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Squarefree density of polynomials

by

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Dedicated with affection and deepest respect to Henryk Iwaniec

Abstract. This paper is concerned with squarefree values of polynomials

$$\mathcal{P}(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_s]$$

where we suppose that for each $j \le s$ we have $|x_j| \le P_j$. Then we define

$$N_{\mathcal{P}}(\mathbf{P}) = \sum_{\substack{\mathbf{x} \\ |x_j| \le P_j \\ \mathcal{P}(\mathbf{x}) \ne 0}} \mu(|\mathcal{P}(\mathbf{x})|)^2$$

and we are interested in its behaviour when $\min_j P_j \to \infty$, and the extent to which this can be approximated by

$$N_{\mathcal{P}}(\mathbf{P}) \sim 2^s P_1 \dots P_s \mathfrak{S}_{\mathcal{P}}$$

where

$$\mathfrak{S}_{\mathcal{P}} = \prod_p \left(1 - \frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\right) \quad \text{and} \quad \rho_{\mathcal{P}}(d) = \operatorname{card}\{\mathbf{x} \in \mathbb{Z}_d^s : \mathcal{P}(\mathbf{x}) \equiv 0 \pmod d\}.$$

We establish this in a number of new cases, and in particular show that if $s \ge 2$ and $\mathfrak{S}_{\mathcal{P}} = 0$, then

$$N_{\mathcal{P}}(\mathbf{P}) = o(P_1 \dots P_s)$$

as $\min_j P_j \to \infty$.

1. Introduction and statement of results. The work described in this paper has its origins in a talk given by Manjul Bhargava at the 60th birthday conference for Krishna Alladi at the University of Florida in March 2016, the main contents of which are in Bhargava (2014) and Bhargava, Shankar and Wang (2022). This was concerned with squarefree values of

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integral forms of degree d in $s \geq 2$ variables $\mathbf{x} = (x_1, \dots, x_s)$,

$$\mathcal{F}(\mathbf{x}) = \sum_{\substack{i_1,\dots,i_d \\ i_1 \leq \dots \leq i_d \leq s}} c_{i_1 \dots i_d} x_{i_1} \dots x_{i_d}, \tag{1.1}$$

in various special cases.

The main result of this paper, Theorem 1.1 below, was intended as part of an attack on the cubic case. However, very recently this was resolved by Lapkova and Xiao (2020) by a different method. Nevertheless Theorem 1.1 has many other uses and we give some examples here.

THEOREM 1.1. *Suppose that* $s \geq 2$ *and* $\mathcal{C}(\mathbf{x})$ *is an integral cubic form not of the shape* $a(b_1x_1 + \dots + b_sx_s)^3$ *where* $a, b_1, \dots, b_s \in \mathbb{Q}$, *and let* $M(P)$ *denote the number of solutions of* $\mathcal{C}(\mathbf{x}) = \mathcal{C}(\mathbf{X})$ *with* $\mathbf{x}, \mathbf{X} \in [-P, P]^s$. *Then*

$$M(P) \ll P^{2s-2+\varepsilon}.$$

Apart possibly from the ε, this is best possible, as can be seen with the example

$$cx_1^3 + c'(x_2 + \dots + x_s)^3.$$

COROLLARY 1.2. *Suppose that* $s \geq 3$, $k = 3$ *or* 4, $c \in \mathbb{Z} \setminus \{0\}$ *and the integral cubic form* $\mathcal{C}^*(x_2, \dots, x_s)$ *is not of the shape* $a(b_2x_2 + \dots + b_sx_s)^3$ *where* $a, b_2, \dots, b_s \in \mathbb{Q}$, *let* $Q = P^{3/k}$, *and let* $L(P)$ *denote the number of solutions of*

$$cx_1^k + \mathcal{C}^*(x_2, \dots, x_s) = cX_1^k + \mathcal{C}^*(X_2, \dots, X_s)$$

with $|x_1|, |X_1| \leq Q$ *and* $|x_j|, |X_j| \leq P$ $(2 \leq j \leq s)$. *Then*

$$L(P) \ll P^{2s-3+\varepsilon} Q^{1/2}.$$

There is, of course, no reason to restrict the original question to forms. Instead one can consider general integral polynomials

$$\mathcal{P}(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_s]$$

where we suppose that for each $j \leq s$ we have $|x_j| \leq P_j$. We extend the definition of the Möbius function by taking $\mu(0) = 0$. Then we define

$$N_{\mathcal{P}}(\mathbf{P}) = \sum_{\substack{\mathbf{x} \\ |x_j| \leq P_j}} \mu(|\mathcal{P}(\mathbf{x})|)^2 \tag{1.2}$$

and we are interested in its behaviour when $\min_j P_j \to \infty$, and the extent to which this can be approximated by

$$N_{\mathcal{P}}(\mathbf{P}) \sim 2^s P_1 \dots P_s \mathfrak{S}_{\mathcal{P}} \tag{1.3}$$

where

$$\mathfrak{S}_{\mathcal{P}} = \prod_{p} \left(1 - \frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\right), \tag{1.4}$$

$$\rho_{\mathcal{P}}(d) = \mathrm{card}\,\{\mathbf{x} \in \mathbb{Z}_d^s : \mathcal{P}(\mathbf{x}) \equiv 0 \pmod{d}\}. \tag{1.5}$$

Note that

$$\prod_{p \le n} \left(1 - \frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\right)$$

is a non-negative decreasing sequence so it converges as $n \to \infty$ to a non-negative limit, although when the limit is 0 it is usual to describe such a product as diverging!

It seems that (1.3) should hold in all cases. Thus if $\mathcal{P}$ has a shortage of squarefree values, then we expect that

$$\mathfrak{S}_{\mathcal{P}} = 0. \tag{1.6}$$

In this case, (1.3) is easy to prove, although we have not seen it in the extant literature.

THEOREM 1.3. *Suppose that $s \ge 2$ and $\mathcal{P} \in \mathbb{Z}[x_1, \ldots, x_s]$ is an integral polynomial. If $\mathfrak{S}_{\mathcal{P}} = 0$, then*

$$N_{\mathcal{P}}(\mathbf{P}) = o(P_1 \ldots P_s)$$

as $\min_j P_j \to \infty$.

It is sometimes asserted that for large s such results should follow by an application of the Hardy–Littlewood method. However, as far as we are aware, there is nothing in the extant literature for general $\mathcal{P}$. Moreover, it is not entirely clear how methods currently available for showing the existence of non-trivial representations of 0 by even cubic forms can be adapted to this situation. First of all, such methods take a small region about a non-trivial real zero of the form and expand it homothetically, but in some sense we need to approximate the number of solutions to

$$\mathcal{P}(\mathbf{x}) = n$$

for most n and there is no clear way of adapting the method to cover the s-dimensional boxes considered here. Secondly, those methods either require the form to satisfy a condition such as being non-singular in order to show that the concomitant exponential sum

$$S(\alpha) = \sum_{|\mathbf{x}| \le P} e(\alpha \mathcal{P}(\mathbf{x}))$$

is relatively small on the minor arcs, or have to use an alternative argument when $S(\alpha)$ is not always small on the minor arcs, which only ensures the existence of solutions rather than gives a good approximation for their number.

We apply Corollary 1.2 above to establish the following.

