ACTA ARITHMETICA
Online First version

Correlation and lower bounds of arithmetic expressions

by

Fernando Chamizo (Madrid)

Dedicated to Henryk Iwaniec

Abstract. We explore the use of correlation with simple functions to get lower bounds for arithmetic quantities. In particular, we apply this idea to the power moments of the error term when counting visible lattice points in large spheres.

1. Average and amplification. Very often in analytic number theory one has to deal with a certain expression Q that, considering some harmonic companions [12], can be included in a kind of spectral family $\{Q_j\}_{j\in J}$, say $Q = Q_{j_0}$. In this situation, average results are usually easier to get than bounds or asymptotic formulas for Q. For instance, a variant of Parseval's identity can lead to something of the form

$$\sum_{j\in J} |Q_j|^2 \sim F,$$

which suggests that $|Q_j|^2$ is typically like $F/|J|$ (if J is not finite there is a chance to introduce weights in the summation) and implies the Ω-result stating that at least one of the $|Q_j|^2$ cannot be asymptotically below $F/|J|$.

Average results do not give good individual estimates because dropping all the terms except Q is wasteful. The amplification method, developed by H. Iwaniec and collaborators to get a number of conspicuous results (e.g. [6, 5, 11, 14]), circumvents this problem. As a guide for the reader, to our taste [7] is its most transparent application and [14] the most impressive one (cf. [1, §10], [4, pp. 93–100]). Let us review briefly the simple and powerful schematic idea in a somewhat restricted setting. A family of linear forms $\mathcal{L}_j(\vec{a}) = \sum_n a_n\lambda_j(n)$ is introduced in such a way that there is a quantifiable cancellation in $\sum_j |Q_j|^2\lambda_j(n)\overline{\lambda}_j(m)$ allowing one to prove a nontrivial

2020 *Mathematics Subject Classification*: Primary 11N37; Secondary 11P21, 11L07.
Key words and phrases: arithmetic functions, visible points, exponential sums.
Received 11 May 2023; revised 12 September 2023.
Published online 2 April 2024.

DOI: 10.4064/aa230511-23-9

average result of the form

$$\sum_j |Q_j|^2 |\mathcal{L}_j(\vec{a})|^2 \le K^2 \|\vec{a}\|^2.$$

A choice $\vec{a} = \vec{a}_0$ giving a large value of $|\mathcal{L}_{j_0}(\vec{a})|$ *amplifies* the contribution of $|Q_{j_0}|^2$ to the sum showing

$$|Q| = |Q_{j_0}| \le \frac{K\|\vec{a}_0\|}{|\mathcal{L}_{j_0}(\vec{a}_0)|}.$$

The amplifier will be stronger if the vectors $\{\lambda_j(n)\}_n$ keep certain quasi-orthogonality for different values of j, and perfect orthogonality would ideally allow one to select a single term. This guides our intuition to construct good amplifiers but note that there is no need for proving anything in this direction.

Dealing with a specific lattice point problem, in [2] a method was introduced that bears a point of resemblance to the amplification philosophy but produces lower bounds instead of upper bounds. Let us say that our spectral family is now a continuous one represented by an oscillatory function $Q(t)$ that is too complicated, so that one cannot determine the asymptotics of $\int |Q|^2\, d\mu$. If we find a simpler function g, which could be called a *resonator*, in such a way that there is a provable correlation

$$\int Qg\, d\mu \ge F > 0,$$

then Hölder's inequality gives

$$\|Q\|_p \ge \frac{F}{\|g\|_q} \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1,\ p, q \in [1, \infty].$$

So, lower bounds for the moments of Q follow from upper bounds for the moments of the simpler function g. If μ is normalized as a probability measure, then $\|Q\|_\infty \ge \|Q\|_p$ and an Ω-result for Q is deduced. Ideally, we look for g being a simple proxy of $\overline{Q}$ amplifying the contribution to the integral via some kind of positivity. In contrast, in the amplification method the artificial squared form tries to mimic the delta symbol.

The purpose of this paper is to show some instances of this idea. The main one is a modest improvement on [2] in which the resonator was based on $\sum_{d^2|n} \mu(d) = \mu^2(n) \ge 0$.

2. Main results. As usual, the notation $e(x)$ abbreviates $e^{2\pi i x}$. We also employ $A \ll B$ meaning $A = O(|B|)$ with Landau's notation and $A = \Omega(B)$ as the opposite of $A = o(B)$.

We start by pushing the correlation technique to improve a little our knowledge about the lattice point problem considered in [2].

