Modular properties of locally compact quantum groups

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Oświadczam, że niniejsza rozprawa została napisana przeze mnie samodzielnie.

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(data i podpis)

Oświadczam, że niniejsza rozprawa jest gotowa do oceny przez recenzentów.

dr hab. Piotr Sołtan

....

(data i podpis)

Acknowledgments

I would like to thank dr. hab. Piotr Sołtan and dr. hab. Adam Skalski for their guidance during my PhD studies. Their constant support and numerous suggestions made this time very fruitful and enjoyable.

I also wish to express my gratitude towards dr. Mateusz Wasilewski for sharing his knowledge on operator algebras and towards prof. Roland Vergnioux for creating hospitable atmosphere during my stay in Caen.

Last but not least, I would like to thank my family: mother Bożena, father Damian, sister Agata and grandmother Maria, as I could have always ask for their help.

Wir müssen wissen
Wir werden wissen

David Hilbert

Wprowadzenie

Niech G będzie lokalnie zwartą[1] grupą abelową. Wraz z G możemy związać $\widehat{G}$, zbiór (mocno ciągłych) reprezentacji nieprzywiedlnych. Wszystkie reprezentacje nieprzywiedlne grup abelowych są jedno-wymiarowe, a zatem na $\widehat{G}$ możemy zdefiniować mnożenie poprzez $(\chi\eta)(g) = \chi(g)\eta(g)\,(\chi,\eta \in \widehat{G}, g \in G)$. Wraz z odpowiednią topologią (zwarto – otwartą), $\widehat{G}$ staje się lokalnie zwartą grupą abelową, zwaną grupą dualną do G w sensie Pontryagina. Konstrukcję tę możemy powtórzyć dla $\widehat{G}$ – wówczas wrócimy do grupy wyjściowej: mamy $\widehat{\widehat{G}} \simeq G$ gdzie $g \in G$ odpowiada reprezentacji $\widehat{G}$ danej poprzez $\widehat{G} \ni \chi \mapsto \chi(g) \in \mathbb{T}$. Najbardziej znanymi przykładami grup dualnych w sensie Pontryagina są $\widehat{\mathbb{R}} \simeq \mathbb{R}$ oraz $\widehat{\mathbb{T}} \simeq \mathbb{Z}$. Naturalnym pytaniem jest, czy dualność Pontryagina możemy w jakiś sposób rozszerzyć do klasy grup lokalnie zwartych które niekoniecznie są przemienne. Problem jaki napotykamy próbując zdefiniować $\widehat{G}$ jest następujący: jeśli G jest grupą lokalnie zwartą która nie jest przemienna, to będzie ona miała reprezentacje nieprzywiedlne działające na przestrzeniach Hilberta wyższego wymiaru. Nie mamy w takiej sytuacji naturalnej metody zdefiniowania działania grupowego na zbiorze reprezentacji nieprzywiedlnych – iloczyn tensorowy reprezentacji nieprzywiedlnych nie musi być nieprzywiedlny.

Jednym z wniosków jakie można wyciągnąć z powyższych rozważań jest konkluzja, że aby uzyskać satysfakcjonującą teorię zamkniętą na dualność i rozszerzającą teorię grup abelowych, należy wprowadzić do niej również obiekty innego typu, nie tylko grupy – obiekty takie nazywane są zwykle grupami kwantowymi. Proponowano różne definicje grup kwantowych: wspomnijmy tutaj algebry Kaca [36]. Przełomem okazały się badania Woronowicza, który zdefiniował zwarte grupy kwantowe (oraz dualne do nich dyskretne grupy kwantowe) w języku C*-algebr, oraz skonstruował nietrywialną (tzn. nie będącą klasyczną lub dualną do klasycznej) zwartą grupę kwantową $\mathrm{SU}_q(2)$ [99, 101, 70]. Wielkim sukcesem definicji Woronowicza był wynik pokazujący istnienie i jedyność całki Haara na zwartych grupach kwantowych. Najbardziej popularną definicję lokalnie zwartych grup kwantowych zaproponowali Kustermans oraz Vaes [57, 56], i to z ich definicji będziemy korzystać (alternatywną definicję zaproponował również Woronowicz [100]). W przeciwieństwie do definicji Woronowicza, tym razem punktem wyjścia jest język algebr von Neumanna, natomiast istnienie całek Haara jest częścią definicji. Każdą lokalnie zwartą grupę G możemy traktować jako lokalnie zwartą grupę kwantową w sensie Kustermansa–Vaesa, natomiast kwantowa grupa dualna $\widehat{G}$ związana jest z algebrami operatorów badanymi w abstrakcyjnej analizie harmonicznej: grupowymi C*-algebrami $\mathrm{C}^*_r(G), \mathrm{C}^*(G)$ oraz grupową algebrą von Neumanna $\mathrm{L}(G)$.

Niezwykle interesującym fenomenem który pojawia się w teorii grup kwantowych jest nieśladowość całek Haara; jeśli oznaczymy całkę Haara na $\mathrm{SU}_q(2)\,(0 < q < 1)$ symbolem h, to możemy znaleźć takie $a, b \in \mathrm{C}(\mathrm{SU}_q(2))$ dla których $h(ab) \neq h(ba)$. Nieśladowość całek Haara jest źródłem wielu interesujących problemów i zjawisk które są w sercu tej

[1] W tym wprowadzeniu zakładamy, że przestrzenie lokalnie zwarte są Hausdorffa.

rozprawy. Korzystając z teorii Tomity–Takesakiego możemy skonstruować grupy automorfizmów modularnych $(\sigma_t^{\varphi})_{t\in\mathbb{R}}, (\sigma_t^{\psi})_{t\in\mathbb{R}}$ związane z lewą φ oraz prawą ψ całką Haara. Poza tym, istnieje również trzecia grupa automorfizmów – grupa skalowania $(\tau_t)_{t\in\mathbb{R}}$. Odwzorowania te działają na algebrze von Neumanna $\mathrm{L}^{\infty}(\mathbb{G})$, zwanej algebrą funkcji ograniczonych na lokalnie zwartej grupie kwantowej $\mathbb{G}$. Tych grup automorfizmów nie widzimy (są trywialne) w przypadku grup klasycznych. Grupy automorfizmów modularnych obecne są jednak już w przypadku niektórych grup kwantowych dualnych do klasycznych. Jeśli G jest klasyczną grupą lokalnie zwartą to całka Haara $\widehat{\varphi}$ na $\widehat{G}$ jest śladowa (równoważnie: automorfizmy modularne $(\sigma_t^{\widehat{\varphi}})_{t\in\mathbb{R}}$ są trywialne) wtedy i tylko wtedy gdy G jest unimodularna. Widzimy więc na tym przykładzie, że istnieje związek między unimodularnością grupy kwantowej, a śladowością całek Haara na kwantowej grupie dualnej. W ogólnym przypadku związek ten jest nieco bardziej skomplikowany – między innymi relacje tego typu badać będziemy w Rozdziale 3.

W Rozdziale 1 wprowadzamy notację której będziemy używać w pracy. W kolejnym rozdziale przypominamy podstawowe wyniki dotyczące teorii wag na C*-algebrach i algebrach von Neumanna, w tym wyniki pochodzące z teorii Tomity–Takesakiego. Następnie wprowadzamy niezbędne pojęcia i rezultaty teorii Woronowicza oraz Kustermansa–Vaesa. Przedstawiamy również kilka przykładów lokalnie zwartych grup kwantowych.

Rozdział 3 poświęcony jest teorii grup kwantowych typu I. Zaczynamy od wprowadzenia kluczowego rezultatu Desmedta z [31] mówiącego o istnieniu miary Plancherela oraz stowarzyszonych z nią obiektów. W szczególności, daje nam on unitarny operator $\mathcal{Q}_L\colon \mathrm{L}^2(\mathbb{G}) \to \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathsf{HS}(\mathsf{H}_\pi)\,\mathrm{d}\mu(\pi)$ który przenosi algebrę von Neumanna $\mathrm{L}^{\infty}(\widehat{\mathbb{G}})$ na całkę prostą $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{B}(\mathsf{H}_\pi)\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)$ oraz pozwala wyrazić lewą całkę Haara $\widehat{\varphi}$ na $\widehat{\mathbb{G}}$ przy pomocy mierzalnego pola ściśle dodatnich samosprzężonych operatorów $(D_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$ (analogiczny wynik mamy również dla prawej całki $\widehat{\psi}$ – dla niej pojawia się pole $(E_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$). Po przypomnieniu twierdzenia Desmedta, rozwijamy wyniki uzyskane przez Caspersa w [17, 18]. Między innymi, uzyskujemy wyrażenia na pewne operatory związane kanonicznie z $\mathbb{G},\widehat{\mathbb{G}}$ wyrażone na poziomie całek prostych (twierdzenia 3.24, 3.25):

$$
\begin{aligned}
\nabla_{\widehat{\psi}}^{-it} &= \mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{2it}\otimes (E_\pi^{-2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L,\\
\nabla_{\widehat{\varphi}}^{it} &= \mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{-2it}\otimes (D_\pi^{2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L, \qquad (0.1)\\
\hat{\delta}^{it} &= \nu^{-\frac{i}{2}t^2}\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{2it}E_\pi^{-2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L
\end{aligned}
$$

dla $t\in\mathbb{R}$, gdzie $\nabla_{\widehat{\psi}},\nabla_{\widehat{\varphi}}$ to operatory modularne związane z całkami $\widehat{\psi},\widehat{\varphi}$, ν to stała skalowania, a $\hat{\delta}$ to element modularny $\widehat{\mathbb{G}}$ (zobacz Rozdział 2.2). W dalszej części tego rozdziału badamy implikacje między warunkami takimi jak unimodularność czy śladowość całek Haara. Wykorzystujemy również wcześniej udowodnione równania (takie jak te zgromadzone w (0.1)) aby dla grup kwantowych typu I wyrazić te własności w terminach operatorów D_π, E_π ($\pi \in \operatorname{Irr}(\mathbb{G})$). W ostatniej części tego rozdziału opisujemy dwa przykłady grup kwantowych typu I: dyskretną grupę kwantową $\widehat{\mathrm{SU}_q(2)}$ oraz kwantową grupę $az+b$. Wyniki zgromadzone w tym rozdziale pochodzą z pracy [49].

W Rozdziale 4 przedstawiamy wynik uzyskany wraz z Piotrem Sołtanem w [51]: mówi on o tym, że dysk kwantowy (opisywany przez algebrę Toeplitza) nie ma struktury zwartej grupy kwantowej. Dowód, który przedstawimy korzysta z teorii grup kwantowych typu I i wyników uzyskanych w Rozdziale 3.

Kolejny rozdział zawiera rezultaty uzyskane wraz z Mateuszem Wasilewskim w [52]. Problemem który badaliśmy jest pytanie, czy algebra von Neumanna $\mathscr{C}_{O_F^+}$ generowana przez charaktery jest maksymalnie przemienna w $\mathrm{L}^\infty(O_F^+)$, algebrze funkcji ograniczonych na kwantowej grupie ortogonalnej O_F^+, w przypadku gdy grupa ta nie jest typu Kaca (tzn. całka Haara nie jest śladowa). Uzyskaliśmy odpowiedź przeczącą. Nasze techniki pozwoliły również udowodnić interesujące wyniki dotyczące algebry von Neumanna $\mathrm{L}^\infty(U_F^+)$ funkcji ograniczonych na kwantowej grupie unitarnej U_F^+: (pod pewnymi warunkami) pokazaliśmy, że relatywny komutant $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+)$ nie jest zawarty w $\mathscr{C}_{U_F^+}$. Rezultaty te uzyskaliśmy korzystając z pojęcia quasi-rozszczepialności włożenia $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$. W rozdziale tym przedstawimy również konstrukcję zwartej grupy kwantowej $\mathbb{H}$, powstającej jako iloczyn bikrzyżowy $\mathbb{H} = \mathrm{SU}_q(2) \bowtie \mathbb{Q}$. Ma ona ciekawe własności: niektóre automorfizmy skalowania $\mathbb{H}$ są wewnętrzne, a algebra von Neumanna $\mathrm{L}^\infty(\mathbb{H})$ jest injektywnym faktorem typu II_∞.

W Rozdziale 6 przedstawiamy wyniki łączące własności aproksymacyjne grupy kwantowej (zwykle dyskretnej) $\mathbb{G}$ oraz algebry von Neumanna $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. Skupiamy się na średniowalności dla grupy kwantowej $\mathbb{G}$ oraz w*-w pełni dodatniej własności aproksymacyjnej (w*-CPAP) dla $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. Związki takie znane są w literaturze w sytuacji gdy $\mathbb{G}$ ma śladowe całki Haara (czyli $\widehat{\mathbb{G}}$ jest typu Kaca), jednak dla ogólnych kwantowych grup dyskretnych równoważność między średniowalnością $\mathbb{G}$ a w*-CPAP algebry von Neumanna $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ jest problemem otwartym. Uzyskaliśmy wynik częściowy: równoważność ta zachodzi jeśli zmodyfikujemy w*-CPAP tak aby brała pod uwagę również algebrę $\ell^\infty(\mathbb{G})$.

Rozdziałem 7 jest dodatek - zawiera on podstawowe informacje dotyczące teorii całek prostych oraz lematy z teorii operatorów i grup kwantowych.

Introduction

Let G be a locally compact[2] abelian group. With G we can associate $\widehat{G}$, the set of (strongly continuous) irreducible representations. Since irreducible representations of abelian groups are one-dimensional, we can introduce on $\widehat{G}$ a multiplication via $(\chi\eta)(g) = \chi(g)\eta(g)$ $(\chi, \eta \in \widehat{G}, g \in G)$. Once equipped with the appropriate (compact – open) topology, $\widehat{G}$ becomes a locally compact abelian group, known as the dual of G in the sense of Pontryagin. We can perform this construction also for $\widehat{G}$ – we will end up with the original group: we have $\widehat{\widehat{G}} \simeq G$, where $g \in G$ corresponds to the representation of $\widehat{G}$ given by $\widehat{G} \ni \chi \mapsto \chi(g) \in \mathbb{T}$. The most known examples of groups dual in the sense of Pontryagin are $\widehat{\mathbb{R}} \simeq \mathbb{R}$ and $\widehat{\mathbb{T}} \simeq \mathbb{Z}$.

It is a natural question to ask, whether the Pontryagin duality can be extended in some way to the larger class of all locally compact (not necesarilly abelian) groups. A problem that arises when one tries to define $\widehat{G}$ is as follows: if G is a locally compact group which is not abelian, then it has irreducible representations acting on Hilbert spaces of higher dimension. In this situation we do not have a natural way of defining multiplication on the set of irreducible representations – tensor product of irreducible representations does not need to be irreducible.

One of the conclusions that we may draw from these considerations is the constatation, that in order to obtain a satisfactory theory closed under duality and extending the theory of abelian groups, one has to include also objects which are not groups – such objects are usually called quantum groups. Various definitions of quantum groups were proposed: let us mention the theory Kac algebras [36]. The work of Woronowicz turned out to be a breakthrough. He defined compact quantum groups (and dual discrete quantum groups) in the language of C^*-algebras, and constructed a non-trivial (i.e. not classical or dual to classical) compact quantum group $\mathrm{SU}_q(2)$ [99, 101, 70]. A great success of Woronowicz's definition was the result showing existence and uniqueness of the Haar integral on any compact quantum group. The most popular definition of locally compact quantum groups was proposed by Kustermans and Vaes [57, 56], it is this definition we will use in our dissertation (an alternative definition was proposed by Woronowicz [100]). Unlike in the definition of Woronowicz, this time the starting point is the language of von Neumann algebras and existence of Haar integrals has to be postulated as a part of definition. Every locally compact group G can be treated as a locally compact quantum group in the sense of Kustermans and Vaes, wheras the dual quantum group $\widehat{G}$ is associated with the operator algebras studied in the abstract harmonic analysis: group C^*-algebras $\mathrm{C}^*_r(G), \mathrm{C}^*(G)$ and the group von Neumann algebra $\mathrm{L}(G)$.