THEOREM 1.4. *Suppose that* $s \geq 3$, $k = 3$ *or* 4, $c \in \mathbb{Z} \setminus \{0\}$ *and the integral cubic form* $\mathcal{C}^*(x_2, \ldots, x_s)$ *is not of the shape* $a(b_2x_2 + \cdots + b_sx_s)^3$ *where* $a, b_1, \ldots, b_s \in \mathbb{Q}$. *Let*

$$\mathcal{P}(\mathbf{x}) = cx_1^k + \mathcal{C}^*(x_2, \ldots, x_s) \tag{1.7}$$

with $c \neq 0$. *Suppose* P *is large and let* $Q = P_1 = P^{3/k}$, $P_j = P$ $(2 \leq j \leq s)$, *and*

$$N_{\mathcal{P}}(\mathbf{P}) = \sum_{|x_1| \leq Q} \sum_{\substack{x_2, \ldots, x_s \\ |x_j| \leq P}} \mu(|\mathcal{P}(\mathbf{x})|)^2.$$

Then, as $P \to \infty$,

$$N_{\mathcal{P}}(\mathbf{P}) = 2^s Q P^{s-1} \mathfrak{S}_{\mathcal{P}} + E_k$$

where

$$E_3 \ll P^{s - \frac{1}{8s-2} + \varepsilon}, \qquad E_4 \ll \frac{QP^{s-1}}{(\log P) \log\log P}.$$

We remark that the case

$$\mathcal{C}^*(x_2, \ldots, x_s) = a(b_2x_2 + \cdots + b_sx_s)^3$$

can be dealt with by the methods of Filaseta (1994) and Sanjaya and Wang (2023).

There is a useful survey on the squarefree density for this and various concomitant questions by Tsvetkov (2019).

Generally formulæ involving ε will hold for every $\varepsilon > 0$ and the stated ranges of the other variables, albeit any implicit constant may well depend on ε. Unless stated otherwise, implicit constants will not depend on the other variables.

2. Lemmata. We begin with a pair of results from the appendix of Estermann (1931).

LEMMA 2.1. *Suppose that* a, b, n *are positive integers and let* $Q(n; a, b)$ *denote the number of solutions of* $ax^2 + by^2 = n$ *in positive integers* x *and* y. *Then*

$$Q(n; a, b) \leq 2d(n).$$

LEMMA 2.2. *Suppose that* a, b, m, n *are positive integers and let* $R(n; a, b)$ *denote the number of solutions of* $ax^2 - by^2 = n$ *in positive integers* x *and* y *with* $ax^2 \leq m$. *Then*

$$R(n; a, b) \leq 2d(n)(1 + \log m).$$

The following is probably in the literature, but we do not know where.

Lemma 2.3. *Suppose that $n \in \mathbb{N}$, $b \in \mathbb{Z}$ and $S(P; n, b)$ denotes the number of integers $y \in [-P, P]$ such that $ny + b$ is a perfect square. Let k^2 be the largest square dividing (n, b). Then*

$$S(P; n, b) \ll n^{\varepsilon}\left(1 + k\sqrt{\frac{P}{n}}\right).$$

It is clear from the example $n = 1$, $b = 0$ that this is essentially best possible. Gallagher's larger sieve gives a somewhat sharper bound but the above is adequate for our purposes.

Proof of Lemma 2.3. Write $(n, b) = k^2 l$ so that l is squarefree, and let $n_1 = n/(n, b)$ and $b_1 = b/(n, b)$, so that $(n_1, b_1) = 1$. If x satisfies $x^2 = ny + b$ then $kl \mid x$. We wish to bound the number of pairs z, y with $z \geq 0$, $y \in [-P, P]$ and $lz^2 = n_1 y + b_1$. Since $(n_1, b_1) = 1$, we have $(l, n_1) = 1$. Let $\mathcal{R}$ denote the set of residue classes r modulo n_1 such that $lr^2 \equiv b_1 \pmod{n_1}$. As $(lb_1, n_1) = 1$, we have $\operatorname{card} \mathcal{R} \ll n_1^{\varepsilon}$. Let $r \in \mathcal{R}$. Then it suffices to bound the number of solutions with $z \equiv r \pmod{n_1}$. Let z_0 be the least such solution and y_0 the corresponding value of y. Thus for any other solution z we have $z = z_0 + n_1 v$ where $v \geq 0$. Hence $lv(2z_0 + n_1 v) = y - y_0$ and so $v|2z_0 + n_1 v| \leq 2P/l$. The number of possible $v \leq (P/(ln_1))^{1/2}$ is

$$\ll 1 + \sqrt{\frac{P}{ln_1}}, \tag{2.1}$$

and when $v > (P/ln_1)^{1/2}$ we have $|2z_0 + n_1 v| \leq 2(Pn_1/l)^{1/2}$ and again the number of such v satisfies (2.1). ∎

We apply these results to obtain bounds for a general quadratic polynomial in two variables. In the lemma below the results depend on whether the three quantities $a^2 + c^2$, Δ and Θ are zero or not. As discussed in the remark below the lemma, the lemma takes care of all eight possibilities.

Lemma 2.4. *Let $P, Q \geq 2$, suppose that a, b, c, d, e, f are integers in $[-Q, Q]$, and let $\Delta = 4ac - b^2$ and $\Theta = 4acf + ebd - ae^2 - cd^2 - fb^2$. Let $N(P, Q)$ denote the number of solutions of*

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

in integers x, y with $|x|, |y| \leq P$.

(i) *If $(a^2 + c^2)\Delta\Theta \neq 0$, then $N(P, Q) \ll (PQ)^{\varepsilon}$.*
(ii) *If $(a^2 + c^2)\Delta \neq 0$, $\Theta = 0$ and $-\Delta$ is not a perfect square, then $N(P, Q) \ll 1$.*
(iii) *If $a = c = 0$ and $\Delta\Theta \neq 0$, then $N(P, Q) \ll (PQ)^{\varepsilon}$.*
(iv) *If $(a^2 + c^2)\Delta \neq 0$, $\Theta = 0$ and $-\Delta$ is a perfect square, then $N(P, Q) \ll P$.*

(v) *If $a \neq 0$, $2ae - bd = \Delta = 0$ and $d^2 - 4af$ is not a perfect square, or if $c \neq 0$, $2cd - be = \Delta = 0$ and $e^2 - 4cf$ is not a perfect square, then $N(P,Q) = 0$.*

(vi) *If $a \neq 0$, $l = 2ae - bd \neq 0$, $\Delta = 0$ and k^2 is the largest square dividing $|l|$, or if $c \neq 0$, $l = 2cd - be \neq 0$, $\Delta = 0$ and k^2 is the largest square dividing $|l|$, then*

$$N(P,Q) \ll Q^{\varepsilon}\left(1 + k\sqrt{\frac{P}{|l|}}\right).$$

(vii) *If $a \neq 0$, $2ae - bd = \Delta = 0$ and $d^2 - 4af$ is a perfect square, or if $c \neq 0$, $2cd - be = \Delta = 0$ and $e^2 - 4cf$ is a perfect square, then $N(P,Q) \ll P$.*

(viii) *If $a = c = \Theta = 0$ and $\Delta \neq 0$, then $N(P,Q) \ll P$.*

For completeness we also state:

(ix) *In all other cases $N(P,Q) \ll P^2$.*

À propos our previous remark we see that (i) takes care of the case $(a^2+c^2)\Delta\Theta \neq 0$, (ii) and (iv) collectively take care of the case $(a^2+c^2)\Delta \neq 0$, $\Theta = 0$, and (iii) takes care of the case $\Delta\Theta \neq 0$, $a^2+c^2 = 0$. When $a^2+c^2 \neq 0$ and $\Delta = 0$ there are various subcases and these are taken care of by (v), (vi), and (vii). Since these subcases are indifferent to the value of Θ, they deal with both the case $(a^2+c^2)\Theta \neq 0$, $\Delta = 0$ and the case $a^2+c^2 \neq 0$, $\Delta = \Theta = 0$. The case $\Delta \neq 0$, $a = c = \Theta = 0$ is taken care of in (viii). Finally, if $a^2 + c^2 = \Delta = 0$, then automatically $\Theta = 0$, so the remaining cases are dealt with by (ix). It is not necessary for our purposes but in this last situation, when d or e is non-zero, the bound could be replaced by $2P+1$.

