),u_\Gamma(T))=(u_T^0,u_{T,\Gamma}^0),\ (\partial_t u(T),\partial_t u_\Gamma(T))=(u_T^1,u_{T,\Gamma}^1) \quad \text{in } \Omega\times\Gamma^1. \end{cases} \tag{1.4}$$

Motivated by [10], this nonlocal system will be primarily viewed as a perturbation of its local counterpart

$$(1.5)\quad \begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega_T, \\ \partial_t^2 u_\Gamma + \partial_\nu u - \Delta_\Gamma u_\Gamma = 0,\ u = u_\Gamma \quad \text{on } \Gamma_T^1, \\ u = 0 \quad \text{on } \Gamma_T^2, \\ (u(0), u_\Gamma(0)) = (u^0, u_\Gamma^0),\ (\partial_t u(0), \partial_t u_\Gamma(0)) = (u^1, u_\Gamma^1) \quad \text{in } \Omega \times \Gamma^1. \end{cases}$$

Then we need to prove an observability estimate for (1.5) (see [10]), which together with a compactness-uniqueness argument will yield the observability inequality for (1.4). To the best of our knowledge, there are no papers in the literature addressing the distributed control of hyperbolic problems with Ventcel-type conditions. We fill this gap by providing the proof for distributed observability of system (1.5) in the two aforementioned geometric settings. To this end, we begin by proving some energy estimates using multipliers; then a contradiction argument yields the desired results [24]. This leads us to the reason behind our making the distinction between a bounded domain with a smooth boundary and a rectangular one. In both cases, we make use of the existing boundary observability results [12, 2]. However, for a smooth domain, it is necessary to observe the whole boundary to recover complete information on the state of the system, whereas in a rectangle, observing a portion of the boundary does the trick.

We point out that, depending on the regularity of the solutions to (1.4) and the region of observation, the second analyticity hypothesis is not always essential. In a smooth domain, since we are observing everywhere on Ventcel's boundary, the first hypothesis in (1.3) suffices to make system (1.4) observable. On the other hand, without the second hypothesis, we do not have unique continuation of the solutions to (1.4) in a rectangular domain. This is true for all kernels K_Ω, K_Γ lying respectively in $L^2(\Omega\times\Omega)$, $L^2(\Gamma^1\times\Gamma^1)$. Yet if we assume that these two are symmetric, we can still achieve observability without the analyticity condition for K_Γ. We prove this by splitting our system into two parts. One part is finite-dimensional and we get an observability estimate using a Hautus-type test [22]. The other observability result follows from the observability of the local system (1.5). Finally, we bring together these two parts employing a result on simultaneous observability [22].

The outline of this paper is as follows. In Section 2, we introduce the functional setup and prove well-posedness. In the three subsequent sections, we first show that the local system (1.5) is observable in finite time, in the two geometries under consideration, and then we establish the same property for the nonlocal problem (1.4).

2. Functional setup and well-posedness. Let $L^2(\Omega)$, $L^2(\Gamma^1)$ be the standard Lebesgue spaces equipped with the usual inner products and norms denoted respectively by $(\cdot,\cdot)_\Omega$, $(\cdot,\cdot)_\Gamma$, $\|\cdot\|_\Omega$ and $\|\cdot\|_\Gamma$. We introduce the spaces of functions

$$\begin{aligned}\mathcal{H} &= L^2(\Omega) \times L^2(\Gamma^1),\\ \mathcal{V} &= \{(u,v) \in H^1_{\Gamma^2}(\Omega) \times H^1(\Gamma^1);\ v = u|_{\Gamma^1}\},\end{aligned}$$

where

$$H^1_{\Gamma^2}(\Omega) = \{u \in H^1(\Omega);\ u|_{\Gamma^2} = 0\}.$$

These are Hilbert spaces endowed with the norms

$$\begin{aligned}\|(u,v)\|^2_{\mathcal{H}} &= \int_\Omega |u|^2\,dx + \int_{\Gamma^1} |v|^2\,d\Gamma,\\ \|(u,v)\|^2_{\mathcal{V}} &= \int_\Omega |\nabla u|^2\,dx + \int_{\Gamma^1} |\nabla_\Gamma v|^2\,d\Gamma,\end{aligned}$$

where ∇_Γ denotes the boundary gradient on Γ.

Hereafter, all the given facts are valid whether we are considering problem (1.1) in a regular bounded domain or in a rectangular one.

Define the linear differential operator A_0 on $\mathcal{H}$ by

$$A_0\begin{pmatrix}u_1\\ u_2\end{pmatrix} := \begin{pmatrix}-\Delta u_1\\ -\Delta_\Gamma u_2 + \partial_\nu u_2\end{pmatrix} \tag{2.1}$$

with domain

$$D(A_0) = \{(u_1,u_2) \in \mathcal{V};\ \Delta u_1 \in L^2(\Omega),\ -\Delta_\Gamma u_2 + \partial_\nu u_2 \in L^2(\Gamma^1)\}.$$

This operator, called the *Ventcel Laplacian*, is self-adjoint and positive on $\mathcal{H}$, with compact resolvent (see [11, Theorem 2.2] and [2]). Thus, it is diagonizable, with real positive eigenvalues μ_k, $k \in \mathbb{N}^*$, satisfying $\lim_{k\to\infty} \mu_k = \infty$ and eigenvectors Φ_k, $k \in \mathbb{N}^*$, forming an orthonormal basis for $\mathcal{H}$ [22, Proposition 3.2.12].

We also define the bounded operator $B_0 \in \mathcal{L}(\mathcal{H})$ by

$$B_0\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} := \begin{pmatrix}-\int_\Omega K_\Omega(x,y)\varphi_1(y)\,dy\\ -\int_{\Gamma^1} K_\Gamma(x,y)\varphi_2(y)\,d\Gamma(y)\end{pmatrix}. \tag{2.2}$$

Setting $V = (v, v_\Gamma, \partial_t v, \partial_t v_\Gamma)$, $V_0 = (v^0, v^0_\Gamma, v^1, v^1_\Gamma)$, system (1.1) can be written in the form of an abstract Cauchy problem in $\mathcal{V}\times\mathcal{H}$,

$$\partial_t V = AV + BV + F, \qquad V(0) = V_0,$$

where A is defined on $\mathcal{V}\times\mathcal{H}$ by

$$D(A) = D(A_0)\times\mathcal{V}, \qquad A\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} = \begin{pmatrix}\Psi\\ -A_0\Phi\end{pmatrix}, \quad \forall(\Phi,\Psi) \in D(A), \tag{2.3}$$

B is a bounded operator on $\mathcal{V} \times \mathcal{H}$ such that

$$B\begin{pmatrix}\Phi \\ \Psi\end{pmatrix} = \begin{pmatrix}0 \\ B_0\Phi\end{pmatrix}, \quad \forall(\Phi, \Psi) \in \mathcal{V} \times \mathcal{H}, \tag{2.4}$$

and $F^\top = (0, 0, w_1, w_2)$.

We already know that the operator A generates a strongly continuous group on $\mathcal{V} \times \mathcal{H}$ [12, 2]. Then, regarding B as a bounded perturbation of the generator A, we deduce from [8, Theorem 1.3] that the sum $A + B$ also generates a strongly continuous group on $\mathcal{V}\times\mathcal{H}$. As a result, our system (1.1) is well-posed in the following sense:

(i) Given $(v^0, v_\Gamma^0, v^1, v_\Gamma^1) \in D(A_0) \times \mathcal{V}$ and $(w_1, w_2) \in L^2(0, T; \mathcal{V})$, there exists a unique strong solution to (1.1) such that

$$(v, v_\Gamma) \in C(0, T; D(A_0)) \cap C^1(0, T; \mathcal{V}).$$

(ii) Given $(v^0, v_\Gamma^0, v^1, v_\Gamma^1) \in \mathcal{V} \times \mathcal{H}$ and $(w_1, w_2) \in L^2(0, T; \mathcal{H})$, there exists a unique weak solution to (1.1) such that

$$(v, v_\Gamma) \in C(0, T; \mathcal{V}) \cap C^1(0, T; \mathcal{H}).$$

Further, we will need the well-posedness of system (1.4) with less regular data belonging to $\mathcal{H} \times \mathcal{V}'$, where $\mathcal{V}'$ represents the dual of $\mathcal{V}$ and we have identified the space $\mathcal{H}$ with its dual. The solutions, in this case, are defined through the transposition method. In fact, one can proceed as in [12, 2] to establish existence and uniqueness of solutions to (1.1) such that

$$(v, v_\Gamma) \in C(0, T; \mathcal{H}) \cap C^1(0, T; \mathcal{V}')$$

for all $(v^0, v_\Gamma^0, v^1, v_\Gamma^1) \in \mathcal{H} \times \mathcal{V}'$ and $(w_1, w_2) \in L^2(0, T; \mathcal{H})$.

In what follows, we define on the dual space $\mathcal{V}'$ a norm that will come useful later on.

It is clear that $\mathcal{V} \subset \mathcal{H}$ densely. Moreover, identifying $\mathcal{H}$ with its dual, we have

$$\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}' \hookrightarrow \mathcal{V}'$$

with dense and compact injections. Define the operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ such that

$$\langle \mathcal{A}U, V\rangle_{\mathcal{V}',\mathcal{V}} := a(U, V), \quad \forall U = (u, u_\Gamma), V = (v, v_\Gamma) \in \mathcal{V},$$

where a is the bilinear form

$$a(U, V) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma^1} \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma v_\Gamma \, d\Gamma = (U, V)_{\mathcal{V}}. \tag{2.5}$$

This operator is a bounded extension, onto $\mathcal{V}$, of the operator A_0 (see (2.1)). The proof of this is detailed in [11]. Further, we can show that the bilinear form a is bounded, symmetric and coercive, which implies that the operator $\mathcal{A}$ is an isometric isomorphism from $\mathcal{V}$ onto $\mathcal{V}'$. Therefore, we have

$$(F,G)_{\mathcal{V}'} = (\mathcal{A}^{-1}F, \mathcal{A}^{-1}G)_{\mathcal{V}}, \qquad \forall F,G \in \mathcal{V}'. \tag{2.6}$$

3. Observability of the local problem (1.5) in a smooth domain. Throughout this section, we work towards showing the internal observability of the local problem (1.5) in a bounded domain Ω with a sufficiently smooth boundary (see Figure 1 below). This is an essential ingredient in the proof of the same property for the nonlocal system (1.4). For that goal, we recall that system (1.5) is observable from the boundary Γ. In [12], the authors succeeded in establishing the exact boundary controllability of systems similar to (1.5) by acting on the whole boundary, Dirichlet's and Ventcel's portions both included. This was done, under a suitable geometric condition, employing an elaborate Carleman estimate. Here, we only recall the final result, that is, the boundary observability inequality [12, Proposition 3.2].