An exceptionally interesting phenomenon that appears in the theory of quantum groups is non-traciality of Haar integrals; if we denote the Haar integral on $\mathrm{SU}_q(2)$ $(0 < q < 1)$ by h, then we can find such $a, b \in \mathrm{C}(\mathrm{SU}_q(2))$ for which $h(ab) \neq h(ba)$. Non-traciality of Haar integrals is a source of many intriguing problems which are at the heart of this disserta-

[2] In this introduction we assume that locally compact spaces are Hausdorff.

tion. Using the Tomita–Takesaki theory we can define groups of modular automorphisms $(\sigma_t^{\varphi})_{t\in\mathbb{R}}, (\sigma_t^{\psi})_{t\in\mathbb{R}}$ associated with the left φ and the right ψ Haar integral. Besides those, there is also a third group of automorphisms – the scaling group $(\tau_t)_{t\in\mathbb{R}}$. These maps act on the von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$, called the algebra of bounded functions on the locally compact quantum group $\mathbb{G}$. We do not see these groups of automorphisms in the classical case (they are trivial). However, groups of modular automorphisms appear already in the case of quantum groups dual to classical ones. If G is a classical locally compact group, then the Haar integral $\widehat{\varphi}$ on $\widehat{G}$ is tracial (equivalently: the modular automorphisms $(\sigma_t^{\widehat{\varphi}})_{t\in\mathbb{R}}$ are trivial) if, and only if G is unimodular. We can see in this example, that there is a relation between unimodularity of a quantum group, and traciality of Haar integrals on its dual. In the general case, this connection is more complicated – among others, we will study relations of this type in the Section 3.

In Section 1 we introduce the notation that will be used in the dissertation. In the next section we recall the basic results concerning the theory of weights on C*-algebras and von Neumann algebras, including results coming from the Tomita–Takesaki theory. Next, we introduce necessary notions and results from the theory of Woronowicz and Kustermans–Vaes. We also present a couple of examples of locally compact quantum groups.

Section 3 is devoted to the theory of type I quantum groups. We start with introducing the seminal result of Desmedt from [31]. It establishes an existence of the Plancherel measure and associated objects. In particular, it gives us a unitary operator $\mathcal{Q}_L\colon \mathrm{L}^2(\mathbb{G}) \to \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{HS}(\mathsf{H}_\pi)\,\mathrm{d}\mu(\pi)$ which transports the von Neumann algebra $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ onto the direct integral $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{B}(\mathsf{H}_\pi)\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)$ and allows us to express the left Haar integral $\widehat{\varphi}$ on $\widehat{\mathbb{G}}$ using a measurable field of strictly positive, self-adjoint operators $(D_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$ (we also have an analogous result for the right Haar integral $\widehat{\psi}$ – it uses another field of operators $(E_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$). After recalling the result of Desmedt, we further develop results of Caspers obtained in [17, 18]. Among others, we obtain expressions for the operators associated with $\mathbb{G}, \widehat{\mathbb{G}}$ on the level of direct integrals (theorems 3.24, 3.25):

$$\begin{aligned}
\nabla_{\widehat{\psi}}^{-it} &= \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{2it}\otimes (E_\pi^{-2it})^{\top}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L,\\
\nabla_{\widehat{\varphi}}^{it} &= \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{-2it}\otimes (D_\pi^{2it})^{\top}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L \qquad (0.2)\\
\hat{\delta}^{it} &= \nu^{-\frac{i}{2}t^2}\mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{2it}E_\pi^{-2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L
\end{aligned}$$

for $t\in\mathbb{R}$, where $\nabla_{\widehat{\psi}}, \nabla_{\widehat{\varphi}}$ are the modular operators associated with integrals $\widehat{\psi}, \widehat{\varphi}$, ν is the scaling constant, and $\hat{\delta}$ is the modular element of $\widehat{\mathbb{G}}$ (see Section 2.2). In the next part of this section, we study implications between conditions like unimodularity or traciality of the Haar integrals. In the case of type I quantum groups, we use previously obtained equations (like these in (0.2)) to express these properties in terms of operators D_π, E_π ($\pi \in \operatorname{Irr}(\mathbb{G})$). In the last part we describe two examples of type I quantum groups: the discrete quantum group $\widehat{\mathrm{SU}_q(2)}$ and the quantum group $az+b$. Results collected in this section are taken from the paper [49].

In Section 4 we present the theorem obtained together with Piotr Sołtan in [51]: it says that the quantum disc (described by the Toeplitz algebra) does not admit a structure of a compact quantum group. The proof we present uses theory of type I quantum groups and results obtained in Section 3.

The next section contains results obtained together with Mateusz Wasilewski in [52]. The problem we were studying is a question whether the von Neumann algebra $\mathscr{C}_{O_F^+}$ generated by characters is maximal abelian in $\mathrm{L}^\infty(O_F^+)$, the algebra of bounded functions on the quantum orthogonal group O_F^+, in the non-Kac case (i.e. when the Haar integral is not tracial). We obtained a negative answer. Our techniques allowed us to obtain also an interesting result concerning the von Neumann algebra $\mathrm{L}^\infty(U_F^+)$ of bounded functions on a quantum unitary group U_F^+: (under some conditions) we showed that the relative commutant $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+)$ is not contained in $\mathscr{C}_{U_F^+}$. These results were obtained using the notion of quasi-split inclusion $\mathscr{C}_\mathbb{G} \subseteq \mathrm{L}^\infty(\mathbb{G})$. In this section we also present a construction of a compact quantum group $\mathbb{H}$, which appears as the bicrossed product $\mathbb{H} = \mathrm{SU}_q(2) \bowtie \mathbb{Q}$. It has interesting properties: some of its scaling automorphisms are inner, and its von Neumann algebra $\mathrm{L}^\infty(\mathbb{H})$ is the injective type II_∞ factor.

In Section 6 we present results connecting approximation properties of a (usually discrete) quantum group $\mathbb{G}$ and the von Neumann algebra $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. We focus on amenability for the quantum group $\mathbb{G}$ and w*-completely positive approximation property (w*-CPAP) for $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. Connections like this are present in the literature in the case when $\mathbb{G}$ has tracial Haar integrals (that is, when $\widehat{\mathbb{G}}$ is of Kac type), but in the general case of discrete quantum groups, equivalence between the amenability of $\mathbb{G}$ and w*-CPAP of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is an open problem [13, 7]. We obtained a partial result: equivalence of this type is true, provided we modify the w*-CPAP in a such way that it takes into consideration also the von Neumann algebra $\ell^\infty(\mathbb{G})$.

Section 7 is an appendix - it contains basic information regarding theory of direct integrals and some lemmas from the theory of operators and quantum groups.

Table of Contents

1 Notation

We will write $\otimes_{alg}, \otimes, \bar{\otimes}$ for respectively: the (algebraic) tensor product of vector spaces, the minimal (spatial) tensor product of C*-algebras and the spatial tensor product of von Neumann algebras. The tensor flip will be denoted by χ.

Whenever we have an unbounded operator x on a Hilbert space H, its domain will be denoted by $\operatorname{Dom}(x) \subseteq \mathsf{H}$. If x, y are unbounded operators on H, then $x \circ y$ is given by

$$x \circ y\colon \operatorname{Dom}(x \circ y) = \{\xi \in \operatorname{Dom}(y) \,|\, y\xi \in \operatorname{Dom}(x)\} \ni \xi \mapsto x(y\xi) \in \mathsf{H}.$$

If $x \circ y$ is closable, its closure will be denoted by xy.

Let H be a Hilbert space. The identity operator on H will be denoted by $\mathbb{1}_{\mathsf{H}}$ (or $\mathbb{1}$ if it is clear from the context on which space $\mathbb{1}$ acts) and $\mathcal{K}(\mathsf{H})$ will be the C*-algebra of compact operators on H. Scalar products are linear from the right. We denote by $\overline{\mathsf{H}}$ the complex conjugate of H, i.e. the Hilbert space consisting of symbols $\overline{\xi}$ ($\xi \in \mathsf{H}$) and the Hilbert space structure given by

$$\alpha\overline{\xi} = \overline{\overline{\alpha}\xi}, \quad \overline{\xi} + \overline{\eta} = \overline{\xi + \eta}, \quad \langle \overline{\xi} \,|\, \overline{\eta} \rangle = \langle \eta \,|\, \xi \rangle \quad (\alpha \in \mathbb{C}, \xi, \eta \in \mathsf{H}).$$

The canonical antilinear map $\mathsf{H} \ni \xi \mapsto \overline{\xi} \in \overline{\mathsf{H}}$ will be denoted by $\mathcal{J}_{\mathsf{H}}$. With $T \in \mathrm{B}(\mathsf{H})$ we associate the operator $T^{\top} \in \mathrm{B}(\overline{\mathsf{H}})$ acting via

$$T^{\top}\overline{\xi} = \overline{T^{*}\xi} \quad (\xi \in \mathsf{H}).$$

The map $\mathrm{B}(\mathsf{H}) \ni T \mapsto T^{\top} \in \mathrm{B}(\overline{\mathsf{H}})$ is easily seen to be linear, bijective, antimultiplicative and $\star$-preserivng. If π is a representation of a C*-algebra $\mathcal{A}$ on H, then π^{c} will denote the representation of the opposite algebra $\mathcal{A}^{op}$ on $\overline{\mathsf{H}}$ given by

$$\pi^{c}(a) = \pi(a)^{\top} \quad (a \in \mathcal{A}).$$

For a C*-algebra $\mathcal{A}$, its spectrum will be denoted by $\operatorname{Irr}(\mathcal{A})$. If θ is a weight on $\mathcal{A}$, its GNS Hilbert space, GNS map and representation will be denoted by H_{θ}, Λ_{θ}, π_{θ}. The modular conjugation and the modular operator will be denoted by $J_{\theta}, \nabla_{\theta}$ (for details and assumptions see Section 2.1).

All von Neumann subalgebras are assumed to be unital unless said otherwise.

If (X, μ) is a measure space, then $\sup_{x \in X}$ (or sup) will mean the essential supremum.

2 Preliminaries

In this disseration we will work in the language of operator algebras: C*-algebras and von Neumann algebras. We refer the reader to the sources [33, 34, 78, 81, 82, 10] for their basic theory.

Let $\mathcal{A}$ be a C*-algebra. According to the Gelfand-Naimark theorem, there is a faithful nondegenerate representation π of $\mathcal{A}$ on some Hilbert space H (nondegeneracy of π means that the set $\{\pi(a)\xi \,|\, a \in \mathcal{A}, \xi \in \mathsf{H}\}$ is linearly dense in H). Because π is faithful, let us write a instead of $\pi(a)$ $(a \in \mathcal{A})$. Define

$$\mathrm{M}(\mathcal{A}) = \{T \in \mathrm{B}(\mathsf{H}) \,|\, T\mathcal{A}, \mathcal{A}T \subseteq \mathcal{A}\}.$$

This is a unital C*-algebra known as the[3] *multiplier algebra of* $\mathcal{A}$. It is easy to check that $\mathcal{A}$ is an ideal in $\mathrm{M}(\mathcal{A})$. Note that when $\mathcal{A}$ is unital, we have $\mathrm{M}(\mathcal{A}) = \mathcal{A}$ – we will use the C*-algebra $\mathrm{M}(\mathcal{A})$ only when working with non-unital algebras. On $\mathrm{M}(\mathcal{A})$ besides the norm topology, there is also another useful topology known as the *strict topology.* We say that a net $(T_i)_{i\in\mathcal{I}}$ in $\mathrm{M}(\mathcal{A})$ converges strictly to some $T \in \mathrm{M}(\mathcal{A})$ if $T_i a \underset{i\in\mathcal{I}}{\longrightarrow} Ta$ and $aT_i \underset{i\in\mathcal{I}}{\longrightarrow} aT$ for all $a \in \mathcal{A}$. It is not difficult to see (using an approximate identity) that $\mathcal{A}$ is strictly dense in $\mathrm{M}(\mathcal{A})$.

Let $\mathcal{A}, \mathcal{B}$ be two C*-algebras. A *morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a $\star$-homomorphism $\pi\colon \mathcal{A} \to \mathrm{M}(\mathcal{B})$ which is nondegenerate in the sense that $\overline{\operatorname{span}}\,\pi(\mathcal{A})\mathcal{B} = \mathcal{B}$. Then π extends uniquely to a strictly continuous $\star$-homomorphism $\mathrm{M}(\mathcal{A}) \to \mathrm{M}(\mathcal{B})$. We denote this extension by the same letter π. The set of morphisms from $\mathcal{A}$ to $\mathcal{B}$ will be denoted by $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$.

2.1 Theory of weights

Let $\mathcal{A}$ be a C*-algebra (in particular $\mathcal{A}$ can be a von Neumann algebra).

Definition 2.1. A weight on $\mathcal{A}$ is a map $\theta\colon \mathcal{A}_+ \to [0, +\infty]$ such that

- $\theta(a + b) = \theta(a) + \theta(b)$ $(a, b \in \mathcal{A}_+)$,
- $\theta(\lambda a) = \lambda\theta(a)$ $(\lambda \in \mathbb{R}_{\geq 0}, a \in \mathcal{A}_+)$.

A basic and motivational example of a weight is given by the integration $\theta(f) = \int_X f \,\mathrm{d}\mu$, where μ is a measure on a topological space X (and then $\mathcal{A} = \mathrm{C}_0(X)$) or on a measurable space X (and then $\mathcal{A} = \mathrm{L}^\infty(X, \mu)$).

When $\mathcal{A}$ is non-commutative, θ should be thought of as a "non-commutative integral" or an integral on a "non-commutative space". An example of weight on such an algebra is given by the trace $\theta(T) = \mathrm{Tr}(T)$ where $\mathcal{A} = \mathrm{B}(\mathsf{H})$ is the von Neumann algebra of bounded operators on a Hilbert space H. A well known property of Tr is its *traciality*: equation $\mathrm{Tr}(TT^*) = \mathrm{Tr}(T^*T)$ holds for all $T \in \mathrm{B}(\mathsf{H})$. However, not all weights are tracial: for

[3] Other (equivalent) constructions of the multiplier algebra are also possible – this will be the most convenient for us. It does not depend on the choice of a faithful nondegenerate representation.

example, we can take a positive operators $a \in \mathrm{B}(\mathsf{H})^+$ and form a new weight on $\mathrm{B}(\mathsf{H})$ via $\theta(T) = \mathrm{Tr}(aT)$. In general (more precisely, when $a \notin \mathbb{R}_{\geq 0}\mathbb{1}$), such φ will not be tracial. This causes a lot of difficulties and at the same time introduces new intriguing phenomena – in this section we will give an overview of basic tools that one uses to deal with non-traciality of weights.

First, let us introduce more notation associated with a weight θ on $\mathcal{A}$.

Definition 2.2. Let θ be a weight on a C*-algebra $\mathcal{A}$. Define the following subsets

$$\begin{aligned} \mathfrak{M}_\theta^+ &= \{a \in \mathcal{A}_+ \,|\, \theta(a) < +\infty\} \\ \mathfrak{N}_\theta &= \{b \in \mathcal{A} \,|\, \theta(b^*b) < +\infty\} \\ \mathfrak{M}_\theta &= \operatorname{span} \mathfrak{M}_\theta^+ = \mathfrak{N}_\theta^* \mathfrak{N}_\theta. \end{aligned}$$

One can check that $\mathfrak{M}_\theta^+$ is a face in $\mathcal{A}_+$, $\mathfrak{N}_\theta$ is a left ideal and $\mathfrak{M}_\theta$ is a $\star$-subalgebra in $\mathcal{A}$. Furthermore, θ extends uniquely to a linear map $\mathfrak{M}_\theta \to \mathbb{C}$ – following the standard convention, we will denote this map also by θ.

The above subsets are adaptations of subsets used in a (classical) integration theory: if θ is a weight on $\mathrm{L}^\infty(X, \mu)$ given by integration, then

$$\mathfrak{M}_\theta^+ = \mathrm{L}^\infty(X,\mu) \cap \mathrm{L}^1(X,\mu)^+, \quad \mathfrak{N}_\theta = \mathrm{L}^\infty(X,\mu) \cap \mathrm{L}^2(X,\mu), \quad \mathfrak{M}_\theta = \mathrm{L}^\infty(X,\mu) \cap \mathrm{L}^1(X,\mu).$$

Definition 2.3. A weight θ on a C*-algebra $\mathcal{A}$ is said to be faithful if $\theta(a) = 0$ implies $a = 0$ for all $a \in \mathcal{A}_+$.

Besides faithfulness, one usually imposes more conditions on weights. These are however different, depending on whether $\mathcal{A}$ is a von Neumann algebra or a C*-algebra. This is why we postpone them to the next subsections.