Proof of Lemma 2.4. If $a = c = 0$, then $\Delta = -b^2$ and $\Theta = b(ed - fb)$. Hence if $\Delta \neq 0$, so that $b \neq 0$, then multiplication by b gives

$$(bx + e)(by + d) + bf - ed = 0$$

and this has $\ll (PQ)^{\varepsilon}$ solutions when $\Theta \neq 0$ and $\ll P$ solutions when $\Theta = 0$. This deals with cases (iii) and (viii) and so, apart from the trivial case (ix), we can suppose that $a^2+c^2 \neq 0$. Thus, without loss of generality, we suppose that $a \neq 0$. Completing the square gives the equation

$$(2ax + by + d)^2 + \Delta y^2 + 2(2ae - bd)y + 4af - d^2 = 0.$$

Suppose $\Delta = 0$. Then we are asking that $2(bd-2ae)y+d^2-4af$ be a perfect square. If $2ae - bd = 0$ and $d^2 - 4af$ is not a perfect square, then we have no solutions and we are in case (v). If $bd \neq 2ae$, then we are in case (vi) and the assertion follows from Lemma 2.3. If $bd = 2ae$ and $d^2 - 4af$ is a perfect square, then x is determined by y and we get case (vii).

Thus we may suppose that $\Delta \neq 0$. Completing the square once more gives

$$\Delta(2ax+by+d)^2 + (\Delta y + 2ae - bd)^2 + 4a\Theta = 0. \tag{2.2}$$

If $\Theta = 0$ and $-\Delta$ is not a perfect square, then solutions are only possible with

$$2ax+by+d = \Delta y + 2ae - bd = 0$$

and so y is determined by the second equality and then x is determined by the first one. This gives case (ii).

If $\Theta = 0$ and $-\Delta$ is a non-zero perfect square, say m^2, then the equation becomes

$$(2amx + mby + md)^2 - (m^2 y + 2ae - bd)^2 = 0.$$

Taking any choice for y gives $\ll 1$ choices for x and gives case (iv).

Finally, we may suppose that $(a^2+c^2)\Delta\Theta \neq 0$, case (i). We can write equation (2.2) in the form

$$\Delta X^2 + Y^2 = -4a\Theta.$$

If $\Delta > 0$ and $-4a\Theta \le 0$, then there is at most one solution in integers X and Y. If $\Delta > 0$ and $-4a\Theta > 0$, then we can appeal to Lemma 2.1 and we see that the number of solutions in X and Y is $\ll Q^{\varepsilon}$. For each such pair X, Y there are $\ll 1$ pairs x, y with $\Delta y + 2ae - bd = Y$ and $2ax+by+d = X$. If $\Delta < 0$ and $-4a\Theta > 0$, then we write the equation in the form

$$Y^2 - (-\Delta)X^2 = -4a\Theta$$

and use Lemma 2.2 instead. On the other hand, if $4a\Theta > 0$, then we can rewrite the equation as

$$(-\Delta)X^2 - Y^2 = 4a\Theta$$

and proceed in the same way. ∎

We will need a bound for $\rho_{\mathcal{P}}(d^2)$ when d is squarefree.

LEMMA 2.5. *Let $k = 3$ or 4 and let $\mathcal{P}$ be as in* (1.7). *Then*

$$\rho_{\mathcal{P}}(p^2) \ll p^{2s-2},$$

and, for squarefree d,

$$\rho_{\mathcal{P}}(d^2) \ll d^{2s-2+\varepsilon}.$$

Proof. Since ρ is a multiplicative function, it suffices to show the first bound. Furthermore, we may suppose that $p > 3$ and is larger than any of the coefficients of $\mathcal{P}$. We are counting the number of solutions of

$$cx_1^k + \mathcal{C}^*(x_2, \dots, x_s) \equiv 0 \ (\mathrm{mod}\, p^2). \tag{2.3}$$

Consider first the number of solutions of

$$cx_1^k + \mathcal{C}^*(x_2, \dots, x_s) \equiv 0 \ (\mathrm{mod}\, p). \tag{2.4}$$

Let Z denote the number of solutions of $\mathcal{C}^*(x_2,\dots,x_s) \equiv 0 \pmod p$. By Lemma 3.1 of §4.3 of Schmidt (2004), $Z \le 3p^{s-2}$. Thus the number of solutions of (2.3) with $x_1 \equiv 0 \pmod p$ is $\le 3p^s \cdot p^{s-2} = 3p^{2s-2}$. The number of solutions of (2.4) with $x_1 \not\equiv 0$ is at most kp^{s-1}. By the initial assumption on p each solution of (2.4) with $x_1 \not\equiv 0 \pmod p$ is non-singular and so lifts to at most p^{s-1} solutions modulo p^2. Thus the number of solutions to (2.3) with $x_1 \not\equiv 0 \pmod p$ is $\le 3p^{2s-2}$. ∎

3. Proof of Theorem 1.3. If there should be a prime p with $\rho_{\mathcal{P}}(p^2) = p^{2s}$, then $\mathcal{P}$ has p^2 as a fixed divisor, $\mathfrak{S}_{\mathcal{P}}$ is perforce 0 and $N_{\mathcal{P}}(P) = 0$. Hence we can assume that, for every prime p,

$$\rho_{\mathcal{P}}(p^2) < p^{2s}. \tag{3.1}$$

We have $\mathfrak{S}_{\mathcal{P}} = 0$. Thus, by (1.6), given $\eta > 0$, when $\xi > \xi_0(\eta)$ for some $\xi_0(\eta)$ we have

$$0 < \prod_{p\le\xi}\left(1 - \frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\right) < \eta. \tag{3.2}$$

Let

$$D = \prod_{p\le\xi} p. \tag{3.3}$$

By (1.2), we have

$$N_{\mathcal{P}}(\mathbf{P}) \le \sum_{\substack{\mathbf{x}\\ |x_j|\le P_j}} \sum_{m^2|(D^2,\mathcal{P}(\mathbf{x}))} \mu(m) = \sum_{m|D}\mu(m) \sum_{\substack{\mathbf{x}\\ |x_j|\le P_j\\ m^2|\mathcal{P}(\mathbf{x})}} 1.$$

Then by apportioning the x_j into residue classes modulo m^2 we find that the inner sum is

$$\rho_{\mathcal{P}}(m^2)\left(\frac{2^sP_1\dots P_s}{m^{2s}} + O\Big(m^{2-2s}P_1\dots P_s\Big(\min_j P_j\Big)^{-1} + 1\Big)\right).$$

