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On d-complete sequences modulo l

by

XING-WANG JIANG (Luoyang) and BING-LING WU (Nanjing)

Abstract. A sequence $\mathcal{T}$ of positive integers is called *d-complete modulo l* if for every integer $0 \le u \le l-1$, there exists an integer v with $vl+u>0$ such that $vl+u$ can be represented as the sum of distinct terms from $\mathcal{T}$, where no one divides any other. Recently, Chen and Yu (2023) proved that $\{m^a n^b : a, b = 0, 1, 2, \ldots\}$ is d-complete modulo l if l, m, n are pairwise coprime with $l, m, n \ge 2$, and posed the following problem: characterize all positive integers l, m, n such that $\{m^a n^b : a, b = 0, 1, 2, \ldots\}$ is d-complete modulo l. We give an answer to this problem.

1. Introduction. Let $\mathbb{N}_0$ be the set of all non-negative integers. A sequence $\mathcal{T}$ of positive integers is called *complete* if every sufficiently large integer can be represented as the sum of distinct terms from $\mathcal{T}$. It is easy to see that the sequence $\{2^a : a \in \mathbb{N}_0\}$ is complete and for any integer $m > 2$, the sequence $\{m^a : a \in \mathbb{N}_0\}$ is not complete. In 1959, Birch [1] proved that for two coprime integers $m > n > 1$, the sequence $\{m^a n^b : a, b \in \mathbb{N}_0\}$ is complete, which confirmed a conjecture of Erdős. It is interesting to study whether $\{m^a n^b : a, b \in \mathbb{N}_0\}$ is still complete or not with the additional restriction that no summand divides any other. Erdős asked the following question: "Is it true that every integer > 1 is the sum of distinct integers of the form $2^a 3^b$ (for a and b non-negative integers), where no summand divides any other?" He overestimated the difficulty of the problem and communicated it to Jansen, who almost immediately gave a simple proof by induction. This motivated the research on *d-complete* sequences, introduced by Erdős and Lewin [6].

A positive integer n is called *d-representable* for $\mathcal{T}$ if it can be represented as the sum of distinct terms from $\mathcal{T}$ such that no one divides any other. A sequence $\mathcal{T}$ of positive integers is called *d-complete* if every sufficiently large integer is d-representable for $\mathcal{T}$. For convenience, we use the following

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notation introduced by Chen and Yu [5]. For positive integers $n_1, \dots, n_k$, let

$$A(n_1, \dots, n_k) = \{n_1^{c_1} \cdots n_k^{c_k} : c_1, \dots, c_k \in \mathbb{N}_0\}.$$

In 1996, Erdős and Lewin [6] reproduced the proof of the d-completeness of $A(2, 3)$ and proved that the sequence $A(m, n)$ is not d-complete if $m > n > 1$ and $\{m, n\} \neq \{2, 3\}$. It is natural to consider the d-completeness of the sequence $A(l, m, n)$. Erdős and Lewin [6] showed that $A(2, 5, n)$ is d-complete for $n \in \{7, 11, 13, 17, 19\}$ and $A(3, 5, 7)$ is d-complete. In 2016, Ma and Chen [11] established a criterion for the d-completeness of $A(2, 5, n)$ and proved that it is d-complete for $n \in \{9, 21, 23, 27, 29, 31\}$.

Erdős and Lewin [6] conjectured that $A(l, m, n)$ is d-complete if l, m, n are pairwise coprime integers not less than 2. Recently, Chen and Yu [5] considered this conjecture. Let r_h be the least positive integer that is d-representable for $A(m, n)$ and congruent to h modulo l, and let s_h be the least positive integer that can be a term in a d-representation for $A(m, n)$ of r_h. Chen and Yu [5] gave the following criterion for the d-completeness of $A(l, m, n)$.