Before we deal with more advanced theory, let us recall the GNS representation[4]. As usual, let θ be a weight on a C*-algebra $\mathcal{A}$. We introduce a sesquilinear map via

$$\mathfrak{N}_\theta \times \mathfrak{N}_\theta \ni (a,b) \mapsto \langle a \,|\, b \rangle_\theta = \theta(a^*b) \in \mathbb{C}$$

and let $\ker(\theta) = \{a \in \mathcal{A} \,|\, \theta(a^*a) = 0\}$. Next, define a Hilbert space H_θ as the completion of the quotient space $\mathfrak{N}_\theta / \ker(\theta)$ under the norm induced by $\langle \cdot \,|\, \cdot \rangle_\theta$. Let us introduce maps

- $\Lambda_\theta \colon \mathfrak{N}_\theta \to \mathsf{H}_\theta$ the canonical map,
- $\pi_\theta \colon \mathcal{A} \to \mathrm{B}(\mathsf{H}_\theta)$ representation given by $\pi_\theta(a)\Lambda_\theta(b) = \Lambda_\theta(ab)$ $(a \in \mathcal{A}, b \in \mathfrak{N}_\theta)$.

π_θ is called the *GNS representation* associated with θ. It can be showed that π_θ is always non-degenerate and faithful if θ is faithful. In such case, we will not write π_θ and simply treat $\mathcal{A}$ as a C*-subalgebra of $\mathrm{B}(\mathsf{H}_\theta)$.

We refer the reader to [77, 78, 81, 82] for more information about weights and proofs of results presented in this section.

[4]It is named after three mathematicians: Israel Gelfand, Mark Naimark and Irving Segal.

2.1.1 Theory of weights on von Neumann algebras

Let us assume now that $\mathcal{A} = \mathrm{M}$ is a von Neumann algebra with a weight θ.

Definition 2.4. We say that

- θ is *normal* if $\theta(\sup_{i\in\mathcal{I}} a_i) = \sup_{i\in\mathcal{I}} \theta(a_i)$ for all norm-bounded increasing nets $(a_i)_{i\in\mathcal{I}}$ in M^+,
- θ is *semifinite* if $\mathfrak{N}_\theta$ is w*-dense in M.

We will be dealing solely with normal, semifinite, faithful[5] weights (abbreviated n.s.f.) on von Neumann algebras. If θ is normal, its GNS representation $\pi_\theta\colon \mathrm{M} \to \mathrm{B}(\mathsf{H}_\theta)$ is normal, i.e. w*-continuous. By a result of Haagerup (see e.g. [77, Theorem 1.3]) the following conditions are equivalent

- θ is normal,
- θ is *lower* w**-semicontinuous*, i.e. the set $\{a \in \mathrm{M}^+ \,|\, \theta(a) \le \lambda\}$ is w*-closed for all $\lambda \ge 0$,
- $\theta(a) = \sup\{\omega(a) \,|\, \omega \in \mathrm{M}_*^+ \colon \omega \le \theta\}$ for all $a \in \mathrm{M}_+$.

Using this result one can quite easily show ([82, Theorem VII 2.7]) that on every von Neumann algebra there is a n.s.f. weight.

A theory that is indispensable when dealing with non-tracial weights is called the Tomita–Takesaki theory. It has its begginings in the 60's, in the work of Tomita. Initially however, his work did not receive much attention. It was only when Takesaki improved it and gave clearer presentation ([80]), when it was recognised as a fundamental achievement. To name a few consequences, it were these results that lead to a proof of the tensor product commutation theorem ($(\mathrm{M}_1 \bar{\otimes} \mathrm{M}_2)' = \mathrm{M}_1' \bar{\otimes} \mathrm{M}_2'$ for all von Neumann algebras $\mathrm{M}_1, \mathrm{M}_2$) and to classification results due to Connes and Haagerup ([23, 43]).

For a fuller account on the Tomita–Takesaki theory, see e.g. [80, 82, 77, 78], here we will present a very brief overview. Our main motivation for introducing this theory, is to use it for Haar integrals on locally compact quantum groups (which are n.s.f. weights with special features). This is why we will use the more down-to-earth language of weights, rather than more abstract theory of Hilbert algebras.

Assume that θ is a n.s.f. weight on a von Neumann algebra M. Recall that we consider M as a subalgebra of $\mathrm{B}(\mathsf{H}_\theta)$. Consider an (unbounded) antilinear map

$$S_{\theta,0}\colon \operatorname{Dom}(S_{\theta,0}) = \{\Lambda_\theta(a) \,|\, a \in \mathfrak{N}_\theta \cap \mathfrak{N}_\theta{}^*\} \to \mathsf{H}_\theta \colon \Lambda_\theta(a) \mapsto \Lambda_\theta(a^*).$$

[5] One can work with weights on von Neumann algebras which are normal and semifinite but not necessarily faithful, by introducing the support of a weight [82, Section VII]. We do not need this level of generality, hence we will stick to faithful weights for the sake of simplicity.

Because we do not assume that θ is tracial, this map can be unbounded. It is however densely defined and closable. Let $S_\theta\colon \operatorname{Dom}(S_\theta) \to \mathsf{H}_\theta$ be its closure. Next, let us introduce maps J_θ and $\nabla_\theta^{\frac{1}{2}}$ via the polar decomposition of S_θ:

$$S_\theta = J_\theta \nabla_\theta^{\frac{1}{2}}. \tag{2.1}$$

Operator J_θ, called the *modular conjugation* is antilinear, bounded, self-adjoint and involutive. ∇_θ called the *modular operator* on the other hand is (in general) unbounded, linear, strictly positive and self-adjoint. The main result of Tomita–Takesaki theory is the following:

Theorem 2.5.

- $J_\theta \,\mathrm{M}\, J_\theta = \mathrm{M}'$,
- *For all* $t \in \mathbb{R}$ *we have* $\nabla_\theta^{it}\,\mathrm{M}\,\nabla_\theta^{-it} = \mathrm{M}$.

The second part of the above theorem allows us to define maps

$$\sigma_t^\theta\colon\ \mathrm{M} \ni a \mapsto \nabla_\theta^{it} a \nabla_\theta^{-it} \in \mathrm{M} \quad (t \in \mathbb{R}). \tag{2.2}$$

One can easily check that each σ_t^θ is an automorphism – $(\sigma_t^\theta)_{t\in\mathbb{R}}$ is called the *modular group* of θ. It is not difficult to see that θ is tracial if, and only if $\nabla_\theta = \mathbb{1}$ (and consequently $\sigma_t^\theta = \mathrm{id}\,(t \in \mathbb{R})$).

In practical calculations, it is often desirable to "move" y outside Λ_θ in an expression of the form $\Lambda_\theta(xy)$ (we will not care for a moment about domain issues). When θ is tracial, we can do this as follows:

$$\Lambda_\theta(xy) = J_\theta \Lambda_\theta(y^* x^*) = J_\theta y^* \Lambda_\theta(x^*) = J_\theta y^* J_\theta \Lambda_\theta(x)$$

simply because $S_\theta = J_\theta$. However, when θ is non-tracial we need to take into consideration the modular operator of θ, i.e. ∇_θ. Performing similar (informal) calculation and using $J_\theta \nabla_\theta^z = \nabla_\theta^{-\overline{z}} J_\theta\,(z \in \mathbb{C})$ we arrive at

$$\begin{aligned}\Lambda_\theta(xy) &= J_\theta \nabla_\theta^{\frac{1}{2}} \Lambda_\theta(y^* x^*) = J_\theta \nabla_\theta^{\frac{1}{2}} y^* \Lambda_\theta(x^*) = J_\theta \nabla_\theta^{\frac{1}{2}} y^* J_\theta \nabla_\theta^{\frac{1}{2}} \Lambda_\theta(x)\\ &= J_\theta \nabla_\theta^{\frac{1}{2}} y^* \nabla_\theta^{-\frac{1}{2}} J_\theta \Lambda_\theta(x) = J_\theta (\nabla_\theta^{-\frac{1}{2}} y \nabla_\theta^{\frac{1}{2}})^* J_\theta \Lambda_\theta(x).\end{aligned}$$

Notice (looking at expression (2.2)) that $\nabla_\theta^{-\frac{1}{2}} y \nabla_\theta^{\frac{1}{2}}$ looks as $\sigma_t^\theta(y)$ for $t = z = i/2$. However, so far we have defined σ_t^θ only for $t \in \mathbb{R}$. Clearly the definition of $\sigma_{i/2}^\theta(y)$ must raise some difficulties of a technical kind – simply because the operators $\nabla_\theta^{\frac{1}{2}}$, $\nabla_\theta^{-\frac{1}{2}}$ are unbounded.

In order to make sense of the expression $\sigma_z^\theta(a)$ for $z \in \mathbb{C}$, we will use an analytical continuation. We say that an operator $a \in \mathrm{M}$ belongs to the domain $\operatorname{Dom}(\sigma_{-iz}^\theta)$ of σ_{-iz}^θ if there exists a function $F\colon \{w \in \mathbb{C} \mid -\varepsilon_1 \leq \mathrm{Re}(w) \leq \varepsilon_2\} \to \mathrm{M}$ for some $\varepsilon_1, \varepsilon_2 \geq 0$ such that $-\varepsilon_1 \leq \mathrm{Re}(z) \leq \varepsilon_2$, which is w*-continuous on the whole strip, analytical in its interior and satisfies[6]

$$F(it) = \sigma^\theta_t(a) \quad (t \in \mathbb{R}).$$

Then we define $\sigma^\theta_{-iz}(a)$ via

$$F(z) = \sigma^\theta_{-iz}(a).$$

In other words, $a \in \mathrm{Dom}(\sigma^\theta_{-iz})$ if the map $i\mathbb{R} \ni it \mapsto \sigma^\theta_t(a) \in \mathrm{M}$ extends to a w*-continuous map on some vertical strip containing z, which is analytic in its interior.

The family of (linear) maps $(\sigma^\theta_z)_{z\in\mathbb{C}}$ have nice properties, to name a few we have

$$\begin{aligned} a \in \mathrm{Dom}(\sigma^\theta_z) \quad &\Rightarrow \quad a^* \in \mathrm{Dom}(\sigma^\theta_{\bar z}),\ \sigma^\theta_z(a)^* = \sigma^\theta_{\bar z}(a^*),\\ a, b \in \mathrm{Dom}(\sigma^\theta_z) \quad &\Rightarrow \quad ab \in \mathrm{Dom}(\sigma^\theta_z),\ \sigma^\theta_z(ab) = \sigma^\theta_z(a)\sigma^\theta_z(b),\\ a \in \mathrm{Dom}(\sigma^\theta_z),\ \sigma^\theta_z(a) \in \mathrm{Dom}(\sigma^\theta_{z'}) \quad &\Rightarrow \quad a \in \mathrm{Dom}(\sigma^\theta_{z+z'}),\ \sigma^\theta_{z'}(\sigma^\theta_z(a)) = \sigma^\theta_{z+z'}(a) \end{aligned}$$

and

$$a \in \mathrm{Dom}(\sigma^\theta_z) \quad \Rightarrow \quad \sigma^\theta_z(a) = \overline{\nabla^{iz}_\theta a \nabla^{-iz}_\theta \restriction_{\mathrm{Dom}(\nabla^{-iz}_\theta)}}$$

where above $\bar{\cdot}$ stands for a closure of a closable operator (see [77, Section 2.14]). Furthermore, for each $z \in \mathbb{C}$ the domain of σ^θ_z is SOT*-dense in M.

One can rigorously prove ([77, Proposition 2.14]) that if $y \in \mathrm{Dom}(\sigma^\theta_{i/2})$ then

$$\Lambda_\theta(xy) = J_\theta \sigma^\theta_{i/2}(y)^* J_\theta \Lambda_\theta(x) \quad (x \in \mathfrak{N}_\theta).$$

Another useful property we would like to mention is the following: we say that an element $a \in \mathrm{M}$ is called analytic (w.r.t. the modular group $(\sigma^\theta_t)_{t\in\mathbb{R}}$) if $a \in \bigcap_{z\in\mathbb{C}} \mathrm{Dom}(\sigma^\theta_z)$. For such an element we have ([77, Section 2.15])

$$\mathfrak{N}_\theta a \subseteq \mathfrak{N}_\theta, \quad a\mathfrak{M}_\theta,\ \mathfrak{M}_\theta a \subseteq \mathfrak{M}_\theta.$$

We have said that the Tomita–Takesaki theory helps us dealing with non-tracial weights. Let us end this part with a result which is an extension of the trace property: let $z \in \mathbb{C}$. If $x \in {\mathfrak{N}_\theta}^* \cap \mathrm{Dom}(\sigma^\theta_{z-i}), \sigma^\theta_{z-i} \in \mathfrak{N}_\theta, y \in \mathfrak{N}_\theta \cap \mathrm{Dom}(\sigma^\theta_z), \sigma^\theta_z(y) \in {\mathfrak{N}_\theta}^*$ then ([77, Proposition 2.17])

$$\theta(xy) = \theta(\sigma^\theta_z(y)\sigma^\theta_{z-i}(x)).$$

Let θ, η be two n.s.f. weights on a von Neumann algebra M. We will now briefly describe the notion of the *Connes' cocycle derivative* between θ and η, which is a non-commutative analog of the classical Radon-Nikodym derivative.

There exists a unique SOT-continuous family $((D\theta : D\eta)_t)_{t\in\mathbb{R}}$ of unitary operators in M such that

- $(D\theta : D\eta)_{t+s} = (D\theta : D\eta)_t\, \sigma^\eta_t((D\theta : D\eta)_s)$ $(t, s \in \mathbb{R})$,

[6] Setting $F(it) = \sigma^\theta_t(a)$ rather then $F(t) = \sigma^\theta_t(a)$ is a matter of convention.

- $(D\theta : D\eta)_t\, \sigma_t^\eta(\mathfrak{N}_\theta{}^* \cap \mathfrak{N}_\eta) = \mathfrak{N}_\theta{}^* \cap \mathfrak{N}_\eta$ $(t \in \mathbb{R})$,
- for $x \in \mathfrak{N}_\theta \cap \mathfrak{N}_\eta{}^*$, $y \in \mathfrak{N}_\theta{}^* \cap \mathfrak{N}_\eta$ there exists a continuous function $F\colon \{z \in \mathbb{C} \,|\, 0 \leq \operatorname{Im}(z) \leq 1\} \to \mathbb{C}$, analytic in its interior, which satisfies

$$F(t) = \theta((D\theta : D\eta)_t\, \sigma_t^\eta(y)\, x), \quad F(t+i) = \eta(x\, (D\theta : D\eta)_t\, \sigma_t^\eta(y)) \quad (t \in \mathbb{R}),$$

- $\sigma_t^\theta(x) = (D\theta : D\eta)_t\, \sigma_t^\eta(x)\, (D\theta : D\eta)_t^*$ $(t \in \mathbb{R},\, x \in \mathrm{M})$.

For a full discussion, see [82, Section VIII.3].

We will use the cocycle derivative for the left and the right Haar integrals on $\mathbb{G}$ (see Section 2.2). More precisely, we will use the following result: let as before θ, η be two n.s.f. weights on a von Neumann algebra M. Assume that there exists a positive number $\nu > 0$ such that $\theta \circ \sigma_t^\eta = \nu^{-t}\theta$. Then ([87, Proposition 5.5]) there exists a strictly positive, self-adjoint operator δ affiliated with M such that

$$\sigma_t^\eta(\delta^{is}) = \nu^{ist}\delta^{is}, \quad (D\theta : D\eta)_t = \nu^{\frac{1}{2}it^2}\delta^{it} \quad (s, t \in \mathbb{R})$$

and $\theta = \eta_\delta$ (and alternative notation for η_δ is $\eta(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$). Let us explain the meaning of this last assertion (see [87]). The weight η_δ is defined as follows: first, let

$$\mathfrak{N}_0 = \{a \in \mathrm{M} \mid a \circ \delta^{\frac{1}{2}} \text{ is bounded and } a\delta^{\frac{1}{2}} \in \mathfrak{N}_\eta\}.$$

It is a core for Λ_{η_δ} and we have

$$\Lambda_{\eta_\delta} \colon \mathfrak{N}_{\eta_\delta} \supseteq \mathfrak{N}_0 \ni a \mapsto \Lambda_\eta(a\delta^{\frac{1}{2}}) \in \mathsf{H}_\eta.$$

The weight η_δ is n.s.f. and its GNS representation can be identified with this of η. Furthermore, the modular conjugation, modular operator and the modular automorphisms of η_δ are given by

$$J_{\eta_\delta} = \nu^{\frac{i}{4}} J_\eta, \quad \nabla_{\eta_\delta}^{it} = \delta^{it}(J_\eta \delta^{it} J_\eta)\nabla_\eta^{it}, \quad \sigma_t^{\eta_\delta}(a) = \delta^{it}\sigma_t^\eta(a)\delta^{-it} \quad (t \in \mathbb{R}, a \in \mathrm{M}).$$

We note that these results can be extended to a situation when ν is a strictly positive, self-adjoint operator affiliated with $\mathcal{Z}(\mathrm{M})$.