Recall that the *visible points* in $\mathbb{Z}^3 - \{\vec{0}\}$ are those having coprime coordinates. If $\mathcal{N}^*(R)$ is the number of them in the ball $\|\vec{x}\| \le R$, it is fairly easy to get $\mathcal{N}^*(R) \sim \frac{4\pi}{3\zeta(3)} R^3$. The main result in [2] ensures

$$\int_R^{2R} |E^*|^2 \gg R^3 \log R \quad \text{where} \quad E^*(R) = \mathcal{N}^*(R) - \frac{4\pi}{3\zeta(3)} R^3. \tag{2.1}$$

Conjecturally $E^*(R) = O(R^{1+\epsilon})$ for any $\epsilon > 0$ and (2.1) implies that it is false for $\epsilon = 0$ and, in fact, $E^*(R) = \Omega(R\sqrt{\log R})$. Here we treat other moments showing that the logarithmic factors are unavoidable in them.

Theorem 2.1. *For any $p > 1$, we have*

$$\int_R^{2R} |E^*|^p \gg R^{p+1} (\log R)^{p/2}.$$

Of course, this adds new information to (2.1) only for $p < 2$.

The next result has a more analytic flavor. Essentially, it shows that the existence of gaps in the frequency spectrum implies oscillation.

Theorem 2.2. *Let $\{\nu_n\}_{n=1}^N \subset \mathbb{Z}^+$ be strictly increasing with $N < \infty$ or $N = \infty$ and consider its gaps $\Lambda_n = \min_{m\ne n} |\nu_m - \nu_n|$. For each $\alpha \in [0,1]$ and $1 < n < N$ the inequality*

$$\Big|\sum_{k=1}^N a_k \sin(2\pi\nu_k x)\Big| > \frac{1}{4}\mathcal{B}_{n,\alpha} |a_n|\, |x|^\alpha$$

holds for x in a positive measure subset of $[-1/2, 1/2]$, where

$$\mathcal{B}_{n,\alpha} = \frac{\pi^{\alpha-1}(1-\alpha^2)\Lambda_n}{\Lambda_n^{1-\alpha} - (1/5)^{1-\alpha}} \quad \text{for } \alpha \ne 1, \qquad \mathcal{B}_{n,1} = \lim_{\alpha\to 1^-} \mathcal{B}_{n,\alpha}$$

and $\{a_n\}_{n=1}^N \subset \mathbb{C} - \{0\}$ is any sequence ensuring the convergence of the series to an L^∞ function if $N = \infty$.

As an arithmetic-oriented example, we can deduce from this result that the function

$$F(x) = \sum_{n=1}^\infty \frac{\tau(n) e(p_n^3 x)}{n^2 (\log n)^{2023}},$$

with p_n the nth prime and τ the divisor function, does not satisfy the Lipschitz condition at any point. In particular, it is nowhere differentiable.

To see this, consider $f(x) = F(x_0 + x) - F(x_0 - x)$ and note that

$$e(a(x_0 + x)) - e(a(x_0 - x)) = 2i\, e(a x_0) \sin(2\pi a x).$$

If F is Lipschitz at x_0 then $|f(x)| \le C|x|$ for some constant C. When we apply Theorem 2.2 to f with $\alpha = 1$ and $\nu_n = p_n^3$, we have $\Lambda_n \ge p_n^2 \gg n^2(\log n)^2$ and $\mathcal{B}_{n,1} \gg n^2 \log n$. Recalling that $\limsup \tau(n)/(\log n)^K = \infty$ [8, Th. 314], the result shows that $|f(x)/x|$ is unbounded.

If $F : [1,\infty) \to \mathbb{C}$ and f is a bounded arithmetic function, the following kind of convolution is closer to the original formulation of Möbius inversion and to some natural applications in combinatorics:

$$G(x) = \sum_{n\le x} f(n) F\Big(\frac{x}{n}\Big). \tag{2.2}$$

Let us consider the case in which F is a partial sum of a Fourier series except for introducing a smooth transition from $F(1)$ to 0 in order to freely extend the upper bound in (2.2). Namely,

$$F(x) = \phi(x) \sum_{n=1}^{N} a_n e(nx)$$

with $\phi \in C_0^\infty$ such that $\phi(x) = 0$ for $x < 1$ and $\phi(x) = 1$ for $x > 2$.

THEOREM 2.3. *Let $R \ge 4N$ and*

$$\mathcal{B} = \sum_{n=1}^{N} b_n \quad \text{with} \quad b_n = a_n \sum_{d\mid n} f(d).$$

Assume $b_n \ge 0$ and for some $K > 1$ and any $V > 1/6$,

$$\sum_{R/(2V)\le d\le 3R} \ \sum_{dV/R\le r<2dV/R} \ \sum_{\substack{n=1\\ n\equiv \pm r\ (d)}}^{N} |a_n| = o((6V)^K \mathcal{B}).$$

Then

$$\int_R^{2R} |G(x)|^2\,dx \gg R\mathcal{B}^2 N^{-1}.$$

In particular, $G(x) = \Omega(\mathcal{B}N^{-1/2})$.