The trivial bound $\rho_{\mathcal{P}}(m^2) \le m^{2s}$ then gives the approximation

$$2^sP_1\dots P_s\frac{\rho_{\mathcal{P}}(m^2)}{m^{2s}} + O\Big(m^2P_1\dots P_s\Big(\min_j P_j\Big)^{-1} + m^{2s}\Big).$$

Hence

$$0 \le N_{\mathcal{P}}(\mathbf{P}) \le 2^sP_1\dots P_s\sum_{m|D}\rho_{\mathcal{P}}(m^2)\frac{\mu(m)}{m^{2s}} + O_\xi\Big(P_1\dots P_s\Big(\min_j P_j\Big)^{-1}\Big),$$

and so, by (3.2),

$$0 \le \limsup_{\min_j P_j\to\infty}\frac{N_{\mathcal{P}}(\mathbf{P})}{2^sP_1\dots P_s} \le \sum_{m|D}\rho_{\mathcal{P}}(m^2)\frac{\mu(m)}{m^{2s}} = \prod_{p\le\xi}\left(1 - \frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\right) < \eta.$$

4. Proof of Theorem 1.1. The proof is inductive. The case $s = 2$ can be deduced from results in the literature. When the polynomial $\mathcal{C}(x, 1)$ is irreducible over $\mathbb{Q}$, then, by Bombieri and Schmidt (1987), the number of solutions of the Thue equation

$$\mathcal{C}(x, y) = n \tag{4.1}$$

for $n \in \mathbb{Z}\backslash\{0\}$ is $\ll |n|^\varepsilon$. When the polynomial is reducible over $\mathbb{Q}$, since both variables occur explicitly and $\mathcal{C}$ is not of the form $a(b_1x_1 + b_2x_2)^3$, (4.1) is of the form

$$(Ax + By)(Cx^2 + Dxy + Ey^2) = Fn$$

for integral $A, \ldots, F$ with $F \neq 0$ and $Ax+By$ not a factor of $Cx^2+Dxy+Ey^2$. Then the factors are complementary divisors d, d' of Fn. Without loss of generality we may suppose that $A \neq 0$. Then $x = (d - By)/A$ and so by substitution

$$C(d - By)^2 + DA(d - By)y + EA^2y^2 = A^2d'.$$

The coefficient of y^2 here is $CB^2 - DAB + EA^2$ and is $\neq 0$ since $Ax + By$ is not a factor of $Cx^2 + Dxy + Ey^2$. Thus there are at most two choices for y and then x is fixed, so the number of solutions of $\mathcal{P}(x, y) = \mathcal{P}(X, Y) \neq 0$ is $\ll P^\varepsilon$ times the number of choices for X, Y. The remaining situation $\mathcal{P}(x, y) = \mathcal{P}(X, Y) = 0$ clearly has $\ll P^2$ solutions.

Now suppose that $s > 2$. It is useful to first transform the form so that at least two of the variables, for example x_1 and x_2, have non-zero x_1^3 and x_2^3 terms. Suppose on the contrary that $c_{111} = 0$. If there is a j such that $c_{11j} \neq 0$ or $c_{1jj} \neq 0$ or $c_{jjj} \neq 0$, then replace x_j by $x_j + \lambda x_1$ where λ is a non-zero integer at our disposal. Then x_1^3 will now occur as $(c_{11j}\lambda + c_{1jj}\lambda^2 + c_{jjj}\lambda^3)x_1^3$. This has a non-zero coefficient for a sufficiently large λ. If $c_{11j} = c_{1jj} = c_{jjj} = 0$ for every j, then since x_1 occurs explicitly, there will be $1 < j < k$ such that $c_{1jk} \neq 0$. Replace x_j and x_k by $x_j + \lambda x_1$ and $x_k + \mu x_1$ respectively where now λ and μ are at our disposal. Then x_1^3 occurs as

$$(c_{1jk}\lambda\mu + c_{jjk}\lambda^2\mu + c_{jkk}\lambda\mu^2)x_1^3.$$

If $c_{jkk} \neq 0$ then one can take $\lambda = 1$ and μ sufficiently large, and if $c_{jkk} = 0$ and $c_{jjk} \neq 0$ then one can take $\mu = 1$ and λ large. Finally, if $c_{jjk} = c_{jkk} = 0$ then one can take $\lambda = \mu = 1$. The transformations in each case are unimodular, so invertible, and the new forms therefore represent the same numbers as the old ones, and vice versa. The region of interest becomes a parallelepiped, but can certainly be accommodated in $[-cP, cP]^s$ for a suitable constant c. The process can be repeated to ensure that at least two variables x_j appear explicitly as x_j^3. Note that the transformation x_k to $x_k + \lambda_k x_j$ $(k \neq j)$ does not alter the coefficient of x_k^3.

Henceforward we can suppose that two of our variables, say x and y, are such that x^3 and y^3 occur explicitly. Denote the remaining variables by $\mathbf{z}$. Thus

$$\begin{aligned}\mathcal{C}(\mathbf{x}) &= Ax^3 + Bx^2y + Cxy^2 + Dy^3\\ &\qquad + x^2\mathcal{L}_1(\mathbf{z}) + xy\mathcal{L}_2(\mathbf{z}) + y^2\mathcal{L}_3(\mathbf{z}) + x\mathcal{Q}_1(\mathbf{z}) + y\mathcal{Q}_2(\mathbf{z}) + \mathcal{C}_1(\mathbf{z})\end{aligned}$$

where $\mathcal{L}_j$, $\mathcal{Q}_j$ and $\mathcal{C}_1$ are linear, quadratic and cubic forms respectively.

If the $\mathcal{L}_j$, $\mathcal{Q}_j$ are all identically 0, then we have $\mathcal{C}(\mathbf{x}) = \mathcal{C}_0(x,y) + \mathcal{C}_1(\mathbf{z})$ where $\mathcal{C}_0$ is a binary cubic form. Moreover, the number of solutions of $\mathcal{C}_0(x,y) - \mathcal{C}_0(X,Y) = m$ is bounded by the number for $m = 0$, so we can appeal to the case $s = 2$.