2.1.2 Theory of weights on C*-algebras

In the theory of locally compact quantum groups, we will occasionally encounter also weights on C*-algebras. These are usually more difficult to handle than weights on von Neumann algebras, hence – whenever possible – we will try to work with von Neumann algebras. For material presented here, we refer to [53] (see also [61, Appendix C]).

Let $\mathcal{A}$ be a C*-algebra with a weight θ. As before, we start with introducing conditions on θ.

Definition 2.6. We say that θ is

- *densely defined* (sometimes called *locally finite*) if $\mathfrak{N}_\theta$ is dense in $\mathcal{A}$,
- *lower semicontinuous* if $\{a \in \mathcal{A}_+ \mid \theta(a) \le t\}$ is closed in $\mathcal{A}$ for all $t \ge 0$.

Let $(\mathsf{H}_\theta, \pi_\theta, \Lambda_\theta)$ be the objects given by the GNS construction for θ. We would like to know that we can define a n.s.f. weight $\tilde{\theta}$ on the von Neumann algebra $\pi_\theta(\mathcal{A})''$ via $\tilde{\theta}\pi_\theta(a) = \theta(a)$ $(a \in \mathcal{A}_+)$. This is not always the case – we need to introduce an extra condition on θ:

Definition 2.7. Assume that θ is a densely defined, lower semicontinuous weight on $\mathcal{A}$. Furthermore, let $(\sigma_t)_{t\in\mathbb{R}}$ be a continuous group of automorphisms of $\mathcal{A}$, i.e. the map $\mathbb{R} \ni t \mapsto \sigma_t(a) \in \mathcal{A}$ is continuous for each $a \in \mathcal{A}$. We say that θ is a *KMS weight* with respect to $(\sigma_t)_{t\in\mathbb{R}}$ if

- $\theta \circ \sigma_t = \sigma_t$ for all $t \in \mathbb{R}$,
- for every $x, y \in \mathfrak{N}_\theta{}^* \cap \mathfrak{N}_\theta$ there exists a continuous function on a strip $f\colon \{z \in \mathbb{C} \mid 0 \le \operatorname{Re}(z) \le 1\} \to \mathbb{C}$, analytic in its interior, such that

$$f(it) = \theta(\sigma_t(x)y), \quad f(it+1) = \theta(y\sigma_t(x)) \quad (t \in \mathbb{R}).$$

We say that θ is a KMS weight if there exists a continuous group $(\sigma_t)_{t\in\mathbb{R}}$ with respect to which θ is a KMS weight.

(See [53, Theorem 6.36] and [53, Definition 2.8]). Group $(\sigma_t)_{t\in\mathbb{R}}$ is not always unique – however, one can show that it is unique if the weight θ is faithful.

By [53, Theorem 6.20], if θ is a densely defined, lower semicontinuous KMS weight on a C*-algebra $\mathcal{A}$, then there exists a unique n.s.f. weight $\tilde{\theta}$ on $\pi_\theta(A)''$ which extends θ in the sense that $\tilde{\theta} \circ \pi_\theta = \theta$. Furthermore we have [53, Section 6]

$$\pi_\theta(\sigma_t(a)) = \nabla_{\tilde{\theta}}^{it}\pi_\theta(a)\nabla_{\tilde{\theta}}^{-it}, \quad \Lambda_\theta(\sigma_t(b)) = \nabla_{\tilde{\theta}}^{it}\Lambda_\theta(b) \quad (a \in A, b \in \mathfrak{N}_\theta, t \in \mathbb{R}).$$

2.2 Locally compact quantum groups

The definition of a locally compact quantum group which we will use was introduced by Kustermans and Vaes in the seminal paper [56] (see also [57]). Their approach was to define a locally compact quantum group $\mathbb{G}$ via a (possibly non-commutative) von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$ (playing a role of the algebra of (classes of) measurable bounded functions on a "quantum space" $\mathbb{G}$)[7] and a map $\Delta_{\mathbb{G}}\colon \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ called *comultiplication* corresponding to the group operation. We also assume an existence of n.s.f. weights φ, ψ on $\mathrm{L}^\infty(\mathbb{G})$ satisfying the left/right invariance conditions – these are called *Haar integrals* and should be thought of as integrals over $\mathbb{G}$ with respect to Haar measures. The precise definition is as follows:

[7]Following this notational convention, the predual of $\mathrm{L}^\infty(\mathbb{G})$ is denoted by $\mathrm{L}^1(\mathbb{G})$.

Definition 2.8. A *locally compact quantum group* $\mathbb{G}$ is a pair $(\mathrm{L}^\infty(\mathbb{G}), \Delta_{\mathbb{G}})$ consisting of

- a von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$ and
- a normal unital $\star$-homomorphism $\Delta_{\mathbb{G}} \colon \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ satisfying $(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}} = (\mathrm{id} \otimes \Delta_{\mathbb{G}}) \circ \Delta_{\mathbb{G}}$.

Furthermore, we assume existence of n.s.f. weights $\varphi, \psi \colon \mathrm{L}^\infty(\mathbb{G})^+ \to [0, +\infty]$ such that

$$\varphi((\omega \otimes \mathrm{id})\Delta_{\mathbb{G}}(x)) = \omega(\mathbb{1})\varphi(x), \quad \psi((\mathrm{id} \otimes \omega)\Delta_{\mathbb{G}}(y)) = \omega(\mathbb{1})\psi(y)$$

for all $\omega \in \mathrm{L}^1(\mathbb{G})^+, x \in \mathfrak{M}_\varphi^+, y \in \mathfrak{M}_\psi^+$.

(We will sometimes write Δ instead of $\Delta_{\mathbb{G}}$ if there is no risk of confusion).

In this section we will present fundamental results of the theory which will be used throughout the dissertation. We will not present their proofs – it would take up too much space – rather than that, we refer the reader to the literature. Besides the above mentioned papers [57, 56] results presented here come from Van Daele's work [93] and Kustermans' paper concerning universal quantum groups [55]. See also [36].

To begin with, let us denote by $\mathrm{L}^2(\mathbb{G})$ the GNS Hilbert space associated with a left Haar integral φ. Since φ is n.s.f., the von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$ is represented in a faithful way on $\mathrm{L}^2(\mathbb{G})$ – henceforth we will treat $\mathrm{L}^\infty(\mathbb{G})$ as a subset of $\mathrm{B}(\mathrm{L}^2(\mathbb{G}))$. The canonical map $\mathfrak{N}_\varphi \to \mathrm{L}^2(\mathbb{G})$ will be denoted by Λ_φ. The choice of the left Haar integral over the right invariant one was arbitrary – luckily one can prove that there exists a number $\nu > 0$, called the *scaling constant*, such that

$$\varphi \circ \sigma_t^\psi = \nu^t \varphi, \quad \psi \circ \sigma_t^\varphi = \nu^{-t}\psi \quad (t \in \mathbb{R}).$$

This has a number of consequences[8]. First, the GNS Hilbert space of ψ can be identified in a canonical way with $\mathrm{L}^2(\mathbb{G})$. Furthermore,

$$J_\psi = \nu^{\frac{i}{4}} J_\varphi, \quad \nabla_\psi^{it} \Lambda_\varphi(x) = \nu^{-\frac{1}{2}t} \Lambda_\varphi(\sigma_t^\psi(x)) \quad (x \in \mathfrak{N}_\varphi, t \in \mathbb{R}) \tag{2.3}$$

and there exists a strictly positive self-adjoint operator δ affiliated with $\mathrm{L}^\infty(\mathbb{G})$ such that

$$(D\psi \colon D\varphi)_t = \nu^{\frac{i}{2}t^2} \delta^{it}, \quad \psi = \varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}}) \quad \text{and} \quad \sigma_t^\psi(x) = \delta^{it} \sigma_t^\varphi(x) \delta^{-it} \tag{2.4}$$

for $x \in \mathrm{L}^\infty(\mathbb{G}), t \in \mathbb{R}$. The operator δ will be called the *modular element*[9] of $\mathbb{G}$. Whenever $\varphi = \psi$ we say that $\mathbb{G}$ is *unimodular* – this happens if and only if $\delta = \mathbb{1}$. Let us denote by Λ_ψ the analog of map Λ_φ for ψ. Then the identification is given by $\Lambda_\varphi(x) = \Lambda_\psi(x\delta^{\frac{1}{2}})$ for

[8] See a slightly broader discussion of this result in Section 2.1

[9] We use this name rather then "modular function" because in the classical setting the modular function is defined as the Radon–Nikodym derivative "$\mathrm{d}\varphi/\,\mathrm{d}\psi$" – here morally speaking we have "$\delta \approx \mathrm{d}\psi/\,\mathrm{d}\varphi$", which could lead to confusion.

sufficiently nice elements x. To be more precise, this equation holds provided $x\in\mathfrak{N}_\varphi$ is an operator for which the composition $x\circ\delta^{\frac12}$ is closable and its closure $x\delta^{\frac12}$ belongs to $\mathfrak{N}_\psi$.
Starting from the axioms in definition 2.8, one can construct a number of objects. Besides the modular element δ, the most important are the two unitary operators W,V acting on the Hilbert space $\mathrm{L}^2(\mathbb{G})\otimes\mathrm{L}^2(\mathbb{G})$. They are defined by the following equalities:

$$((\omega\otimes\mathrm{id})\mathrm{W}^*)\,\Lambda_\varphi(x)=\Lambda_\varphi((\omega\otimes\mathrm{id})\Delta_{\mathbb{G}}(x)),\quad ((\mathrm{id}\otimes\omega)\mathrm{V})\,\Lambda_\psi(y)=\Lambda_\psi((\mathrm{id}\otimes\omega)\Delta_{\mathbb{G}}(y))$$

which hold for $\omega\in\mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*, x\in\mathfrak{N}_\varphi, y\in\mathfrak{N}_\psi$. It is not dificult to show that the left leg of W belongs to $\mathrm{L}^\infty(\mathbb{G})$ and the right leg of V belongs to $\mathrm{L}^\infty(\mathbb{G})'$ (see equation (2.17)). Furthermore, these operators are related to the comultiplication via

$$\Delta_{\mathbb{G}}(x)=\mathrm{W}^*(\mathbb{1}\otimes x)\mathrm{W},\quad \Delta_{\mathbb{G}}(x)=\mathrm{V}(x\otimes\mathbb{1})\mathrm{V}^*\quad (x\in\mathrm{L}^\infty(\mathbb{G}))\tag{2.5}$$

and (using the leg numbering notation)

$$(\Delta_{\mathbb{G}}\otimes\mathrm{id})\mathrm{W}=\mathrm{W}_{13}\mathrm{W}_{23},\quad (\mathrm{id}\otimes\Delta_{\mathbb{G}})\mathrm{V}=\mathrm{V}_{12}\mathrm{V}_{13}.\tag{2.6}$$

It is also not terribly difficult to establish these properties. In fact, the toughest feature of these operators to show, is the fact that they are unitary. Relations (2.6) are equivalent to the so-called pentagonal equations

$$\mathrm{W}_{23}\mathrm{W}_{12}=\mathrm{W}_{12}\mathrm{W}_{13}\mathrm{W}_{23},\quad \mathrm{V}_{23}\mathrm{V}_{12}=\mathrm{V}_{12}\mathrm{V}_{13}\mathrm{V}_{23}.\tag{2.7}$$

Before we move further, let us mention here a number of useful density results:

$$\begin{aligned}\overline{\{(\mathrm{id}\otimes\omega)\mathrm{W}\,|\,\omega\in\mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*\}}^{\mathrm{w}^*}&=\mathrm{L}^\infty(\mathbb{G}),\\ \overline{\{(\omega\otimes\mathrm{id})\mathrm{V}\,|\,\omega\in\mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*\}}^{\mathrm{w}^*}&=\mathrm{L}^\infty(\mathbb{G}),\\ \overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathrm{id}\otimes\omega)\Delta_{\mathbb{G}}(x)\,|\,\omega\in\mathrm{L}^1(\mathbb{G}),x\in\mathrm{L}^\infty(\mathbb{G})\}&=\mathrm{L}^\infty(\mathbb{G}),\\ \overline{\operatorname{span}}^{\mathrm{w}^*}\{(\omega\otimes\mathrm{id})\Delta_{\mathbb{G}}(x)\,|\,\omega\in\mathrm{L}^1(\mathbb{G}),x\in\mathrm{L}^\infty(\mathbb{G})\}&=\mathrm{L}^\infty(\mathbb{G}).\end{aligned}$$

The property of W,V being unitary turns out to be closely related to the existence of the antipode S. It is a densely defined, w^*– closed operator on $\mathrm{L}^\infty(\mathbb{G})$ such that for any $\omega\in\mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*$ we have $(\mathrm{id}\otimes\omega)\mathrm{W}\in\mathrm{Dom}(S)$ and

$$S((\mathrm{id}\otimes\omega)\mathrm{W})=(\mathrm{id}\otimes\omega)\mathrm{W}^*.$$

The space of operators $(\omega\otimes\mathrm{id})\mathrm{W}\,(\omega\in\mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*)$ forms a w^*– core for S.
It is rather difficult to work directly with the antipode S. Instead, we will use its polar decomposition. First, there exists a linear, normal, $\star$-preserving, antimultiplicative bounded operator $R\colon\mathrm{L}^\infty(\mathbb{G})\to\mathrm{L}^\infty(\mathbb{G})$ satisfying $R^2=\mathrm{id}$ (see also equation (2.15)). This operator is called the *unitary antipode* of $\mathbb{G}$. Next, one can define a point w^*– continuous group of $\star$-automorphism of $\mathrm{L}^\infty(\mathbb{G})$, $(\tau_t)_{t\in\mathbb{R}}$ called the *scaling group* of $\mathbb{G}$ satisfying

$R\circ\tau_t=\tau_t\circ R\,(t\in\mathbb{R})$. Its analytic continuation to the point $z=-\frac{i}{2}$ together with R form the polar decomposition of S, namely

$$S=R\circ\tau_{-\frac{i}{2}}.$$

It turns out that the Haar integrals are relatively invariant under the scaling group, i.e. $\varphi\circ\tau_t=\nu^{-t}\varphi,\ \psi\circ\tau_t=\nu^{-t}\psi$. Using the scaling group of $\mathbb{G}$ one can define a strictly positive self-adjoint operator P via

$$P^{it}\Lambda_\varphi(x)=\nu^{\frac{t}{2}}\Lambda_\varphi(\tau_t(x))\quad(x\in\mathfrak{N}_\varphi,t\in\mathbb{R}).$$

This operator implements $(\tau_t)_{t\in\mathbb{R}}$, i.e. $\tau_t(x)=P^{it}xP^{-it}\,(x\in\mathrm{L}^\infty(\mathbb{G}),t\in\mathbb{R})$ (see also equation (2.13)) and satisfies

$$P^{it}\Lambda_\psi(y)=\nu^{-\frac{t}{2}}\Lambda_\psi(\tau_t(y))\quad(y\in\mathfrak{N}_\psi,t\in\mathbb{R}).$$

Besides the scaling group we have two modular automorphism groups coming from the Haar integrals, $(\sigma^\varphi_t)_{t\in\mathbb{R}},(\sigma^\psi_t)_{t\in\mathbb{R}}$. They are uniquely determined – it is a consequence of the important and non-trivial result that Haar integrals on $\mathbb{G}$ are unique up to positive constants. In particular, we can choose these constants in such a way that the equality $\varphi\circ R=\psi$ holds. Let us end this part of Section 2.2 with a collection of formulas connecting the already defined objects:

$$\begin{gathered}\Delta_{\mathbb{G}}\circ\tau_t=(\tau_t\otimes\tau_t)\circ\Delta_{\mathbb{G}}=(\sigma^\varphi_t\otimes\sigma^\psi_{-t})\circ\Delta_{\mathbb{G}},\\ \Delta_{\mathbb{G}}\circ\sigma^\varphi_t=(\tau_t\otimes\sigma^\varphi_t)\circ\Delta_{\mathbb{G}},\quad \Delta_{\mathbb{G}}\circ\sigma^\psi_t=(\sigma^\psi_t\otimes\tau_{-t})\circ\Delta_{\mathbb{G}},\\ \Delta_{\mathbb{G}}\circ R=(R\otimes R)\circ\Delta^{op}_{\mathbb{G}},\qquad R\circ\sigma^\varphi_t=\sigma^\psi_{-t}\circ R,\end{gathered}\tag{2.8}$$

where $\Delta^{op}_{\mathbb{G}}$ is the comultiplication $\Delta_{\mathbb{G}}$ composed with the tensor flip. Furthermore, groups of $\star-$automorphisms $(\sigma^\varphi_t)_{t\in\mathbb{R}},(\sigma^\psi_t)_{t\in\mathbb{R}},(\tau_t)_{t\in\mathbb{R}}$ commute. There is also a number of formulas concerning the modular element[10] δ:

$$\sigma^\varphi_t(\delta^{is})=\nu^{its}\delta^{is},\quad\sigma^\psi_t(\delta^{is})=\nu^{its}\delta^{is},\quad\tau_t(\delta^{is})=\delta^{is},\quad R(\delta^{is})=\delta^{-is}\quad(t,s\in\mathbb{R}).\tag{2.9}$$

It turns out that we also have $\Delta_{\mathbb{G}}(\delta^{it})=\delta^{it}\otimes\delta^{it}\,(t\in\mathbb{R})$ though it is more difficult to prove and requires (at least in the approach of [93]) passing to the dual quantum group.