The hypothesis $b_n \ge 0$ is satisfied for instance when $a_n \ge 0$ and f is a real character. In some sense, the main result in [2], which we refine here, relies on an anharmonic version of this with $a_n \ge 0$ and $f(d) = \mu(\sqrt{d})$ if d is square and 0 otherwise.

3. Proofs. The proof of Theorem 2.1 is based on showing that the asymptotics in [2, Th. 1.1] are preserved, except for a $\sigma/2$ factor, when replacing g there by the shorter resonator

$$g_\sigma(x) = \sum_{n\le R^\sigma} \frac{\cos(2\pi x\sqrt{n})}{\sqrt{n}} \qquad \text{with } 0 < \sigma < 2.$$

We employ the same notation as in that work, introducing

$$I(R) = \int g_\sigma(t) E^*(t)\, d\nu(t),$$

where $d\nu(x) = R^{-1}\psi(x/R)$ is a probability measure with $\psi \in C_0^\infty((1,2))$ and our goal is to prove the following result.

THEOREM 3.1. *Given* $0 < \sigma < 2$, *as* $R \to \infty$,

$$I(R) \sim -\sigma C R \log R,$$

where

$$C = C_0 \int t\psi(t)\, dt \quad \text{and} \quad C_0 = \frac{7}{8} \prod_{p>2} \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{1}{p} - \frac{1}{p^2}\Big).$$

As a matter of fact, in the statement of the main result of [2], formally corresponding to $\sigma \to 2^-$, the constant C_0 was wrongly substituted by $7/\pi^2$ because v as in the lemma below was not correctly evaluated. We apologize for the inconvenience. Both constants differ by less than 4% and it does not affect the Ω-result which was the main interest.

LEMMA 3.2. *Let* $v(d)$ *be the number of solutions of* $x^2 + y^2 + z^2 \equiv 0 \pmod{d^2}$. *For* d *odd squarefree,*

$$v(2d) = 8v(d) \quad \text{and} \quad v(d) = \prod_{p\mid d} p^2(p^2 + p - 1).$$

In particular, $v(d) = O(d^4 \log\log d)$.

Proof. By the Chinese remainder theorem, v is multiplicative. Since $n^2 \equiv 0, 1 \pmod 4$ for every n, depending on its parity, $x^2 + y^2 + z^2 \equiv 0 \pmod 4$ implies $x \equiv y \equiv z \equiv 0 \pmod 2$, showing $v(2) = 2^3$. We have to prove $v(p) = p^2(p^2 + p - 1)$ for primes $p > 2$. Expanding the cube and changing the order of summation, the following exponential sum representation is obtained:

$$p^2 v(p) = \sum_{a=1}^{p^2} \bigg(\sum_{n=1}^{p^2} e\Big(\frac{an^2}{p^2}\Big) \bigg)^3.$$

The innermost sum is p^2 if $a = p^2$. The classical evaluation of quadratic Gauss sums [13, (3.38)] shows that it is p if $p \nmid a$, and $\big(\frac{k}{p}\big) c_p$ for certain $|c_p| = \sqrt{p}$ if $a = kp$ with $1 \le k < p$. Collecting these contributions yields

$$p^2 v(p) = (p^2)^3 + (p^2 - p)p^3 + c_p^3 \sum_{k=1}^{p-1} \Big(\frac{k}{p}\Big).$$

The sum is null, giving $v(p) = p^2(p^2 + p - 1)$ as expected.

The last claim comes from $v(d) < d^4 \prod_{p\mid d}(1 + p^{-1})$ [8, §18.3]. ∎

The motivation to take a shorter sum is to control higher moments of g_σ. This requires some arithmetic considerations about linear combinations of square roots.

LEMMA 3.3. *For each $k\in\mathbb{Z}^+$ and $0<\sigma<2/(2^{2k-1}-1)$ there exists a positive constant $C_{k,\sigma}$ such that*

$$\int\Big|\sum_{n\le R^\sigma}\frac{e(x\sqrt{n})}{\sqrt{n}}\Big|^{2k}\,d\nu(x)\sim C_{k,\sigma}(\log R)^k.$$

In particular, $\int|g_\sigma|^{2k}\,d\nu\ll(\log R)^k$.

