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Ergodicity in some families of Nevanlinna functions

by

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Abstract. We study *Nevanlinna functions* f, that is, transcendental meromorphic functions having N asymptotic values and no critical values. Keen and Kotus (1999) proved that if the orbits of all the asymptotic values have accumulation sets that are compact and on which f is a repeller, then f acts ergodically on its Julia set. In the present paper we prove that if some but not all of the asymptotic values have this property, while the others are prepoles, the same holds true. This is the first paper to consider this mixed case.

1. Introduction. An early result of McMullen [M] says that if f is a rational map of degree greater than 1, and if $P(f)$ is its post-singular set, one of two things holds: either f's Julia set is the whole Riemann sphere and the action of f is ergodic, or it is not the whole sphere and the spherical distance $d(f^n(z), P(f))$ tends to 0 for almost every z in $J(f)$ as $n \to \infty$, that is, the ω-limit set $\omega(z)$ is a subset of $P(f)$ that varies with z. Bock [B2] proved a similar result for meromorphic functions. This begs the question: under what conditions on a meromorphic function whose Julia set is the whole sphere is the action of the function on the sphere ergodic (or not)? In the realm of entire functions, Lyubich [Lyu] proved that the exponential function e^z, whose Julia set is the sphere, is not ergodic, and Bock [B1] proved that if the set of singular values of an entire function is finite, and all of these are preperiodic but not periodic, then the map is ergodic. In the realm of meromorphic functions, Bock [B2] showed (see also [RVS, Theorem 3.3] for another proof) that if the "radial Julia set", a subset of the Julia set, has positive measure, the action is ergodic. Other earlier results dealt with the particularly simple example of meromorphic functions with two asymptotic values and no critical values. There are partial results on the

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ergodicity question for this family: Let $\lambda, \mu \in \mathbb{C}$, and

$$f = \frac{\lambda e^z - \mu e^{-\mu}}{e^z - e^{-z}},$$

where λ, μ are the two asymptotic values of f. Keen and Kotus [KK] have shown that if the accumulation sets of both λ and μ are compact, and f is a repeller on these sets, then the Julia set is $\widehat{\mathbb{C}}$ and f is ergodic. By way of contrast, Skorulski [S1, S2] has shown that if there exist natural numbers p and q such that $f^p(\lambda) = f^q(\mu) = \infty$, then the Julia set is $\widehat{\mathbb{C}}$ and f is non-ergodic.

Weiyuan Qiu asked one of the authors what happens in the remaining case where one asymptotic value lands on a repelling cycle, and the other is a prepole. In answering his question, we were able to prove a more general result for the full family of functions with finitely many asymptotic values and no critical values, so-called "Nevanlinna functions":

Main Theorem. *If f is a Nevanlinna function with N asymptotic values of which $0 < K < N$ are prepoles, and if the ω-limit sets of the remaining $N - K$ are compact repellers, then the Julia set is $\widehat{\mathbb{C}}$ and f is ergodic.*

Remark. Our proof of this theorem implies that for these Nevanlinna functions, the measure of the radial Julia set is positive.

The case $K = 0$ was analyzed in [KK]. For the case $K = N$, we have the following conjecture which we are still working on and will report on in a future paper.

Conjecture 1. *When $K = N$, the action of f on its Julia set $\widehat{\mathbb{C}}$ is not ergodic.*

The proof of our theorem depends on generalizations of some lemmas in [KK]. After an introductory section in which we give the basic definitions and properties of Nevanlinna functions, we state and prove these lemmas and apply them to the proof of the theorem.

2. Preliminaries. In this section, we recall some of the basic theory of transcendental meromorphic functions. Such a function, $f : \mathbb{C} \to \widehat{\mathbb{C}}$, is holomorphic except at the set of poles, $\{f^{-1}(\infty)\}$, and is a local homeomorphism everywhere except at the set S_f of singular points. In this paper, we will be interested in those functions for which $\#S_f$ is finite and will assume this throughout. For such functions, the singular values are of two types. Let v be a singular value and let V be a neighborhood of v. Then:

- If, for some component U of $f^{-1}(V)$, there is a $u \in U$ such that $f'(u) = 0$, then u is a *critical point* and $v = f(u) \in V$ is the corresponding *critical value*.

- If, for some component U of $f^{-1}(V)$, $f : U \to V \setminus \{v\}$ is a universal covering map, then v is a *logarithmic asymptotic value*. The component U is called an *asymptotic tract* for v. Any path $\gamma(t) \in U$ such that $\lim_{t\to 1} \gamma(t) = \infty$, $\lim_{t\to 1} f(\gamma(t)) = v$ is called an *asymptotic path* for v.

At regular or non-singular points, meromorphic functions are local homeomorphisms. The dynamics of meromorphic functions with finitely many singular values have been the focus of many dynamical studies. In particular, all their asymptotic values are isolated and hence logarithmic. We, therefore, drop the descriptor "logarithmic" below and call them asymptotic values.

An important tool in studying meromorphic functions with finitely many critical points and finitely many asymptotic values is that they can be characterized by their Schwarzian derivatives.

Definition 1. If $f(z)$ is a meromorphic function, its *Schwarzian derivative* is

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarzian differential operator satisfies the chain rule condition

$$S(f \circ g) = S(f) g'^2 + S(g),$$

from which it is easy to deduce that if f is a Möbius transformation, then $S(f) = 0$, so that $f \circ g$ and g have the same Schwarzian derivative.