An important result in the theory of locally compact quantum groups is an existence of the *dual locally compact quantum group* $\widehat{\mathbb{G}}$. Its von Neumann algebra $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is defined via

$$\mathrm{L}^\infty(\widehat{\mathbb{G}})=\overline{\{(\omega\otimes\mathrm{id})\mathrm{W}\,|\,\omega\in\mathrm{L}^1(\mathbb{G})\}}^{\mathrm{w}^*},$$

in particular it is represented on $\mathrm{L}^2(\mathbb{G})$. To see that this subspace is closed under multiplication one has to use the equation $(\Delta_{\mathbb{G}}\otimes\mathrm{id})\mathrm{W}=\mathrm{W}_{13}\mathrm{W}_{23}$. It is also closed under

[10] Whenever possible we will use the unitary operators $\delta^{it}\in\mathrm{L}^\infty(\mathbb{G})\,(t\in\mathbb{R})$ rather than δ to avoid unnecessary technical complications.

the adjoint – we have $((\omega\otimes\mathrm{id})\mathrm{W})^* = (\omega'\otimes\mathrm{id})\mathrm{W}$ for nice enough $\omega\in \mathrm{L}^1(\mathbb{G})$ (namely for ω analytic with respect to the group of automorphisms of $\mathrm{L}^1(\mathbb{G})$, predual to $(\tau_t)_{t\in\mathbb{R}}$) and certain $\omega'\in\mathrm{L}^1(\mathbb{G})$. Recall that χ stands for the flip map on $\mathrm{B}(\mathrm{L}^2(\mathbb{G}))\bar{\otimes}\mathrm{B}(\mathrm{L}^2(\mathbb{G}))$. Define unitary operator[11] $\widehat{\mathrm{W}}=\chi(\mathrm{W}^*)$. Using this operator we define comultiplication on $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ as

$$\Delta_{\widehat{\mathbb{G}}}(x)=\widehat{\mathrm{W}}^*(\mathbb{1}\otimes x)\widehat{\mathrm{W}}\quad (x\in\mathrm{L}^\infty(\widehat{\mathbb{G}})).$$

It turns out that $\widehat{\mathrm{W}}$ is the right "W operator"[12] for $\widehat{\mathbb{G}}$. Showing that there exists a left-invariant weight $\widehat{\varphi}$ on $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is highly non-trivial and is done using the theory of Hilbert algebras. One can identify the GNS Hilbert space for $\widehat{\varphi}$ with $\mathrm{L}^2(\mathbb{G})$ and basically by definition of $\Lambda_{\widehat{\varphi}}$ we have

$$\langle\Lambda_\varphi(x)\,|\,\Lambda_{\widehat{\varphi}}((\omega\otimes\mathrm{id})\mathrm{W})\rangle=\omega(x^*)$$

for $x\in\mathfrak{N}_\varphi$ and "nice enough" $\omega\in\mathrm{L}^1(\mathbb{G})$. The right Haar integral on $\widehat{\mathbb{G}}$ is defined via $\widehat{\psi}=\widehat{\varphi}\circ\widehat{R}$, where

$$\widehat{R}(x)=J_\varphi x^* J_\varphi\quad (x\in\mathrm{L}^\infty(\widehat{\mathbb{G}})). \tag{2.10}$$

is the unitary antipode of $\widehat{\mathbb{G}}$ and, as usual, J_φ is the modular conjugation associated with φ.

There is a number of formulas relating objects associated with $\mathbb{G}$ and $\widehat{\mathbb{G}}$. To begin with, the scaling constant of $\widehat{\mathbb{G}}$ is $\widehat{\nu}=\nu^{-1}$ and $\widehat{P}=P$ holds. Next, the modular conjugation $J_{\widehat{\varphi}}$ can be used to relate operators W and V:

$$\mathrm{V}=(J_{\widehat{\varphi}}\otimes J_{\widehat{\varphi}})\chi(\mathrm{W}^*)(J_{\widehat{\varphi}}\otimes J_{\widehat{\varphi}}), \tag{2.11}$$

we also have

$$(R\otimes\hat{R})\mathrm{W}=\mathrm{W}. \tag{2.12}$$

Another important relation expresses the scaling group of $\mathbb{G}$ using modular operators $\nabla_{\widehat{\varphi}}$:

$$\tau_t(x)=\nabla_{\widehat{\varphi}}^{it}x\nabla_{\widehat{\varphi}}^{-it}\quad (x\in\mathrm{L}^\infty(\mathbb{G}),t\in\mathbb{R}). \tag{2.13}$$

There are also various commutation relations:

$$\begin{aligned} J_{\widehat{\varphi}}J_\varphi &= \nu^{\frac{i}{4}}J_\varphi J_{\widehat{\varphi}}, & \nabla_\psi^{it} &= J_{\widehat{\varphi}}\nabla_\varphi^{-it}J_{\widehat{\varphi}}, & \nabla_\psi^{it} &= \delta^{it}(J_\varphi\delta^{it}J_\varphi)\nabla_\varphi^{it}\\ J_\varphi P^{it} &= J_\varphi P^{it}, & \nabla_\varphi^{is}P^{it} &= P^{it}\nabla_\varphi^{is}, & \nabla_\psi^{is}P^{it} &= P^{it}\nabla_\psi^{is},\\ \nabla_\psi^{it} &= \hat{\delta}^{-it}P^{-it}, & \delta^{it}\hat{\delta}^{is} &= \nu^{ist}\hat{\delta}^{is}\delta^{it}, & P^{-2it} &= \delta^{it}(J_\varphi\delta^{it}J_\varphi)\hat{\delta}^{it}(J_{\widehat{\varphi}}\hat{\delta}^{it}J_{\widehat{\varphi}}). \end{aligned} \tag{2.14}$$

One can show that the bidual of $\mathbb{G}$ is isomorphic to $\mathbb{G}$, so in particular we can put hats in all of the above formulas – for example equation (2.10) implies

$$R(x)=J_{\widehat{\varphi}}x^*J_{\widehat{\varphi}}\quad (x\in\mathrm{L}^\infty(\mathbb{G})). \tag{2.15}$$

[11] We will follow this convention and decorate objects corresponding to the dual group with hats.

[12] Known also as the *Kac-Takesaki operator*.

We can obtain information concerning the relative position of von Neumann algebras $\mathrm{L}^\infty(\mathbb{G}), \mathrm{L}^\infty(\widehat{\mathbb{G}})$ inside $\mathrm{B}(\mathrm{L}^2(\mathbb{G}))$:

$$\begin{aligned}\mathrm{L}^\infty(\mathbb{G}) \cap \mathrm{L}^\infty(\widehat{\mathbb{G}}) = \mathbb{C}\mathbb{1}, \quad \mathrm{L}^\infty(\mathbb{G}) \cap \mathrm{L}^\infty(\widehat{\mathbb{G}})' = \mathbb{C}\mathbb{1},\\ \mathrm{L}^\infty(\mathbb{G})' \cap \mathrm{L}^\infty(\widehat{\mathbb{G}}) = \mathbb{C}\mathbb{1}, \quad \mathrm{L}^\infty(\mathbb{G})' \cap \mathrm{L}^\infty(\widehat{\mathbb{G}})' = \mathbb{C}\mathbb{1}.\end{aligned} \tag{2.16}$$

So far we have considered quantum groups described using von Neumann algebras. Similarly to the classical theory, aside from the algebra of essentialy bounded measurable functions (i.e. $\mathrm{L}^\infty(\mathbb{G})$), there is also the algebra of bounded continuous functions which vanish at infinity: $\mathrm{C}_0(\mathbb{G})$. It is defined via the following formula:

$$\mathrm{C}_0(\mathbb{G}) = \overline{\{(\mathrm{id} \otimes \omega)\mathrm{W} \,|\, \omega \in \mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*\}}.$$

It is not hard to see that

$$\mathrm{W} \in \mathrm{M}(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\widehat{\mathbb{G}})) \subseteq \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{G}}). \tag{2.17}$$

Since W is unitary, $\mathrm{C}_0(\mathbb{G}) \subseteq \mathrm{B}(\mathrm{L}^2(\mathbb{G}))$ is non-degenerate. Note also that (2.17) implies that $\Delta_{\mathbb{G}}$ restricts to a morphism, i.e. an element of $\mathrm{Mor}(\mathrm{C}_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G}))$. It is also not dificult to check, using already introduced relations, that the scaling group, modular automorphisms (associated with φ and ψ) and the unitary antipode preserve $\mathrm{C}_0(\mathbb{G})$. Haar integrals after restriction become faithful, densely defined and lower-semicontinuous KMS weights on the C^*-algebra $\mathrm{C}_0(\mathbb{G})$.
Another important C^*-algebra is the universal version of the C^*-algebra of continuous functions vanishing at infinity: $\mathrm{C}_0^u(\mathbb{G})$. It was introduced by Kustermans in [55], let us mention here only these results which will be of use to us.
First, there is a unitary operator

$$\mathbb{W} \in \mathrm{M}(\mathrm{C}_0^u(\mathbb{G}) \otimes \mathrm{C}_0(\widehat{\mathbb{G}}))$$

satisfying

$$\mathrm{C}_0^u(\mathbb{G}) = \overline{\{(\mathrm{id} \otimes \omega)\mathbb{W} \,|\, \omega \in \mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*\}}.$$

Next, there exists a $\star$-epimorphism

$$\Lambda_{\mathbb{G}} \colon \mathrm{C}_0^u(\mathbb{G}) \to \mathrm{C}_0(\mathbb{G})$$

such that $(\Lambda_{\mathbb{G}} \otimes \mathrm{id})\mathbb{W} = \mathrm{W}$. Similarly, we can define $\mathrm{C}_0^u(\widehat{\mathbb{G}})$ and corresponding

$$\Lambda_{\widehat{\mathbb{G}}} \colon \mathrm{C}_0^u(\widehat{\mathbb{G}}) \to \mathrm{C}_0(\widehat{\mathbb{G}}), \quad \mathcal{W} \in \mathrm{M}(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0^u(\widehat{\mathbb{G}})) : \quad (\mathrm{id} \otimes \Lambda_{\widehat{\mathbb{G}}})\mathcal{W} = \mathrm{W}.$$

There is also an operator $\mathbb{W}\!\!\!\!\mathcal{W}$ with "both legs universal", i.e. $\mathbb{W}\!\!\!\!\mathcal{W} \in \mathrm{M}(\mathrm{C}_0^u(\mathbb{G}) \otimes \mathrm{C}_0^u(\widehat{\mathbb{G}}))$. It satisfies

$$(\Lambda_{\mathbb{G}} \otimes \mathrm{id})\mathbb{W}\!\!\!\!\mathcal{W} = \mathcal{W}, \quad (\mathrm{id} \otimes \Lambda_{\widehat{\mathbb{G}}})\mathbb{W}\!\!\!\!\mathcal{W} = \mathbb{W}.$$

The above objects have similar properties to their reduced versions. Using "W operators" one can define comultiplications on the universal C*-algebras, there are also lifts of the Haar integrals, corresponding modular automorphisms, scaling group, modular element and the unitary antipode. These objects will be decorated with ^{u}, e.g. φ^u is the left Haar integral on $\mathrm{C}_0^u(\mathbb{G})$. Note however that φ^u, ψ^u are not necessarily faithful.
A result of utmost importance is the fact that (non-degenerate) representations of C*-algebra $\mathrm{C}_0^u(\widehat{\mathbb{G}})$ are in one-to-one correspondence with (unitary) representations of $\mathbb{G}$, i.e. unitaries $U \in \mathrm{M}(\mathrm{C}(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}_U))$ (where H_U is a complex Hilbert space) satisfying $(\Delta_{\mathbb{G}} \otimes \mathrm{id})U = U_{12}U_{13}$. This correspondence is given by the following prescription: having a representation $\pi \in \mathrm{Mor}(\mathrm{C}_0^u(\widehat{\mathbb{G}}), \mathcal{K}(\mathsf{H}_U))$, the corresponding representation of $\mathbb{G}$ is given by $U = (\mathrm{id} \otimes \pi)\mathbb{W}$ – one can show that all representations of $\mathbb{G}$ arise in this way.
Let us introduce a useful notation:

$$\lambda(\omega) = (\omega \otimes \mathrm{id})\mathrm{W} \in \mathrm{C}_0(\widehat{\mathbb{G}}), \quad \lambda^u(\omega) = (\omega \otimes \mathrm{id})\mathbb{W} \in \mathrm{C}_0^u(\widehat{\mathbb{G}})$$

for $\omega \in \mathrm{L}^1(\mathbb{G})$. The images of these maps generate $\mathrm{C}_0(\widehat{\mathbb{G}}), \mathrm{C}_0^u(\widehat{\mathbb{G}})$.

In a couple of places (most notably Section 6, but also when discussing examples) we will meet the notions of amenability and coamenability of a locally compact quantum group. They are defined as follows:

Definition 2.9. Let $\mathbb{G}$ be a locally compact quantum group. We say that $\mathbb{G}$ is *amenable*, if there exists a state $m \in \mathrm{L}^\infty(\mathbb{G})$ (called a *mean*) such that

$$m((\omega \otimes \mathrm{id})\Delta_{\mathbb{G}}(x)) = m((\mathrm{id} \otimes \omega)\Delta_{\mathbb{G}}(x)) = m(x)\omega(\mathbb{1}) \quad (x \in \mathrm{L}^\infty(\mathbb{G}), \omega \in \mathrm{L}^1(\mathbb{G})).$$

We say that $\mathbb{G}$ is *coamenable*, when $\Lambda_{\mathbb{G}}$ is an isomorphism – in such case we identify $\mathrm{C}_0(\mathbb{G})$ with $\mathrm{C}_0^u(\mathbb{G})$.

Let us mention that coamenability has a number of equivalent formulations [8, Theorem 3.1].

One can quite easily see that coamenability of $\mathbb{G}$ implies amenability of $\widehat{\mathbb{G}}$ [8, Theorem 3.2], while whether the converse holds is a major open problem. It is known that amenability of $\widehat{\mathbb{G}}$ implies coamenability of $\mathbb{G}$, when $\mathbb{G}$ is compact – this beautiful result was proved by Tomatsu [86] (and independently by Blanchard, Vaes).