From now on, we assume the following geometric condition on the boundary Γ: there exists $x_0 \notin \overline{\Omega}$ such that

$$\begin{cases} (x-x_0)\cdot\nu(x) \le 0 & \text{for all } x \in \Gamma^1, \\ (x-x_0)\cdot\nu(x) \ge 0 & \text{for all } x \in \Gamma^2. \end{cases} \tag{3.1}$$

Before stating the boundary observation result of [12], we recall that the energy of system (1.5),

$$E(t) = \frac{1}{2}\int_{\Omega}(|\partial_t u|^2 + |\nabla u|^2)\ dx + \frac{1}{2}\int_{\Gamma^1}(|\partial_t u_\Gamma|^2 + |\nabla_\Gamma u_\Gamma|^2)\, d\Gamma, \quad 0 \le t \le T,$$

is conserved throughout evolution.

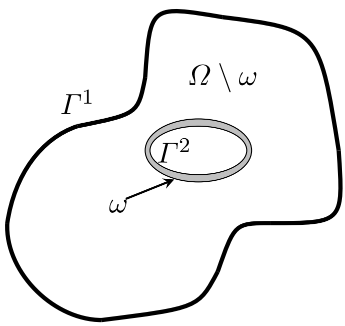

Fig. 1. A smooth domain Ω where we are acting on Γ^1 and on a neighborhood ω of Γ^2 in Ω.

Setting $T_0 = 2\max\{|x-x_0| : x \in \overline{\Omega}\}$, under the geometric hypothesis (3.1) we have the following inverse inequality:

PROPOSITION 3.1. *Let $T > T_0$ and let $(u^0, u^0_\Gamma, u^1, u^1_\Gamma) \in \mathcal{V} \times \mathcal{H}$. Then there exists a constant $c > 0$, independent of the initial data, such that the solution to* (1.5) *satisfies*

$$E(0) \le c\Big[\int_{\Gamma^1_T} (|\partial_t u_\Gamma|^2 + |u_\Gamma|^2)\, d\Gamma\, dt + \int_{\Gamma^2_T} |\partial_\nu u|^2\, d\Gamma\, dt\Big]. \tag{3.2}$$

Now, let $\omega \subset \Omega$ be a neighborhood of Γ^2 in Ω, that is, there exists a neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of Γ^2 such that $\omega = \mathcal{O} \cap \Omega$. The main result of this section is

THEOREM 3.2. *Let $T > T_0$. Let $\omega \subset \Omega$ be a neighborhood of Γ^2 in Ω. Then there exists $c > 0$ such that*

$$\|(u^0, u^0_\Gamma)\|^2_{\mathcal{H}} + \|(u^1, u^1_\Gamma)\|^2_{\mathcal{V}'} \le c\Big(\int_{\omega_T} |u|^2\, dx\, dt + \int_{\Gamma^1_T} |u_\Gamma|^2\, d\Gamma\, dt\Big) \tag{3.3}$$

for all solutions to (1.5) *corresponding to initial data $(u^0, u^0_\Gamma) \in \mathcal{H}$ and $(u^1, u^1_\Gamma) \in \mathcal{V}'$.*

We carry out the proof of Theorem 3.2 in three steps. First, we show that estimate (3.3) follows from another energy estimate satisfied by the solutions to (1.5), whose initial configurations lie in $\mathcal{V} \times \mathcal{H}$. This is provided by the first of the following two lemmas. Then, we give an intermediate energy estimate necessary for the actual proof of Theorem 3.2, that comes in the third and final step.

LEMMA 3.3. *For $(u^0, u^0_\Gamma) \in \mathcal{V}$, $(u^1, u^1_\Gamma) \in \mathcal{H}$, suppose there exists a constant $c > 0$ such that the corresponding solutions (u, u_Γ) to* (1.5) *satisfy the estimate*

$$\|(u^0, u^0_\Gamma)\|^2_{\mathcal{V}} + \|(u^1, u^1_\Gamma)\|^2_{\mathcal{H}} \le c\Big(\int_{\omega_T} |\partial_t u|^2\, dx\, dt + \int_{\Gamma^1_T} |\partial_t u_\Gamma|^2\, d\Gamma\, dt\Big). \tag{3.4}$$

Then inequality (3.3) *holds for solutions to* (1.5) *when the initial datum $(u^0, u^0_\Gamma, u^1, u^1_\Gamma)$ is in $\mathcal{H} \times \mathcal{V}'$.*

Proof. Given $(u^0, u^0_\Gamma) \in \mathcal{H}$ and $(u^1, u^1_\Gamma) \in \mathcal{V}'$, let $(\chi, \chi_\Gamma) \in \mathcal{V}$ be the solution to the steady-state problem

$$\mathcal{A}\begin{pmatrix} \chi \\ \chi_\Gamma \end{pmatrix} = \begin{pmatrix} u^1 \\ u^1_\Gamma \end{pmatrix}.$$

Since the associated bilinear form a (see (2.5)) is bounded, symmetric and coercive, such a function exists in $\mathcal{V}$ and is unique. Now, define

$$\Psi(x,t) = \begin{pmatrix} \psi(x,t) \\ \psi_\Gamma(x,t) \end{pmatrix} = \begin{pmatrix} \int_0^t u(x,s)\, ds - \chi(x) \\ \int_0^t u_\Gamma(x,s)\, ds - \chi_\Gamma(x) \end{pmatrix},$$

where (u, u_Γ) is the solution to (1.5) with $(u^0, u^0_\Gamma) \in \mathcal{H}$, $(u^1, u^1_\Gamma) \in \mathcal{V}'$. Integrating over time the first two equations of (1.5), and taking into account the relations

$$\begin{aligned} \partial_t\psi(x,t) &= u(x,t), & \partial_t^2\psi(x,t) &= \partial_t u(x,t) & &\text{in } \Omega_T, \\ \partial_t\psi_\Gamma(x,t) &= u_\Gamma(x,t), & \partial_t^2\psi_\Gamma(x,t) &= \partial_t u_\Gamma(x,t) & &\text{on } \Gamma^1_T, \end{aligned} \tag{3.5}$$

we can see that $(\psi, \psi_\Gamma) \in C^2(0,T;\mathcal{V}')$ solves

$$\begin{cases} \begin{pmatrix} \partial_t^2\psi \\ \partial_t^2\psi_\Gamma \end{pmatrix} = -\mathcal{A}\begin{pmatrix} \psi \\ \psi_\Gamma \end{pmatrix}, \\ \psi = 0 \quad \text{on } \Gamma^2, \\ (\psi(0), \psi_\Gamma(0)) = -(\chi, \chi_\Gamma),\ (\partial_t\psi(0), \partial_t\psi_\Gamma(0)) = (u^0, u^0_\Gamma) \quad \text{in } \Omega \times \Gamma^1. \end{cases} \tag{3.6}$$

Considering that $(\chi, \chi_\Gamma) \in \mathcal{V}$ and $(u^0, u^0_\Gamma) \in \mathcal{H}$, problem (3.6) is well-posed and

$$(\psi, \psi_\Gamma) \in C(0,T;\mathcal{V}) \cap C^1(0,T;\mathcal{H}).$$

Thus, if we take into account hypothesis (3.4), we obtain

$$\|(\chi, \chi_\Gamma)\|^2_{\mathcal{V}} + \|(u^0, u^0_\Gamma)\|^2_{\mathcal{H}} \le c\Big(\int\limits_{\omega_T} |\partial_t\psi|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |\partial_t\psi_\Gamma|^2\,d\Gamma\,dt\Big). \tag{3.7}$$

Recall that the norm on the dual $\mathcal{V}'$ is induced by the inner product (2.6), which implies that

$$\|(\chi, \chi_\Gamma)\|^2_{\mathcal{V}} = \|\mathcal{A}^{-1}(u^1, u^1_\Gamma)\|^2_{\mathcal{V}} = \|(u^1, u^1_\Gamma)\|^2_{\mathcal{V}'}.$$

Plugging this into (3.7), the estimate becomes

$$\|(u^1, u^1_\Gamma)\|^2_{\mathcal{V}'} + \|(u^0, u^0_\Gamma)\|^2_{\mathcal{H}} \le c\Big(\int\limits_{\omega_T} |u|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |u_\Gamma|^2\,d\Gamma\,dt\Big), \tag{3.8}$$

where we have once more used (3.5). ∎

As a consequence, we can focus on proving (3.4). To that end, we start by establishing, using multipliers techniques and the observability inequality provided in [12], the energy estimate given in the following lemma.

Lemma 3.4. *Let $T > T_0$. For $(u^0, u^0_\Gamma) \in \mathcal{V}$, $(u^1, u^1_\Gamma) \in \mathcal{H}$, solutions (u, u_Γ) to* (1.5) *satisfy*

$$E(0) \le c\int\limits_{\omega_T} (|\partial_t u|^2 + |u|^2)\,dx\,dt + c\int\limits_{\Gamma^1_T} (|\partial_t u_\Gamma|^2 + |u_\Gamma|^2)\,d\Gamma\,dt, \tag{3.9}$$

where c is a positive constant independent of the data (u^0, u^0_Γ) and (u^1, u^1_Γ).

Proof. Let $\varepsilon > 0$ be such that $T - 2\varepsilon > T_0$. Then a simple change of variables $\tau = \frac{T-2\varepsilon}{T}t + \varepsilon$, $0 \le t \le T$, in the second integral on the right-hand side of (3.2) gives

$$E(0) \le c \int_{\Gamma_T^1} (|\partial_t u|^2 + |u|^2)\, d\Gamma\, dt + c \int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma^2} |\partial_\nu u|^2\, d\Gamma\, dt. \tag{3.10}$$

Define the function $h \in (C^1(\overline{\Omega}))^n$ as follows:

$$\begin{cases} h \cdot \nu \ge 0 & \text{on } \Gamma, \\ h = \nu & \text{on } \Gamma^2, \\ h = 0 & \text{in } (\Omega \setminus \omega) \cup \Gamma^1, \end{cases}$$

and the function $\eta \in C^1([0,T])$ such that $\eta(0) = \eta(T) = 0$, $\eta(t) = 1$ in $(\varepsilon, T-\varepsilon)$. Thus, the function $q(x,t) = \eta(t)h(x)$ satisfies

$$\begin{cases} q(x,t) = \nu(x) & \text{on } \Gamma^2 \times (\varepsilon, T-\varepsilon), \\ q(x,t) \cdot \nu(x) \ge 0 & \text{on } \Gamma \times (0,T), \\ q(x,0) = q(x,T) = 0 & \text{in } \Omega, \\ q(x,t) = 0 & \text{in } ((\Omega \setminus \omega) \cup \Gamma^1) \times (0,T). \end{cases}$$

Multiplying the wave equation in (1.5) by $q \cdot \nabla u$ and integrating over Ω_T, we get

$$\begin{aligned} \frac{1}{2} \int_{\Gamma_T^2} q.\nu |\partial_\nu u|^2\, d\Gamma\, dt &= \big[(\partial_t u, q.\nabla u)_\Omega\big]_0^T - \int_{\Omega_T} \partial_t u\ \partial_t q.\nabla u\, dx\, dt \\ &\quad + \frac{1}{2} \int_{\Omega_T} \operatorname{div} q\, (|\partial_t u|^2 - |\nabla u|^2)\, dx\, dt + \int_{\Omega_T} \nabla u^\top \nabla q \nabla u\, dx\, dt. \end{aligned} \tag{3.11}$$

We note that this identity does not have any terms on Γ^1 because the function q vanishes on that part of $\overline{\Omega}$. Thanks to the properties of q, the identity above leads to the inequality

$$\int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma^2} |\partial_\nu u|^2\, d\Gamma\, dt \le c \int_{\varepsilon}^{T-\varepsilon} \int_{\omega} (|\partial_t u|^2 + |\nabla u|^2)\, dx\, dt, \tag{3.12}$$

where $c > 0$ is a constant depending only on q.