Thus we can assume that $AD \neq 0$ and not all of $\mathcal{L}_j$, $\mathcal{Q}_j$ are identically 0. Let $R(n)$ denote the number of solutions of $\mathcal{C}(\mathbf{x}) = n$ with $\mathbf{x} \in [-P,P]^s$, where in this case $\mathbf{x}$ is a shorthand for $(x, y, z_1, \ldots, z_{s-2})$. We sort the solutions according to $\mathbf{z}$. Thus

$$R(n) = \sum_{\mathbf{z}\in[-P,P]^{s-2}} R(n,\mathbf{z})$$

where $R(n,\mathbf{z})$ is the number of solutions x, y of

$$\begin{aligned}&Ax^3 + Bx^2y + Cxy^2 + Dy^3\\ &\qquad + x^2\mathcal{L}_1(\mathbf{z}) + xy\mathcal{L}_2(\mathbf{z}) + y^2\mathcal{L}_3(\mathbf{z}) + x\mathcal{Q}_1(\mathbf{z}) + y\mathcal{Q}_2(\mathbf{z}) + \mathcal{C}_1(\mathbf{z}) = n\end{aligned}$$

with $x, y \in [-P,P]$. The object of the theorem, namely the number $M(P)$ of solutions of $\mathcal{C}(\mathbf{x}) = \mathcal{C}(\mathbf{X})$, can be written as

$$\sum_{n\in\mathbb{Z}} R(n)^2.$$

By Cauchy's inequality this is at most

$$(2P+1)^{s-2} \sum_{\mathbf{z}\in[-P,P]^{2s-2}} \sum_{n\in\mathbb{Z}} R(n,\mathbf{z})^2.$$

Thus the equation to be considered becomes

$$\begin{aligned}&Ax^3 + Bx^2y + Cxy^2 + Dy^3\\ &\qquad\qquad + x^2\mathcal{L}_1(\mathbf{z}) + xy\mathcal{L}_2(\mathbf{z}) + y^2\mathcal{L}_3(\mathbf{z}) + x\mathcal{Q}_1(\mathbf{z}) + y\mathcal{Q}_2(\mathbf{z})\\ &\qquad = AX^3 + BX^2Y + CXY^2 + DY^3\\ &\qquad\qquad + X^2\mathcal{L}_1(\mathbf{z}) + XY\mathcal{L}_2(\mathbf{z}) + Y^2\mathcal{L}_3(\mathbf{z}) + X\mathcal{Q}_1(\mathbf{z}) + Y\mathcal{Q}_2(\mathbf{z}).\end{aligned}$$

This has $s+2$ variables and we desire to show that there are $\ll P^{s+\varepsilon}$ solutions.

We now define $g = X - x$, $h = Y - y$ so that the equation becomes

$$Ag(3x^2 + 3gx + g^2) + B(x^2h + 2gxy + 2ghx + g^2y + g^2h) + C(2hxy + gy^2 + h^2x + 2ghy + gh^2) + Dh(3y^2 + 3hy + h^2) + \mathcal{L}_1(\mathbf{z})(2xg + g^2) + \mathcal{L}_2(\mathbf{z})(xh + yg + gh) + \mathcal{L}_3(\mathbf{z})(2yh + h^2) + g\mathcal{Q}_1(\mathbf{z}) + h\mathcal{Q}_2(\mathbf{z}) = 0.$$

We can rewrite this as

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{4.2}$$

where

$$a = 3Ag + Bh, \quad b = 2Bg + 2Ch, \quad c = Cg + 3Dh, \tag{4.3}$$

$$d = 3Ag^2 + 2Bgh + Ch^2 + 2\mathcal{L}_1(\mathbf{z})g + \mathcal{L}_2(\mathbf{z})h, \tag{4.4}$$

$$e = Bg^2 + 2Cgh + 3Dh^2 + \mathcal{L}_2(\mathbf{z})g + 2\mathcal{L}_3(\mathbf{z})h, \tag{4.5}$$

$$f = Ag^3 + Bg^2h + Cgh^2 + Dh^3 + \mathcal{L}_1(\mathbf{z})g^2 + \mathcal{L}_2(\mathbf{z})gh + \mathcal{L}_3(\mathbf{z})h^2 + \mathcal{Q}_1(\mathbf{z})g + \mathcal{Q}_2(\mathbf{z})h. \tag{4.6}$$

We emphasise that the coefficients a through e depend on g, h, $\mathbf{z}$, but not on x, y.

We will apply Lemma 2.4 multiple times. In the notation of that lemma, if for any given g, h, $\mathbf{z}$ any of the cases (i), (ii), (iii) or (v) hold, then we have a suitable bound for the number of corresponding solutions of (4.2).

Now suppose that case (iv) holds. If $3AC - B^2 \neq 0$, then $\Theta = 0$ and Θ has a term $A(3AC - B^2)g^5$ and so g is determined by $\mathbf{z}$ and h alone. Again we have a suitable bound. If $3AC - B^2 = 0$, then $\Theta = 0$ and Θ has a term $(9A^2D + 2ABC - B^2)g^4h = A(9AD - BC)g^4h$ and so if $9AD - BC \neq 0$, then $h = 0$ or g is determined by h, which once again gives a suitable bound. Finally, if $3AC - B^2 = BC - 9AD = 0$, then by considering the term in h^5 either we have the desired bound or $3AC - B^2 = BC - 9AD = C^2 - 3BD = 0$. Since $AD \neq 0$, we have $BC \neq 0$, and so $a = \frac{B}{C}(Bg + Ch)$, $b = 2(Bg + Ch)$, $c = \frac{C}{B}(Bg + Ch)$. But then $\Delta = 0$ contrary to the assumption in this case.

In the notation of Lemma 2.4 the remaining situations are

1. $a^2 + c^2 \neq 0$ and $\Delta = 0$,
2. $a = c = 0 = \Theta = 0$ and $\Delta \neq 0$.
3. $a = c = \Delta = 0$.

Case 1. One of the cases (v), (vi) or (vii) of Lemma 2.4 applies. In case (v) there are no solutions, so we can forget that case. In cases (vi) and (vii), $\Delta = 0$ gives

$$(B^2 - 3AC)g^2 + (BC - 9AD)gh + (C^2 - 3BD)h^2 = 0.$$

If any of the coefficients $B^2 - 3AC$, $BC - 9AD$, $C^2 - 3BD$ are non-zero, then g or h is 0 or g is determined by h or vice versa. That leaves the situation, as in case (iv), of

$$B^2 - 3AC = BC - 9AD = C^2 - 3BD = 0.$$

Since $AD \neq 0$, we also have $BC \neq 0$. Now

$$A = \frac{B^2}{3C}, \qquad D = \frac{C^2}{3B}$$

and so

$$a = BC^{-1}(Bg + Ch), \qquad c = CB^{-1}(Bg + Ch) \tag{4.7}$$

and

$$ax^2 + bxy + cy^2 = B^{-1}C^{-1}(Bg + Ch)(Bx + Cy)^2.$$

The above implies, *inter alia*, that $3C \mid B^2$ and $3B \mid C^2$. Therefore, in what follows all of our expressions are integer valued.

Since $a \neq 0$ or $c \neq 0$ we have $Bg + Ch \neq 0$. We also have

$$\begin{aligned} d &= C^{-1}(Bg + Ch)^2 + 2\mathcal{L}_1(\mathbf{z})g + \mathcal{L}_2(\mathbf{z})h, \\ e &= B^{-1}(Bg + Ch)^2 + \mathcal{L}_2(\mathbf{z})g + 2\mathcal{L}_3(\mathbf{z})h, \\ f &= 3^{-1}B^{-1}C^{-1}(Bg + Ch)^3 \\ &\quad + \mathcal{L}_1(\mathbf{z})g^2 + \mathcal{L}_2(\mathbf{z})gh + \mathcal{L}_3(\mathbf{z})h^2 + \mathcal{Q}_1(\mathbf{z})g + \mathcal{Q}_2(\mathbf{z})h. \end{aligned}$$