THEOREM A ([5, Theorem 1.1]). *Let l, m, n be pairwise coprime integers not less than 2, let t be a positive integer, and let*

$$\{a_1 < a_2 < \cdots\} = \{m^b n^c : b, c \in \mathbb{N}_0,\ m^b n^c \equiv 1 \pmod{l}\}.$$

(i) *There exists an explicit integer $i_0 = i(l, m, n, t)$ such that*

$$r_h a_{i+1} + lt < (r_h + l s_h) a_i$$

for all $i \geq i_0$ and all $1 \leq h \leq l - 1$.

(ii) *If every integer k with*

$$t < k \leq R a_{i_0} + lt$$

is d-representable for $A(l, m, n)$, where $R = \max\{r_h : 1 \leq h \leq l - 1\}$, then $A(l, m, n)$ is d-complete.

As applications of Theorem A, Chen and Yu [5] showed that $A(2, 5, n)$ is d-complete for $1 \leq n \leq 87$ with $\gcd(n, 10) = 1$, $A(2, 7, n)$ is d-complete for $1 \leq n \leq 33$ with $\gcd(n, 14) = 1$, and $A(3, 5, n)$ is d-complete for $1 \leq n \leq 14$ with $\gcd(n, 15) = 1$. For more related results, one may refer to [1–4, 6–10, 12, 13].

Chen and Yu [5] also considered *d-complete sequences modulo l*.

DEFINITION 1.1. A sequence $\mathcal{T}$ of positive integers is called *d-complete modulo l* if for every integer $0 \leq u \leq l - 1$, there exists an integer v with $vl + u > 0$ such that $vl + u$ is d-representable for $\mathcal{T}$.

It is easy to see that a sequence $\mathcal{T}$ of positive integers is d-complete modulo l if and only if for every integer $0 \leq u \leq l - 1$, u is congruent modulo l to a sum of distinct terms from $\mathcal{T}$ such that no one divides any other. Chen and Yu [5] proved the following results.

Theorem B ([5, Theorem 5.2]). *Suppose that* $\{2,3\} \not\subseteq \{l,m,n\}$. *If* $A(l,m,n)$ *is d-complete, then* $A(m,n)$ *is d-complete modulo* l.

Theorem C ([5, Theorem 5.3]). *If* l,m,n *are pairwise coprime with* $l,m,n \geq 2$, *then* $A(m,n)$ *is d-complete modulo* l.

Chen and Yu [5] posed the following problem:

Problem ([5, Problem 5.4]). *Characterize all positive integers* l,m,n *such that* $A(m,n)$ *is d-complete modulo* l.

In this paper, we solve this problem and prove the following result.

Theorem 1.2. *Let* l,m,n *be three integers with* $l,m,n \geq 2$. *Then* $A(m,n)$ *is d-complete modulo* l *if and only if at least one of the following conditions holds:*

(1) $\gcd(l,mn) = 1$, $m \neq n^{\alpha}$ *for any rational number* α;
(2) $\gcd(l,m) = 1$, $\gcd(l,n)$ *is a prime and* m *is a primitive root of* $\gcd(l,n)$;
(3) $\gcd(l,n) = 1$, $\gcd(l,m)$ *is a prime and* n *is a primitive root of* $\gcd(l,m)$;
(4) $\gcd(l,m)$ *and* $\gcd(l,n)$ *are distinct primes, and* m,n *are primitive roots of* $\gcd(l,n)$ *and* $\gcd(l,m)$, *respectively.*

Remark 1.3. It is easy to see that

(1) for any positive integers m,n, $A(m,n)$ is d-complete modulo 1;
(2) for $l \geq 2$, neither $A(m,1)$ nor $A(1,n)$ is d-complete modulo l.