Next, we work on replacing the integral of the gradient ∇u on the right-hand side by that of u. To this end, let ω_0 be a neighborhood of Γ^2 such that $\omega_0 \subset \omega$. Since the neighborhood ω in (3.12) is arbitrary, this estimate remains valid for ω_0. Define the functions $\rho \in C^\infty(\overline{\Omega})$, $p \in C^1(\overline{\Omega} \times (0,T))$ such that

$$\begin{cases} \rho(x) = 1 & \text{in } \omega_0, \\ \rho(x) = 0 & \text{in } (\Omega \setminus \omega) \cup \Gamma^1, \\ \rho(x) \ge 0 & \text{in } \overline{\Omega}, \end{cases} \tag{3.13}$$

and $p(x,t) = \eta(t)\rho(x)$. We multiply the first equation in (1.5) by pu and integrate over Ω_T to find

$$\int\limits_{\omega_T} p|\nabla u|^2\,dx\,dt$$
$$= \int\limits_{\omega_T} p|\partial_t u|^2\,dx\,dt + \int\limits_{\omega_T} \partial_t p u \partial_t u\,dx\,dt - \int\limits_{\omega_T} (\nabla p\cdot\nabla u)u\,dx\,dt$$
$$= \int\limits_{\omega_T} p|\partial_t u|^2\,dx\,dt + \int\limits_{\omega_T} \partial_t p u \partial_t u\,dx\,dt + \frac{1}{2}\int\limits_{\omega_T} \Delta p|u|^2\,dx\,dt.$$

The properties of the function p together with the Young inequality enable us to see that the terms on the right-hand side of the identity above can be bounded from above so that we have

$$\int\limits_{\varepsilon}^{T-\varepsilon}\int\limits_{\omega_0} |\nabla u|^2\,d\Gamma\,dt \le c\int\limits_{\omega_T} (|\partial_t u|^2 + |u|^2)\,dx\,dt.$$

Thus, using estimate (3.12) for the neighborhood ω_0 yields

$$\int\limits_{\varepsilon}^{T-\varepsilon}\int\limits_{\Gamma^2} |\partial_\nu u|^2\,d\Gamma\,dt \le c\int\limits_{\omega_T} (|\partial_t u|^2 + |u|^2)\,dx\,dt.$$

At this stage, we add the terms on Γ_T^1 to obtain

$$\int\limits_{\Gamma_T^1} (|\partial_t u|^2 + |u|^2)d\Gamma\,dt + \int\limits_{\varepsilon}^{T-\varepsilon}\int\limits_{\Gamma^2} |\partial_\nu u|^2 d\Gamma\,dt$$
$$\le c\int\limits_{\omega_T} (|\partial_t u|^2 + |u|^2)\,dx\,dt + c\int\limits_{\Gamma_T^1} (|\partial_t u|^2 + |u|^2)\,d\Gamma\,dt.$$

Taking into account the inverse estimate (3.10), we conclude the proof. ■

Now, it remains to remove the integrals of u over ω_T and Γ_T from inequality (3.9). In order to achieve this, we argue by contradiction and use standard unique continuation results for the wave equation [17, Theorem 8.1 and Lemma 8.1].

Proof of Theorem 3.2. Suppose that estimate (3.4) does not hold. Then there exists a sequence of initial data $(u_n^0, u_{\Gamma,n}^0, u_n^1, u_{\Gamma,n}^1) \in \mathcal{V}\times\mathcal{H}$ such that the corresponding solutions to (1.5) satisfy

$$n\Big(\int\limits_{\omega_T} |\partial_t u_n|^2\,dx + \int\limits_{\Gamma_T^1} |\partial_t u_{\Gamma,n}|^2\,d\Gamma\Big)$$
$$< \int\limits_{\Omega}(|u_n^1|^2 + |\nabla u_n^0|^2)\,dx + \int\limits_{\Gamma^1} (|u_{\Gamma,n}^1|^2 + |\nabla_\Gamma u_{\Gamma,n}^0|^2)\,d\Gamma.$$

We can assume without loss of generality that

$$\int_{\Omega}(|u_n^1|^2 + |\nabla u_n^0|^2)\,dx + \int_{\Gamma^1}(|u_{\Gamma,n}^1|^2 + |\nabla_\Gamma u_{\Gamma,n}^0|^2)\,d\Gamma = 1. \tag{3.14}$$

Then

$$\lim_{n\to\infty}\Big(\int_{\omega_T}|\partial_t u_n|^2\,dx + \int_{\Gamma_T^1}|\partial_t u_{\Gamma,n}|^2\,d\Gamma\Big) = 0. \tag{3.15}$$

Also, since the sequences $(u_n^0, u_{\Gamma,n}^0)$, $(u_n^1, u_{\Gamma,n}^1)$ are bounded in $\mathcal{V}$ and $\mathcal{H}$ respectively, we can find subsequences such that

$$\begin{aligned} (u_n^0, u_{\Gamma,n}^0) &\rightharpoonup (u^0, u^0|_\Gamma) && \text{in } \mathcal{V},\\ (u_n^1, u_{\Gamma,n}^1) &\rightharpoonup (u^1, u_\Gamma^1) && \text{in } \mathcal{H}. \end{aligned}$$

On the other hand, we know that solutions to (1.5) with initial data $(u_n^0, u_{\Gamma,n}^0) \in \mathcal{V}$, $(u_n^1, u_{\Gamma,n}^1) \in \mathcal{H}$ are bounded. Precisely,

$$\begin{aligned} &(u_n, u_{\Gamma,n}) \text{ is bounded in } L^\infty(0,T;\mathcal{V}),\\ &(\partial_t u_n, \partial_t u_{\Gamma,n}) \text{ is bounded in } L^\infty(0,T;\mathcal{H}), \end{aligned}$$

which implies that there exists a convergent subsequence such that

$$\begin{aligned} (u_n, u_{\Gamma,n}) &\overset{*}{\rightharpoonup} (u, u_\Gamma) && \text{in } L^\infty(0,T;\mathcal{V}),\\ (\partial_t u_n, \partial_t u_{\Gamma,n}) &\overset{*}{\rightharpoonup} (\bar{u}, \bar{u}_\Gamma) && \text{in } L^\infty(0,T;\mathcal{H}). \end{aligned} \tag{3.16}$$

Then, for all $(w, w_\Gamma) \in \mathcal{H}$, we also have

$$\begin{aligned} ((u_n, u_{\Gamma,n}), (w, w_\Gamma))_{\mathcal{H}} &\to ((u, u_\Gamma), (w, w_\Gamma))_{\mathcal{H}} && \text{in } \mathcal{D}'(0,T),\\ ((\partial_t u_n, \partial_t u_{\Gamma,n}), (w, w_\Gamma))_{\mathcal{H}} &\to ((\bar{u}, \bar{u}_\Gamma), (w, w_\Gamma))_{\mathcal{H}} && \text{in } \mathcal{D}'(0,T). \end{aligned} \tag{3.17}$$

Hence, the first convergence in (3.17) together with uniqueness of the limit yields

$$((\partial_t u_n, \partial_t u_{\Gamma,n}), (w, w_\Gamma))_{\mathcal{H}} \to ((\partial_t u, \partial_t u_\Gamma), (w, w_\Gamma))_{\mathcal{H}} \quad \text{in } \mathcal{D}'(0,T),$$

which in turn leads to

$$(\partial_t u_n, \partial_t u_{\Gamma,n}) \overset{*}{\rightharpoonup} (\partial_t u, \partial_t u_\Gamma) \quad \text{in } L^\infty(0,T;\mathcal{H}). \tag{3.18}$$

Moreover, the boundedness in L^∞-spaces along with the fact that $\mathcal{V}$ is compactly embedded in $\mathcal{H}$ allows us to apply the Aubin–Lions Lemma, providing a strongly converging subsequence

$$(u_n, u_{\Gamma,n}) \to (u, u_\Gamma) \quad \text{in } L^2(0,T;L^2(\omega)\times L^2(\Gamma^1)). \tag{3.19}$$

Applying the Banach–Steinhaus Theorem to the weak-star convergent sequence $(\partial_t u_n, \partial_t u_{\Gamma,n})$, we get, on account of (3.15),

$$\begin{cases} \partial_t u = 0 & \text{in } \omega\times(0,T),\\ \partial_t u_\Gamma = 0 & \text{on } \Gamma^1\times(0,T). \end{cases} \tag{3.20}$$

Since $(u_n^0, u_{\Gamma,n}^0, u_n^1, u_{\Gamma,n}^1) \in \mathcal{V} \times \mathcal{H}$, we have the solution $U_n := (u_n, u_{\Gamma,n})$ satisfying

$$\partial_t^2 U_n + \mathcal{A} U_n = 0 \quad \text{in } L^1(0,T;\mathcal{V}'),$$

or equivalently, for all $W = (w, w_\Gamma) \in \mathcal{V}$ we have

$$\frac{d}{dt}(\partial_t U_n(t), W)_{\mathcal{H}} + a(U_n(t), W) = 0,$$

in the sense of $\mathcal{D}'(0,T)$. Passing to the limit as $n \to \infty$, we see that in $\mathcal{D}'(0,T)$ the limit $U := (u, u_\Gamma)$ (3.16) satisfies

$$\frac{d}{dt}(\partial_t U(t), W)_{\mathcal{H}} + a(U(t), W) = 0.$$

Hence, on account of (3.20),

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma_T^2, \\ \partial_\nu u_\Gamma - \Delta_\Gamma u_\Gamma = 0 & \text{on } \Gamma_T^1, \\ \partial_t u = 0 & \text{in } \omega_T. \end{cases} \tag{3.21}$$

If we put $v = \partial_t u$, we obtain the following equation in the sense of distributions:

$$\partial_t^2 v - \Delta v = 0 \quad \text{in } \Omega_T.$$

However, we already know that $v = 0$ in ω_T, hence, by Holmgren's Theorem [17, Theorem 8.1 and Lemma 8.1], we obtain $v = \partial_t u = 0$ on Ω. Therefore, (3.21) turns into