Proof. For $\vec{n}\in\mathbb{Z}^{2k}_{>0}$ let $L(\vec{n})=\sum_{j=1}^k(\sqrt{n_j}-\sqrt{n_{j+k}})$. Opening the power gives

$$\int\Big|\sum_{n\le R^\sigma}\frac{e(x\sqrt{n})}{\sqrt{n}}\Big|^{2k}\,d\nu=\sum_{\|\vec{n}\|_\infty\le R^\sigma}\frac{\widehat{\psi}(RL(\vec{n}))}{\sqrt{n_1\cdots n_{2k}}}.$$

By [15, Lemma 2.2], if $L(\vec{n})\ne 0$ then

$$|L(\vec{n})|\gg R^{-\delta}\qquad\text{with}\qquad \delta=(2^{2k-1}-1)\sigma/2<1$$

and $\widehat{\psi}(RL(\vec{n}))\ll R^{-K}$ for any $K>0$, giving a negligible contribution. Hence we have to show

$$\sum_{\substack{\|\vec{n}\|_\infty\le R^\sigma\\ L(\vec{n})=0}}\frac{1}{\sqrt{n_1\cdots n_{2k}}}\sim C_{k,\sigma}(\log R)^k. \tag{3.1}$$

It is clear that the terms with $\{n_1,\dots,n_k\}=\{n_{k+1},\dots,n_{2k}\}$ give the expected asymptotics. We are going to check that the rest of the sum in (3.1) is $O((\log R)^{k-1})$.

Any $n\in\mathbb{Z}^+$ can be decomposed uniquely as $n=s^2m$ with m squarefree and $s\in\mathbb{Z}^+$. A well-known result due to Besicovitch states that the square roots of squarefree numbers are linearly independent over $\mathbb{Q}$. Hence, for each $\vec{n}$ with $L(\vec{n})=0$ there are partitions

$$\bigcup_{i=1}^{\ell}A_i=\{1,\dots,k\}\quad\text{and}\quad\bigcup_{i=1}^{\ell}B_i=\{k+1,\dots,2k\}$$

selecting the coordinates with the same squarefree part, which must cancel the squared parts. In formulas, $n_j=s_j^2m_i$ for every $j\in A_i\cup B_i$ with m_i distinct and $\sum_{j\in A_i}s_j=\sum_{j\in B_i}s_j$.

The case $\ell=k$ corresponds to $\#A_i=\#B_i=1$, hence $A_i=\{\tau(i)\}$, $B_i=\{k+\lambda(i)\}$ for some permutations τ and λ of $\{1,\dots,k\}$. In particular, $n_{\tau(i)}=s^2_{\tau(i)}m_i=n_{k+\lambda(i)}$ and $\{n_1,\dots,n_k\}=\{n_{k+1},\dots,n_{2k}\}$. Consequently, if these sets are not equal then $\ell<k$ and the contribution to the sum (3.1)

is bounded by

$$\sum_{\ell=1}^{k-1} \sum_{\{A_i\}_{i=1}^{\ell}} \sum_{\{B_i\}_{i=1}^{\ell}} \sum \sum_{m_1<\cdots<m_\ell\le R^\sigma} \cdots \sum \frac{1}{m_1\cdots m_\ell} \sum \frac{1}{s_1\cdots s_{2k}}, \tag{3.2}$$

where in the inner sum we have the restrictions $s_j \le \sqrt{R^\sigma/m_i}$ for $j\in A_i\cup B_i$ and $\sum_{j\in A_i} s_j = \sum_{j\in B_i} s_j$. Let us see that this inner sum is bounded. Say that the largest s_j is s_{j_1} with $j_1\in A_{i_1}$ (the case $j_1\in B_{i_1}$ is symmetric). Then $u=\sum_{j\in A_{i_1}} s_j = \sum_{j\in B_{i_1}} s_j < ks_{j_1}$ and the same holds after replacing s_{j_1} by the greatest s_j with $j\in B_{i_1}$. Obviously, any other variable is at most u and we have

$$\sum \frac{1}{s_1\cdots s_{2k}} < k^2 \sum_{u=1}^{\infty} \frac{1}{u^2}\Big(\sum_{s\le u}\frac{1}{s}\Big)^{2k-2} \ll \sum_{u=1}^{\infty} \frac{(\log u)^{2k-2}}{u^2} \ll 1.$$

Then (3.2) is $O((\log R)^{k-1})$. ∎

Proof of Theorem 2.1. Take $k=\lceil p/(2p-2)\rceil$. By Theorem 3.1 and Hölder's inequality,

$$R^p(\log R)^p \ll \Big(\int |g_\sigma E^*|\,d\nu\Big)^p \le \int |E^*|^p\,d\nu\cdot\Big(\int |g_\sigma|^q\,d\nu\Big)^{p/q}.$$

The last factor is at most $(\int |g_\sigma|^{2k}\,d\nu)^{p/2k} \ll (\log R)^{p/2}$ by Lemma 3.3 with σ small enough. ∎

Before entering into the proof of Theorem 3.1, let us recall some notation and results of [2], namely Lemmas 2.1 and 2.2 there.