The proof of Theorem 1.2 proceeds by applying the three lemmas proved in Section 2. Condition (1) of Theorem 1.2 follows from Lemma 2.1. If $\gcd(l,m) = 1$ and $\gcd(l,n) > 1$, we point out that $\gcd(l,n) = p$ is prime when $A(m,n)$ is d-complete modulo l. Let $l = l_1 p^r$ with $\gcd(l_1,p) = 1$. Then $\gcd(l_1,mn) = 1$. The arguments for the d-completeness modulo l_1 and modulo p^r of $A(m,n)$ are given in Lemmas 2.1 and 2.2, respectively. Combining this with Lemma 2.3, we obtain condition (2). Conditions (3) and (4) can be obtained by a similar discussion.

2. Proof of Theorem 1.2. First, we prove some lemmas which will be used to prove Theorem 1.2.

Lemma 2.1. *Let* $l,m,n \geq 2$ *be integers with* $\gcd(mn,l) = 1$. *Then* $A(m,n)$ *is d-complete modulo* l *if and only if* $m \neq n^{\alpha}$ *for any rational number* α.

Proof. Firstly, we prove the necessity. Since $A(m,n)$ is d-complete modulo l, there exist non-negative integers a_i and b_i such that

$$\sum_{i=1}^{r} m^{a_i}n^{b_i} \equiv 0 \pmod{l}$$

with

$$m^{a_i}n^{b_i} \nmid m^{a_j}n^{b_j}, \quad i \neq j. \tag{2.1}$$

By $\gcd(mn,l)=1$, we have $r \geq 2$. Let

$$m = p_1^{\alpha_1}\cdots p_s^{\alpha_s}, \quad n = p_1^{\beta_1}\cdots p_s^{\beta_s},$$

where $\alpha_i, \beta_i \geq 0$ $(1 \leq i \leq s)$. If $m = n^{b/a}$ for some positive integers a, b with $\gcd(a,b)=1$, then $\alpha_i = \beta_i \cdot \frac{b}{a}$ which implies that $a \mid \beta_i$. Since $\alpha_i a_1 + \beta_i b_1 = \frac{\beta_i}{a}(ba_1 + ab_1)$ and $\alpha_i a_2 + \beta_i b_2 = \frac{\beta_i}{a}(ba_2 + ab_2)$, we have

$$p_i^{\alpha_i a_1+\beta_i b_1} \mid p_i^{\alpha_i a_2+\beta_i b_2} (1 \leq i \leq s) \quad \text{or} \quad p_i^{\alpha_i a_2+\beta_i b_2} \mid p_i^{\alpha_i a_1+\beta_i b_1} \quad (1 \leq i \leq s).$$

It follows that

$$m^{a_1}n^{b_1} \mid m^{a_2}n^{b_2} \quad \text{or} \quad m^{a_2}n^{b_2} \mid m^{a_1}n^{b_1},$$

a contradiction with (2.1). Therefore, $m \neq n^{\alpha}$ for any rational number α.

Now, we prove the sufficiency. By Theorem C, it suffices to deal with the case $\gcd(m,n) > 1$. Let

$$m = p_1^{\alpha_1}\cdots p_s^{\alpha_s}, \quad n = p_1^{\beta_1}\cdots p_s^{\beta_s},$$

where $\alpha_i, \beta_i \geq 0$ $(1 \leq i \leq s)$. Since $m \neq n^{\alpha}$ for any rational number α, it follows from $\gcd(m,n) > 1$ that either m or n has at least two prime divisors and there are two integers $1 \leq i_1, i_2 \leq s$ with $\alpha_{i_1}/\beta_{i_1} \neq \alpha_{i_2}/\beta_{i_2}$. Without loss of generality, we may assume that $\alpha_1/\beta_1 > \alpha_2/\beta_2$, where $\beta_1, \beta_2 \geq 1$. Then there exists an irreducible fraction d/c such that

$$\frac{\alpha_1}{\beta_1} > \frac{d}{c} > \frac{\alpha_2}{\beta_2}. \tag{2.2}$$

By Euler's theorem, for any integer $1 \leq u \leq l$,

$$\sum_{i=1}^{u} m^{(u-i)c\varphi(l)}n^{id\varphi(l)} \equiv u \pmod{l}.$$

Now, we shall show that

$$m^{(u-i)c\varphi(l)}n^{id\varphi(l)} \nmid m^{(u-j)c\varphi(l)}n^{jd\varphi(l)}, \quad i \neq j. \tag{2.3}$$