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma^2, \\ \partial_\nu u - \Delta_\Gamma u = 0 & \text{on } \Gamma^1. \end{cases}$$

Consequently, $\int_\Omega |\nabla u|^2 + \int_{\Gamma^1} |\nabla_\Gamma u|^2 = 0$ and we deduce that

$$u \equiv 0 \quad \text{on } \overline{\Omega}.$$

Thus, we have

$$(u_n, u_{\Gamma,n}) \to (0,0) \quad \text{strongly in } L^2(0,T; L^2(\omega) \times L^2(\Gamma^1)).$$

At this point estimate (3.9) derived in Lemma 3.4, applied to the sequence $(u_n, u_{\Gamma,n})$, implies that the corresponding energy $E_n(0)$ goes to 0 as $n \to \infty$, in contradiction to (3.14). ■

Remark 3.5. We note that the two controls are equally needed to control the dynamics of both the inside of the domain and Ventcel's portion of the boundary; see the final remarks in [12]. The action in the neighborhood ω serves to dampen the vibrations inside the domain Ω, and the action on Ventcel's boundary is needed to take care of its vibrations since we are dealing with dynamic boundary conditions. If we use just one control acting only in the domain Ω, then we get $u = 0$ in Ω. This implies that $\partial_\nu u = 0$ on Γ^1, which means that the wave equation and Ventcel's condition are no longer coupled and we have nothing to control the wave on Ventcel's part of the boundary. On the other hand, if we have one control on the latter, it will lead to $u_\Gamma = 0$ on Γ^1, and we lose the coupling between the inside and the boundary. Hence, we obtain the wave equation with Dirichlet condition on one part and Neumann condition on the other part of the boundary, with no control to steer the evolution toward equilibrium. That is to say, we cannot control the whole system with a single control.

4. Observability of the local problem (1.5) in a rectangular domain. For simplicity, we consider problem (1.5) in a two-dimensional rectangle $\Omega = (0, l_1) \times (0, l_2)$. In this case, we have Ventcel's condition on $\Gamma^1 = (0, l_1) \times \{l_2\}$ and Dirichlet's condition on the remainder of the boundary $\Gamma^2 = \Gamma^{2,1} \cup \Gamma^{2,2} \cup \Gamma^{2,3}$ where $\Gamma^{2,1} = \{0\} \times (0, l_2)$, $\Gamma^{2,2} = (0, l_1) \times \{0\}$, $\Gamma^{2,3} = \{l_1\} \times (0, l_2)$ as illustrated in Figure 2. Let τ represent the unit tangent vector to Γ^1 at the endpoints.

As in the preceding section, we begin by recalling a result on the boundary observability of (1.5) in the above geometry; it has been established in a previous work [2], using Ingham's type estimates. Under the geometric control condition of [1], the endpoint $(0, l_2)$ and the sides $\Gamma^{2,1} \cup \Gamma^{2,2}$ make up the observed region. Denote $T_{R,0} = 2(\sqrt{2} + 1)\sqrt{l_1^2 + 4l_2^2}$. Then we have the following

THEOREM 4.1. *Let $(u^0, u^0_\Gamma) \in \mathcal{V}$ and $(u^1, u^1_\Gamma) \in \mathcal{H}$. Then, for $T > T_{R,0}$ there exists a constant $c > 0$ such that the solution to (1.5) satisfies*

$$\|(u^0, u^0_\Gamma)\|^2_{\mathcal{V}} + \|(u^1, u^1_\Gamma)\|^2_{\mathcal{H}} \le c\Big(\int_0^T \int_{\Gamma^{2,1}\cup\Gamma^{2,2}} |\partial_\nu u(x,t)|^2 d\Gamma\, dt + \int_0^T |\partial_\tau u(0, l_2, t)|^2 dt\Big). \tag{4.1}$$

Let $\mathcal{O} \subset \mathbb{R}^2$ be a neighborhood of the observed region $\Gamma^{2,1} \cup \Gamma^{2,2} \cup \{(0, l_2)\}$. We denote by ω^1, ω^2 the intersections $\mathcal{O} \cap \Gamma^1$, $\mathcal{O} \cap \Omega$, respectively; consequently, $\omega = \omega^1 \cup \omega^2$ is a neighborhood of $\Gamma^{2,1} \cup \Gamma^{2,2} \cup \{(0, l_2)\}$ in $\Omega \cup \Gamma^1$.

Our goal is to show that, starting from the boundary observability (4.1), we can prove the counterpart of Theorem 3.2 for rectangular domains. However, we point out that this time around we will not act everywhere on Ventcel's boundary. Observation on a small part of the latter suffices. In fact, system (1.5) is observable in any time $T > T_{R,0}$.

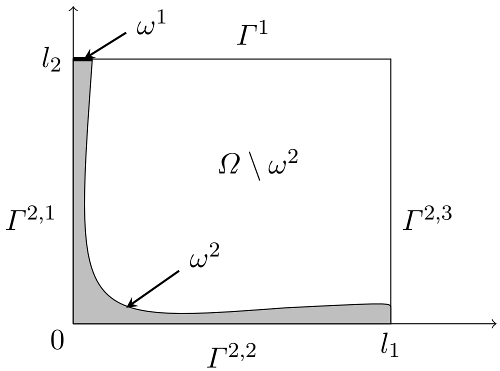

Fig. 2. A rectangular domain Ω where we are acting on $\omega = \omega^2 \cup \omega^1$, a neighborhood of $\Gamma^{2,1} \cup \Gamma^{2,2} \cup \{(0, l_2)\}$ in $\Omega \cup \Gamma^1$.

THEOREM 4.2. *Given* $(u^0, u^0_\Gamma) \in \mathcal{H}$ *and* $(u^1, u^1_\Gamma) \in \mathcal{V}'$, *for* $T > T_{R,0}$, *the corresponding solution of* (1.5) *satisfies*

$$\|(u^0, u^0_\Gamma)\|^2_{\mathcal{H}} + \|(u^1, u^1_\Gamma)\|^2_{\mathcal{V}'} \leq c\Big(\int\limits_{\omega^2_T} |u|^2 \, dx \, dt + \int\limits_{\omega^1_T} |u_\Gamma|^2 \, d\Gamma \, dt\Big), \tag{4.2}$$

where $\omega = \omega^1 \cup \omega^2$ *is a neighborhood of* $\Gamma^{2,1} \cup \Gamma^{2,2} \cup \{(0, l_2)\}$ *in* $\Omega \cup \Gamma^1$ *and* $c > 0$ *is a positive constant.*

We invoke Lemma 3.3 as earlier, since it is more convenient to work with regular data. It can be seen that the geometry under consideration is not restrictive. We can follow the proof of the lemma with some changes, stemming from the fact that we are observing a small part of Γ^1 near the point $(0, l_2)$. So if one has

$$\|(u^0, u^0_\Gamma)\|^2_{\mathcal{V}} + \|(u^1, u^1_\Gamma)\|^2_{\mathcal{H}} \leq c\Big(\int\limits_{\omega^2_T} |\partial_t u|^2 \, dx \, dt + \int\limits_{\omega^1_T} |\partial_t u_\Gamma|^2 \, d\Gamma \, dt\Big) \tag{4.3}$$

for data in $\mathcal{V} \times \mathcal{H}$, one will have estimate (4.2) as well for data belonging to $\mathcal{H} \times \mathcal{V}'$. On account of this, we now focus on proving that inequality (4.3) holds, and the first step toward this is the following.

LEMMA 4.3. *Let* $T > T_{R,0}$. *Then there exists a positive constant* $c > 0$ *such that the solution* (u, u_Γ) *to* (1.5) *satisfies*

$$E(0) \leq c\Big(\int\limits_{\omega^2_T} (|\partial_t u|^2 + |u|^2) \, dx \, dt + \int\limits_{\omega^1_T} (|\partial_t u|^2 + |u|^2) \, d\Gamma \, dt\Big) \tag{4.4}$$

for all initial data $(u^0, u^0_\Gamma) \in \mathcal{V}$ *and* $(u^1, u^1_\Gamma) \in \mathcal{H}$.

Proof. Let $\varepsilon > 0$ be such that $T - 2\varepsilon > T_{R,0}$. A change of variables in the right-hand side of (4.1) yields

$$E(0) \leq c \int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma^{2,1}\cup\Gamma^{2,2}} |\partial_\nu u(x,t)|^2 \, d\Gamma \, dt + c \int_{\varepsilon}^{T-\varepsilon} |\partial_\tau u(0,l_2,t)|^2 \, dt. \tag{4.5}$$

Let $\tilde{\omega}$ be a neighborhood of $\Gamma^{2,1} \cup \Gamma^{2,2} \cup \{(0,l_2)\}$ in $\Omega \cup \Gamma^1$ such that $\tilde{\omega} \subset \omega$. We define the function $\rho \in C^\infty(\overline{\Omega})$ satisfying

$$\begin{cases} \rho = 1 & \text{on } \tilde{\omega}, \\ \rho = 0 & \text{on } (\Omega \cup \Gamma^1) \setminus \omega, \\ \rho \geq 0 & \text{on } \overline{\Omega}, \end{cases}$$

and let $q(x,t) = \rho(x)\eta(t)$ for $x \in \overline{\Omega}$ and $t \in [0,T]$, where η is as in the proof of Lemma 3.4. Then, multiplying the first two equations in (1.5) by $q \cdot \nabla u$ and $q \cdot \nabla u_\Gamma$ respectively and integrating over Ω_T and Γ_T^1 leads to the identity

$$\begin{aligned}
\frac{1}{2}\int_0^T \int_{\Gamma^2} q \cdot \nu |\partial_\nu u|^2 \, d\Gamma \, dt + \frac{1}{2}\int_0^T \int_{\partial\Gamma^1} q.\tau |\partial_\tau u|^2 \, d\sigma \, dt &= \big[(\partial_t u(t), q \cdot \nabla u(t))_\Omega\big]_0^T \\
&+ [(\partial_t u(t), q \cdot \nabla u_\Gamma(t))_{\Gamma^1}]_0^T - \int_0^T \int_\Omega \partial_t u \, \partial_t q \cdot \nabla u \, dx \, dt \\
&- \int_0^T \int_{\Gamma^1} \partial_t u \partial_t q \cdot \nabla_\Gamma u \, d\Gamma \, dt + \frac{1}{2}\int_0^T \int_\Omega \operatorname{div} q(|\partial_t u|^2 - |\nabla u|^2) \, dx \, dt \\
&+ \frac{1}{2}\int_0^T \int_{\Gamma^1} \operatorname{div}_\Gamma q(|\partial_t u|^2 - |\nabla_\Gamma u|^2) \, d\Gamma \, dt + \int_0^T \int_\Omega \nabla u^\top \nabla q \nabla u \, dx \, dt \\
&+ \int_0^T \int_{\Gamma^1} \nabla_\Gamma u_\Gamma^\top \nabla_\Gamma q \nabla_\Gamma u_\Gamma \, d\Gamma \, dt - \frac{1}{2}\int_0^T \int_{\Gamma^1} q \cdot \nu(|\partial_t u|^2 - |\nabla_\Gamma u|^2) \, d\Gamma \, dt.
\end{aligned}$$