2.2.1 Example: classical locally compact quantum group and its dual

Let us describe here the motivating example of a locally compact quantum group, namely the quantum group associated with a classical group. Let G be a locally compact (Hausdorff) topological group, with Haar measures μ_L, μ_R chosen in such a way that $\mu_R(E) = \mu_L(E^{-1})$ for Borel $E \subseteq G$. For the sake of simplicty, assume that G is second countable. The associated locally compact quantum group $\mathbb{G}$ (denoted also by G) is described via $\mathrm{L}^\infty(G)$, the L^∞-space associated with the left Haar measure, comultiplication Δ_G given by

$$\Delta_G(f)(x, y) = f(xy) \quad (f \in \mathrm{L}^\infty(G),\, x, y \in G)$$

and the Haar integrals

$$\varphi(f)=\int_G f\,\mathrm{d}\mu_L,\quad \psi(f)=\int_G f\,\mathrm{d}\mu_R \quad (f\in \mathrm{L}^\infty(G)^+).$$

The associated C*-algebras $\mathrm{C}_0(\mathbb{G}),\mathrm{C}_0^u(\mathbb{G})$ are both equal to[13] $\mathrm{C}_0(G)$ and $\Lambda_{\mathbb{G}}$ is the identity map. Since $\mathrm{L}^\infty(G)$ is commutative, the modular automorphisms of φ,ψ are trivial. The scaling group is also trivial and we have $\nu=1$. Consequently, $P=\mathbb{1}$ and the antipode is equal to the unitary antipode given by

$$R(f)(x)=f(x^{-1})\quad (f\in\mathrm{L}^\infty(G),\, x\in G).$$

The modular element δ is equal to the Radon-Nikodym derivative $\frac{\mathrm{d}\mu_R}{\mathrm{d}\mu_L}$, i.e. the inverse of the usual modular function.
The dual locally compact quantum group $\widehat{\mathbb{G}}=\widehat{G}$ is more interesting. Its von Neumann algebra $\mathrm{L}^\infty(\widehat{G})$ is equal to the group von Neumann algebra $\mathrm{L}(G)$, i.e. the von Neumann subalgebra of $\mathrm{B}(\mathrm{L}^2(G))$ generated by the image of the left regular representation. Next, we have $\mathrm{C}_0(\widehat{\mathbb{G}})=\mathrm{C}^*_r(G)$, the reduced group C*-algebra and $\mathrm{C}_0^u(\widehat{\mathbb{G}})$ is equal to $\mathrm{C}^*(G)$, the full group C*-algebra of G. Comultiplication on $\mathrm{L}(G)$ and $\mathrm{C}^*(G)$ is given by

$$\Delta_{\widehat{G}}(\lambda_x)=\lambda_x\otimes\lambda_x \quad\text{and}\quad \Delta^u_{\widehat{G}}(u_x)=u_x\otimes u_x\quad (x\in G)$$

where $\lambda_x, u_x\,(x\in G)$ are the canonical unitaries in $\mathrm{L}(G),\mathrm{M}(\mathrm{C}^*(G))$. Both Haar integrals on $\widehat{G}$ are equal to the Plancherel weight (see [82, Section VII.3]). It is tracial if and only if G is unimodular. Indeed, we already know that $P=\mathbb{1}$, hence $\nabla_{\widehat{\psi}}=\delta^{-1}$ – see equation (2.14). N.b., the dual version of this equation (i.e. after applying hats) implies that $\widehat{G}$ is unimodular. However, as $\widehat{P}=P=\mathbb{1}$ the scaling group (and the scaling constant) of $\widehat{G}$ are always trivial. Consequently

$$\widehat{S}(\lambda_x)=\widehat{R}(\lambda_x)=\lambda_{x^{-1}}\quad (x\in G).$$

We will see more examples of locally compact quantum groups in sections 2.3.1 – 2.3.4, 3.6, 3.7 and 5.4.

2.3 Compact/discrete quantum groups

Compact quantum groups were introduced by Woronowicz: in [99] he defined his famous $\mathrm{SU}_q(2)$ quantum group and later developed the general theory of compact quantum groups [98, 101]. In this section we will describe the basic theory of compact quantum groups as well as their duals, i.e. discrete quantum groups [70]. Besides the above mentioned references we also refer the reader to [85] and a very well written book by Neshveyev and Tuset [64].

[13] This means that $\mathbb{G}=G$ is coamenable.

Definition 2.10. A compact quantum group $\mathbb{G}$ is a pair $(\mathrm{C}(\mathbb{G}), \Delta)$ consisting of

- a unital C^*-algebra $\mathrm{C}(\mathbb{G})$,
- a unital $\star$-homomorphism $\Delta\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ satisfying $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and

$$\overline{\operatorname{span}}\, \Delta(\mathrm{C}(\mathbb{G}))(\mathrm{C}(\mathbb{G}) \otimes \mathbb{1}) = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) = \overline{\operatorname{span}}\, \Delta(\mathrm{C}(\mathbb{G}))(\mathbb{1} \otimes \mathrm{C}(\mathbb{G})).$$

The fundamental result concerning a compact quantum group $\mathbb{G}$ is the existence of the Haar integral, i.e. the unique state $h \in \mathrm{C}(\mathbb{G})^*$ which is left and right invariant:

$$(h \otimes \mathrm{id})\Delta(a) = h(a)\mathbb{1} = (\mathrm{id} \otimes h)\Delta(a) \quad (a \in \mathrm{C}(\mathbb{G})).$$

This result was proved by Woronowicz under the separability assumption on $\mathrm{C}(\mathbb{G})$ and in general by Van Daele in [91].
The Haar integral provided by the above result does not need to be faithful. Let $\mathrm{L}^2(\mathbb{G})$ be the Hilbert space obtained by the GNS representation associated with h and let $\mathrm{C}^r(\mathbb{G})$ be the image of $\mathrm{C}(\mathbb{G})$ in $\mathrm{B}(\mathrm{L}^2(\mathbb{G}))$. Taking the bicommutant of $\mathrm{C}^r(\mathbb{G})$ we get a von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$. One can show that the comultiplication descends in a canonical way to $\mathrm{C}^r(\mathbb{G})$ and then extends to a normal map Δ^r on $\mathrm{L}^\infty(\mathbb{G})$. In the GNS representation, the Haar integral becomes a vector state which we will denote by[14] h^r. It turns out that the quadruple $(\mathrm{L}^\infty(\mathbb{G}), \Delta^r, h^r, h^r)$ is a locally compact quantum group in the sense of Definition 2.8. In particular, the left and the right Haar integrals are equal hence $\delta^r = \mathbb{1}$ and $\nu^r = 1$.

There is also a more intrinsic characterisation of compact quantum groups among locally compact quantum groups: a locally compact quantum group $\mathbb{G}$ is compact if and only if $\mathrm{C}_0(\mathbb{G})$ is unital or equivalently one of the Haar integrals is a state [8, Proposition 3.1].

A feature that makes compact quantum groups especially amenable to concrete calculations is their very tractable and powerful representation theory – we will see in Section 3 that it is the case (to some degree) also for type I quantum groups.
Let $\mathbb{G}$ be a compact quantum group. Recall that a unitary representation of $\mathbb{G}$ on a Hilbert space H_U is a unitary element $U \in \mathrm{M}(\mathrm{C}(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}_U))$ such that $(\Delta \otimes \mathrm{id})U = U_{13}U_{23}$. A matrix element of U is an operator of the form $(\mathrm{id} \otimes \omega)U \in \mathrm{C}(\mathbb{G})\,(\omega \in \mathcal{K}(\mathsf{H}_U)^*)$. An intertwiner between two representations U, V is an operator $T \in \mathrm{B}(\mathsf{H}_U, \mathsf{H}_V)$ satisfying $(\mathbb{1} \otimes T)U = V(\mathbb{1} \otimes T)$. The space of intertwiners between U and V will be denoted by $\mathrm{Mor}(U, V)$, we will also write $\mathrm{End}(U) = \mathrm{Mor}(U, U)$. We say that U is irreducible if the only self-intertwiners are proportional to the identity operator $\mathbb{1}_U \in \mathrm{B}(\mathsf{H}_U)$. Equivalently, U is not equivalent to a direct sum of two representations. The fundamental results of the representation theory are as follows:

[14] In other sections we will be dealing only with compact quantum groups with $\mathrm{C}(\mathbb{G}) = \mathrm{C}^u(\mathbb{G})$ (the universal form) or $\mathrm{C}(\mathbb{G}) = \mathrm{C}^r(\mathbb{G})$ (the reduced form). In the latter case we will simply write $h = h^r$, etc.

- irreducible representations of $\mathbb{G}$ are finite dimensional, i.e. $\dim(U) = \dim(\mathsf{H}_U) < +\infty$,
- every unitary representation of $\mathbb{G}$ is equivalent to a direct sum of irreducible representations,
- the subset $\mathrm{Pol}(\mathbb{G}) \subseteq \mathrm{C}(\mathbb{G})$ formed by the matrix elements of unitary finite dimensional representations of $\mathbb{G}$ is a dense $\star$-subalgebra.

In fact, more can be said about $\mathrm{Pol}(\mathbb{G})$: together with the restricted comultiplication it is a Hopf $\star$-algebra. The counit ε and the antipode S act as follows:

$$(\varepsilon \otimes \mathrm{id})U = \mathbb{1}_U, \quad (S \otimes \mathrm{id})U = U^{-1}$$

for any finite dimensional unitary representation U. In Section 2.2 we have said that with any locally compact quantum group $\mathbb{G}$ we can associate a C*-algebra $\mathrm{C}_0^u(\mathbb{G})$ which is the "universal version of the algebra of C_0 functions on $\mathbb{G}$". When $\mathbb{G}$ is compact, the definition of $\mathrm{C}_0^u(\mathbb{G})$ (denoted then by $\mathrm{C}^u(\mathbb{G})$ due to obvious reasons) is much simpler – $\mathrm{C}^u(\mathbb{G})$ is simply the enveloping C*-algebra of $\mathrm{Pol}(\mathbb{G})$ [7, Section 3].
We have seen in Section 2.2 that the modular theory of Haar integrals can be very interesting: it gives rise to modular automorphisms, scaling group and the scaling constant. It turns out that in the compact case it has its roots in the representation theory. Let U be a finite dimensional unitary representation of $\mathbb{G}$. We define the *contragradient representation*

$$U^c = (S \otimes j_U)U$$

(recall that $j_U \colon \mathrm{B}(\mathsf{H}_U) \to \mathrm{B}(\overline{\mathsf{H}_U})$ is the canonical antimultiplicative isomorphism) and the conjugate representation

$$\overline{U} = (R \otimes j_U)U.$$

One can prove that both these elements are equvalent representations of $\mathbb{G}$ on $\overline{\mathsf{H}_U}$ and $\overline{U}$ is unitary, but U^c not necesarilly so. One can perform the contragradient construction once again and arrive at the representation U^{cc} – it turns out that it is equivalent to U. If U is irreducible, we define ρ_U to be the unique positive invertible intertwiner $\rho_U \in \mathrm{Mor}(U, U^{cc})$ satisfying $\mathrm{Tr}(\rho_U) = \mathrm{Tr}(\rho_U^{-1})$. For general unitary representation U, one defines ρ_U by decomposing U into irreducible summands and then taking a direct sum of corresponding operators (see [64, Proposition 1.4.4]) – it is a positive and invertible operator. The number $\mathrm{Tr}(\rho_U)$ is called the *quantum dimension* of U, $\dim_q(U)$ – in general it is greater or equal to the usual dimension $\dim(U)$.
Let us denote by $\mathrm{Irr}(\mathbb{G})$ the set of (equivalence classes[15] of) irreducible representations of $\mathbb{G}$. The family of operators $\{\rho_\alpha\}_{\alpha \in \mathrm{Irr}(\mathbb{G})}$ is of utmost importance. Using them we can express the action of the modular and scaling automorphism groups introduced in the previous section. First, let us define a family $\{f_z\}_{z \in \mathbb{C}}$ of functionals on $\mathrm{Pol}(\mathbb{G})$ via

$$(f_z \otimes \mathrm{id})U = \rho_U^z \quad (z \in \mathbb{C}) \tag{2.18}$$

[15]We will follow the common abuse of notation and identify in notation a class of representations with its representative. For example, we will write ρ_α to denote the operator associated with a chosen representation $U^\alpha \in \alpha \in \mathrm{Irr}(\mathbb{G})$.

for any finite dimensional unitary representation U. These functionals have a number of useful properties (see [64, Proposition 1.7.2]), we will only mention here the following[16]:

$$\tau_t(a) = f_{-it} \star a \star f_{it} \quad \sigma^h_t(a) = f_{it} \star a \star f_{it} \quad (t \in \mathbb{R}, a \in \mathrm{Pol}(\mathbb{G})), \tag{2.19}$$

where the convolution of functionals ϕ, ϕ' and an element a is defined via

$$\phi \star a = (\mathrm{id} \otimes \phi)\Delta(a), \quad a \star \phi = (\phi \otimes \mathrm{id})\Delta(a), \quad (\phi \star a) \star \phi' = \phi \star (a \star \phi') = \phi \star a \star \phi'.$$

Another very useful result is the fact the using operators $\{\mathsf{\rho}_\alpha\}_{\alpha \in \mathrm{Irr}(\mathbb{G})}$ we can express the action of h on products of two matrix coefficients – for $\alpha, \beta \in \mathrm{Irr}(\mathbb{G}), \xi, \xi' \in \mathsf{H}_\alpha, \eta, \eta' \in \mathsf{H}_\beta$ we have ([64, Theorem 1.4.3])

$$h(U^{\alpha}_{\xi,\xi'}{}^* U^{\beta}_{\eta,\eta'}) = \tfrac{\delta_{\alpha,\beta} \langle \xi' \,|\, \eta' \rangle \langle \eta \,|\, \mathsf{\rho}_\alpha^{-1} \xi \rangle}{\dim_q(\alpha)}, \quad h(U^{\alpha}_{\xi,\xi'} U^{\beta}_{\eta,\eta'}{}^*) = \tfrac{\delta_{\alpha,\beta} \langle \xi \,|\, \eta \rangle \langle \eta' \,|\, \mathsf{\rho}_\alpha \xi' \rangle}{\dim_q(\alpha)},$$

where $U^\alpha_{\xi,\xi'} = (\mathrm{id} \otimes \omega_{\xi,\xi'}) U^\alpha \in \mathrm{C}(\mathbb{G})$. These equations are called the *orthogonality relations.*

Let $\widehat{\mathbb{G}}$ be the locally compact quantum group dual to $\mathbb{G}$, as described in Section 2.2. Any quantum group which arises in this way will be called *discrete.* Using the representation theory of $\mathbb{G}$, we can describe in detail some of the structure of $\widehat{\mathbb{G}}$. First, we have

$$\mathrm{L}^\infty(\widehat{\mathbb{G}}) = \prod_{\alpha \in \mathrm{Irr}(\mathbb{G})} \mathrm{B}(\mathsf{H}_\alpha), \quad \mathrm{C}^u_0(\widehat{\mathbb{G}}) = \mathrm{C}_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} \mathrm{B}(\mathsf{H}_\alpha).$$

which in some sense explains name "discrete" (when $\widehat{\mathbb{G}}$ is a classical discrete group we have $\dim(\mathsf{H}_\alpha) = 1$, here we only know that $\dim(\mathsf{H}_\alpha) < +\infty$). We will henceforth write

$$\ell^\infty(\widehat{\mathbb{G}}) = \mathrm{L}^\infty(\widehat{\mathbb{G}}), \quad \mathrm{c}_0(\widehat{\mathbb{G}}) = \mathrm{C}_0(\widehat{\mathbb{G}}).$$

We already know that the scaling constant of $\widehat{\mathbb{G}}$ is trivial (because it is for $\mathbb{G}$), but the modular element need not be trivial. Indeed, the left and the right Haar integrals are given by

$$\widehat{\varphi} = \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} \dim_q(\alpha) \, \mathrm{Tr}_\alpha(\mathsf{\rho}_\alpha^{-1} \pi_\alpha(\cdot)), \quad \widehat{\psi} = \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} \dim_q(\alpha) \, \mathrm{Tr}_\alpha(\mathsf{\rho}_\alpha \pi_\alpha(\cdot))$$

(where $\pi_\alpha \colon \ell^\infty(\widehat{\mathbb{G}}) \to \mathrm{B}(\mathsf{H}_\alpha) \, (\alpha \in \mathrm{Irr}(\mathbb{G}))$ are the canonical projections) and the modular element is equal to

$$\hat{\delta} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} \mathsf{\rho}_\alpha^2.$$

Since $\ell^\infty(\widehat{\mathbb{G}})$ and the Haar integrals $\widehat{\varphi}, \widehat{\psi}$ are given by such simple formulas, one can easily show that the GNS Hilbert space $\mathrm{L}^2(\widehat{\mathbb{G}}) \simeq \mathrm{L}^2(\mathbb{G})$ can be identified in a canonical way with

[16]Note that these formulas make sense also for complex $z = t$ – one can show that elements of $\mathrm{Pol}(\mathbb{G})$ are analytic with respect to $(\tau_t)_{t \in \mathbb{R}}, (\sigma^h_t)_{t \in \mathbb{R}}$

$\bigoplus_{\alpha\in\mathrm{Irr}(\mathbb{G})}\mathsf{HS}(\mathsf{H}_\alpha)$ – we will discuss it further in Section 3. The modular automorphisms of $\widehat{\varphi},\widehat{\psi}$ are given by

$$\sigma_t^{\widehat{\varphi}}((x_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})})=(\rho_\alpha^{-it}x_\alpha\rho_\alpha^{it})_{\alpha\in\mathrm{Irr}(\mathbb{G})},\quad \sigma_t^{\widehat{\psi}}((x_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})})=(\rho_\alpha^{it}x_\alpha\rho_\alpha^{-it})_{\alpha\in\mathrm{Irr}(\mathbb{G})} \tag{2.20}$$

and the scaling group acts as follows

$$\widehat{\tau}_t((x_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})})=(\rho_\alpha^{-it}x_\alpha\rho_\alpha^{it})_{\alpha\in\mathrm{Irr}(\mathbb{G})} \tag{2.21}$$

where $t\in\mathbb{R}$ and $(x_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\in\ell^\infty(\widehat{\mathbb{G}})$. Note that the above equations show $\widehat{\tau}_t=\sigma_t^{\widehat{\varphi}}$ for all $t\in\mathbb{R}$. Indeed, it follows from equations (2.14) that $\nabla_{\widehat{\varphi}}^{it}=P^{it}$. It is a consequence of unimodularity of $\mathbb{G}$ (see also Proposition 3.32).
We can also identify the Kac-Takesaki operator for $\mathbb{G}$:

$$\mathrm{W}=\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}U^\alpha\in\mathrm{M}(\mathrm{C}(\mathbb{G})\otimes\mathrm{c}_0(\widehat{\mathbb{G}})) \tag{2.22}$$

(this series converges in the strict topology).