In [N, Chap. XI, §3], Nevanlinna, using a technique he calls rational approximation, shows how to, given a finite set of points in the plane and finite or infinite branching data for these points, construct a meromorphic function whose topological covering properties are determined by this data. The function is defined up to Möbius transformations. He proves

Theorem 1. *The Schwarzian derivative of a meromorphic function with finitely many critical points and finitely many asymptotic values is a rational function. If there are no critical points, it is a polynomial. Conversely, if a meromorphic function has a rational Schwarzian derivative, then it has finitely many critical points and finitely many asymptotic values. If the Schwarzian derivative is a polynomial of degree m, then the meromorphic function has $m+2$ asymptotic values and no critical points.*

In the literature, meromorphic functions with polynomial Schwarzian derivative are often called *Nevanlinna functions* (see e.g. [C, EM]). These are the focus of this paper.

To prove our results, we will need estimates on the asymptotic behavior of the poles and residues of Nevanlinna functions, summarized in Proposition 5 at the end of this section. These are well-known, and there is extensive literature; see, e.g., [H, Chap. 5], [L, Chap. 4], or [C] for details.

We begin by recalling the connection between Nevanlinna functions and the second-order differential equation

$$w'' + P(z)w = 0, \tag{1}$$

where $P(z)$ is a polynomial of degree m. The solutions of (1) are holomorphic and form a two-dimensional linear space. A straightforward calculation shows that if w_1, w_2 are linearly independent solutions of (1), then $f = w_1/w_2$ is meromorphic and $S(f) = 2P(z)$.

To develop some intuition for discussing Nevanlinna functions, let us begin with a "toy" example. Let $P(z) = 1$ so that (1) becomes

$$w'' + w = 0. \tag{2}$$

It is easy to check that its solutions lie in the two-dimensional space generated by the "principal" solutions $w_1 = e^{iz}$ and $w_2 = e^{-iz}$ and that the quotient f of these two, or of any pair of linearly independent solutions of (2), satisfies $S(f) = 2$.

Define the ray $\rho_0(t) = \{z = t > 0\}$, and for any $R > 0$ and $\epsilon \in (0, \pi)$ define the sector

$$\mathcal{S}(R, \epsilon) = \mathcal{S} = \{z \colon |z| > R,\ |\arg z - \pi| > \epsilon\}.$$

Obviously, $\mathcal{S} \setminus \rho_0(t)$ consists of two components, $\mathcal{U}^+$ containing an infinite segment of the positive imaginary axis and $\mathcal{U}^-$ containing an infinite segment of the negative imaginary axis. Note that w_1 maps $\mathcal{U}^+$ to a punctured neighborhood of zero and maps $\mathcal{U}^-$ to a punctured neighborhood of ∞ while w_2 interchanges the images. Thus, 0 and ∞ are asymptotic values of w_i, and these two components are their respective asymptotic tracts. Note that along the ray $\rho_0(t)$ separating these two asymptotic tracts, the function w_1/w_2 assumes every value in the unit circle infinitely often.

Similarly, for any $\delta > 0$ there is a large T such that in the neighborhood $\{z \colon |z| > T,\ |\arg z| < \delta\}$, $f = w_1/w_2$ takes every value in $\mathbb{C}^*$ infinitely often. The ray $\rho_0(t)$ is called the *critical ray* and the argument $\theta_0 = 0$ of $\rho_0(t)$ is called the *critical direction*.

For a general Nevanlinna function, we define its *critical rays* and *critical directions* as follows:

Definition 2. Let f be a Nevanlinna function with Schwarzian derivative $2P(z)$ of degree $m \geq 0$ and suppose the leading coefficient of $P(z)$ is a. Set $N = m + 2$. Then each of the solutions θ_k, $k \in \{0, 1, \ldots, N-1\}$, of the congruence

$$\arg a + N\theta \equiv 0 \bmod 2\pi$$

determines a direction and a ray $\rho_k(t) = \{te^{\theta_k i} \colon t > T\}$ at infinity. Thus, each solution determines a *critical direction* and a *critical ray* of f [1].

[1] See e.g. [C, p. 11772]. These are often also called Julia directions and Julia rays of f.

We will show that, as in the toy example, there is a sector containing the critical ray $\rho_k(t) = \{te^{\theta_k i}\colon t > 0\}$, for each $0 \le k < N$, on which every solution f of the Schwarzian equation takes on infinitely many values infinitely often.

To do this, for each k, defined $\bmod N$, we make a change of variable that essentially turns the sector

$$S_k = \{\theta_{k-1} < \arg z < \theta_{k+1}\}$$

of the z-plane into a sector of a $Z(z)$-plane on which the transported function acts like the toy example. More precisely, in the rest of this section, assume that both $R_0 > 0$ and the solution θ_k are fixed and define

$$Z(z) = \int_{R_0 e^{i\theta_k}}^{z} P(s)^{1/2}\, ds, \qquad z \in S_k,$$

where the branch of the square root is chosen so that after integration $(az^N)^{1/2}$ is real and positive on $\rho_k(t)$.

LEMMA 2. *For some small $\epsilon_0 > 0$, the function $Z = Z(z)$ satisfies*

$$Z(z) = \frac{2a^{1/2}}{N} z^{N/2}(1 + \mathrm{o}(1)) \quad \text{as } z \to \infty, \qquad |\arg z - \theta_k| \le \frac{2\pi}{N} - \epsilon_0.$$

Moreover, for any $R > R_0$ and $\epsilon > \epsilon_0$, $Z(z)$ is univalent on the sector

$$\mathcal{S} = \{z\colon |z| > R,\ |\arg z - \theta_k| < 2\pi/N - \epsilon\}$$

and Z maps $\mathcal{S}$ onto a region in the Z-plane containing the sector

$$\mathcal{T} = \{Z\colon |Z| > R',\ |\arg Z - \pi| > \epsilon'\},$$

where R' is large and $\epsilon' > N\epsilon/2$.