On the one hand, from the definition of q, we deduce

$$\begin{aligned}
\frac{1}{2}\int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma^{2,1}\cup\Gamma^{2,2}} |\partial_\nu u|^2 \, d\Gamma \, dt + \frac{1}{2}\int_{\varepsilon}^{T-\varepsilon} |\partial_\tau u(0,l_2)|^2 \, d\sigma \, dt \\
\leq \frac{1}{2}\int_0^T \int_{\Gamma^2} q \cdot \nu |\partial_\nu u|^2 \, d\Gamma \, dt + \frac{1}{2}\int_0^T \int_{\partial\Gamma^1} q.\tau |\partial_\tau u|^2 \, d\sigma \, dt.
\end{aligned} \tag{4.6}$$

On the other hand, since the derivatives $\partial q_k/\partial t$, $\partial q_k/\partial x_k$ are bounded, it is clear that the terms on the right side of the identity above are all bounded from above by the integrals

$$c\int_0^T \int_{\omega^1} (|\partial_t u|^2 + |\nabla_\Gamma u|^2) \, d\Gamma \, dt + c\int_0^T \int_{\omega^2} (|\partial_t u|^2 + |\nabla u|^2) \, dx \, dt.$$

Together with estimate (4.6) this gives

$$\begin{aligned}(4.7)\qquad \frac{1}{2}\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma^{2,1}\cup\Gamma^{2,2}} |\partial_\nu u|^2\,d\Gamma\,dt + \frac{1}{2}\int_{\varepsilon}^{T-\varepsilon} |\partial_\tau u(0,l_2)|^2\,d\sigma\,dt \\ \le c\int_{\varepsilon}^{T-\varepsilon}\int_{\omega^1}(|\partial_t u|^2 + |\nabla_\Gamma u|^2)\,d\Gamma\,dt \\ + c\int_{\varepsilon}^{T-\varepsilon}\int_{\omega^2}(|\partial_t u|^2 + |\nabla u|^2)\,dx\,dt.\end{aligned}$$

Now, since this inequality is true regardless of the choice of the neighborhood $\omega = \omega^1 \cup \omega^2$, it is also true for a neighborhood $\omega_0 = \omega_0^1 \cup \omega_0^2$ such that $\omega_0 \subset \omega$.

This time, we use the multipliers pu and pu_Γ where $p(x,t) = \psi(x)\eta(t)$ for $x \in \overline{\Omega}$ and $t \in [0,T]$. The function η is as in Lemma 3.4 and $\psi \in C^\infty(\overline{\Omega})$ is such that

$$\begin{cases}\psi = 1 & \text{on } \omega_0,\\ \psi = 0 & \text{on } (\Omega\cup\Gamma^1)\setminus\omega,\\ \psi \ge 0 & \text{on } \overline{\Omega}.\end{cases}$$

We integrate by parts over Ω_T and Γ_T^1 respectively to obtain

$$\begin{aligned}&\int_0^T\int_{\omega^2} p|\nabla u|^2\,dx\,dt + \int_0^T\int_{\omega^1} p|\nabla_\Gamma u|^2\,d\Gamma\,dt\\ &\quad= \int_0^T\int_{\omega^1} p|\partial_t u|^2\,d\Gamma\,dt + \int_0^T\int_{\omega^2} p|\partial_t u|^2\,dx\,dt + \int_0^T\int_{\omega^1} \partial_t p u\partial_t u\,d\Gamma\,dt\\ &\qquad+ \int_0^T\int_{\omega^2} \partial_t p u\partial_t u\,dx\,dt - \int_0^T\int_{\omega^2}(\nabla p\cdot\nabla u)u\,dx\,dt\\ &\qquad- \int_0^T\int_{\omega^1}(\nabla_\Gamma p\cdot\nabla_\Gamma u)u\,d\Gamma\,dt\\ &\quad= \int_0^T\int_{\omega^1} p|\partial_t u|^2\,d\Gamma\,dt + \int_0^T\int_{\omega^2} p|\partial_t u|^2\,dx\,dt + \int_0^T\int_{\omega^1} \partial_t p u\partial_t u\,d\Gamma\,dt\\ &\qquad+ \int_0^T\int_{\omega^2} \partial_t p u\partial_t u\,dx\,dt + \frac{1}{2}\int_0^T\int_{\omega^2} \Delta p|u|^2\,dx\,dt\end{aligned}$$

$$-\frac{1}{2}\int_0^T\int_{\omega^1}\partial_\nu p|u|^2\,d\Gamma\,dt+\frac{1}{2}\int_0^T\int_{\omega^1}\Delta_\Gamma p|u|^2\,d\Gamma\,dt.$$

We can readily show that there exists a constant $c>0$ depending on p such that

$$\int_0^T\int_{\omega^2}p|\nabla u|^2\,dx\,dt+\int_0^T\int_{\omega^1}p|\nabla_\Gamma u|^2\,d\Gamma\,dt \\ \le c\int_0^T\int_{\omega^1}(|\partial_t u|^2+|u|^2)\,d\Gamma\,dt+c\int_0^T\int_{\omega^2}(|\partial_t u|^2+|u|^2)\,dx\,dt.$$

Therefore,

$$\int_\varepsilon^{T-\varepsilon}\int_{\omega_0^2}|\nabla u|^2\,dx\,dt+\int_\varepsilon^{T-\varepsilon}\int_{\omega_0^1}|\nabla_\Gamma u|^2\,d\Gamma\,dt \\ \le c\int_0^T\int_{\omega^1}(|\partial_t u|^2+|u|^2)\,d\Gamma\,dt+c\int_0^T\int_{\omega^2}(|\partial_t u|^2+|u|^2)\,dx\,dt.$$

From this estimate and (4.7), it follows that

$$\frac{1}{2}\int_\varepsilon^{T-\varepsilon}\int_{\Gamma^{2,1}\cup\Gamma^{2,2}}|\partial_\nu u|^2\,d\Gamma\,dt+\frac{1}{2}\int_\varepsilon^{T-\varepsilon}|\partial_\tau u(0,l_2)|^2\,d\sigma\,dt \\ \le c\int_0^T\int_{\omega^1}(|\partial_t u|^2+|u|^2)\,d\Gamma\,dt+c\int_0^T\int_{\omega^2}(|\partial_t u|^2+|u|^2)\,dx\,dt.$$

Finally, taking into account the inverse inequality (4.5), we reach the conclusion (4.4). ∎

Proof of Theorem 4.2. Assume that estimate (4.3) does not hold, so there exists a sequence $(u_n^0,u_{\Gamma,n}^0,u_n^1,u_{\Gamma,n}^1)\in\mathcal{V}\times\mathcal{H}$ of initial data such that

$$1=\int_\Omega(|u_n^1|^2+|\nabla u_n^0|^2)\,dx+\int_{\Gamma^1}(|u_{\Gamma,n}^1|^2+|\nabla_\Gamma u_{\Gamma,n}^0|^2)\,d\Gamma \\ >n\Big(\int_{\omega_T^2}|\partial_t u_n|^2\,dx+\int_{\omega_T^1}|\partial_t u_{\Gamma,n}|^2\,d\Gamma\Big). \tag{4.8}$$

Following the proof of Theorem 3.2, we find a subsequence (not renamed) such that

$$(u_n,u_{\Gamma,n})\to(u,u_\Gamma)\quad\text{in } L^2(0,T;L^2(\omega^2)\times L^2(\omega^1)).$$

and

$$\begin{cases} \partial_t u = 0 & \text{in } \omega_T^2, \\ \partial_t u_\Gamma = 0 & \text{on } \omega_T^1. \end{cases}$$

Passing to the limit in system (1.5), as $n \to \infty$, we find that the limit (u, u_Γ) solves in the sense of distributions

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega_T, \\ \partial_t u = 0 & \text{in } \omega_T^2, \\ \partial_t^2 u_\Gamma + \partial_\nu u - \Delta_\Gamma u_\Gamma = 0 & \text{on } \Gamma_T^1, \\ \partial_t u_\Gamma = 0 & \text{in } \omega_T^1, \\ u = 0 & \text{on } \Gamma^2. \end{cases} \tag{4.9}$$

Now, take $v = \partial_t u$ and $v_\Gamma = \partial_t u_\Gamma$. Then (v, v_Γ) satisfies in the distributional sense

$$\begin{cases} \partial_t^2 v - \Delta v = 0 & \text{in } \Omega_T, \\ \partial_t^2 v_\Gamma + \partial_\nu v - \Delta_\Gamma v_\Gamma = 0 & \text{on } \Gamma_T^1. \end{cases}$$

As $v = 0$ in ω_T^2, Holmgren's Theorem gives $v = 0$ in Ω, which also implies that $\partial_\nu v = 0$. So the domain Ω and Ventcel's portion of the boundary are no longer coupled. Hence,

$$\begin{cases} \partial_t^2 v_\Gamma - \Delta_\Gamma v_\Gamma = 0 & \text{on } \Gamma_T^1, \\ v_\Gamma = 0 & \text{in } \omega_T^1. \end{cases}$$

Again, according to Holmgren's Theorem, we get $v_\Gamma = 0$ on Γ^1. Thus, (4.9) becomes

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u - \Delta_\Gamma u = 0 & \text{on } \Gamma^1, \\ u = 0 & \text{on } \Gamma^2, \end{cases}$$

which yields $u \equiv 0$ on $\overline{\Omega}$. Furthermore, as $n \to \infty$ in (4.4), the energy $E_n(0)$ goes to zero. However, $E_n(0) = 1$ for all $n \in \mathbb{N}$. Thus, we have arrived at a contradiction. ∎

