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Report on the doctoral dissertation by Simeng Wang
entitled "Some problems in harmonic analysis on quantum groups".

1. General remarks

The dissertation has been written under the co-supervision of Quanhua Xu (Université de Franche-Comté, Besançon) and Adam Skalski (IM PAN, Warszawa). It starts with the english summary (*Abstract*) and the polish summary (*Streszczenie*), and consists of Introduction, 4 Chapters (Preliminaries and 3 Chapters with main results) and the Bibliography, with the total of 97 pages (three of which: 6, 8, 10 are empty). As the Author declares, the thesis is based on two papers: "*L_p-improving convolution operators on finite quantum groups*" (Chapter 3) accepted for publication in the Indiana University Mathematical Journal and "*Lacunary Fourier series for compact quantum groups*" (Chapter 4), accepted for publication in the Communications in Mathematical Physics. This has to be considered as a significant achievement of the Author, as both journals are prestigious.

2. Description of the Dissertation

In the Introduction the Author collects basic information about the topics in classical harmonic analysis on compact groups, and explains how one introduces their generalizations into the context of compact *quantum* groups (CQG) like the Fourier transform, Fourier multipliers on noncommutative L^p-spaces, analogues of the Hausdorff-Young inequality, various notions of lacunary sets, including Sidon sets and $\Lambda(p)$ sets in the dual objects. The Author mentions there some of his main results: characterizations of such lacunary sets, as well as some examples, produced on CQG being products of either free unitary quantum groups or special unitary quantum groups.

The Chapter 1, "Preliminaries", presents necessary information about the noncommutative L^p-spaces as described by Haagerup and by Kosaki. It also gives introduction to the theory of CQG in the sense of Woronowicz, discussing the modular properties, co-representation theory as well as the free and tensor product constructions.

The Chapter 2, named "Introduction to Fourier analysis on compact quantum groups", starts with discussion of the (generalization of) notion of Fourier series and Fourier algebra (Proposition 2.1.5) for CQG. In particular the Fourier inversion formula and the Plancherel formula are presented in Proposition 2.1.2., generalized Young's inequality (Proposition 2.2.1), proved for the maximal range of parameters (p, q, r) in the tracial case of Kac type CQG, and with some restriction in the general (non-tracial) case. Then, in Subsection 2.3, the left and right L^p-Fourier multipliers are defined and the relation of the L^2-norms of both is shown in the general case. In the tracial case the Author proves in Lemma 2.3.3 important and useful result, that the equality of the L^p-norms of left and right multipliers holds if one properly incorporates the modular function.

In the Chapter 3, entitled "Convolution of states and L^p-improving operators", the Author considers the existence of the L^p-improving operators, which map $T : L^p \mapsto L^2$ for some $1 < p < 2$. This result is formulated in Theorem 3.1.6 for finite dimensional C^*-algebras with faithful tracial state, and trace preserving maps T. The applications of this Theorem for multipliers is obtained in Theorem 3.3.3, and for convolutions (on finite quantum groups) in Theorem 3.3.4. In particular the equivalent condition for a state φ to be L^p-improving convolution operator (for some $1 < p < 2$) is that the Fourier transform satisfies $\|\widehat{\varphi}(\pi)\| < 1$ for all irreducible nontrivial representations π. In the proof an important role is played by the recent result of Ricard-Xu (provided in Theorem 3.1.5) about convexity of noncommutative L^p-spaces.

Another interesting original application of the developed techniques is the Theorem 3.2.4, which generalizes earlier result by Banica-Franz-Skalski and also by Collins-Vernioux-Brannan, and provides method for computing indempotent states for Hopf images. The significance of this original result of the Dissertation has been confirmed in the very recent works by Banica, who has used it already several times in his studies of the *inner faithful matrix models* for CQG's.

The Chapter 4, "Lacunarity", is very interesting and constitutes an important achievement of the Dissertation. The first Subsection 4.1. "*Sidon sets*", starts with the natural and original Author's Definition 4.1.1 of a *Sidon* set in a CQG, which is a subset $\mathbf{E}$ of irreducible representations of the CQG such that the inequality $\|x\|_1 \leq K\|x\|_\infty$ holds for all polynomials x with $\widehat{x}(\pi) = 0$ if $\pi \notin \mathbf{E}$. Then there is the main characterization of the Sidon sets, given in Theorem 4.1.3, which shows that several of the seven equivalent conditions from the classical situation (for which the Author refers to the monograph II by Hewitt and Ross, Theorem 37.2, although they were shown by Ch. Akemann in his Pacific J. Math 1967 paper, Theorem 2) can be adapted to the CQG situation. The proofs are motivated, to some extend, by the classical ones and contain some similar and some different chains of steps, but they have very different meaning and context. In particular, the proof is new and original in the non-coamenable case. As a Corollary 4.1.8, the Author shows that surjectivity of the Fourier transform is equivalent to finiteness of the given CQG. Example of constructing infinite Sidon set in a product of CQG's is presented, with motivations comming from the classical case, and it is also shown in Proposition 4.1.11 that the sum of two Sidon sets (in dual objects of the two given CQG's) is a Sidon set in the (dual object of the) free product of the quantum groups. The Author considers also weakened versions of the notion of Sidon set, and gives his Definition 4.1.13 with generalizations, to CQG's, of the notions of (w) *weak Sidon set*, (i) *interpolation set of bounded multipliers on* L^∞ and (u) *unconditional Sidon set*. Moreover he proves in Theorem 4.1.15 that $(w) \Rightarrow (i) \Rightarrow (u)$ for a general CQG, and that these conditions are all equivalent with the notion of the *Sidon set* if the given CQG is coamenable. The Author shows in Remark 4.1.16 (2) that the coamenability condition for the equivalence is necessary, since the set of free generators of the free infinite group is not weak Sidon but is interpolating and unconditional Sidon set.