Proof (see [L, Lemma 4.3.6]). Since $P(s)^{1/2} = a^{1/2}s^{m/2}(1 + \mathrm{o}(1))$ for large $|s|$, it follows that

$$(3) \qquad Z(z) = \int_{R_0 e^{i\theta_k}}^{z} P(s)^{1/2}\, ds = \frac{2a^{1/2}}{N} z^{N/2}(1 + \mathrm{o}(z)) \quad \text{for large } |z|.$$

Thus, the auxiliary map

$$\xi = \frac{2a^{1/2}}{N} z^{N/2}$$

maps the sector

$$\mathcal{S}_1 = \left\{z\colon |z| > R_0,\ |\arg z - \theta_k| < \frac{2\pi}{N} - \frac{\epsilon}{2}\right\}$$

univalently onto the sector

$$\Sigma' = \left\{\xi\colon |\xi| > R_0',\ |\arg \xi| < \pi - \frac{N\epsilon}{4}\right\}$$

in the ξ-plane for some large $R_0' > 0$. From (3), it follows that $|Z(z)-\xi(z)| = o(|z|^{N/2})$ on $\mathcal{S}$ and therefore $Z(z)$ is univalent on $\mathcal{S}$, so that its image in the Z-plane contains a sector of the form $\mathcal{T}$ as well. ∎

Next, the *Liouville transformation*

$$W(Z) = P(z)^{1/4} w(z)$$

transforms equation (1) into a new one for $W(Z)$ as follows:

$$\text{(4)} \quad W''(Z)+(1-F(Z))W(Z)=0, \quad \text{where} \quad F(Z)=\frac{1}{4}\frac{P''(z)}{P(z)^2}-\frac{5}{16}\frac{P'(z)^2}{P(z)^3}.$$

For $R' \gg 0$ and $\delta \in (0,\pi)$, let $\mathcal{T} = \{Z\colon |Z| > R',\ |\arg Z - \pi| > \delta\}$ be a sector in the Z-plane. On $\mathcal{T}$, $F(Z) = O(1/Z^2)$ so that the solutions to (4) are asymptotic to the solutions of (2). The solution space is generated by

$$\text{(5)} \qquad W_1(Z) = e^{iZ}(1 + O(1/|Z|)) \quad \text{and} \quad W_2(Z) = e^{-iZ}(1 + O(1/|Z|)).$$

Each $W_i(z)$ has two asymptotic values, and the sector $\mathcal{T}$ contains their asymptotic tracts separated by the critical ray: the positive real line. Pulling back to the $z-$plane by the map $Z(z)$, we obtain two linearly independent "principal solutions"

$$w_i(z) = P(z)^{-1/4} W_i(Z), \qquad i = 1, 2,$$

of the original second order equation (1), defined in the sector

$$\mathcal{S}_k = \{z\colon |z| > R,\ |\arg z - \theta_k| < 2\pi/N - \delta'\},$$

where R is a large constant and δ' is a small constant depending on δ.

Using the asymptotic expressions in (5), we see that

$$\text{(6)} \qquad F(Z) = \frac{AW_1(Z) + BW_2(Z)}{CW_1(Z) + DW_2(Z)} \sim \frac{Ae^{iZ} + Be^{-iZ}}{Ce^{iZ} + De^{-iZ}}$$

has two asymptotic values with asymptotic tracts separated by the positive real line. Since, by Lemma 2, $Z(z)$ is univalent, we see that

$$f(z) = F(Z(z)) = \frac{Aw_1(z) + Bw_2(z)}{Cw_1(z) + Dw_2(z)} = \frac{AW_1(Z) + BW_2(Z)}{CW_1(Z) + DW_2(Z)}$$

has two asymptotic values with asymptotic tracts separated by the critical ray $\rho_k(t)$ in the sector $\mathcal{S}_k$.

Remark 2.1. Since there are N solutions to the congruence, there are N possible choices for θ_k. Applying the above transformation theory to each solution defines a sector in the z-plane containing a central critical ray and bounded by its adjacent rays. The pullback solutions for each solution have two asymptotic values with asymptotic tracts in the complement of the critical ray. Pairs of adjacent sectors overlap on one asymptotic tract. Thus, f has N asymptotic values.

Equation (5) also shows that the w_i have no zeros in the sectors where they are defined but, for any $A, B \in \mathbb{C}^*$, the equation $Aw_1 + Bw_2 = 0$ has infinitely many zeros. We next show that these zeros accumulate along the critical rays.

PROPOSITION 3. *Let w_1, w_2 be the principal solutions defined in the sector $\mathcal{S}$ containing the critical ray $\rho_k(t)$. If A, B are non-zero constants and $w = Aw_1 + Bw_2$ then $w = 0$ has infinitely many solutions s_j in $\mathcal{S}$. Label them so that $\cdots \le |s_j| \le |s_{j+1}| \le \cdots$. Then $|s_j| \sim \mathrm{O}(|j|^{2/N})$ and $\lim_{j\to\infty} |\arg s_j - \theta_k| = 0$.*

Proof (see also [L, p. 64]). Set $G(z) = \frac{1}{2i}\log\frac{W_1}{W_2}$. By Lemma 2, if $z \in \mathcal{S}$ is sufficiently large,

$$G(z) = Z(z) + \mathrm{o}(1) = \frac{2a^{1/2}}{N} z^{N/2}(1 + \mathrm{o}(1)).$$

Furthermore, the zeros s_j of w satisfy

$$2iG(s_j) = \log\left(-\frac{B}{A}\right) + 2j\pi i \quad \text{or} \quad G(s_j) = \frac{1}{2i}\log\left(-\frac{B}{A}\right) + j\pi;$$

that is, $G(s_j)$ lies near the positive line.