5. The nonlocal wave system (1.4). Now, we are in a position to prove the observability of the nonlocal system (1.4), in the two geometries under consideration. We will do so using compactness-uniqueness arguments as in [10]. First, we observe that, for any $(u_T^0, u_{T,\Gamma}^0) \in \mathcal{H}$ and $(u_T^1, u_{T,\Gamma}^1) \in \mathcal{V}'$, the solution to (1.4) can be written as the sum $(u, u_\Gamma) = (p + q, p_\Gamma + q_\Gamma)$

where the functions p and q are solutions, respectively, to

(5.1)
$$\begin{cases} \partial_t^2 p - \varDelta p = 0 & \text{in } \varOmega_T, \\ \partial_t^2 p_\varGamma + \partial_\nu p - \varDelta_\varGamma p_\varGamma = 0,\ p = p_\varGamma & \text{on } \varGamma_T^1, \\ p = 0 & \text{on } \varGamma_T^2, \\ (p(T), p_\varGamma(T)) = (u_T^0, u_{T,\varGamma}^0),\ (\partial_t p(T), \partial_t p_\varGamma(T)) = (u_T^1, u_{T,\varGamma}^1) & \text{in } \varOmega \times \varGamma^1, \end{cases}$$

and

$$\begin{cases} \partial_t^2 q - \varDelta q + \int_\varOmega K_\varOmega(x,y) q(x,t)\, d\xi & \\ \quad = -\int_\varOmega K_\varOmega(x,y) p(x,t)\, dx & \text{in } \varOmega_T, \\ \partial_t^2 q_\varGamma + \partial_\nu q - \varDelta_\varGamma q_\varGamma + \int_{\varGamma^1} K_\varGamma(\zeta,\xi) q_\varGamma(\zeta,t)\, d\varGamma & \\ \quad = -\int_{\varGamma^1} K_\varGamma(\zeta,\xi) p_\varGamma(\zeta,t)\, d\varGamma,\ q = q_\varGamma & \text{on } \varGamma_T^1, \\ q = 0 & \text{on } \varGamma_T^2, \\ (q(T), q_\varGamma(T)) = (0,0),\ (\partial_t q(T), \partial_t q_\varGamma(T)) = (0,0) & \text{in } \varOmega \times \varGamma^1. \end{cases} \tag{5.2}$$

5.1. A smooth domain. We shall show system (1.4) is observable in time $T > T_0$ (cf. Theorem 3.2). In fact, we have the following

THEOREM 5.1. *Let $T > T_0$. For all solutions to* (1.4) *associated to final data $(u_T^0, u_{T,\varGamma}^0) \in \mathcal{H}$, $(u_T^1, u_{T,\varGamma}^1) \in \mathcal{V}'$, we have the estimate*

$$\|(u,u_\varGamma)(0)\|_{\mathcal{H}}^2 + \|(\partial_t u, \partial_t u_\varGamma)(0)\|_{\mathcal{V}'}^2 \le c\Big(\int\limits_{\omega_T} |u|^2\, dx\, dt + \int\limits_{\varGamma_T^1} |u_\varGamma|^2\, d\varGamma\, dt\Big). \tag{5.3}$$

Proof. We know from Theorem 3.2 that the solutions to the homogeneous problem (5.1) are observable in any time $T > T_0$, that is,

$$\|(u_T^0, u_{T,\varGamma}^0)\|_{\mathcal{H}}^2 + \|(u_T^1, u_{T,\varGamma}^1)\|_{\mathcal{V}'}^2 \le c\Big(\int\limits_{\omega_T} |p|^2\, dx\, dt + \int\limits_{\varGamma_T^1} |p_\varGamma|^2\, d\varGamma\, dt\Big). \tag{5.4}$$

Using Young's inequality, we have

$$\begin{aligned} \|(u,u_\varGamma)(0)\|_{\mathcal{H}}^2 &+ \|(\partial_t u, \partial_t u_\varGamma)(0)\|_{\mathcal{V}'}^2 \\ &\le 2\big(\|(p,p_\varGamma)(0)\|_{\mathcal{H}}^2 + \|(\partial_t p, \partial_t p_\varGamma)(0)\|_{\mathcal{V}'}^2 \\ &\quad + \|(q,q_\varGamma)(0)\|_{\mathcal{H}}^2 + \|(\partial_t q, \partial_t q_\varGamma)(0)\|_{\mathcal{V}'}^2\big). \end{aligned} \tag{5.5}$$

Making use of the observability inequality (3.3), we obtain

$$\begin{aligned} \|(p,p_\varGamma)(0)\|_{\mathcal{H}}^2 &+ \|(\partial_t p, \partial_t p_\varGamma)(0)\|_{\mathcal{V}'}^2 \\ &\le c\Big(\int\limits_{\omega_T} |p|^2\, dx\, dt + \int\limits_{\varGamma_T^1} |p_\varGamma|^2\, d\varGamma\, dt\Big). \end{aligned} \tag{5.6}$$

Moreover, the continuity of the solutions p, q with respect to given data yields

$$\begin{aligned}\|(q, q_\Gamma)(0)\|^2_{\mathcal{H}} &+ \|(\partial_t q, \partial_t q_\Gamma)(0)\|^2_{\mathcal{V}'} \\ &\le c(K_\Omega, K_\Gamma)\Big(\int\limits_{\Omega_T} |p|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |p_\Gamma|^2\,d\Gamma\,dt\Big) \\ &\le c\big(\|(u^0_T, u^0_{T,\Gamma})\|^2_{\mathcal{H}} + \|(u^1_T, u^1_{T,\Gamma})\|^2_{\mathcal{V}'}\big).\end{aligned}$$

Again, we employ the observability estimate (3.3) to get

$$\|(q, q_\Gamma)(0)\|^2_{\mathcal{H}} + \|(\partial_t q, \partial_t q_\Gamma)(0)\|^2_{\mathcal{V}'} \le c\Big(\int\limits_{\omega_T} |p|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |p_\Gamma|^2\,d\Gamma\,dt\Big). \tag{5.7}$$

Combining the estimates (5.5)–(5.7), we find

$$\|(u, u_\Gamma)(0)\|^2_{\mathcal{H}} + \|(\partial_t u, \partial_t u_\Gamma)(0)\|^2_{\mathcal{V}'} \le c\Big(\int\limits_{\omega_T} |p|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |p_\Gamma|^2\,d\Gamma\,dt\Big).$$

Thus, in order to obtain (5.3) it suffices to show that

$$\int\limits_{\omega_T} |p|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |p_\Gamma|^2\,d\Gamma\,dt \le c\Big(\int\limits_{\omega_T} |u|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |u_\Gamma|^2\,d\Gamma\,dt\Big).$$

Indeed, suppose the opposite is true, so we can find a sequence $(u^0_{T,n}, u^0_{T,\Gamma,n}, u^1_{T,n}, u^1_{T,\Gamma,n}) \in \mathcal{H} \times \mathcal{V}'$ such that

$$\int\limits_{\omega_T} |p_n|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |p_{\Gamma,n}|^2\,d\Gamma\,dt = 1, \tag{5.8}$$

$$\lim_{n\to\infty} \int\limits_{\omega_T} |u_n|^2\,dx\,dt + \int\limits_{\Gamma^1_T} |u_{\Gamma,n}|^2\,d\Gamma\,dt = 0, \tag{5.9}$$

where $(u_n, u_{\Gamma,n})$ are the solutions to (1.4) corresponding to the final states $(u^0_{T,n}, u^0_{T,\Gamma,n}, u^1_{T,n}, u^1_{T,\Gamma,n})$, and p_n (resp. q_n) are the solutions to (5.1) (resp. (5.2)) such that $(u_n, u_{\Gamma,n}) = (p_n + q_n, p_{\Gamma,n} + q_{\Gamma,n})$.

Taking into account (5.4) and (5.8), we find that $(u^0_{T,n}, u^0_{T,\Gamma,n}, u^1_{T,n}, u^1_{T,\Gamma,n})$ is uniformly bounded in $\mathcal{H} \times \mathcal{V}'$. Hence,

$$(p_n, p_{\Gamma,n}, \partial_t p_n, \partial_t p_{\Gamma,n}) \text{ is uniformly bounded in } C^0(0, T; \mathcal{H} \times \mathcal{V}'), \tag{5.10}$$

which also means that so are the source terms in problem (5.2). Then

$$(q_n, q_{\Gamma,n}, \partial_t q_n, \partial_t q_{\Gamma,n}) \text{ is uniformly bounded in } C^0(0, T; \mathcal{H} \times \mathcal{V}'), \tag{5.11}$$

and this leads to $(u_n, u_{\Gamma,n})$ being uniformly bounded in $C^0(0, T; \mathcal{H})$. Thus, there exists a subsequence such that

$$(u_n, u_{\Gamma,n}) \overset{*}{\rightharpoonup} (u, u_\Gamma) \quad \text{in } L^\infty(0, T; \mathcal{H}).$$

Combined with (5.9), this gives

$$\begin{cases} u = 0 & \text{in } \omega_T, \\ u_\Gamma = 0 & \text{on } \Gamma_T^1. \end{cases}$$

Now, letting $n \to \infty$ in (1.4) we have, in the sense of distributions,

$$\begin{cases} \partial_t^2 u - \Delta u + \int_\Omega K_\Omega(x,y)u(x,t)\,dx = 0 & \text{in } \Omega_T, \\ u = 0 \quad \text{in } \omega_T. \end{cases} \tag{5.12}$$

Since $u = 0$ in ω_T, we see that also $\partial_t^2 u = \Delta u = 0$ in ω_T. Then, from the first equation in (5.12), it follows that

$$\int_\Omega K_\Omega(x,y)u(x,t)\,dx = 0 \quad \text{in } \omega_T. \tag{5.13}$$

The analyticity hypothesis on the function $y \mapsto \int_\Omega K_\Omega(x,y)u(x,t)\,dx$ implies that

$$\int_\Omega K_\Omega(x,y)u(x,t)\,dx = 0 \quad \text{in } \Omega_T. \tag{5.14}$$

Thus, (5.12) becomes

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{in } \omega_T, \end{cases} \tag{5.15}$$

and we can now apply classical uniqueness results [17, Theorem 8.1 and Lemma 8.1] to deduce that $u = 0$ in Ω. Therefore,

$$(u_n, u_{\Gamma,n}) \overset{*}{\rightharpoonup} (0,0) \quad \text{in } L^\infty(0,T;\mathcal{H}). \tag{5.16}$$

Next, since the sequences $(q_n, q_{\Gamma,n}, \partial_t q_n, \partial_t q_{\Gamma,n})$, $(p_n, p_{\Gamma,n}, \partial_t p_n, \partial_t p_{\Gamma,n})$ are uniformly bounded and the embedding $\mathcal{H} \hookrightarrow \mathcal{V}'$ is compact, we can apply the Aubin–Lions Lemma to get (up to subsequences)

$$(p_n, p_{\Gamma,n}) \to (p, p_\Gamma) \quad \text{in } L^2(0,T;\mathcal{H}), \qquad (q_n, q_{\Gamma,n}) \to (q, q_\Gamma) \quad \text{in } L^2(0,T;\mathcal{H}).$$

Letting $n \to \infty$, we find that the limits (q, q_Γ) and (p, p_Γ) satisfies the system (5.2) in the distributional sense. However, recalling that $(u_n, u_{\Gamma,n}) = (p_n + q_n, p_{\Gamma,n} + q_{\Gamma,n})$, it follows that

$$(u, u_\Gamma) = (p + q, p_\Gamma + q_\Gamma).$$

Thus, we have

$$\begin{cases} \partial_t^2 q - \Delta q + \int_\Omega K_\Omega(x,y)u(x,t)d\xi = 0 & \text{in } \Omega_T, \\ \partial_t^2 q_\Gamma + \partial_\nu q - \Delta_\Gamma q_\Gamma + \int_{\Gamma^1} K_\Gamma(\zeta,\xi)u_\Gamma(\zeta,t)d\Gamma = 0, \; q = q_\Gamma & \text{on } \Gamma_T^1, \\ q = 0 & \text{on } \Gamma_T^2, \\ (q(T), q_\Gamma(T)) = (0,0), \; (\partial_t q(T), \partial_t q_\Gamma(T)) = (0,0) & \text{in } \Omega \times \Gamma^1. \end{cases}$$

Then, taking into account (5.16), the integral terms are null. Consequently, we obtain $(q, q_\Gamma) = (0, 0)$.