Then, by (4.3) and (4.7),

$$\begin{aligned} 2ae - bd = 2BC^{-1}&(Bg + Ch)(B^{-1}(Bg + Ch)^2 + \mathcal{L}_2(\mathbf{z})g + 2\mathcal{L}_3(\mathbf{z})h) \\ &- 2(Bg + Ch)(C^{-1}(Bg + Ch)^2 + 2\mathcal{L}_1(\mathbf{z})g + \mathcal{L}_2(\mathbf{z})h), \end{aligned}$$

so that

$$2ae - bd = 2(Bg + Ch)\big(2BC^{-1}\mathcal{L}_3(\mathbf{z})h + (BgC^{-1} - h)\mathcal{L}_2(\mathbf{z}) - 2\mathcal{L}_1(\mathbf{z})g\big). \tag{4.8}$$

Likewise

$$2cd - be = 2(Bg + Ch)\big(2CHB^{-1}\mathcal{L}_1(\mathbf{z})g + (ChB^{-1} - g)\mathcal{L}_2(\mathbf{z}) - 2\mathcal{L}_3(\mathbf{z})h\big). \tag{4.9}$$

In case (vi) consider first the possibility $a \neq 0$, $2ae - bd \neq 0$, $\Delta = 0$. Let $l = 2ae - bd$, so that $0 < |l| \ll P^3$. Then g, h, $\mathbf{z}$ satisfy

$$2(Bg + Ch)\big(2B\mathcal{L}_3(\mathbf{z})h + (Bg - Ch)\mathcal{L}_2(\mathbf{z}) - 2C\mathcal{L}_1(\mathbf{z})g\big) = Cl.$$

Given such g, h, $\mathbf{z}$, by Lemma 2.4(vi) the number of choices of x and y is

$$\ll P^{\varepsilon}(1 + k(P/|l|)^{1/2})$$

where $k^2 \mid l$. The total contribution from the P^{ε} term is $\ll P^{s+\varepsilon}$, which is acceptable. Thus it suffices to consider the contribution from the $P^{\varepsilon}k(P/|l|)^{1/2}$ term. Therefore we can presume that there is an l with $|l| \in \mathbb{N}$ and a $u \in \mathbb{Z} \setminus \{0\}$ such that $2(Bg + Ch) = u$ and

$$(B\mathcal{L}_2(\mathbf{z}) - 2C\mathcal{L}_1(\mathbf{z}))g + (2B\mathcal{L}_3(\mathbf{z}) - C\mathcal{L}_2(\mathbf{z}))h = Cl/u.$$

Given g, the first of these equations determines h. In the second equation we have $Cl/u \neq 0$ and the left side is a linear form in $\mathbf{z}$. Thus at least one of the z_j appears explicitly and so it is determined by the other variables. Hence, given l, the total number of possible g, h and $\mathbf{z}$ is $\ll P^{s-2+\varepsilon}$. Thus the total contribution is

$$\begin{aligned}
&\ll P^{s-2+\varepsilon} \sum_{0<|l|\ll P^3} \sum_{k^2|l} k(P/|l|)^{1/2} \\
&\ll P^{s-2+\varepsilon} \sum_{k\ll P^{3/2}} P^{1/2} \sum_{0<j\ll P^3/k^2} j^{-1/2} \ll P^{s+2\varepsilon}
\end{aligned}$$

and we have an acceptable bound for the number of solutions in this case.

Alternatively, if $a = 0$ but $c \neq 0$, then a concomitant argument where $2ae - bd$ is replaced by $2cd - be$ gives the desired bound.

Now consider case (vii), and to begin with suppose that $a \neq 0$, $2ae-bd = \Delta = 0$, and $d^2 - 4af$ is a perfect square. Then by (4.8) we have

$$(Bg + Ch)\bigl(2BC^{-1}\mathcal{L}_3(\mathbf{z})h + (BgC^{-1} - h)\mathcal{L}_2(\mathbf{z}) - 2\mathcal{L}_1(\mathbf{z})g\bigr) = 0.$$

Since $a \neq 0$, by (4.7) we have $Bg + Ch \neq 0$. Thus

$$2BC^{-1}\mathcal{L}_3(\mathbf{z})h + (BgC^{-1} - h)\mathcal{L}_2(\mathbf{z}) - 2\mathcal{L}_1(\mathbf{z})g = 0,$$

so that

$$(B\mathcal{L}_2(\mathbf{z}) - 2C\mathcal{L}_1(\mathbf{z}))g + (2B\mathcal{L}_3(\mathbf{z}) - C\mathcal{L}_2(\mathbf{x}))h = 0.$$

If $B\mathcal{L}_2(\mathbf{z}) - 2C\mathcal{L}_1(\mathbf{z})$ or $2B\mathcal{L}_3(\mathbf{z}) - C\mathcal{L}_2(\mathbf{x})$ is not identically 0, then either g is determined by h and $\mathbf{z}$, or h is by g and $\mathbf{z}$, or there are $\ll P^{s-3}$ choices of $\mathbf{z}$ for which $B\mathcal{L}_2(\mathbf{z}) - 2C\mathcal{L}_1(\mathbf{z}) = 0$ or $2B\mathcal{L}_3(\mathbf{z}) - C\mathcal{L}_2(\mathbf{x}) = 0$. Thus we can then appeal to case (vii) of the lemma. If $B\mathcal{L}_2(\mathbf{z}) - 2C\mathcal{L}_1(\mathbf{z})$ and $2B\mathcal{L}_3(\mathbf{z}) - C\mathcal{L}_2(\mathbf{x})$ are identically 0, then we have

$$\begin{aligned}
0 &= ax^2 + bxy + cy^2 + dx + ey + f \\
&= (Bg + Ch)3^{-1}B^{-1}C^{-1} \\
&\quad \times \bigl(3(Bx + Cy)^2 + 3(Bx + Cy)(Bg + Ch) + (Bg + Ch)^2\bigr) \\
&\quad + \mathcal{L}_1(\mathbf{z})g^2 + \mathcal{L}_2(\mathbf{z})gh + \mathcal{L}_3(\mathbf{z})h^2 + \mathcal{Q}_1(\mathbf{z})g + \mathcal{Q}_2(\mathbf{z})h.
\end{aligned}$$

Since $B\mathcal{L}_2(\mathbf{z}) - 2C\mathcal{L}_1(\mathbf{z})$ and $2B\mathcal{L}_3(\mathbf{z}) - C\mathcal{L}_2(\mathbf{x})$ are identically 0, we obtain

$$\begin{aligned}
G(3X^2 + 3XG + G^2) + G^2B^{-2}\mathcal{L}_1(\mathbf{z}) + GB^{-1}\mathcal{Q}_1(\mathbf{z})& \\
+ hB^{-1}(B\mathcal{Q}_2(\mathbf{z}) - C\mathcal{Q}_1(\mathbf{z})) &= 0
\end{aligned}$$

where $X = Bx + Cy$ and $G = Bg + Ch$. If $B\mathcal{Q}_2(\mathbf{z}) - C\mathcal{Q}_1(\mathbf{z})$ is not identically 0, then that are at most $\ll P^{s-3}$ values of $\mathbf{z}$ for which it is 0. There are also $\ll P$ values of g and h for which $G = 0$ or $h = 0$. Thus we can suppose that $B\mathcal{Q}_2(\mathbf{z}) - C\mathcal{Q}_1(\mathbf{z})$ is not identically 0 and $hG \neq 0$. But then $G|h(B\mathcal{Q}_2(\mathbf{z}) - C\mathcal{Q}_1(\mathbf{z}))$ and therefore given h and $\mathbf{z}$ there are $\ll P^{\varepsilon}$ choices for g, so we have a suitable bound in case (vii). The alternative case $c \neq 0$, $2cd - be = \Delta = 0$, $e^2 - 4cf$ a perfect square is similar. That completes the analysis of Case 1.