Express m, n as $m = p_1^{\alpha_1}p_2^{\alpha_2}m_1$, $n = p_1^{\beta_1}p_2^{\beta_2}n_1$. Then

$$m^{(u-i)c\varphi(l)}n^{id\varphi(l)} = p_1^{\varphi(l)(\alpha_1(u-i)c+\beta_1 id)}p_2^{\varphi(l)(\alpha_2(u-i)c+\beta_2 id)}m_1^{(u-i)c\varphi(l)}n_1^{id\varphi(l)},$$

$$m^{(u-j)c\varphi(l)}n^{jd\varphi(l)} = p_1^{\varphi(l)(\alpha_1(u-j)c+\beta_1 jd)}p_2^{\varphi(l)(\alpha_2(u-j)c+\beta_2 jd)}m_1^{(u-j)c\varphi(l)}n_1^{jd\varphi(l)}.$$

By (2.2), when $i < j$,

$$\alpha_1(u-i)c+\beta_1 id > \alpha_1(u-j)c+\beta_1 jd, \qquad \alpha_2(u-i)c+\beta_2 id < \alpha_2(u-j)c+\beta_2 jd,$$

from which one can immediately get (2.3). Therefore, $A(m,n)$ is d-complete modulo l. ∎

Lemma 2.2. *Let m, n be integers with $m, n \geq 2$ and p be a prime with $p \mid n$. Then*

(i) *$A(m,n)$ is d-complete modulo p if and only if m is a primitive root of p;*
(ii) *when $r \geq 2$, $A(m,n)$ is d-complete modulo p^r if and only if m is a primitive root of p and $p^2 \nmid n$.*

Proof. First, we prove the necessity of (i) and (ii). Obviously, if $A(m,n)$ is d-complete modulo p^r, then it is also d-complete modulo p^i $(1 \leq i \leq r)$. It follows from $p \mid n$ that for any integer $1 \leq u \leq p-1$, there exists a non-negative integer a_u with

$$m^{a_u} \equiv u \pmod{p},$$

which shows that m is a primitive root of p.

Suppose that $p^2 \mid n$ when $r \geq 2$. Then since $A(m,n)$ is d-complete modulo p^2,

$$m^{a_p} \equiv p \pmod{p^2}$$

for some positive integer a_p, and so $p \mid m$, which is impossible since m is a primitive root of p. Thus, $p^2 \nmid n$ when $r \geq 2$.

Now, we prove the sufficiency of (i) and (ii). Since m is a primitive root of p, for every integer $1 \leq u \leq p-1$ there is a non-negative integer a_u with

$$m^{a_u} \equiv u \pmod{p}.$$

It follows from $n \equiv 0 \pmod{p}$ that $A(m,n)$ is d-complete modulo p. Hence, Lemma 2.2(i) holds.

Next, we assume that $r \geq 2$, and so $p^2 \nmid n$. Let $n = pn_1$ with $\gcd(p, n_1) = 1$. We shall use induction to prove that $A(m,n)$ is d-complete modulo p^r.

By the above argument, $A(m,n)$ is d-complete modulo p. Suppose that $A(m,n)$ is d-complete modulo p^s; we will prove that it is d-complete modulo p^{s+1}.