Let $E(R)$ be the lattice point error for the ball, i.e.,

$$E(R) = \#\{\vec{n}\in\mathbb{Z}^3 : \|\vec{x}\|\le R\} - \tfrac{4}{3}\pi R^3.$$

This quantity is related to E^* through the formula

$$E^*(t) = \sum_{d\le 2R} \mu(d)E(t/d) + o(t) \quad \text{for } 1<t<2R. \tag{3.3}$$

A smoothed Voronoi formula for $E(R)$ is (cf. [13, §4.4])

$$E(t) = -\frac{R}{\pi}\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}\cos\big(2\pi t\sqrt{n}\,\big) + T(t) + U(t), \tag{3.4}$$

where

$$a_n = \frac{r_3(n)}{\sqrt{n}}\widehat{\phi}\Big(\frac{\sqrt{n}}{M}\Big), \quad M=\frac{R}{(\log R)^{1/3}}, \quad \phi\in C_0^\infty((-1,1)) \text{ even}, \ \widehat{\phi}(0)=1,$$

and T and U are less important terms.

Using (3.3) and (3.4), when computing $I(R)$ there appears a term of the form

$$(3.5)\qquad -\frac{1}{\pi}\sum_{d<2R}\sum_{m\le R^\sigma}\sum_{n=1}^{\infty}\frac{\mu(d)a_n}{d\sqrt{mn}}\int t\cos\big(2\pi t\sqrt{m}\,\big)\cos\Big(2\pi\frac{t}{d}\sqrt{n}\Big)\,d\nu(t).$$

In [2, Propositions 3.1–3.3] it is proved that the terms with $\sqrt{m}\neq\sqrt{n}/d$ as well as those coming from the integration of T and U contribute $O\big(R(\log R)^{5/6}\big)$ when R^σ is replaced by M^2 in the definition of g_σ. The important point is that these terms are estimated in absolute value and then the bound still holds with our g_σ because $\sigma<2$ ensures $R^\sigma=o(M^2)$ and there are fewer terms.

Proof of Theorem 3.1. After the previous comments, we have to prove that the terms in (3.5) with $\sqrt{m}=\sqrt{n}/d$ contribute $-\sigma CR\log R$ asymptotically. For them the integral in (3.5) is

$$\int t\cos^2\big(2\pi t\sqrt{m}\,\big)\,d\nu(t)=\frac{1}{2}R\int_1^2 t\psi(t)\,dt+\frac{R}{2}\int_1^2 t\psi(t)\cos\big(4\pi Rt\sqrt{m}\,\big)\,dt.$$

By repeated partial integration, the last integral decays faster than any negative power of R. Then the result is deduced if we prove that

$$(3.6)\qquad M_\sigma(R)\sim 2\pi C_0\sigma\log R\quad\text{with}\quad M_\sigma(R)=\sum_{d<2R}\sum_{\substack{m\le R^\sigma\\ d\sqrt{m}=\sqrt{n}}}\sum_{n=1}^{\infty}\frac{\mu(d)a_n}{d\sqrt{mn}}.$$

In [2], the range of m and the range of n in which a_n is not negligible were balanced and d was essentially only subject to $d^2\mid n$. In this situation, the identity $\sum_{d^2|n}\mu(d)=\mu^2(n)$ and its positivity play an important role. Now, the ranges of m and n are unbalanced and small values of d are forbidden for n large, ruining the application of the exact identity. This forces one to take a roundabout way with similar ingredients.

Substituting $n=md^2$ and the definition of a_n we get

$$M_\sigma(R)=\sum_{d<2R}\frac{\mu(d)}{d^6}\sum_{m\le R^\sigma}r_3(md^2)f_d(m)\quad\text{with}\quad f_d(x)=\Big(\frac{d}{\sqrt{x}}\Big)^3\widehat{\phi}\Big(\frac{d\sqrt{x}}{M}\Big).$$

The properties of the Hecke operators [9, §7] give, for d squarefree,

$$r_3(md^2)\le r_3(m)\prod_{p|d}(p+2)<r_3(m)d\prod_{p|d}(1+p^{-1})^2\ll r_3(m)(\log\log d)^2d.$$

Then the contribution to $M_\sigma(R)$ of $m\le d^2$ is bounded by

$$\sum_{d<2R}\frac{1}{d^6}\sum_{m\le d^2}\frac{r_3(m)(\log\log d)^2d^4}{m^{3/2}}\ll\sum_{d<2R}\frac{(\log\log d)^2}{d^2}\log d\ll 1,$$

because $\sum_{m\le N} r_3(m) \ll N^{3/2}$. Consequently, we can restrict the sum in $M_\sigma(R)$ to $d^2 < m \le R^\sigma$, losing $O(1)$.