Finally, observe that

$$\begin{aligned} 1 &= \int\limits_{\omega_T} |p_n|^2 \, dx \, dt + \int\limits_{\Gamma_T^1} |p_{\Gamma,n}|^2 \, d\Gamma \, dt \\ &\le 2 \int\limits_{\omega_T} (|u_n|^2 + |q_n|^2) \, dx \, dt + 2 \int\limits_{\Gamma_T^1} (|u_{\Gamma,n}|^2 + |q_{\Gamma,n}|^2) \, d\Gamma \, dt, \end{aligned}$$

which is in contradiction with what we have just established above. This completes the proof. ∎

Remark 5.2. 1. It is worth noting that unlike the case of a rectangular geometry, we cannot only act on a portion of the boundary. In the case of the rectangle, the boundary is flat. The rays that start from the controlled region, being tangent to the surface, cover the entire Ventcel portion and therefore allow one to obtain exact control, because the GCC condition of Bardos–Lebeau–Rauch [1] is satisfied. On the other hand, for a curved boundary, the rays starting from any portion and tangent to the surface leave the boundary. Therefore, no points of Ventcel's portion can be reached and the GCC condition cannot be satisfied. Thus, a portion of the curved boundary cannot give exact controllability.

2. We emphasize that the kernels K_Ω, K_Γ being bounded is not enough to yield unique continuation for the solution of the homogeneous system (1.4). Indeed, we can find counterexamples where we do not have unique continuation, if no additional assumptions on the kernels are imposed apart from just being bounded. One such example is given in [3, Section 5].

3. The boundary observability result recalled in Proposition 3.1 is also true when the wave equation and Ventcel's condition contain lower-order terms (see [12, Section 3])

$$(5.17) \qquad \begin{cases} \partial_t^2 u - \Delta u + q(t,x)u = 0 & \text{in } \Omega_T, \\ \partial_t^2 u_\Gamma + \partial_\nu u_\Gamma - \Delta_\Gamma u_\Gamma + q_\Gamma(t,x)u = 0 & \text{on } \Gamma_T^1, \end{cases}$$

where $q \in L^\infty(\Omega_T)$ and $q_\Gamma \in L^\infty(\Gamma_T^1)$. Then the observability we have just established holds also for the nonlocal version of (5.17), namely

$$(5.18) \qquad \begin{cases} \partial_t^2 u - \Delta u + q(t,x)u + \int\limits_\Omega K_\Omega(x,y)u(t,x)\, dx = 0 & \text{in } \Omega_T, \\ \partial_t^2 u_\Gamma + \partial_\nu u_\Gamma - \Delta_\Gamma u_\Gamma + q_\Gamma(t,x)u_\Gamma & \\ \quad + \int\limits_{\Gamma^1} K_\Gamma(x,y)u_\Gamma(t,x)\, d\Gamma(x) = 0 & \text{on } \Gamma_T^1. \end{cases}$$

4. In the case of time-dependent kernels $K_\Omega = K_\Omega(t,x,y)$, $K_\Gamma = K_\Gamma(t,x,y)$, it is clear that we can recover the result in Theorem 5.1 using

the approach above, under the condition that

$$y \mapsto \int_{\Omega} K_{\Omega}(t,x,y) f(x)\,dx \tag{5.19}$$

is an analytic function for all $t \in [0,T]$ and $f \in L^2(\Omega)$.

5.2. Rectangular domain. In a similar fashion, we establish an observability result for system (1.4) in the geometric setting of Section 4 where Ω is a rectangular domain.

THEOREM 5.3. *Let $T > T_{R,0}$ (see Theorem 4.2). Then there exists a constant $c > 0$ such that the solution to problem (1.4) satisfies*

$$\|(u,u_\Gamma)(0)\|^2_{\mathcal{H}} + \|(\partial_t u, \partial_t u_\Gamma)(0)\|^2_{\mathcal{V}'} \le c\Big(\int_{\omega_T^2} |u|^2\,dx\,dt + \int_{\omega_T^1} |u_\Gamma|^2 d\Gamma\,dt\Big) \tag{5.20}$$

for all final data $(u_T^0, u_{T,\Gamma}^0) \in \mathcal{H}$ and $(u_T^1, u_{T,\Gamma}^1) \in \mathcal{V}'$.

The proof of this theorem mostly goes along the same lines as that of Theorem 5.1. We just replace the integral over Γ_T^1 with the integral over ω_T^1 from the beginning up to where we obtain

$$\begin{cases} u = 0 & \text{in } \omega_T^2, \\ u_\Gamma = 0 & \text{on } \omega_T^1. \end{cases}$$

However, since for the data considered the solution (u,u_Γ) is not sufficiently regular for u and u_Γ to be connected, the analyticity of the function $y \mapsto \int_\Omega K_\Omega(x,y)u(x,t)\,dx$ is no longer enough to yield both $u = 0$ in Ω_T and $u_\Gamma = 0$ on Γ_T^1. Thus, we need to also assume that the function $\zeta \mapsto \int_{\Gamma^1} K_\Gamma(\xi,\zeta)u_\Gamma(\xi,t)\,d\Gamma$ is analytic on Γ^1. Under these two analyticity assumptions, the limit (u,u_Γ) is a solution to the local problem (1.5), and the uniqueness result provided by Theorem 4.2 gives $(u,u_\Gamma) = (0,0)$. The remainder of the proof follows as previously.

REMARK 5.4. 1. Here too we can extend the result of Theorem 5.3 to the case where the kernels K_Ω, K_Γ depend on time. We just need the assumption that the functions

$$y \mapsto \int_{\Omega} K_\Omega(t,x,y) f(x)\,dx, \qquad y \mapsto \int_{\Gamma^1} K_\Gamma(t,x,y) g(x)\,d\Gamma(x) \tag{5.21}$$

are analytic for all $t \in [0,T]$, $f \in L^2(\Omega)$ and $g \in L^2(\Gamma^1)$.

2. Again, we can consider the system (1.4) with potentials as in (5.18). However, since the local result in Theorem 4.1 does not cover lower-order terms (cf. (5.17)), we need $q \in L^\infty(\Omega_T), q_\Gamma \in L^\infty(\Gamma_T^1)$ to be analytic [21, Chapter 6, Proposition 4.3] for the unique continuation of the solution to (5.18) to hold.

5.2.1. *The case where the kernel functions K_Ω and K_Γ are symmetric.* It is worth pointing out that, though we need the second analyticity assumption in (1.3) to recover estimate (5.20), it is not essential. In fact, if we consider two symmetric kernel functions K_Ω and K_Γ, we can show that system (1.4) is observable in time $T > T_{R,0}$ whenever $(u_T^0, u_{T,\Gamma}^0) \in \mathcal{V}$ and $(u_T^1, u_{T,\Gamma}^1) \in \mathcal{H}$, without any additional hypotheses. More precisely, we shall see that given $T > T_{R,0}$, the solution to (1.4) corresponding to these final data satisfies

$$\tag{5.22} \begin{aligned} &\|(u, u_\Gamma)(0)\|_{\mathcal{V}}^2 + \|(\partial_t u, \partial_t u_\Gamma)(0)\|_{\mathcal{H}}^2 \\ &\qquad\le c\Big(\int_{\omega_T^2} |\partial_t u|^2\,dx\,dt + \int_{\omega_T^1} |\partial_t u_\Gamma|^2\,d\Gamma\,dt\Big). \end{aligned}$$

To reach this estimate, we take the approach of [22, Chapter 7]. We break down our system into two parts using an appropriate decomposition of $\mathcal{H}$. We establish observability for each part. Then, employing a result on simultaneous observability given in [22, Proposition 6.4.2], we get inequality (5.22).

First, recall the bounded operator $B_0 \in \mathcal{L}(\mathcal{H})$ given by

$$B_0 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} := \begin{pmatrix} -\int_\Omega K_\Omega(x,y)\varphi_1(x)\,dx \\ -\int_{\Gamma^1} K_\Gamma(x,y)\varphi_2(x)\,d\Gamma(x) \end{pmatrix}.$$

We can readily see that $\|B_0\| \le K$, where

$$K := \Big(\int_\Omega\int_\Omega |K_\Omega(x,y)|^2\,dx\,dy + \int_{\Gamma^1}\int_{\Gamma^1} |K_\Gamma(x,y)|^2\,d\Gamma(x)\,d\Gamma(y)\Big)^{1/2}.$$

Later on, we will also need the observation operator, defined in this case as follows:

$$CU = C\begin{pmatrix} \varphi \\ \varphi_\Gamma \\ \psi \\ \psi_\Gamma \end{pmatrix} = \begin{pmatrix} \psi|_{\omega^2} \\ \psi_\Gamma|_{\omega^1} \end{pmatrix}, \qquad \forall U \in \mathcal{V}\times\mathcal{H}.$$

Obviously, the operator $C \in \mathcal{L}(\mathcal{V}\times\mathcal{H}, \mathcal{H})$ is an admissible observation operator for system (1.4).

The kernel functions K_Ω, K_Γ being symmetric, the operator $A_0 - B_0$ is clearly self-adjoint on $\mathcal{H}$ with compact resolvent. Thus, we have a real spectrum $\{\mu_k : k \in \mathbb{N}^*\}$ and the associated eigenvectors $\{\Phi_k\} \in D(A_0)$ constitute an orthonormal basis of $\mathcal{H}$. In addition, the eigenvalues of the operator $A + B$ (cf. (2.3), (2.4)) are closely related to those of $A_0 - B_0$. Indeed, setting $\lambda_k = i\sqrt{|\mu_k|}$ for $k > 0$ and $\lambda_k = -\lambda_{-k}$ for $k < 0$, the terms of the sequence $\{\lambda_k\}_k$ are the eigenvalues of $A + B$, and the corresponding eigenvectors are $U_{\pm k} = (\frac{1}{i\sqrt{|\mu_k|}}\Phi_k, \Phi_k)$, $k \in \mathbb{Z}^*$ (see [22, Proposition 7.3.3]).