For an integer $0 \leq u \leq p^{s+1}-1$, u can be written as

$$u = vp^s + w$$

with $0 \leq v \leq p-1$ and $0 \leq w \leq p^s - 1$. Clearly, $n^{s+1} \equiv 0 \pmod{p^{s+1}}$. Now, we deal with the case $u \geq 1$, that is, either $v > 0$ or $w > 0$. Since $A(m,n)$ is d-complete modulo p^s, there exist non-negative integers a_i and b_i such that

$$\sum_{i=1}^{t_w} m^{a_i} n^{b_i} \equiv w \pmod{p^s}$$

with

$$m^{a_i}n^{b_i} \nmid m^{a_j}n^{b_j}, \quad 1 \le i \ne j \le t_w. \tag{2.4}$$

Here, we define $t_0 = 0$ and $\sum_{i=1}^{0} m^{a_i}n^{b_i} = 0$, and for $w \ge 0$, we may require that

$$a_i > p + n^s \quad \text{and} \quad 0 \le b_i \le s-1 \tag{2.5}$$

since $p \mid n$ and $m^{a_i+k\varphi(p^s)} \equiv m^{a_i} \pmod{p^s}$. Let

$$\sum_{i=1}^{t_w} m^{a_i}n^{b_i} = v'p^s + w.$$

If $v' \equiv v \pmod{p}$, then

$$\sum_{i=1}^{t_w} m^{a_i}n^{b_i} \equiv vp^s + w \pmod{p^{s+1}}$$

with $m^{a_i}n^{b_i} \nmid m^{a_j}n^{b_j}$ $(i \ne j)$. If $v' \not\equiv v \pmod{p}$, then by the d-completeness modulo p of $A(m,n)$, there exists an integer $0 \le a_{t_w+1} < p$ such that

$$m^{a_{t_w+1}} \equiv (v - v')\bar{n}_1^s \pmod{p},$$

where $n_1\bar{n}_1 \equiv 1 \pmod{p^{s+1}}$ (such an $\bar{n}_1$ exists since $\gcd(n_1, p) = 1$). Thus

$$\sum_{i=1}^{t_w} m^{a_i}n^{b_i} + m^{a_{t_w+1}}n^s \equiv v'p^s + w + (v - v')p^s n_1^s \bar{n}_1^s \equiv vp^s + w \pmod{p^{s+1}}.$$

By (2.5) and $a_{t_w+1} < p$,

$$m^{a_{t_w+1}}n^s \nmid m^{a_i}n^{b_i}, \quad m^{a_i}n^{b_i} \nmid m^{a_{t_w+1}}n^s, \quad 1 \le i \le t_w.$$

It follows from (2.4) that

$$m^{a_i}n^{b_i} \nmid m^{a_j}n^{b_j}, \quad 1 \le i \ne j \le t_w + 1,$$

where $b_{t_w+1} = s$. Hence $A(m,n)$ is d-complete modulo p^{s+1}. Therefore, Lemma 2.2(ii) holds. ∎

Lemma 2.3. *Let r, l, m, n be integers with $r \ge 1$, $l, m, n \ge 2$, $\gcd(l, n) = 1$ and p be a prime with $p \mid n$. If $A(m,n)$ is d-complete both modulo l and modulo p^r, then $A(m,n)$ is d-complete modulo lp^r.*

Proof. The proof is by induction on r. First, we prove that Lemma 2.3 is true for $r = 1$.

For an integer $0 \le u \le lp - 1$, u can be written as

$$u = vp + w$$

with $0 \le v \le l-1$ and $0 \le w \le p-1$. Since $A(m,n)$ is d-complete modulo p, we have $p \nmid m$ and for $1 \le w \le p-1$, there exists a sufficiently large integer $a_1 = a_1(w)$ such that

$$m^{a_1} \equiv w \pmod{p}.$$

Define $I_w = m^{a_1}$ if $1 \le w \le p-1$ and $I_w = 0$ if $w = 0$. Let $I_w = v'p + w$ and $n = n_1 p$. Noting that $A(m,n)$ is d-complete modulo l and $\gcd(l,n) = 1$, there exist non-negative integers a_i, b_i $(i \ge 2)$ such that