On the other hand, Gauss' elementary geometric argument to count lattice points [10] (cf. [2, §2]) applied to the lattice $(d^2\mathbb{Z})^3$ shows that for $N \ge d^2$

$$R_d(N) = \frac{4\pi}{3}\left(\frac{N}{d^2}\right)^{3/2} + O\left(\frac{N}{d^2}\right), \quad \text{where} \quad R_d(N) = \frac{1}{v(d)} \sum_{0\le m\le N} r_3(md^2),$$

with $v(d)$ the number of solutions of $x^2+y^2+z^2 \equiv 0 \pmod{d^2}$.

By Abel's summation formula,

$$\sum_{d^2<m\le R^\sigma} f_d(m)\frac{r_3(md^2)}{v(d)} = R_d(R^\sigma)f_d(R^\sigma) - R_d(d^2)f_d(d^2) - \int_{d^2}^{R^\sigma} R_d(t)f_d'(t)\,dt.$$

It is easy to see $f_d(t)$, $tf_d'(t) \ll d^3t^{-3/2}$, because $\widehat{\phi}(z)$ and $z\widehat{\phi}'(z)$ are bounded (and rapidly decreasing). This implies that $R_d(t)f_d(t)$ and $d^{-2}\int_{d^2}^{R^\sigma} t|f_d'(t)|\,dt$ are $O(1)$. Recalling the last part of Lemma 3.2, we get

$$M_\sigma(R) = -\frac{4\pi}{3}\sum_{d<R^{\sigma/2}} \frac{\mu(d)v(d)}{d^9}\int_{d^2}^{R^\sigma} t^{3/2}f_d'(t)\,dt + O(1).$$

The range $d < R^{\sigma/2}$ is forced by the previous restriction to $d^2 < m \le R^\sigma$. Integrating by parts shows that the above equals

$$2\pi\sum_{d<R^{\sigma/2}} \frac{\mu(d)v(d)}{d^9}\int_{d^2}^{R^\sigma} t^{1/2}f_d(t)\,dt + O(1).$$

Unwrapping the definition of $f_d(t)$ and introducing

$$s_d = 2\pi\chi(d)\int_{d^2}^{R^\sigma} t^{-1}\widehat{\phi}\left(\frac{d\sqrt{t}}{M}\right)dt - 2\pi\sigma\log R$$

with χ the characteristic function of $[1, R^{\sigma/2}]$, we have

$$M_\sigma(R) = 2\pi\sigma(\log R)\sum_{d=1}^{\infty}\frac{\mu(d)v(d)}{d^6} + \sum_{d=1}^{\infty}\frac{\mu(d)v(d)}{d^6}s_d + O(1).$$

The first sum is, by Lemma 3.2,

$$\prod_p\left(1-\frac{v(p)}{p^6}\right) = \left(1-\frac{8}{2^6}\right)\prod_p\left(1-\frac{p^2+p-1}{p^4}\right) = C_0.$$

It remains to check that the second sum is negligible. Choose

$$\delta = \tfrac{1}{4}\min(\sigma, 2-\sigma).$$

Recall that $\widehat{\phi}$ is regular, even and $\widehat{\phi}(0)=1$, so $\widehat{\phi}(x)=1+O(x^2)$. Substituting this in s_d, we get

$$s_d \ll \log d + \int_{d^2}^{R^\sigma} \frac{d^2}{M^2}\,dt \ll \log d \qquad \text{for } d \le R^\delta,$$

giving a bounded contribution to the sum. For $d \ge R^\delta$ we use the trivial bound $s_d = O(\log R)$ and Lemma 3.2 to get

$$(\log R)\sum_{d>R^\delta} \frac{\log\log d}{d^2} \ll \frac{\log\log R}{R^\delta}\log R = o(1).$$

Summing up, $M_\sigma(R) = 2\pi\sigma C_0 \log R + O(1)$, which is a strong form of the required formula (3.6). ∎

The resonator giving Theorem 2.2 is a variant of the Fejér kernel and the gaps between the frequencies ensure that we can capture one of them without interferences, obtaining the expected correlation. Namely, we choose

$$g(x) = \frac{e(-\nu_n x)}{\Lambda_n}\Big(\frac{\sin(\pi\Lambda_n x)}{\sin(\pi x)}\Big)^2 = \sum_{k=-\Lambda_n}^{\Lambda_n}\Big(1-\frac{|k|}{\Lambda_n}\Big)e((k-\nu_n)x).$$

This idea appeared in a different context in [3, Prop. 3.3].