Combining these, we have

$$\frac{2a^{1/2}}{N} s_j^{N/2}(1 + \mathrm{o}(1)) = \frac{1}{2i}\log\left(-\frac{B}{A}\right) + j\pi.$$

Hence as $j \to \infty$, $\arg s_j \sim \theta_k$; and there is a constant c_1 such that $s_j \sim c_1|j|^{2/N}$. ■

For any function

$$f(z) = \frac{aw_1 + bw_2}{cw_1 + dw_2} \tag{7}$$

its poles are zeros of $cw_1 + dw_2$, and the estimate of the residue at each pole can be computed using the definition: $\mathrm{Res}(f, s_j) = \lim_{z\to s_j}(z - s_j)f(z)$.

PROPOSITION 4 ([C, p. 6]). *Let f be as in (7). Denote the poles of f in $\mathcal{S}$ and their respective residues by s_j and r_j, and assume the poles are labeled so that $\cdots \le |s_j| \le |s_{j+1}| \le \cdots$. Then*

$$r_j = \frac{1}{2i}\left(\frac{a}{c} - \frac{b}{d}\right)P(s_j)^{-1/2} \sim c_2 \cdot s_j^{-(N-2)/2}$$

for some constant c_2.

From the two propositions above, it follows that the relation between the residues and the poles is

$$|r_j| \sim c_2|s_j|^{-(N-2)/2} \sim c_3|j|^{-(N-2)/N}. \tag{8}$$

Proposition 5. *As above, let f be a solution of $S(f) = 2P$ where P is a polynomial of degree m. Set $N = m+2$. For any $z_0 \in \mathbb{C}$, denote its preimages in a given sector $\mathcal{S}$ by p_j and label them so that $\cdots \le |p_j| \le |p_{j+1}| \le \cdots$. Then there exists a constant $c_4 > 0$ such that $|f'(p_j)| \sim c_4 lj|^{(N-2)/N}$.*

Proof. Let

$$g(z) = \frac{1}{f(z) - z_0};$$

then $S(g) = S(f) = 2P$. If p_j are solutions of $f(z) = z_0$, they are poles of $g(z)$ so that Proposition 4 implies $|\mathrm{Res}(g, p_j)| \sim c_3|j|^{-(N-2)/N}$. Furthermore, since $g(z) = 1/(f(z) - z_0)$, a simple computation shows that

$$|f'(p_j)| = \frac{1}{|\mathrm{Res}(g, p_j)|} \sim c_4|j|^{\frac{N-2}{N}}. \ \blacksquare$$

In the proofs of our results we will repeatedly use the Koebe distortion theorems to obtain estimates on the behavior of the Nevanlinna functions at regular points. Many proofs exist in the standard literature on conformal mappings (see e.g. [A, Theorem 5.3]). For the reader's convenience, we state the theorems here without proof.

Theorem 6 (Koebe distortion theorem). *Let $f : D(z_0, r) \to \mathbb{C}$ be a univalent function. Then for any $\eta < 1$,*

(i) $|f'(z_0)|\dfrac{\eta r}{(1+\eta)^2} \le |f(z) - f(z_0)| \le |f'(z_0)|\dfrac{\eta r}{(1-\eta)^2}$, $z \in D(z_0, \eta r)$,

(ii) *if $T(\eta) = (1+\eta)^4/(1-\eta)^4$, then*

$$\frac{|f'(z)|}{|f'(w)|} \le T(\eta) \quad \text{for any } z, w \in D(z_0, \eta r).$$

Theorem 7 (Koebe 1/4 theorem). *Let $f : D(z_0, r) \to \mathbb{C}$ be a univalent function. Then*

$$D(f(z_0), r|f'(0)|/4) \subset f(D(z_0, r)).$$

3. The main theorem. Let $\mathcal{F}_N$ be the set of Nevanlinna functions with N asymptotic values. For $f \in \mathcal{F}_N$ and $i = 1, \ldots, N$ denote the asymptotic values by λ_i and the corresponding asymptotic tracts by T_i. Assume there is an integer K, $1 \le K < N$, and integers $p_i \ge 0$, $i = 1, \ldots, K$, such that

$$f^{p_i}(\lambda_i) = \infty, \qquad i = 1, \ldots, K.$$

If $\lambda_i = \infty$, then $p_i = 0$. This can happen for at most $N/2$ asymptotic values and the asymptotic tracts of these infinite asymptotic values must be separated by the asymptotic tract of a finite asymptotic value. Also assume that for each $i = K+1, \ldots, N$, the accumulation set $\omega(\lambda_i)$ of the orbit of λ_i is a compact repeller; that is, there exists a $\kappa > 1$ such that, for each

$z \in \omega(\lambda_i)$, there exists an $n = n(z)$ such that $|(f^n)'(z)| > \kappa$. Note that this implies that these asymptotic values are finite.

Define

$$I = I(f) = \{z \in \mathbb{C}\colon f^n(z) \to \infty\},$$
$$L = L(f) = \Big\{z \in \mathbb{C}\colon \omega(z) = \bigcup_{i=K+1}^{N} \omega(\lambda_i)\Big\}.$$

The proof of the Main Theorem depends on the following theorems:

THEOREM 8. *The set I is of measure zero.*

THEOREM 9. *The set L is of measure zero.*

Versions of these theorems are proved in [KK] under the assumption that all of the asymptotic values accumulate on a compact repeller.

3.1. The measure of the set I. For each $1 \le i \le K$, define the orbit of the prepole asymptotic value λ_i by

$$\mathrm{Orb}(\lambda_i) = \{\lambda_i, f(\lambda_i), \ldots, f^{p_i-1}(\lambda_i), \infty\}.$$

If $\lambda_i = \infty$ for some i, then $\mathrm{Orb}(\lambda_i) = \{\infty\}$.