$$\sum_{i=2}^{t} m^{a_i} n^{b_i} \equiv (v - v')\bar{n}_1 \pmod{l},$$

where $\bar{n}_1 n_1 \equiv 1 \pmod{l}$ (such an $\bar{n}_1$ exists since $\gcd(n_1, l) = 1$) and

$$m^{a_i} n^{b_i} \nmid m^{a_j} n^{b_j}, \qquad 2 \le i \ne j \le t. \tag{2.6}$$

Hence

$$\sum_{i=2}^{t} m^{a_i} n^{b_i+1} \equiv (v - v')\bar{n}_1 n_1 p \equiv (v - v')p \pmod{l},$$

and so

$$I_w + \sum_{i=2}^{t} m^{a_i} n^{b_i+1} \equiv (v - v')p + v'p + w = vp + w \pmod{l}.$$

In view of $p \,|\, n$ and $I_w \equiv w \pmod{p}$,

$$I_w + \sum_{i=2}^{t} m^{a_i} n^{b_i+1} \equiv w \equiv vp + w \pmod{p}.$$

Since $\gcd(l,p) = 1$, it follows that

$$I_w + \sum_{i=2}^{t} m^{a_i} n^{b_i+1} \equiv vp + w \pmod{lp}.$$

By (2.6),

$$m^{a_i} n^{b_i+1} \nmid m^{a_j} n^{b_j+1}, \qquad 2 \le i \ne j \le t.$$

In addition, for $1 \le w \le p-1$, we have both $m^{a_i} n^{b_i+1} \nmid m^{a_1}$ and $m^{a_1} \nmid m^{a_i} n^{b_i+1}$ $(2 \le i \le t)$ since $p \,|\, n$, $p \nmid m$ and a_1 is sufficiently large. Hence, $A(m,n)$ is d-complete modulo lp. Thus, the conclusion of Lemma 2.3 is true for $r = 1$.

Now, we assume that $r \ge 2$ and Lemma 2.3 holds for $r - 1$. We shall prove that Lemma 2.3 holds for r. The proof is similar to that for $r = 1$.

For an integer $0 \le u \le lp^r - 1$, u can be written as

$$u = vp + w,$$

with $0 \le v \le lp^{r-1} - 1$ and $0 \le w \le p - 1$. Note that $A(m,n)$ is d-complete modulo p^r, so it is d-complete modulo p^s $(1 \le s \le r)$. Thus, for $1 \le w \le p-1$, there exists a sufficiently large integer a_1 such that

$$m^{a_1} \equiv w \pmod{p}.$$

Define $I_w = m^{a_1}$ if $1 \le w \le p-1$ and $I_w = 0$ if $w = 0$. Let $I_w = v'p + w$. By Lemma 2.2, $p^2 \nmid n$. We can express n as $n = n_1 p$ with $\gcd(p, n_1) = 1$. By inductive hypothesis, $A(m,n)$ is d-complete modulo lp^{r-1}. Hence, there exist non-negative integers a_i, b_i $(i \ge 2)$ such that

$$\sum_{i=2}^{t} m^{a_i} n^{b_i} \equiv (v - v')\bar{n}_1 \pmod{lp^{r-1}},$$

where $\bar{n}_1 n_1 \equiv 1 \pmod{lp^r}$ (such an $\bar{n}_1$ exists since $\gcd(n_1, lp^r) = 1$) and

$$m^{a_i} n^{b_i} \nmid m^{a_j} n^{b_j}, \qquad 2 \le i \ne j \le t.$$

It follows that

$$\sum_{i=2}^{t} m^{a_i} n^{b_i+1} \equiv (v - v')\bar{n}_1 n_1 p \equiv (v - v')p \pmod{lp^r}$$

and

$$I_w + \sum_{i=2}^{t} m^{a_i} n^{b_i+1} \equiv (v - v')p + v'p + w = vp + w \pmod{lp^r}.$$