Let us end this section with a definition: one says that a compact quantum group $\mathbb{G}$ is of Kac type if the Haar integral h is a trace. Equivalently: the scaling group of $\mathbb{G}$ (or $\widehat{\mathbb{G}}$) is trivial, (one of the) Haar integrals on $\mathbb{G}$ (or $\widehat{\mathbb{G}}$) is tracial, $\widehat{\mathbb{G}}$ is unimodular (i.e. $\widehat{\varphi}=\widehat{\psi}$) or $\rho_U=\mathbb{1}_U$ for all finite dimensional unitary representations U of $\mathbb{G}$. We will obtain more conditions in this spirit for type I quantum groups in Section 3.3.

2.3.1 Example: the quantum group $\mathrm{SU}_q(2)$

In this section we will describe the quantum version of the group SU(2), introduced by Woronowicz in [99]. Let $q\in\,]-1,1[\setminus\{0\}$ be a fixed parameter. The C*-algebra $\mathrm{C}^u(\mathrm{SU}_q(2))$ is defined as the universal unital C*-algebra generated by elements α,γ such that

$$\alpha^*\alpha+\gamma^*\gamma=\mathbb{1},\quad \alpha\alpha^*+q^2\gamma^*\gamma=\mathbb{1},\quad \gamma\gamma^*=\gamma^*\gamma$$
$$\alpha\gamma=q\gamma\alpha,\quad \alpha\gamma^*=q\gamma^*\alpha.$$

Alternatively, one can impose the condition that the matrix $U^{1/2}=\begin{bmatrix}\alpha & -q\gamma^*\\ \gamma & \alpha^*\end{bmatrix}$ is unitary.
Comultiplication is defined by

$$\Delta(\alpha)=\alpha\otimes\alpha-q\gamma^*\otimes\gamma,\quad \Delta(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma$$

and its existence follows from the universal property of $\mathrm{C}^u(\mathrm{SU}_q(2))$.
The representation theory of $\mathrm{SU}_q(2)$ is quite simple: we have $\mathrm{Irr}(\mathrm{SU}_q(2))\simeq\frac{1}{2}\mathbb{Z}_+$ in such a way that U^0 is the trivial representation, $U^{1/2}$ is the above two dimensional representation, $\overline{U^n}\simeq U^n$ ($n\in\frac{1}{2}\mathbb{Z}_+$) and the fusion rules (i.e. rules of tensor product multiplication) are given by

$$U^n\mathbin{\bigcirc\!\!\!\!\top} U^m\simeq U^{|n-m|}\oplus\cdots\oplus U^{n+m}\quad (n,m\in\tfrac{1}{2}\mathbb{Z}_+). \tag{2.23}$$

It follows that $\mathrm{Pol}(\mathrm{SU}_q(2))$ is the unital $\star$-algebra generated by α, γ. Conesquently $\mathrm{C}^u(\mathrm{SU}_q(2))$ is indeed universal in the sense of Section 2.3. It turns out that the Haar integral on $\mathrm{C}^u(\mathrm{SU}_q(2))$ is faithful – the quantum group $\mathrm{SU}_q(2)$ is coamenable [7, Theorem 2.12] and we will henceforth write $\mathrm{C}^u(\mathrm{SU}_q(2)) = \mathrm{C}(\mathrm{SU}_q(2))$.
$\mathrm{SU}_q(2)$ is the fundamental example of a compact quantum group which is not of Kac type. Indeed, the modular and scaling automorphism groups are given by ([64, Section 1.7])

$$\sigma_t^h(\alpha) = |q|^{-2it}\alpha, \quad \sigma_t^h(\gamma) = \gamma, \quad \tau_t(\alpha) = \alpha, \quad \tau_t(\gamma) = |q|^{2it}\gamma \quad (t \in \mathbb{R}).$$

We can also give an explicit formula for the unitary antipode:

$$R(\alpha) = \alpha^*, \quad R(\gamma) = -\operatorname{sgn}(q)\gamma.$$

We will discuss $\mathrm{SU}_q(2)$ and its dual in greater detail in sections 3.6, 5.3.

2.3.2 Example: the quantum group O_F^+ group

The next class of examples we will introduce are the *free orthogonal quantum groups* O_F^+. They were introduced by Van Daele and Wang [94] (see also [3]). One starts with an invertible matrix $F \in \mathrm{M}_n(\mathbb{C})\,(n \geq 2)$ satisfying $F\overline{F} \in \mathbb{R}\mathbb{1}$, where $\overline{F}$ is the matrix obtained from F by taking the complex conjugate of every entry. Then $\mathrm{C}^u(O_F^+)$ is defined[17] as the universal unital C^*-algebra generated by elements $\{U_{i,j} \,|\, 1 \leq i, j \leq n\}$ satisfying

$$U \text{ is unitary and } \; U = FU^cF^{-1}$$

where $U = (U_{i,j})_{i,j=1}^n$ and $U^c = (U_{i,j}^*)_{i,j=1}^n$ are matrices in $\mathrm{M}_n(\mathrm{C}^u(O_F^+))$.
Comultiplication Δ on $\mathrm{C}^u(O_F^+)$ is obtained by declaring that U is a representation, i.e.

$$\Delta(U_{i,j}) = \sum_{k=1}^n U_{i,k} \otimes U_{k,j} \quad (1 \leq i, j \leq n). \tag{2.24}$$

The representation theory of O_F^+ is very similar to that of $\mathrm{SU}_q(2)$: we have $\mathrm{Irr}(O_F^+) \simeq \frac{1}{2}\mathbb{Z}_+$ where 0 and $\frac{1}{2}$ correspond respectively to the trivial representation and U, each irreducible representation is self-conjugate up to equivalence and the fusion rules are the same as for $\mathrm{SU}_q(2)$ (i.e. the analog of equation (2.23) holds). In fact, $\mathrm{SU}_q(2) = O_F^+$ for $F = \begin{bmatrix} 0 & 1 \\ -q^{-1} & 0 \end{bmatrix}$ and every quantum group O_F^+ with $F \in \mathrm{M}_2(\mathbb{C})$ is isomorphic to one of the $\mathrm{SU}_q(2)$ quantum groups [85, Proposition 6.4.8].
We note here the intriguing property that whenever $n \geq 3$, the Haar integral on $\mathrm{C}^u(O_F^+)$ is not faithful and henceforth quantum group O_F^+ is not coamenable [4, Corollaire 1] (see also [94, Proposition 2.2]).
We will obtain some interesting information concerning the von Neumann algebra $\mathrm{L}^\infty(O_F^+)$ in Section 5.

[17] This C^*-algebra is often denoted also by $A_o(F)$.

2.3.3 Example: U_F^+ group

The next class of examples we wish to discuss in this section are the free unitary quantum groups U_F^+ introduced in [94]. Similarly to the case of O_F^+, we start with an invertible matrix $F \in \mathrm{M}_n(\mathbb{C})$ but this time we do not impose additional conditions on F. $\mathrm{C}^u(U_F^+)$ is defined[18] as the universal unital C*-algebra generated by $\{U_{i,j} \,|\, 1 \leq i,j \leq n\}$ such that

$$U \quad \text{and} \quad FU^cF^{-1} \quad \text{are unitary,}$$

where, as previously, $U = (U_{i,j})_{i,j=1}^n$ and $U^c = (U_{i,j}^*)_{i,j=1}^n$. The comultiplication is defined in such a way that U is a representation, i.e. by formula (2.24).
Representation theory of U_F^+ was determined by Banica in [4], let us describe its elements. Let $\mathbb{Z}_+ \star \mathbb{Z}_+$ be the free product of monoids $\mathbb{Z}_+$ with generators α, β and the neutral element e. There is a unique antimultiplicative involution $x \mapsto \overline{x}$ on $\mathbb{Z}_+ \star \mathbb{Z}_+$ satisfying $\overline{e} = e, \overline{\alpha} = \beta$. We can identify $\mathbb{Z}_+ \star \mathbb{Z}_+$ with $\mathrm{Irr}(U_F^+)$ in such a way that denoting this identification by $x \mapsto U_x$, U_e is the trivial representation, $U_\alpha = U$ and $\overline{U_x} \simeq U_{\overline{x}}$ for all $x \in \mathbb{Z}_+ \star \mathbb{Z}_+$. Furthermore, the fusion rules of U_F^+ are as follows: for any $x, y \in \mathbb{Z}_+ \star \mathbb{Z}_+$ we have

$$U_x \tiny{\textcircled{\tiny T}} U_y \simeq \bigoplus_{\substack{a,b,c \in \mathbb{Z}_+ \star \mathbb{Z}_+: \\ x = ac,\ \overline{c}b = y}} U_{ab}.$$

Let us end this section with a remark that the quantum group U_F^+ is not coamenable for any $F \in \mathrm{M}_n(\mathbb{C})$ [4] (see also [94, Proposition 2.2]). We will study the von Neumann algebra $\mathrm{L}^\infty(U_F^+)$ in Section 5.6.

In the case $F = \mathbb{1}$ it is common to denote the resulting compact quantum groups of Kac type O_F^+, U_F^+ by O_n^+, U_n^+.

2.3.4 Example: $\mathbb{G}_{Aut}(B,\psi)$ group

The last class of examples we will discuss here is formed by *quantum automorphism groups* $\mathbb{G}_{Aut}(B,\psi)$. They were introduced by Wang in [96] and studied by many authors, let us mention here papers of Banica [6, 5] and Brannan [12]. We start with the following auxiliary definition:

Definition 2.11.

- A unital $\star$-homomorphism $\alpha\colon \mathcal{A} \to \mathcal{A} \otimes \mathrm{C}(\mathbb{G})$ is a right action of a compact quantum group $\mathbb{G}$ on a unital C*-algebra $\mathcal{A}$ if

$$(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha, \quad \overline{\mathrm{span}}\{\alpha(a)(\mathbb{1} \otimes x) \,|\, a \in \mathcal{A},\, x \in \mathrm{C}(\mathbb{G})\} = \mathcal{A} \otimes \mathrm{C}(\mathbb{G}).$$

- Functional $\varphi \in \mathcal{A}^*$ is preserved by the action of $\mathbb{G}$ is

$$(\varphi \otimes \mathrm{id})\alpha(a) = \varphi(a)\mathbb{1} \quad (a \in \mathcal{A}).$$

[18] $\mathrm{C}^u(U_F^+)$ is known also as $A_u(F)$.

Now we can give the definition of the quantum automorphism group $\mathbb{G}_{Aut}(B,\psi)$.

Definition 2.12. Let B be a finite dimensional C^*-algebra with a faithful state $\psi \in B^*$. The *quantum automorphism group* of (B,ψ), denoted $\mathbb{G}_{Aut}(B,\psi)$, is the compact quantum group with a right action $\alpha\colon B \to B \otimes \mathrm{C}^u(\mathbb{G}_{Aut}(B,\psi))$ preserving ψ such that

- $\mathrm{C}^u(\mathbb{G}_{Aut}(B,\psi))$ is the universal C^*-algebra generated by $\{(\omega \otimes \mathrm{id})\alpha(b) \mid b \in B, \omega \in B^*\}$,
- if $\beta\colon B \to B \otimes \mathrm{C}(\mathbb{H})$ is a right action of a compact quantum group $\mathbb{H}$ on B which preserves ψ, then there exists a unital $\star$-homomorphism $\pi\colon \mathrm{C}^u(\mathbb{G}_{Aut}(B,\psi)) \to \mathrm{C}(\mathbb{H})$ such that $\beta = (\mathrm{id} \otimes \pi) \circ \alpha$.

We will not need this result, let us mention however that a more concrete description of $\mathrm{C}^u(\mathbb{G}_{Aut}(B,\psi))$, using the multiplication map $m\colon B \otimes B \to B$ and the unit map $\nu\colon \mathbb{C} \to B$, is possible. To be more precise, $\mathrm{C}(\mathbb{G}_{Aut}(B,\psi))$ is the universal C^*-algebra generated by elements $\{U_{i,j}\}_{i,j=1}^{\dim(B)}$ such that

$$\text{the matrix } U = (U_{i,j})_{i,j=1}^{\dim(B)} \text{ is unitary,} \quad m \in \mathrm{Mor}(U^{\oplus 2}, U), \quad \text{and} \quad \nu \in \mathrm{Mor}(1, U)$$

(we treat U as acting on the Hilbert space B, with inner product defined by ψ).

The most studied examples are given by (B,ψ) where ψ is a so-called δ-form:

Definition 2.13. Let B be a finite dimensional C^*-algebra, $\psi \in B^*$ a faithful state and $\delta > 0$ a positive number. Functionals ψ and $\psi \otimes \psi$ give us a Hilbert space structure on B and $B \otimes B$. We say that ψ is a *δ-form* if the multiplication map $m\colon B \otimes B \to B$ satisfies $mm^* = \delta^2 \mathrm{id}$, where m^* is the (Hilbert space) adjoint of m.

Banica was able to describe the representation theory of $\mathbb{G}_{Aut}(B,\psi)$ (see [12, Theorem 3.8]): we can identify $\mathrm{Irr}(\mathbb{G}_{Aut}(B,\psi))$ with $\mathbb{Z}_+$: every $k \in \mathbb{Z}_+$ corresponds to a finite dimensional unitary representation U^k and $\{U^k\}_{k \in \mathbb{Z}_+}$ have the following properties:

- U^0 is the trivial representation, $U \simeq U^0 \oplus U^1$,
- $\overline{U^k} \simeq U^k$ for all $k \in \mathbb{Z}_+$,
- the fusion rules are given by

$$U^n \oplus U^k \simeq \bigoplus_{m=0}^{2\min(n,k)} U^{k+n-m} \quad (n,k \in \mathbb{Z}_+),$$

  i.e. $\mathbb{G}_{Aut}(B,\psi)$ has the same fusion rules as $\mathrm{SO}(3)$.

When $\dim(B) \leq 3$, $\mathbb{G}_{Aut}(B,\psi)$ is the finite permutation group $S_{\dim(B)}$. Furthermore for $\dim(B) \geq 4$, $\mathbb{G}_{Aut}(B,\psi)$ is coamenable only when $\dim(B) = 4$.

We will obtain new information about the von Neumann algebra $\mathrm{L}^\infty(\mathbb{G}_{Aut}(B,\psi))$ in Section 5.

3 Type I locally compact quantum groups

In Section 2.3 we have seen that compact quantum groups have a very nice representation theory: one has a family of irreducible representations α which are "building blocks" of arbitrary representations. With every α comes a positive invertible operator $\rho_\alpha \in \mathrm{B}(\mathsf{H}_\alpha)$, and using these operators, we can express objects (modular automorphisms, scaling group, etc.) related to the compact quantum group or its discrete dual in a very explicit way. Furthermore, matrix elements of finite dimensional unitary representations form a w*-dense subspace in $\mathrm{L}^\infty(\mathbb{G})$ where calculations are more attainable.

These properties make compact quantum groups a class of locally compact quantum groups especially amenable to precise analysis. In this section we will introduce a class of type I locally compact quantum groups which is significantly larger then the class of compact quantum groups, but nevertheless preserve some of the properties mentioned above. This section for the most part is based on the seminal PhD dissertation of Desmedt [31], results of Caspers [17, 18] as well as the author's work [50, 49].