We decompose the space $\mathcal{V} \times \mathcal{H}$ according to the eigenvalues of $A+B$. Let $N>0$ be such that $\mu_N>0$. We define

$$W_0 = \operatorname{span}\left\{\begin{pmatrix}\frac{1}{i\operatorname{sign}(k)}\Phi_k\\ \Phi_k\end{pmatrix} : k\in\mathbb{Z}^*,\ \lambda_k=0\right\},$$

$$W_N = \operatorname{span}\left\{\begin{pmatrix}\frac{1}{\lambda_k}\Phi_k\\ \Phi_k\end{pmatrix} : k\in\mathbb{Z}^*,\ |k|<N,\ \lambda_k\neq 0\right\}, \tag{5.23}$$

$$V_N = \overline{\operatorname{span}\left\{\begin{pmatrix}\frac{1}{\lambda_k}\Phi_k\\ \Phi_k\end{pmatrix} : |k|\geq N\right\}}.$$

We can follow the lines of the proof of [22, Lemma 7.3.4] to show that these closed subspaces form a direct sum decomposition of $\mathcal{V}\times\mathcal{H}$, that is, $\mathcal{V}\times\mathcal{H} = Y_N\oplus V_N$ where $Y_N = W_0+W_N$. Moreover, if the initial state of our system belongs to one of these subspaces, then for $t\in(0,T]$ the state remains in that subspace.

Let $(u_T^0, u_{T,\Gamma}^0)\in\mathcal{V}$ and $(u_T^1, u_{T,\Gamma}^1)\in\mathcal{H}$. We have

$$\begin{aligned}\int_{\omega_T^2} |\partial_t u|^2\,dx\,dt + \int_{\omega_T^1} |\partial_t u_\Gamma|^2\,d\Gamma\,dt \\ \geq \frac{1}{2}\Big(\int_{\omega_T^2} |\partial_t p|^2\,dx\,dt + \int_{\omega_T^1} |\partial_t p_\Gamma|^2\,d\Gamma\,dt\Big) \\ - \Big(\int_{\omega_T^2} |\partial_t q|^2\,dx\,dt + \int_{\omega_T^1} |\partial_t q_\Gamma|^2\,d\Gamma\,dt\Big),\end{aligned}$$

where (p,p_Γ) (resp. (q,q_Γ)) is the solution to problem (5.1) (resp. (5.2)). Observe that the solution (q,q_Γ) satisfies

$$\begin{aligned}\int_{\omega_T^2} |\partial_t q|^2\,dx\,dt + \int_{\omega_T^1} |\partial_t q_\Gamma|^2\,d\Gamma\,dt \\ \leq c\|B_0\|^2\Big(\int_{\Omega_T} |p|^2\,dx\,dt + \int_{\Gamma_T^1} |p_\Gamma|^2\,d\Gamma\,dt\Big) \\ \leq c_0\|B_0\|^2\big(\|(u_T^0,u_{T,\Gamma}^0)\|_{\mathcal{V}}^2 + \|(u_T^1,u_{T,\Gamma}^1)\|_{\mathcal{H}}^2\big).\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\omega_T^2} |\partial_t u|^2\,dx\,dt + \int_{\omega_T^1} |\partial_t u_\Gamma|^2\,d\Gamma\,dt \\ &\quad\geq \left(\frac{1}{2c_{\mathrm{obs}}} - c_0\|B_0\|^2\right)\big(\|(u_T^0,u_{T,\Gamma}^0)\|_{\mathcal{V}}^2 + \|(u_T^1,u_{T,\Gamma}^1)\|_{\mathcal{H}}^2\big) \\ &\quad\geq c\left(\frac{1}{2c_{\mathrm{obs}}} - c_0\|B_0\|^2\right)\big(\|(u,u_\Gamma)(0)\|_{\mathcal{V}}^2 + \|(\partial_t u,\partial_t u_\Gamma)(0)\|_{\mathcal{H}}^2\big),\end{aligned} \tag{5.24}$$

where c_{obs} stands for the constant in the observability estimate (4.3). This implies that (5.22) holds if $\|B_0\| < \sqrt{1/(2c_0c_{\text{obs}})}$.

If we take $N > 0$ such that $\mu_N > K$, the restriction of $B \in \mathcal{L}(\mathcal{V} \times \mathcal{H}, \mathcal{H})$ to the subspace V_N satisfies

$$\|B_{V_N}\| \leq \frac{K}{\sqrt{\mu_N - K}}. \tag{5.25}$$

To prove this, let us take $\Psi = (\psi, \psi_\Gamma)$ such that $\Psi = \sum_{k=N}^{M} \alpha_k \Phi_k$. Then

$$\|\Psi\|_{\mathcal{H}}^2 = \sum_{k=N}^{M} |\alpha_k|^2.$$

On the other hand,

$$\begin{aligned}
&\|\nabla\psi\|_\Omega^2 + \|\nabla_\Gamma\psi_\Gamma\|_\Gamma^2 + \int_\Omega \Big(\int_\Omega K_\Omega(x,y)\psi(x)\,dx\Big)\psi(y)\,dy \\
&\qquad + \int_{\Gamma^1} \Big(\int_{\Gamma^1} K_\Gamma(x,y)\psi_\Gamma(x)\,d\Gamma\Big)\psi_\Gamma(y)\,d\Gamma \\
&\quad = (-\Delta\psi, \psi)_\Omega + (\partial_\nu\psi - \Delta_\Gamma\psi_\Gamma, \psi_\Gamma)_\Gamma + \Big(\int_\Omega K_\Omega(x,\cdot)\psi(y)\,dy, \psi\Big)_\Omega \\
&\qquad + \Big(\int_{\Gamma^1} K_\Gamma(x,\cdot)\psi_\Gamma(x)\,d\Gamma, \psi_\Gamma\Big)_\Gamma \\
&\quad = \sum_{k=N}^{M} \mu_k|\alpha_k|^2 \geq \mu_N\|\Psi\|_{\mathcal{H}}^2,
\end{aligned}$$

which leads to

$$\|\nabla\psi\|_\Omega^2 + \|\nabla_\Gamma\psi_\Gamma\|_\Gamma^2 \geq (\mu_N - K)\|\Psi\|_{\mathcal{H}}^2. \tag{5.26}$$

Let

$$U = \begin{pmatrix} F \\ G \end{pmatrix} \in \operatorname{span}\Big\{ \begin{pmatrix} \frac{1}{\lambda_k}\Phi_k \\ \Phi_k \end{pmatrix} : |k| \geq N \Big\}.$$

Since such vectors are dense in V_N, what follows also holds for all $U \in V_N$. In fact, making use of inequality (5.26), we get (5.25):

$$\|BU\|_{\mathcal{V}\times\mathcal{H}}^2 = \|B_0F\|_{\mathcal{H}}^2 \leq K^2\|F\|_{\mathcal{H}}^2 \leq \frac{K^2}{\mu_N - K}\|U\|_{\mathcal{V}\times\mathcal{H}}^2.$$

Now, we are in a position to establish the observability estimate (5.22). Let $(u_T^0, u_{T,\Gamma}^0, u_T^1, u_{T,\Gamma}^1) \in V_N$. Taking $\mu_N > K$ large enough for $\|B_{V_N}\| < \sqrt{1/(2c_0c_{\text{obs}})}$ to be true, it follows that system (1.4) is observable in time $T > T_{0,R}$.

Next, let $(u_T^0, u_{T,\Gamma}^0, u_T^1, u_{T,\Gamma}^1)$ be in the finite-dimensional space Y_N. To prove that the restriction of system (1.4) is observable, we apply the finite-dimensional Hautus test [22]. More precisely, we use one of its consequences that asserts that the system under consideration is observable if and only if $CU \neq 0$ for every eigenvector U of the operator $A + B$ (see [22, Remark 1.5.2]). To prove this, we argue by contradiction.

Let $U = (\varphi, \varphi_\Gamma, \psi, \psi_\Gamma) \in D(A)$ be an eigenvector of the operator $A + B$ corresponding to the eigenvalue $i\sqrt{|\mu|}$. Then the functions φ, φ_Γ satisfy the equations

$$\begin{cases} \Delta\varphi - \int_\Omega K_\Omega(x,y)\varphi(x)dx + \mu\varphi = 0 & \text{in } \Omega, \\ \Delta_\Gamma\varphi - \partial_\nu\varphi - \int_{\Gamma^1} K_\Gamma(x,y)\varphi_\Gamma(x)d\Gamma(x) + \mu\varphi = 0 & \text{on } \Gamma^1. \end{cases}$$

We assume that $CU = 0$, so $\psi = 0$ in ω^2 and $\psi_\Gamma = 0$ on ω^1, which in turn implies that

$$\varphi = \frac{1}{i\sqrt{|\mu|}}\psi = 0 \quad \text{in } \omega^2, \qquad \varphi_\Gamma = \frac{1}{i\sqrt{|\mu|}}\psi_\Gamma = 0 \quad \text{on } \omega^1.$$

Then, whenever $U \in W_0$ or $U \in W_N$, under the first analyticity assumption in (1.3), the function $\varphi \in H^2(\Omega)$ satisfies

$$\Delta\varphi + \mu\varphi = 0 \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{in } \omega^2.$$

Therefore, according to Holmgren's Theorem, we see that $\varphi = 0$ in Ω. As for the boundary, since φ lies in $H^2(\Omega)$, we can deduce immediately that $\varphi_\Gamma = 0$. In turn, this implies that $\psi = 0$ in Ω and $\psi_\Gamma = 0$ on Γ^1. In other words, we have $U = 0$. This contradicts U being an eigenvector of $A + B$, which shows that $CU \neq 0$ for every eigenvector of $A + B$. Consequently, by the finite-dimensional Hautus test (in particular [22, Remark 1.5.2]), the restriction of our system to the finite-dimensional subspace Y_N is observable. Since $Y_N \cap V_N = \emptyset$, combining the observability on V_N and Y_N (or equivalently applying [22, Theorem 6.4.2]), we obtain inequality (5.22).

Remark 5.5. We note that this result could not be proved using the compactness-uniqueness method without the analyticity condition for the kernel function K_Γ. Moreover, whether we can extend this result to weak solutions belonging to $\mathcal{H} \times \mathcal{V}'$ depends on the ability to provide a result analogous to that given in Lemma 3.3 in this particular case of symmetric kernels. In other words, we need to prove well-posedness in $\mathcal{V}$ of the steady-state problem

$$(\mathcal{A} - B_0)\begin{pmatrix} \chi \\ \chi_\Gamma \end{pmatrix} = \begin{pmatrix} u_T^1 \\ u_{T,\Gamma}^1 \end{pmatrix} \tag{5.27}$$

for every $(u_T^1, u_{T,\Gamma}^1) \in \mathcal{V}'$. This is not guaranteed for arbitrary symmetric kernels; we need additional requirements on K_Ω and K_Γ. For example (see [15]), if

$$\|B_0\| \leq \frac{1}{\|\mathcal{A}^{-1}\|}, \tag{5.28}$$

then $\mathcal{A} - B_0 : \mathcal{V} \to \mathcal{V}'$ is an isomorphism and problem (5.27) admits a unique solution.

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