Similar to the argument for $r = 1$, we have

$$m^{a_i} n^{b_i+1} \nmid m^{a_j} n^{b_j+1}, \qquad 2 \le i \ne j \le t$$

and for $1 \le w \le p-1$,

$$m^{a_i} n^{b_i+1} \nmid m^{a_1}, \qquad m^{a_1} \nmid m^{a_i} n^{b_i+1}, \qquad 2 \le i \le t.$$

Therefore, $A(m,n)$ is d-complete modulo lp^r. ■

Proof of Theorem 1.2. Firstly, we prove the necessity. If $\gcd(l, mn) = 1$, then (1) is true by Lemma 2.1. If $\gcd(l, mn) > 1$, without loss of generality we may assume $\gcd(l, n) = \gamma > 1$. Since $A(m,n)$ is d-complete modulo l, it is d-complete modulo γ. It follows from $\gamma \mid n$ that for every integer $1 \le u \le \gamma - 1$, there is an integer α_u such that

$$m^{\alpha_u} \equiv u \pmod{\gamma}.$$

Since $m^{\alpha_1} \equiv 1 \pmod{\gamma}$, we see that $\gcd(m, \gamma) = 1$. If γ is composite, then there exists a prime p with $p \mid \gamma$ and $p < \gamma$. However, in view of $m^{\alpha_p} \equiv p \pmod{\gamma}$, we have $p \mid m$, a contradiction to $\gcd(m, \gamma) = 1$. Hence, if $\gcd(l, n) > 1$, then $\gcd(l, n)$ is prime and m is a primitive root of $\gcd(l, n)$, from which one immediately deduces (2)–(4).

Now, we prove the sufficiency. If condition (1) is true, then we infer that $A(m,n)$ is d-complete modulo l by Lemma 2.1.

Suppose that condition (2) holds. Then $\gcd(l, n) = p$ with p prime and m is a primitive root of p. Clearly, $m \ne n^\alpha$ for any rational number α. Let $l = l_1 p^r$ and $n = n_1 p^s$, where $\gcd(l_1, n_1) = 1$ and $\gcd(l_1 n_1, p) = 1$. Since $\gcd(l, n) = p$, we have $r = 1$ or $s = 1$. By Lemma 2.2, $A(m,n)$ is d-complete modulo p^r. Since $\gcd(l,m)=1$, it follows from $\gcd(l_1,n_1)=\gcd(l_1,p)=1$ that $\gcd(l_1,mn)=1$. By Lemma 2.1, $A(m,n)$ is d-complete modulo l_1. From Lemma 2.3, we see that $A(m,n)$ is d-complete modulo l. When condition (3) holds, one can prove similarly that $A(m,n)$ is d-complete modulo l.

Suppose that condition (4) holds. Let $\gcd(l,n)=p$ and $\gcd(l,m)=q$ be two distinct primes. Then l,m,n can be expressed as $l=l_1p^{r_1}q^{r_2}$, $n=n_1p^s$ and $m=m_1q^t$, where $\gcd(p,l_1n_1)=1$, $\gcd(q,l_1m_1)=1$ and $\gcd(l_1,mn)=1$. We have $r_1=1$ or $s=1$ by $\gcd(l,n)=p$, and $r_2=1$ or $t=1$ by $\gcd(l,m)=q$. By an argument similar to that when (2) holds, we deduce that $A(m,n)$ is d-complete modulo all of p^{r_1}, q^{r_2} and l_1. By Lemma 2.3, $A(m,n)$ is d-complete modulo l. ∎

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Xing-Wang Jiang
Department of Mathematics
Luoyang Normal University
Luoyang 471934, P.R. China
E-mail: xwjiangnj@sina.com

Bing-Ling Wu
School of Science
Nanjing University of Posts and Telecommunications
Nanjing 210023, P.R. China
E-mail: wbl.math@njupt.edu.cn