Let $S = \{1, \ldots, K\}$. Since there are $2^K - 1$ distinct non-empty subsets of S, label them S_l, $l = 1, \ldots, 2^K - 1$, and denote the collection by Σ. For any S_l, define

$$\mathrm{Orb}_l = \bigcup_{i \in S_l} \mathrm{Orb}(\lambda_i).$$

For $S_l \in \Sigma$, where $l = 1, \ldots, 2^K - 1$, define

$$I_l = I_l(f) = \{z \in \mathbb{C}\colon \omega(z) = \mathrm{Orb}_l\}.$$

THEOREM 10. *Each of the sets I_l is of measure zero.*

The proof of this theorem depends on the next two lemmas.

Fix $R \gg 0$, and let $\mathcal{A}_R = \{z \in \mathbb{C}\colon |z| > R\}$. For each $1 \le i \le K$ and $j \in \mathbb{Z}$, let $b_{ij} = f^{-1}(\lambda_i)$.

Because λ_i is a prepole of order p_i, one component of $f^{-p_i}(\mathcal{A}_R)$ is the topological disk D_i punctured at λ_i. Therefore the set of components of $f^{-1}(D_i)$ consists of the asymptotic tract T_i of λ_i and the topological disks V_{ij} punctured at b_{ij}. For each $S_l \in \Sigma$ and $z \in \bigcup_{i \in S_l}(\bigcup_j V_{ij} \cup T_i)$, define the map $\sigma_l(z) = f^{p_i+1}(z)$.

LEMMA 11. *If $z \in \bigcup_{i=1}^{K} T_i$, then*

$$|\sigma_l'(z)| > \frac{\big|\log|\sigma_l(z)| - \log R\big|}{4\pi} \cdot \frac{|\sigma_l(z)|}{|z|}.$$

Proof. Since $f^{p_i} : D_i \to \mathcal{A}_R$ is conformal and $f : T_i \to D_i$ is a universal covering, it follows that $\sigma_l : T_i \to \mathcal{A}_R$ is also a universal covering. The rest of the proof given below follows along the lines of the corresponding proof in [Lyu].

Consider $H_R = \log \mathcal{A}_R$, the right half-plane with real part greater than R, and let $\mathcal{U}_l = \log(\bigcup_{i\in S_l} T_i)$. Then $\mathcal{U}_l \subset H_R$ and $\mathcal{U}_l$ consists of infinitely many disjoint simply connected components U_{im}, $i \in S_l$, $m \in \mathbb{Z}$; moreover, there is an $\epsilon_{im} > 0$, depending on R, such that each U_{im} is fully contained inside a strip of height $2\pi - \epsilon_{im}$. Because there are at most K sets U_{im}, the sum of their heights is less than $2\pi K/N - \epsilon_R$ where $\epsilon_R = \sum \epsilon_{im}$ depends on R and S_l.

For each U_{im} there is a conformal map F_{im} to H_R such that the following diagram commutes:

$$\begin{array}{ccc} U_{im} & \xrightarrow{F_{im}} & H_R \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ T_i & \xrightarrow{\sigma_l} & \mathcal{A}_R \end{array}$$

For each point $z_0 \in \bigcup T_i$, denote the lifts of z_0 and $\sigma_l(z_0)$ by $w_0 \in U_{im}$ and $w_1 \in H_R$ respectively. Note that $\Re w_1 = \log|\sigma_l(z)|$. Consider $D = D(w_1, \Re w_1 - \log R)$ and its preimage under F_{im}. By the Koebe 1/4 theorem, the preimage contains a disk of radius

$$\frac{\Re w_1 - \log R}{4|F'_{im}(w)|}.$$

As the width of each strip is less than 2π, we have

$$|F'_{im}(w)| \geq \frac{\Re w_1 - \log R}{4\pi}.$$

The lemma now follows from the chain rule. ∎

Theorem 8 is a direct consequence since $I \subset \bigcup_{S_l \in \Sigma} I_l$.

The next lemma is the analog of Lemma 11 in the case that $S_l \in \Sigma$ and $z \in \bigcup_{i\in S_l}(\bigcup_j V_{ij})$. Fix $i \in S_l$ and, suppressing the index i for readability, denote the zeros of $f(z) - \lambda_i = 0$ by b_j. Note that by Proposition 5, $|f'(b_j)| \sim c|j|^{(N-2)/N}$.

Lemma 12. *There exists a neighborhood V'_j of b_j and a constant $b > 0$ such that $V'_j \subset \overline{V'_j} \subset V_j$,*

$$V'_j \subset D\biggl(b_j, \frac{b}{|j|^{(N-2)/N} R}\biggr),$$

and for $z \in U_j$ and some constant $B > 0$,

$$|\sigma'_l(z)| > BR|j|^{(N-2)/N}.$$

Proof. For each $\lambda_i \in S_l$, denote the pole $f^{p_i-1}(\lambda_i)$ by s_i. Then expanding f at s_i, we get

$$f(z) = \frac{r_i}{z - s_i}(1 + \phi_i(z)),$$

where r_i is the residue of f at s_i, and ϕ_i is analytic at s_i. Consider the annular region $\mathcal{A}_{2R} \subset \mathcal{A}_R$, and denote by g the branch of f^{-1} such that $g(\mathcal{A}_R)$ is a punctured neighborhood of s_i. Set $h = g(1/z) : D(0, 1/R) \to \mathbb{C}$, so that 0 is a removable singularity, and let $U = h(D(0, 1/(2R))) = g(\mathcal{A}_{2R}) \cup \{\infty\}$. Then h is conformal and $h'(0) = r_i$. The Koebe distortion theorem applied to h proves that for any $z \in D(0, 1/(2R))$,

$$|h(z) - h(0)| \le |h'(0)| \frac{\frac{1}{2} \cdot \frac{1}{R}}{\left(1 - \frac{1}{2}\right)^2} = \frac{2r_i}{R}.$$