CASE 2. This corresponds exactly to (viii) of Lemma 2.4. Then by (4.3) and the fact that $A \neq 0$ we see that g is determined by h, so that for any given $\mathbf{z}$, by (viii), the total number of choices of x, y, g, h is $\ll P^2$ and this is sufficient.

CASE 3. This corresponds to (ix) of Lemma 2.4. It is similar to case (vii) but now $a = b = c = 0$. Then $g = -Bh/(3A)$, $h = -Cg/(3D)$. Thus $B = 0$ implies $g = 0$ and so $h = 0$ and this gives an adequate bound for the total number of such solutions. Likewise for $C = 0$. Hence we can suppose $BC \neq 0$. Moreover, anyway g is fixed by h and vice versa. It follows that

$$g = \lambda h$$

where

$$\lambda = -B/(3A) = -3D/C = -C/B$$

and

$$\begin{aligned} 3Ag^2 + 2Bgh + Ch^2 &= 0, \\ Bg^2 + 2Cgh + 3Dh^2 &= 0, \\ Ag^3 + Bg^2h + Cgh^2 + Dh^3 &= 0, \end{aligned}$$

and so our equation reduces to

$$dx + ey + f = 0$$

where

$$\begin{aligned} d &= 2\mathcal{L}_1(\mathbf{z})g + \mathcal{L}_2(\mathbf{z})h, \\ e &= \mathcal{L}_2(\mathbf{z})g + 2\mathcal{L}_3(\mathbf{z})h, \\ f &= \mathcal{L}_1(\mathbf{z})g^2 + \mathcal{L}_2(\mathbf{z})gh + \mathcal{L}_3(\mathbf{z})h^2 + \mathcal{Q}_1(\mathbf{z})g + \mathcal{Q}_2(\mathbf{z})h. \end{aligned}$$

Recall that g is fixed by h and vice versa. If $d \neq 0$ or $e \neq 0$, then x or y is fixed by the other variables. Thus we can suppose that $d = e = 0$ and hence $f = 0$. Thus

$$2\mathcal{L}_1(\mathbf{z})\lambda + \mathcal{L}_2(\mathbf{z}) = \mathcal{L}_2(\mathbf{z})\lambda + 2\mathcal{L}_3(\mathbf{z}) = 0, \tag{4.10}$$

$$\mathcal{L}_1(\mathbf{z})\lambda^2 + \mathcal{L}_2(\mathbf{z})\lambda + \mathcal{L}_3(\mathbf{z}) + \mathcal{Q}_1(\mathbf{z})\lambda + \mathcal{Q}_2(\mathbf{z}) = 0. \tag{4.11}$$

Substituting from (4.10) into (4.11) gives

$$\mathcal{Q}_1(\mathbf{z})\lambda + \mathcal{Q}_2(\mathbf{z}) = 0. \tag{4.12}$$

If any one of the equations (4.10) and (4.12) does not hold identically, then it holds for at most $\ll P^{s-3}$ choices of $\mathbf{z}$ and we are done. If they all hold identically, then $d = e = f = 0$. Returning to the original equation $\mathcal{C}(\mathbf{x}) = \mathcal{C}(\mathbf{X})$

this gives $\mathcal{C}(x+\lambda h, y+h, \mathbf{z}) = \mathcal{C}(x, y, \mathbf{z})$ identically for all x, y, h, $\mathbf{z}$. Taking $h = -y$ gives $\mathcal{C}(x, y, \mathbf{z}) = \mathcal{C}(x-\lambda y, 0, \mathbf{z})$ and then we can appeal to our inductive hypothesis since this reduces to the case $s-1$.

5. The proof of Corollary 1.2. We adapt the proof of Hua's Lemma (Lemma 2.5 of Vaughan (1997)). Let

$$f(\alpha) = \sum_{|x|\le Q} e(\alpha x^k) \quad \text{and} \quad S(x) = \sum_{\substack{x_2,\dots x_s\\ |x_j|\le P}} e(\alpha \mathcal{C}^*(x_2,\dots,x_s)).$$

Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned} L(P) &= \int_0^1 |f(c\alpha)S(\alpha)|^2\,d\alpha \\ &\le \Big(\int_0^1 |S(\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int_0^1 |f(c\alpha)^2S(\alpha)|^2\,d\alpha\Big)^{1/2}. \end{aligned} \tag{5.1}$$

By Weyl differencing and the Cauchy–Schwarz inequality,

$$|f(c\alpha)|^4 \le \sum_{|h|\ll P^3} Q(h)e(\alpha h)$$

where $Q(0) \ll Q^3$, $Q(h) \ll QP^\varepsilon$ $(h\ne 0)$.

For $n\in\mathbb{Z}$, let $R(n)$ denote the number of solutions of $\mathcal{C}^*(x_2,\dots,x_s) = n$ with $|x_j|\le P$. Then, by Theorem 1.1,

$$\int_0^1 |S(\alpha)|^2\,d\alpha = \sum_n R(n)^2 \ll P^{2s-4+\varepsilon}.$$

The conclusion then follows from (5.1) and the observation

$$\begin{aligned} \int_0^1 |f(c\alpha)^2S(\alpha)|^2\,d\alpha &\le \sum_{n_1,n_2} R(n_1)R(n_2)Q(n_2-n_1) \\ &\ll Q^3\sum_n R(n)^2 + QP^\varepsilon\Big(\sum_n R(n)\Big)^2 \ll QP^{2s-2+2\varepsilon}. \end{aligned}$$

6. The proof of Theorem 1.4. For a given integer n, let $R(n)$ denote the number of choices of $\mathbf{x}$ with $|x_1|\le Q$, $|x_j|\le P$ $(2\le j\le s)$ and $\mathcal{P}(\mathbf{x}) = cx_1^k + \mathcal{C}^*(x_2,\dots,x_s) = n$. Then

$$N_{\mathcal{P}}(\mathbf{P}) = \sum_{n\ne 0}\mu(|n|)^2R(n) = \sum_d \mu(d)\sum_{\substack{n\ne 0\\ d^2|n}} R(n).$$

Let $T \geq 1$ be at our disposal. By Cauchy's inequality and Corollary 1.2 we have

$$\sum_{d>T}\sum_{\substack{n\neq 0\\ d^2|n}} R(n) \ll \Big(\sum_{T<d\ll P^{3/2}} \frac{P^3}{d^2}\Big)^{1/2}\Big(\sum_{n}\sum_{d^2|n} R(n)^2\Big)^{1/2} \tag{6.1}$$
$$\ll T^{-1/2}P^{s+\varepsilon}Q^{1/4}.$$

Hence

$$\sum_{d>T}\mu(d)\sum_{\substack{n\neq 0\\ d^2|n}} R(n) \ll QP^{s-1}PT^{-1/2}Q^{-3/4}. \tag{6.2}$$

Suppose first that $k=3$, so that $Q=P$. For $d\leq T$ we have

$$\sum_{\substack{n\\ d^2|n}} R(n) = \Big(\frac{2P}{d^2}+O(1)\Big)^s \rho_{\mathcal{P}}(d^2)$$
$$= \rho_{\mathcal{P}}(d^2)\frac{(2P)^s}{d^{2s}} + O\Big(\rho_{\mathcal{P}}(d^2)\frac{P^{s-1}}{d^{2s-2}}+\rho_{\mathcal{P}}(d^2)\Big).$$