Then the Author presents two main examples of CQG's on which he tests the aforementioned properties. The first Example 4.1.17 is the infinite product of the free unitary compact quantum groups $U_{N_k}^+$ of dimensions $N_k \geq 2$ and the set $\mathbf{E} := \{u^{(k)}, k = 1, 2, \ldots\}$ is the collection of fundamental (co-)representations of these groups. This $\mathbf{E}$ turns out to be the weak Sidon set and thus also interpolation set and unconditional Sidon set, but **it is not a Sidon set**. The second Example 4.1.18 is the infinite product of the quantum $SU_{q_n}(2)$ groups of Woronowicz, with $0 \leq q_n \leq 1$, and similarly the Author considers the set $\mathbf{E} := \{u^{(n)}, n \geq 1\}$ where $u^{(n)}$ is the fundamental (co-)representation of $SU_{q_n}(2)$. In this case, using the coamenability of the quantum group, the Author shows that the set $\mathbf{E}$ is a Sidon set if the parameters are separated from zero, i.e. if $q_n \geq q > 0$, and is not a Sidon set if $q_n \to 0$.

The second Subection 4.2 "$\Lambda(p)$-sets" is devoted to the study of the $\Lambda(p)$-sets in (the dual object of a) CQG. The Definition 4.2.1 says that a subset $\mathbf{E}$ of irreducible representations of a CQG is a $\Lambda(p)$-set if the inequality $\|x\|_p \leq K\|x\|_1$ holds for all such polynomials x, which satisfy $\widehat{x}(\pi) = 0$ if $\pi \notin \mathbf{E}$. Moreover, the Author considers also the *central* $\Lambda(p)$*-sets* which satisfy the above condition for central polynomials (i.e. finite linear combination of characters). In classical harmonic analysis the $\Lambda(p)$-sets were defined by Rudin in 1960, who showed in particular that every Sidon set is a $\Lambda(p)$-set for all $1 < p < \infty$. He proved that if E is a Sidon set with constant K, i.e. if $\sum |\widehat{f}(n)| \leq K\|f\|_\infty$ for every polynomial f with spectrum in E (i.e. with $\widehat{f}(n) = 0$ if $n \notin E$), then the inequality $\|g\|_p \leq K\sqrt{p}\|g\|_2$ holds for all $2 < p < \infty$ and every polynomial g with spectrum in E. The converse of this was proved by G. Pisier in 1978. Rudin also showed that there are sets which are $\Lambda(p)$ for all $1 < p < \infty$ and which are not Sidon sets, and that a polynomial f with spectrum in the Sidon set E satisfies also $\|f\|_2 \leq 2K\|f\|_1$ (for the above constant K). In the Dissertation the Author gives in Theorem 4.2.7 new methods, even for classical compact groups, of proving characterisations of the $\Lambda(p)$ sets. The proof encounters many difficulties not appearing in the classical context, and uses some "heavy machinery" like the Proposition 2.3.5 (due to Junge) or Proposition 4.2.5, which provides control of the norm of modular functions for irreducible representations taken from an interpolation set of bouded multipliers on L^p-space (given $2 < p < \infty$) – the notion introduced by the Author, which is crucial in the non-Kac type CQG (in general this modular function is unbounded). The crucial estimate (4.11) in the proof of Theorem 4.2.7 requires careful and delicate treatment of the modular function (inevitable in the non-tracial case). In my opinion this is one of the most significant achievements of the Dissertation. The Author proves also natural generalisation of the classical Rudin result that a Sidon set is a $\Lambda(p)$ set for all $1 < p < \infty$, in fact as a Corollary 4.2.9 of his original Theorem 4.2.8 for an interpolation set of bounded multipliers on L^∞.

3. Remarks

In my opinion the Dissertation is an important step in the studies of lacunary sets in quantum groups, it also provides interesting new results in the theory of L^p-multipliers on CQG. To achieve these the Author had to adopt broad range of advanced techiques from the theory of noncommutative L^p-spaces, quantum groups, classical harmonic analysis, and he proved his skills of making efficient use of these. Definitely Simeng Wang is a very talented young mathematician and soon (if not already) he might become an expert in these areas.

The Dissertation is well written, the first part nicely introduces into the main topics of the second part, the proofs are correct and clear, though neither easy nor trivial – they required a lot of invention from the Author.

As far as I am able to judge, the use of the english language is adequate and proper, with just a few minor missprints (or small errors), which do not affect the pleasure of reading the Dissertation.

4. Conclusion

The dissertation fulfills all the legal requirements for the doctoral degree in mathematics, and thus I recommend it for further proceedings.

Moreover, I strongly suppoort the suggestion for awarding the Dissertation with distinctions.

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