The Koebe 1/4 theorem applied to h on $D(0, 1/(2R))$ proves $D(s_i, r_i/8R) \subset U$. Combining these gives

$$D\left(s_i, \frac{|r_i|}{8R}\right) \subset U \subset D\left(s_i, \frac{2|r_i|}{R}\right).$$

Therefore, for any $z \in U$,

$$|f'(z)| = \left| -\frac{r_i(1 + \phi_i(z))}{(z - s_i)^2} + \frac{r_i \phi_i'(z)}{z - s_i} \right| \ge \left| \frac{r_i}{z - s_i} \right| \ge \frac{R}{2}. \tag{9}$$

Since f has no critical points, the disk $D(s_i, 4|r_i|/R)$ at the pole s_i is mapped univalently by the respective branches of f^{-p_i} onto neighborhoods of the points $b_j = f^{-p_i}(s_i)$. Let V_j' be the component of $f^{-p_j}(U) = f^{-(p_i+1)}(\mathcal{A}_{2R})$ at s_j. It is obvious that $\overline{V_j'}$ is contained in V_j, a component of $f^{-(p_i+1)}(\mathcal{A}_R)$. Since

$$U_i \subset D\left(s_i, \frac{2|r_i|}{R}\right) \subset D\left(s_i, \frac{4|r_i|}{R}\right),$$

the Koebe distortion theorem implies that for any $z, w \in \widetilde{V}_{ij}$,

$$\frac{(f^{p_i})'(z)}{(f^{p_i})'(w)} \le T(1/2). \tag{10}$$

By Proposition 5, for some $c_1 > 0$, we have $|f'(b_j)| \sim c_1 |j|^{(N-2)/N}$. Since f is univalent on the orbit of λ_i, there exists $c_2 > 0$ such that $|(f^{p_i})'(b_j)| \sim c_2 |j|^{(N-2)/N}$ and thus

$$|(f^{p_i})'(z)| > c_2 T^{-1}(1/2)|j|^{(N-2)/N} \quad \text{for any } z \in V_j'.$$

Since $U \subset D(s_i, 2|r_i|/R)$, this implies

$$V_j' \subset D\left(b_j, \frac{2T(1/2)|r_i|}{c_2 |j|^{(N-2)/N} R}\right),$$

which, combined with (9), also implies that for all $z \in V_j'$,

$$\sigma_l'(z) \geq \frac{c_2 R |j|^{(N-2)/N}}{2T(1/2)}$$

and thus completes the proof. ∎

3.2. Proof of Theorem 10. Let $E = \{z\colon \sigma_l^n(z) \to \infty \text{ as } n \to \infty\}$; then

$$I_l(f) = \bigcup_{n=0}^{\infty} f^{-n}(E).$$

To prove Theorem 10, it suffices to show that the measure of the set E is zero. We assume otherwise and obtain a contradiction. Let z_0 be a Lebesgue density point of E, and let $z_n = \sigma_l^n(z_0)$. As $z_n \to \infty$, without loss of generality, we may assume that for each n, $|z_{n+1}| \geq |z_n| \geq R$. Set $\mathcal{A}_{r,s} = \{z\colon r < |z| < s\}$.

Since by hypothesis $K < N$, it follows that $\bigcup_{i=K+1}^{N} T_i \neq \emptyset$ but $(\bigcup_{i=K+1}^{N} T_i) \cap \sigma_l^{-1}(\mathcal{A}_R) = \emptyset$. Therefore, for any $s > r > R$, there is a $\tau > 0$ such that

$$\frac{m(\mathcal{A}_{r,s} \cap \sigma_l^{-1}(\mathcal{A}_{r,s}))}{m(\mathcal{A}_{r,s})} < 1 - \tau.$$

Note that if $K = N$, the asymptotic tracts fill up $\sigma_l^{-1}(\mathcal{A}_R)$. The proof of non-ergodicity for $K = N = 2$ in [S2] uses this fact and lends support to Conjecture 1.

The proof of Theorem 10 splits into two parts depending on the orbit of z_0.

Part 1. Assume that for all n, $z_n \in \bigcup_{i=1}^{K} T_i$. This part of the proof depends on Lemma 11 and uses the notation in that lemma.

As in the lemma, for $z_n \in T_i$, set $w_n = \log z_n \in U_{im}$ and $r_n = \Re w_n$. Then $F_{im}^{-1} : H_R \to U_{im}$ is the inverse branch such that $F_{im}^{-1}(w_n) = w_{n-1}$. The function F_{im}^{-1} is univalent in the disk $D(w_n, r_n - \log R)$. By Lemma 11, it follows that

$$|(F_{im}^{-1})'(w_n)| \leq \frac{4\pi}{r_n - \log R}.$$

Note that it may be that $z_{n-1} \in T_j$, $i \neq j \in S_l$, and similarly, w_{n-1} may be in a different U_{jm}. For the sake of readability, we will ignore these indices and write U for whichever U_{im} is meant and write F^{-1} for whichever inverse branch is meant.