Hence, by Lemma 2.5,

$$\sum_{d\leq T}\mu(d)\sum_{\substack{n\neq 0\\ d^2|n}} R(n) = (2P)^s\sum_{d=1}^{\infty}\mu(d)\frac{\rho_{\mathcal{P}}(d^2)}{d^{2s}} + E$$

where $E \ll TR(0)+P^sT^{\varepsilon-1}+T^{1+\varepsilon}P^{s-1}+T^{2s-1+\varepsilon}$. Clearly $R(0)\ll P^{s-1}$, so by (6.2),

$$N_{\mathcal{P}}(\mathbf{P}) = (2P)^s\sum_{d=1}^{\infty}\mu(d)\frac{\rho_{\mathcal{P}}(d^2)}{d^{2s}} + E'$$

where $E' \ll P^{s+1/4+\varepsilon}T^{-1/2}+P^sT^{\varepsilon-1}+T^{1+\varepsilon}P^{s-1}+T^{2s-1+\varepsilon}$. The equality $T=P^{\frac{4s+1}{8s-2}}$ gives

$$E' \ll P^{s-\frac{1}{8s-2}+\varepsilon}.$$

Also, by Lemma 2.5, and since $\rho_{\mathcal{P}}(d^2)$ is multiplicative, we have

$$\sum_{d=1}^{\infty}\mu(d)\frac{\rho_{\mathcal{P}}(d^2)}{d^{2s}} = \prod_p\Big(1-\frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\Big).$$

Now suppose that $k=4$, so that $Q=P^{3/4}$. This is more delicate. Let

$$X=\frac{1}{3}\log P, \qquad D=\prod_{p\leq X} p$$

and define

$$N_0 = \sum_{n \neq 0} \sum_{d^2 | (D^2, n)} \mu(d) R(n), \qquad N(U, V) = \sum_{U < p \leq V} \sum_{\substack{n \\ p^2 | n}} R(n).$$

Then clearly every n counted by $N_{\mathcal{P}}(\mathbf{P})$ is counted by N_0, and those n counted by N_0 but not by $N_{\mathcal{P}}(\mathbf{P})$ will have a factor p^2 with $p > X$. Thus

$$N_0 - N(P^{8/9}, \infty) - N(|c|Q, P^{8/9}) - N(P^{1/2}, |c|Q) - N(X, P^{1/2}) \\ \leq N_{\mathcal{P}}(\mathbf{P}) \leq N_0.$$

Inequality (6.1) gives

$$N(P^{8/9}, \infty) \ll QP^{s - 4/9 - 9/16 + \varepsilon} = QP^{s - 1/144 + \varepsilon}.$$

In the inner sum in $N(|c|Q, P^{8/9})$, given p with $|c|Q < p \leq P^{8/9}$ we are counting those $\mathbf{x}$ such that $|x_1| \leq Q$, $|x_j| \leq P$ $(2 \leq j \leq s)$ and $p^2 \mid cx_1^4 + \mathcal{C}^*(x_2, \ldots, x_s)$. When $j \geq 2$, write $x_j = u_j + pv_j$ where $1 \leq u_j \leq p$, and so $|v_j| \ll P/p$. When $x_1 = 0$, the number of choices for $u_2, \ldots, u_s$ is $\ll p^{s-2}$ and so the number of choices for $\mathbf{x}$ with $x_1 = 0$ is

$$\ll P^{s-1} p^{-1}.$$

When $x_1 \neq 0$, from $p > c|x_1|$ we deduce $p \nmid \mathcal{C}^*(u_2, \ldots, u_s)$. Hence, by the Euler relation, $u_2, \ldots, u_s$ is a non-singular point modulo p of $\mathcal{C}^*$. Moreover, given $x_1 \neq 0$ the number of choices for $u_2, \ldots, u_s$ is $\ll p^{s-2}$, and for each such, since $p^2 > P$, the number of choices for $v_2, \ldots, v_s$ is $\ll (P/p)^{s-2}$. Thus the total number of choices for $\mathbf{x}$ with $x_1 \neq 0$ is

$$\ll Qp^{s-2}(P/p)^{s-2} = QP^{s-2}.$$

Therefore

$$N(|c|Q, P^{8/9}) \ll \sum_{|c|Q < p \leq P^{8/9}} (P^{s-1}p^{-1} + QP^{s-2}) \ll QP^{s-10/9}.$$

When $P^{1/2} < p \leq |c|Q$, we proceed in the same way, but now the number of choices for $x_1 \equiv 0 \pmod{p}$ is $\ll Q/p$. Thus

$$N(P^{1/2}, |c|Q) \ll \sum_{P^{1/2} < p \leq |c|Q} (QP^{s-1}p^{-2} + QP^{s-2}) \ll QP^{s-5/4}.$$

For the sum $N(X, P^{1/2})$, for a given prime p with $X < p \leq P^{1/2}$ we divide the $\mathbf{x}$ into residue classes modulo p^2 and obtain the bound

$$\sum_{X < p \leq P^{1/2}} \left(\frac{Q}{p^2} + 1 \right) \frac{P^{s-1}}{p^{2s-2}} \rho_{\mathcal{P}}(p^2) \\ \ll QP^{s-1} \sum_{X < p \leq P^{1/2}} \frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}} + P^{s-1} \sum_{X < p \leq P^{1/2}} \frac{\rho_{\mathcal{P}}(p^2)}{p^{2s-2}},$$

and by Lemma 2.5 this gives

$$N(X,P^{1/2}) \ll \frac{QP^{s-1}}{X\log X} + QP^{s-5/4}.$$

This leaves N_0. By the prime number theorem, $D = \exp(\vartheta(X)) \le P^{1/2}$. Hence

$$N_0 = \sum_{d|D}\mu(d)\sum_{m\neq 0}R(md^2) = \sum_{d|D}\mu(d)\sum_{m}R(md^2) + O(P^{s-1/2}).$$

The new inner sum is the number of $\mathbf{x}$ with $\mathcal{P}(\mathbf{x}) \equiv 0 \pmod{d^2}$ and this is

$$\begin{aligned}\left(\frac{Q}{d^2}+O(1)\right)&\left(\frac{P}{d^2}+O(1)\right)^{s-1}\rho_{\mathcal{P}}(d^2)\\ &= QP^{s-1}\frac{\rho_{\mathcal{P}}(d^2)}{d^{2s}} + O\biggl(P^{s-1}\frac{\rho_{\mathcal{P}}(d^2)}{d^{2s-2}} + \rho_{\mathcal{P}}(d^2)\biggr).\end{aligned}$$

Hence, by Lemma 2.5,

$$N_0 = QP^{s-1}\prod_{p\le X}\left(1-\frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\right) + O(P^{s-1/2+\varepsilon}).$$

To complete the proof we observe that by Lemma 2.5 again,

$$\prod_{p>X}\left(1-\frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\right) = \exp\biggl(-\sum_{p>X}\log\biggl(1-\frac{\rho_{\mathcal{P}}(p^2)}{p^{2s}}\biggr)\biggr) = 1 + O\biggl(\frac{1}{X\log X}\biggr).$$

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