Proof of Theorem 2.2. Let S be the sine sum in the statement. Since

$$e(x)+e(-x) = 2i\sin(2\pi x)$$

and $\Lambda_n \ge |\nu_n-\nu_m|$ for $m \ne n$, we have

$$\int_{-1/2}^{1/2} S(t)g(t)\,dt = \frac{1}{2}a_n, \qquad \text{where } g \text{ is as above.}$$

Hence for x in a positive measure subset of $[-1/2,1/2]$ we have

$$|x|^{-\alpha}|S(x)| \int_{-1/2}^{1/2} |t|^\alpha |g(t)|\,dt \ge \frac{1}{2}|a_n|$$

and the result follows if we check that (note $|g|$ is even)

$$\mathcal{B}_{n,\alpha}\int_0^{1/2} t^\alpha \Lambda_n |g(t)|\,dt < \Lambda_n. \tag{3.7}$$

Let I_1 and I_2 be the contributions to the integral of $t \le (\pi\Lambda_n)^{-1}$ and $t \ge (\pi\Lambda_n)^{-1}$. In I_1 we use the trivial bound $\Lambda_n|g(t)| \le \Lambda_n^2$, and in I_2, $\Lambda_n|g(t)| < \csc^2(\pi t) \le (\pi t)^{-2}+0.6$ because $\csc^2 t - t^{-2}$ is increasing in $(0,\pi/2]$.

Substituting these bounds gives

$$I_1+I_2<\frac{1}{\pi^{\alpha+1}}\left(\frac{2\Lambda^{1-\alpha}}{1-\alpha^2}-\frac{(2/\pi)^{1-\alpha}}{1-\alpha}\right)+0.3<\frac{(\pi\Lambda)^{1-\alpha}}{1-\alpha^2}-\frac{2^{-\alpha}}{1-\alpha}+0.3$$

for $\alpha<1$ and the limit $\alpha\to 1^+$ makes sense. The function

$$f(\alpha)=2^{-\alpha}+0.3(\alpha-1)-(1+\alpha)^{-1}(\pi/5)^{1-\alpha}$$

is decreasing in $[0,1]$ (a tedious proof consists in subdividing the interval into a number of pieces and using trivial bounds to get $f'<0$ in each of them). Then $f(\alpha)\ge f(1)=0$ and we can add $f(\alpha)/(1-\alpha)$ to the previous bound for I_1+I_2 to obtain

$$I_1+I_2<\frac{\Lambda_n^{1-\alpha}-(1/5)^{1-\alpha}}{(1-\alpha^2)\pi^{\alpha-1}},$$

and (3.7) follows. ■

Proof of Theorem 2.3. We take as resonator the shifted Dirichlet kernel

$$g(t)=\sum_{n=1}^{N}e(-nt).$$

Consider a probability measure $d\mu=R^{-1}\psi(x/R)\,dx$ with $\psi(x)\neq 0$ for $x\in[1,2]$, $\psi\in C_0^\infty((1/2,5/2))$. We are going to prove that

$$\int Gg\,d\mu\sim\mathcal{B}. \tag{3.8}$$

Then Cauchy's inequality and $\int|g|^2\,d\mu\ll N$ imply $\int|G|^2\,d\mu\gg\mathcal{B}^2/N$, giving the result.

Substituting the definitions of G and g and changing the variable $t=Rx$ shows that the integral (3.8) is

$$I=\sum_{m=1}^{N}\sum_{n=1}^{N}\sum_{d=1}^{\infty}f(d)a_n\int\alpha_d(x)e(R(n/d-m)x)\,dx$$

with $\alpha_d(x)=\phi(Rx/d)\psi(x)$.

The terms with $m=n/d$ contribute

$$\sum_{n=1}^{N}a_n\sum_{d|n}f(d)\int\alpha_d(x)\,dx=\mathcal{B}$$

because $R\ge 4N$ ensures $\alpha_d=\psi$ for $d<R/4$ and $\int\psi=1$.