Next, consider the disk $D(w_n, r_n/4)$. First note that since U does not intersect any of the preimages, $f^{-1}(T_i)$, $i = K+1, \ldots, N$, there exists a $\tau' > 0$ such that

$$\frac{m(D(w_n, r_n/4) \cap U)}{m(D(w_n, r_n/4))} < 1 - \tau'.$$

Moreover, for $w \in D(w_n, r_n/4)$, the Koebe distortion theorem implies that

$$|F^{-1}(w) - F^{-1}(w_n)| \le \frac{4\pi}{r_n - \log R} \cdot \frac{\eta(r_n - \log R)}{(1-\eta)^2},$$

where

$$\eta = \frac{r_n}{4(r_n - \log R))} < \frac{1}{2}.$$

Therefore,

$$F^{-1}(D(w_n, r_n/4)) \subset D(w_{n-1}, d) \quad \text{where} \quad d = 8\pi.$$

For each k with $1 \le k \le n-1$, F^{-1} is univalent in the disk $D(w_k, 2d)$ and $(F^{-1})'(w_k) \le 1/8$. The Koebe 1/4 theorem applies so that

$$F^{-1}(D(w_k, d)) \subset D(w_{k-1}, d/2).$$

Next, iterate F^{-1} and set $B_n = F^{-n}(D(w_n, r_n/4)) \subset D(w_0, 2^{-m+1}d)$. Now since the iterated function F^{-n} is univalent on $D(w_n, \Re w_n - \log R)$, apply Koebe distortion again to get

$$D(w_0, t\rho_n) \subset B_n \subset D(w_0, \rho_n),$$

where t is independent of n, and ρ_n is the radius of the smallest disk centered at w_0 containing B_n. It follows that $\rho_n \le 2^{-n+1}d$, which in turn implies that $\rho_n \to 0$ as $n \to \infty$.

Part (ii) of the Koebe distortion theorem applied to F^{-n} implies there exists a τ'' such that

$$\frac{m(B_n \cap E)}{m(E)} \le 1 - T^{-2}(1/2)\tau''$$

for all n. In other words, the Lebesgue density of the point w_0 is less than 1, which contradicts the assumption that w_0 is a density point.

Part 2. Now consider a subsequence $z_{n_k} \in \bigcup_{i \in S_l}(\bigcup_j V_j)$. Suppose that

$$\frac{m(\mathcal{A}_{2R} \cap W)}{m(\mathcal{A}_{2R})} < 1 - \tau'''.$$

By Lemma 12,

$$V_j' \subset D\left(b_j, \frac{b}{R|j|^{(N-2)/N}}\right),$$

and for any $z \in V_j'$,

$$\sigma_l'(z) \ge BR|j|^{(N-2)/N} \quad \text{and} \quad \frac{(f^{p_i})'(z)}{(f^{p_i})'(w)} \le T(1/2).$$

Therefore,

$$\frac{m(V_j' \cap \sigma_l^{-1}(W))}{m(V_j')} < 1 - T(1/2)^{-2}\tau'''.$$

Without loss of generality assume that $|z_{n+1}| \geq |z_n| \gg R$ for all n. Then the above inequality, together with Lemmas 11 and 12, show that $|\sigma_l'(z_n)| > M > 1$ for all n.

Let $B_{n_k} = \sigma_l^{-n_k}(V_j')$. Then

$$B_{n_k} \subset D\left(z_0, M^{-n_k}\frac{b}{R|j|^{(N-2)/N}}\right).$$

Since $\sigma_l^{-n_k}$ is univalent on $V_j \supset V_j'$, this implies

$$D(z_0, t\rho_{n_k}) \subset B_{n_j-1} \subset D(z_0, \rho_k),$$

where t is independent of n_j, and ρ_{n_k} is the radius of the smallest disk centered at z_0 containing B_{n_j-1}. It follows that $\rho_{n_k} \to 0$ as $n_k \to \infty$.

Applying part (ii) of the Koebe distortion theorem, we get

$$\frac{m(B_{n_k} \cap E)}{m(E)} \leq 1 - T^{-4}(1/2)\tau'''$$

for all n_k, which implies that the Lebesgue density of the point z_0 is less than 1. This contradicts the assumption that z_0 is a density point and completes the proof of Theorem 10.

3.3. Proof of Theorem 9. To prove $m(L) = 0$, note first that by assumption $\Omega = \bigcup_{i=K+1}^{N} \omega(\lambda_i)$ is a finite union of compact repellers, so it is again a compact repeller. This implies that the orbits of the non-prepolar asymptotic values do not accumulate on Ω, but actually land on it. The proof in [KK] assumes $K = 0$. Although the present proof is similar, here we modify it to take account of the prepole asymptotic values.

Let $\mathcal{K}_\epsilon = \{z\colon \operatorname{dist}(z, \Omega) < \epsilon\}$. We claim there is an $\epsilon > 0$ and an integer $M > 0$ such that if $y = \lambda_i$, $i = K+1, \ldots, N$, $n > M$ and $f^n(y) \in \mathcal{K}_{\epsilon/2}$, then $f^n(y) \in \Omega$.

If Ω is finite, the claim is obviously true, so assume it is infinite. By compactness, there are no prepoles in Ω and there are constants $\kappa > 1$ and $\epsilon > 0$ such that $|(f^n)'(w)| \geq \kappa$ for some m and all $w \in \mathcal{K}_\epsilon$, and thus for all $w \in \overline{\mathcal{K}_{\epsilon/2}}$. By the forward invariance of Ω and this expansion property, $\overline{\mathcal{K}_{\epsilon/2}} \subset f^n(\overline{\mathcal{K}_{\epsilon/2}})$. Let g be the inverse branch of f^n reversing this inclusion. Then set

$$\mathcal{A}_0 = \overline{\mathcal{K}_{\epsilon/2}} - g(\overline{\mathcal{K}_{\epsilon/2}}) \quad \text{and} \quad \mathcal{A}_{n+1} = g^n(\mathcal{A}_0), \quad n \to \infty.$$

These disjoint annuli are nested, and since the inverse branches are univalent, the annuli have the same moduli. Therefore, if for some n, $f^n(y) \in \overline{\mathcal{K}_\epsilon} \setminus \Omega$, then by compactness there are subsequences of its iterates that converge both to points in $\overline{\mathcal{A}_0}$ and to points in Ω. This is a contradiction because these sets are disjoint, and the claim is proved.

Choose ϵ as above and set

$$\mathcal{L} = \bigcap_{n\geq 0} f^{-n}(\mathcal{K}_{\epsilon/2}).$$

A point z is in $\mathcal{L} \setminus \Omega$ if its full forward orbit is in $\mathcal{K}_{\epsilon/2}$. We will show $m(\mathcal{L} \setminus \Omega) = 0$. Since $L \subset \bigcup_{n=0}^{\infty} f^{-n}(\mathcal{L})$ and Ω is countable, this will imply that $m(L) = 0$.

Suppose $m(\mathcal{L} \setminus \Omega) > 0$ and let z_0 be a density point of $\mathcal{L} \setminus \Omega$. Since Ω is compact, a subsequence $z_k = f^{n_k}(z_0)$ converges to a point $y_0 \in \overline{\mathcal{K}_{\epsilon/2}} \subset \mathcal{K}_\epsilon$. Denote the respective inverse branches by g_k. Set $D_k = D(z_k, \epsilon/4)$; then $D_k \subset \mathcal{K}_\epsilon$ and g_k is univalent on D_k. Applying Koebe distortion we obtain

$$\frac{m(g_k(D_k) \cap \mathcal{L})}{m(g_k(D_k))} \to 1 \quad \text{and} \quad \frac{m(D_k \cap f^{n_k}(\mathcal{L}))}{m(g_k(D_k))} \to 1.$$

Finally, let U be an open set with compact closure contained in $\mathbb{C} \setminus \overline{\mathcal{K}}$. Since the Julia set is the whole sphere, there is an integer M such that $f^M(D_k) \supset \overline{U}$ so that $m(f^{M+n_k}(\mathcal{L} \cap U)) > 0$. For all $k \in \mathbb{N}$, however, $f^k(\mathcal{L}) \subset \mathcal{K}_{\epsilon/2}$ so that $f^{M+n_k}(\mathcal{L} \cap U) = \emptyset$. This contradiction shows $m(L) = 0$ and completes the proof of Theorem 9. ∎

3.4. Proof of the Main Theorem. We reword the theorem:

THEOREM 13. *If f is a Nevanlinna function with $1 \leq K < N$ prepole* ([2]) *asymptotic values and $N-K$ asymptotic values that accumulate on a compact repeller, then f acts ergodically on its Julia set.*

Proof. Let A be an f-invariant subset of the Julia set with positive measure. We will show that $A = \widehat{\mathbb{C}}$ up to a set of measure zero. Let z_0 be a Lebesgue density point of A and denote its orbit by $z_n = f^n(z)$, $n = 0, 1, \ldots$. We proved above that the measure of each of the sets I, I_l and L is zero. Since these three sets together contain all points whose orbits accumulate on $\bigcup_{i=1}^{N} \omega(\lambda_i)$, we assume that z_0 is not among them.

By the above, the density point z_0 of A has an accumulation point $y \in \mathbb{C} \setminus \bigcup_{i=1}^{N} \omega(\lambda_i)$. Recall that $\Omega = \cup_{i=K+1}^{N} \omega(\lambda_i)$. Hence there is an $\epsilon > 0$ such that $2\epsilon = \operatorname{dist}(y, \Omega) > 0$. Thus, there is a subsequence $\{n_j\}$ in $\mathbb{N}$ such that $z_{n_j} \to y$ as $j \to \infty$ and $\operatorname{dist}(z_{n_i}, \Omega) \geq \epsilon$.

Let $B_j = B(z_{n_j}, \epsilon)$ and $V_j = B(z_{n_j}, \epsilon/2)$. Let g_j be the inverse branch of f^{-n_j} that sends z_{n_j} to z; it is a univalent function on B_j. Let $U_j = g_j(V_j)$. All of the inverse branches of f are contracting with respect to the hyperbolic metric on $\mathbb{C} \setminus \Omega$; this implies that $g_j' \to 0$ on V_j as j goes to ∞, which in turn implies that the diameter of U_j tends to 0. Since g_j is univalent on B_j, the Koebe distortion theorem shows that U_j is almost a disk. Since z is a

([2]) If ∞ is an asymptotic value, we consider it a "prepole of order 0".

density point of A,

$$\lim_{j\to\infty} \frac{m(A\cap U_j)}{m(U_j)} = 1.$$

Applying Koebe distortion again, since A is an invariant subset, we get

$$\lim_{j\to\infty} \frac{m(A\cap V_j)}{m(V_j)} = 1.$$

This and the fact that V_j approaches $B_y = B(y, \epsilon/2)$ as j goes to ∞ together imply that $B_y \subset A$ up to a set of measure zero. Since A is an invariant subset of the Julia set J, $f^n(B_y) \subset A \subset J$. Since A is in the Julia set, B_y is also in the Julia set and $f^n(B_0)$ approaches $\mathbb{C}$ as n goes to ∞. Therefore, $A = \widehat{\mathbb{C}}$ up to a zero-measure set and the proof of Theorem 13 is complete. ∎

Remark 3.1. The assumption that the ω-limit sets of the non-prepolar asymptotic values are compact repellers says the Julia set is the whole sphere and gives us the expansion we need to prove our theorem. We could replace this by assuming that the Julia set is the sphere and that the orbits of the non-prepolar orbits are bounded. Then the main theorem of [GKS] implies that the expansion we need exists.

Remark 3.2. Another application of the results in [GKS] and [RVS, Theorem 1.1] to the Nevanlinna functions f of our Main Theorem is that such an f supports no invariant line field.

Remark 3.3. Finally, the results in [KU] applied to the Nevanlinna functions f of our Main Theorem prove that f has a σ-finite ergodic conservative f-invariant measure absolutely continuous with respect to the Lebesgue measure.

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