On the other hand, if $d>3R$ then α_d is identically zero. Hence, for any d, by partial integration, $\widehat{\alpha}_d(\xi)\ll(1+|\xi|)^{-2K}$. Then the contribution of the terms with $m\neq n/d$ is bounded by

$$\sum_{d\le 3R}\sum_{\substack{m=1\\ n/d\ne m}}^{N}\sum_{n=1}^{N}|a_n|\Big(1+R\Big|\frac{n}{d}-m\Big|\Big)^{-2K}$$
$$=\sum_{\substack{j=-\infty\\ V=2^j}}^{\infty}\ \sum_{d\le 3R}\ \sum_{m=1}^{N}\ \sum_{\substack{n=1\\ dV/R\le|n-md|<2dV/R}}^{N}|a_n|(1+V)^{-2K}.$$

Write $r=|n-md|$. Then $n\equiv\pm r \pmod d$ and given n, r and d, the value of m remains determined. Note that the sum is empty if $d\le R/(2V)$, which forces $V>1/6$. So, we can write the sum as

$$\sum_{\substack{V=2^j\\ V>1/6}}^{\infty}\ \sum_{R/(2V)\le d\le 3R}\ \sum_{dV/R\le r<2dV/R}\ \sum_{\substack{n=1\\ n\equiv\pm r\,(d)}}^{N}|a_n|(1+V)^{-2K}.$$

Using that $(6V)^K/(1+V)^{2K}$ is bounded, the assumption in the statement ensures that this is $o(\mathcal{B})$ proving (3.8). ■

Acknowledgements. I am deeply indebted to Professor Iwaniec for his decisive help along my career and for his outstanding mathematical contributions, his books and his beautiful lecture notes that are a gift for the number-theoretical community. I also want to take the opportunity to thank E. Valenti. I am grateful to A. Ubis for pointing out some corrections.

This work was partially supported by the PID2020-113350GB-I00 grant of the MICINN (Spain), by "Severo Ochoa Programme for Centres of Excellence in R&D" (CEX2019-000904-S) and by the Madrid Government agreement with UAM in the context of the V PRICIT.

Added in proof. I have known that there is a "resonance method" introduced by Professor Soundararajan in his paper *Extreme values of zeta and L-functions* in which a device to prove Ω-results is also called a "resonator". I apologize for the unintentional clash of notation and for being unaware of the method. Although the resonators in the aforementioned paper and here are in the same vein, there are some differences, the former being based on a more elaborate theory. Here, they are not assumed to be positive, in fact they are already oscillatory in [2] (which was published a little earlier than the "resonance method"), and they do not involve any optimization.

References

[1] V. Blomer and R. Holowinsky. *Bounding sup-norms of cusp forms of large level*, Invent. Math. 179 (2010), 645–681.

[2] F. Chamizo, E. Cristóbal, and A. Ubis, *Visible lattice points in the sphere*, J. Number Theory 126 (2007), 200–211.

[3] F. Chamizo and A. Ubis, *Some Fourier series with gaps*, J. Anal. Math. 101 (2007), 179–197.

[4] A. Córdoba, J. L. Fernández, and P. Fernández (eds.), *All That Math: Portraits of Mathematicians as Young Readers*, Rev. Mat. Iberoamer., Madrid, 2011.

[5] W. Duke, J. B. Friedlander, and H. Iwaniec, *Bounds for automorphic L-functions. II*, Invent. Math. 115 (1994), 219–239.

[6] J. Friedlander and H. Iwaniec, *A mean-value theorem for character sums*, Michigan Math. J. 39 (1992), 153–159.

[7] J. B. Friedlander and H. Iwaniec, *A note on character sums*, in: The Rademacher Legacy to Mathematics (University Park, PA, 1992), Contemp. Math. 166, Amer. Math. Soc., Providence, RI, 1994, 295–299.

[8] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford Univ. Press, Oxford, 2008.

[9] M. D. Hirschhorn and J. A. Sellers, *On representations of a number as a sum of three squares*, Discrete Math. 199 (1999), 85–101.

[10] A. Ivić, E. Krätzel, M. Kühleitner, and W. G. Nowak, *Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic*, in: Elementare und analytische Zahlentheorie, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006. 89–128.

[11] H. Iwaniec, *The spectral growth of automorphic L-functions*, J. Reine Angew. Math. 428 (1992), 139–159.

[12] H. Iwaniec, *Harmonic analysis in number theory*, in: Prospects in Mathematics (Princeton, NJ, 1996), Amer. Math. Soc., Providence, RI, 1999, 51–68.

[13] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. 53, Amer. Math. Soc., Providence, RI, 2004.

[14] H. Iwaniec and P. Sarnak, *L^∞ norms of eigenfunctions of arithmetic surfaces*, Ann. of Math. (2) 141 (1995), 301–320.

[15] W. Zhai, *On higher-power moments of $\Delta(x)$. II*, Acta Arith. 114 (2004), 35–54.

Fernando Chamizo
Departamento de Matemáticas and ICMAT
Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: fernando.chamizo@uam.es