3.1 Plancherel measure

Recall [33, Definition 5.5.1] that a C*-algebra A is of *type I* if for every representation $\pi\colon A \to \mathrm{B}(\mathsf{H}_\pi)$, the von Neumann algebra $\pi(A)''$ is of type I (see also [33, Theorem 9.1, 9.5.6] for equivalent characterisations). Let us introduce the definition of type I locally compact quantum group.

Definition 3.1. Let $\mathbb{G}$ be a locally compact quantum group. We say that $\mathbb{G}$ is *type I* if the C*-algebra $\mathrm{C}_0^u(\widehat{\mathbb{G}})$ is of type I.

This definition is a direct generalisation of the classical notion of type I locally compact group: such a group G is type I if and only if its full C*-algebra[19] $\mathrm{C}^*(G)$ is of type I. For more information and examples of classical type I groups see for example [39, Section 7]. In particular, let us mention [39, Theorem 7.8]: if G is a connected Lie group which is nilpotent or semi-simple then it is of type I.
The principal reason we are interested in those quantum groups $\mathbb{G}$ for which $\mathrm{C}_0^u(\widehat{\mathbb{G}})$ is a C*-algebra of type I is the fact that (non-degenerate) representations of type I C*-algebras decompose in a unique way into direct integrals over the spectrum $\mathrm{Irr}(A)$, which is a standard Borel space (see [33, Theorem 8.6.6], we have gathered basic results concerning direct integrals in Appendix 7.1). We will use this fact to deduce existence of the Plancherel measure, a result due to Desmedt (Theorem 3.3).
We will be working with direct integrals, to avoid unnecessary technical difficulties we will impose some separability conditions:

[19]For a locally compact quantum group $\mathbb{G}$, the C*-algebra $\mathrm{C}_0^u(\widehat{\mathbb{G}})$ plays a role of the full group C*-algebra, see Section 2.2.

Lemma 3.2. *Let $\mathbb{G}$ be a locally compact quantum group. The following conditions are equivalent:*

1) $\mathrm{C}_0(\mathbb{G})$ *is a separable* C^**-algebra,*

2) $\mathrm{C}_0^u(\mathbb{G})$ *is a separable* C^**-algebra,*

3) $\mathrm{L}^2(\mathbb{G})$ *is a separable Hilbert space*

4) $\mathrm{L}^1(\widehat{\mathbb{G}})$ *is a separable Banach space.*

If these conditions hold, we say that $\mathbb{G}$ is second countable. Note that since $\mathrm{L}^2(\mathbb{G}) \simeq \mathrm{L}^2(\widehat{\mathbb{G}})$, $\mathbb{G}$ is second countable if and only if $\widehat{\mathbb{G}}$ is second countable.

Proof. The GNS Hilbert spaces of φ, φ^u can be identified with $\mathrm{L}^2(\mathbb{G})$, hence we get 1) $\Rightarrow$ 3) and 2) $\Rightarrow$ 3) (see [61, Theorem C.2]). Since $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is a von Neumann algebra acting on $\mathrm{L}^2(\mathbb{G})$, point 3) implies 4). Next, since $\mathrm{C}_0(\mathbb{G})$ is the norm closure of $\{(\mathrm{id}\otimes\omega)\mathrm{W} \,|\, \omega \in \mathrm{L}^1(\widehat{\mathbb{G}})\}$ we get 3) $\Rightarrow$ 1). Implication 3) $\Rightarrow$ 2) is analogous. $\square$

The fundamental result concerning type I, second countable, locally compact quantum groups is the Plancherel theorem proved by Desmedt (a similar result for possibly non-unimodular classical groups was derived by Tatsuuma in [84]).

Theorem 3.3. *Let $\mathbb{G}$ be a second countable, type I locally compact quantum group. There exists a standard measure μ on $\mathrm{Irr}(\mathbb{G})$, a measurable field of Hilbert spaces $(\mathsf{H}_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$, measurable field of representations*[20]*, measurable fields of strictly positive self-adjoint operators $(D_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}, (E_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$ and unitary operators $\mathcal{Q}_L, \mathcal{Q}_R \colon \mathrm{L}^2(\mathbb{G}) \to \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{HS}(\mathsf{H}_\pi)\,\mathrm{d}\mu(\pi)$ such that:*

1a) *For all $\alpha \in \mathrm{L}^1(\mathbb{G})$ such that $\lambda(\alpha) \in \mathfrak{N}_{\widehat{\varphi}}$ and μ-almost every $\pi \in \mathrm{Irr}(\mathbb{G})$ the operator $(\alpha\otimes\mathrm{id})(U^\pi)\circ D_\pi^{-1}$ is bounded and its closure $(\alpha\otimes\mathrm{id})(U^\pi)D_\pi^{-1}$ is Hilbert-Schmidt.*

1b) *For all $\alpha \in \mathrm{L}^1(\mathbb{G})$ such that $\lambda(\alpha) \in \mathfrak{N}_{\widehat{\psi}}$ and μ-almost every $\pi \in \mathrm{Irr}(\mathbb{G})$ the operator $(\alpha\otimes\mathrm{id})(U^\pi)\circ E_\pi^{-1}$ is bounded and its closure $(\alpha\otimes\mathrm{id})(U^\pi)E_\pi^{-1}$ is Hilbert-Schmidt.*

2a) *The operator $\mathcal{Q}_L$ is the isometric extension of*

$$\Lambda_{\widehat{\varphi}}(\lambda(\mathrm{L}^1(\mathbb{G}))\cap\mathfrak{N}_{\widehat{\varphi}}) \ni \Lambda_{\widehat{\varphi}}(\lambda(\alpha)) \mapsto \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\alpha\otimes\mathrm{id})(U^\pi)D_\pi^{-1}\,\mathrm{d}\mu(\pi) \in \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{HS}(\mathsf{H}_\pi)\,\mathrm{d}\mu(\pi).$$

2b) *Similarly, $\mathcal{Q}_R$ is the isometric extension of*

$$J_{\widehat{\varphi}}J_\varphi\Lambda_{\widehat{\psi}}(\lambda(\mathrm{L}^1(\mathbb{G}))\cap\mathfrak{N}_{\widehat{\psi}}) \ni J_{\widehat{\varphi}}J_\varphi\Lambda_{\widehat{\psi}}(\lambda(\alpha)) \mapsto$$
$$\mapsto \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\alpha\otimes\mathrm{id})(U^\pi)E_\pi^{-1}\,\mathrm{d}\mu(\pi) \in \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{HS}(\mathsf{H}_\pi)\,\mathrm{d}\mu(\pi),$$

[20] We will often abuse the notation and write π for a representation as well as its class in $\mathrm{Irr}(\mathbb{G})$.

3a) The operator $\mathcal{Q}_L$ satisfies

$$\mathcal{Q}_L(\omega\otimes\mathrm{id})\mathrm{W} = \Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})U^{\pi}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L$$

and

$$\mathcal{Q}_L(\omega\otimes\mathrm{id})\chi(\mathrm{V}) = \Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}_\pi}\otimes \pi^c((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L$$

for every $\omega\in\mathrm{L}^1(\mathbb{G})$.

3b) The operator $\mathcal{Q}_R$ satisfies

$$\mathcal{Q}_R J_{\widehat{\varphi}}J_{\varphi}(\omega\otimes\mathrm{id})\mathrm{W} = \Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})U^{\pi}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_R J_{\widehat{\varphi}}J_{\varphi}$$

and

$$\mathcal{Q}_R J_{\widehat{\varphi}}J_{\varphi}(\omega\otimes\mathrm{id})\chi(\mathrm{V}) = \Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}_\pi}\otimes \pi^c((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_R J_{\widehat{\varphi}}J_{\varphi}$$

for every $\omega\in\mathrm{L}^1(\mathbb{G})$.

4) Haar integrals on $\widehat{\mathbb{G}}$ are tracial if and only if almost all D_π are multiples of the identity and this happens if and only if almost all E_π are multiples of the identity.

5) Operators $\mathcal{Q}_L,\mathcal{Q}_R$ transform $\mathcal{Z}(\mathrm{L}^\infty(\widehat{\mathbb{G}}))$ onto diagonalisable operators.

6) We can assume that $(\mathsf{H}_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$ is the canonical measurable field of Hilbert spaces.

Remark. If $\widehat{\mathbb{G}}$ is unimodular, we have $E_\pi = D_\pi$ $(\pi\in\mathrm{Irr}(\mathbb{G}))$ and $\mathcal{Q}_R=\mathcal{Q}_L J_\varphi J_{\widehat{\varphi}}$.

The above result is based on a result concerning lower semicontinuous, densely defined KMS weights (see Section 2.1) on C*-algebras of type I [31, Theorem 3.3.5].
If A is a separable C*-algebra of type I, let $(K_\sigma)_{\sigma\in\mathrm{Irr}(A)}$ be the canonical measurable field of Hilbert spaces, i.e. $K_\sigma=\mathbb{C}^{\dim(\sigma)}$ $(\sigma\in\mathrm{Irr}(A))$ [33, Section 8.6.1]. Next, for a GNS representation π_θ associated with a weight θ define a representation ρ_θ of the opposite algebra on H_θ via $\rho_\theta = J_\theta\pi_\theta(\cdot^*)J_\theta$.

Theorem 3.4. *Let A be a separable C^*-algebra of type I and θ a lower semicontinuous, densely defined, KMS weight on A. There exists a measure μ on $\mathrm{Irr}(A)$, a measurable field of representations $(\pi_\sigma)_{\sigma\in\mathrm{Irr}(A)}$ on $(K_\sigma)_{\sigma\in\mathrm{Irr}(A)}$ such that $\pi_\sigma\in\sigma$ for every σ, a measurable field of strictly positive, self-adjoint operators $(D_\sigma)_{\sigma\in\mathrm{Irr}(A)}$ and a unitary operator $\mathcal{P}\colon \mathsf{H}_\theta\to\int_{\mathrm{Irr}(A)}^{\oplus}\mathrm{HS}(\mathsf{K}_\sigma)\,\mathrm{d}\mu(\sigma)$ with the following properties:*

1) For all $x\in\mathfrak{N}_\theta$ and μ-almost all $\sigma\in\mathrm{Irr}(A)$ the operator $\pi_\sigma(x)\circ D_\sigma^{-1}$ is bounded and its closure $\pi_\sigma(x)D_\sigma^{-1}$ is Hilbert-Schmidt.

2) The operator $\mathcal{P}$ is the isometric extension of

$$\Lambda_\theta(\mathfrak{N}_\theta) \ni \Lambda_\theta(x) \mapsto \int_{\mathrm{Irr}(A)}^{\oplus} \pi_\sigma(x) D_\sigma^{-1} \,\mathrm{d}\mu(\sigma) \in \int_{\mathrm{Irr}(A)}^{\oplus} \mathrm{HS}(\mathsf{K}_\sigma) \,\mathrm{d}\mu(\sigma)$$

3) Let J_σ be the antilinear map $\mathrm{HS}(\mathsf{K}_\sigma) \ni T \mapsto T^* \in \mathrm{HS}(\mathsf{K}_\sigma)$ $(\sigma \in \mathrm{Irr}(A))$. *The operator $\mathcal{P}$ transforms*

- J_θ *onto* $\int_{\mathrm{Irr}(A)}^{\oplus} J_\sigma \,\mathrm{d}\mu(\sigma)$,
- π_θ *onto* $\int_{\mathrm{Irr}(A)}^{\oplus} \pi_\sigma \otimes \mathbb{1}_{\overline{\mathsf{K}_\sigma}} \,\mathrm{d}\mu(\sigma)$ *and* ρ_θ *onto* $\int_{\mathrm{Irr}(A)}^{\oplus} \mathbb{1}_{\mathsf{K}_\sigma} \otimes \pi_\sigma^{c} \,\mathrm{d}\mu(\sigma)$,
- $\pi_\theta(A)''$ *onto* $\int_{\mathrm{Irr}(A)}^{\oplus} \mathrm{B}(\mathsf{K}_\sigma) \otimes \mathbb{1}_{\overline{\mathsf{K}_\sigma}} \,\mathrm{d}\mu(\sigma)$ *and* $\pi_\theta(A)'$ *onto* $\int_{\mathrm{Irr}(A)}^{\oplus} \mathbb{1}_{\mathsf{K}_\sigma} \otimes \mathrm{B}(\overline{\mathsf{K}_\sigma}) \,\mathrm{d}\mu(\sigma)$,
- $\mathcal{Z}(\pi_\theta(A)'')$ *onto* $\mathrm{Diag}\big(\int_{\mathrm{Irr}(A)}^{\oplus} \mathrm{HS}(\mathsf{K}_\sigma) \,\mathrm{d}\mu(\sigma)\big)$.

4) Using the operator $\mathcal{P}$ we can write θ as a composition[21]

$$\begin{aligned} A^+ &\xrightarrow{\pi_\theta} \Big(\int_{\mathrm{Irr}(A)}^{\oplus} \mathrm{B}(\mathsf{K}_\sigma) \otimes \mathbb{1}_{\overline{\mathsf{K}}_\sigma} \,\mathrm{d}\mu(\sigma)\Big)^+ \\ &\simeq \Big(\int_{\mathrm{Irr}(A)}^{\oplus} \mathrm{B}(\mathsf{K}_\sigma) \,\mathrm{d}\mu(\sigma)\Big)^+ \xrightarrow{\int_{\mathrm{Irr}(A)}^{\oplus} (\mathrm{Tr}_\sigma)_{D_\sigma^{-1}} \,\mathrm{d}\mu(\sigma)} [0,+\infty] \end{aligned}$$

5) θ is tracial if and only if $D_\sigma \in \mathbb{R}_{>0}\mathbb{1}_{\mathsf{K}_\sigma}$ for almost all $\sigma \in \mathrm{Irr}(A)$.

This (slightly modified) result of Desmedt is an extension of [33, Theorem 8.8.5] which is an analogous result for tracial weights. Due to its importance, we present its proof which is almost entirely taken from [31].

Proof. The proof will be divided into a number of claims.
Let $\tilde{\theta}$ be the canonical extension of θ to a n.s.f. weight on $\pi_\theta(A)''$ (see Section 2.1). Recall that a measure space (Y, ϖ) is called standard if there exists a ϖ-null set Y_0 such that $Y \setminus Y_0$ is a Borel space of a separable, completely metrizable topological space [34, Section I.1].

Claim 1. There exists a standard measure space (X, μ), a measurable field of Hilbert spaces $(K_x)_{x\in X}$, a measurable field of strictly positive, self-adjoint operators $(D_x)_{x\in X}$ and a unitary operator $\mathcal{P}\colon \mathsf{H}_\theta \to \int_X^{\oplus} \mathrm{HS}(\mathsf{K}_x) \,\mathrm{d}\mu(x)$ such that:

- the operator $\mathcal{P}$ transforms $\pi_\theta(A)''$ onto $\int_X^{\oplus} \mathrm{B}(\mathsf{K}_x) \otimes \mathbb{1}_{\overline{\mathsf{K}}_x} \,\mathrm{d}\mu(x)$: every $y \in \pi_\theta(A)''$ corresponds to $\int_X^{\oplus} y_x \otimes \mathbb{1}_{\overline{\mathsf{K}}_x} \,\mathrm{d}\mu(x)$,
- for all $y \in \mathfrak{N}_{\tilde{\theta}}$ and μ-almost all $x \in X$, the operator $y_x \circ D_x^{-1}$ is closable and its closure is Hilbert-Schmidt,

[21] For the definition of $\int_{\mathrm{Irr}(A)}^{\oplus} (\mathrm{Tr}_\sigma)_{D_\sigma^{-1}} \,\mathrm{d}\mu(\sigma)$ see Appendix 7.1 and Section 2.1.

- the operator $\mathcal{P}$ acts on $\Lambda_{\tilde\theta}(\mathfrak{N}_{\tilde\theta})$ via $\Lambda_{\tilde\theta}(y) \mapsto \int_X^{\oplus} y_x D_x^{-1}\,\mathrm{d}\mu(x)$,
- the operator $\mathcal{P}$ transforms $J_{\tilde\theta}$ onto $\int_X^{\oplus} J_x\,\mathrm{d}\mu(x)$, where J_x is given by $\mathrm{HS}(\mathsf{K}_x) \ni T_x \mapsto T_x^* \in \mathrm{HS}(\mathsf{K}_x)\,(x \in X)$,
- $\mathcal{Z}(\pi_\theta(A)'')$ is transformed via $\mathcal{P}$ onto the algebra of diagonalisable operators.

Note that since $(K_x)_{x\in X}$ is measurable, so is $(\mathrm{HS}(\mathsf{K}_x))_{x\in X}$ (see Section 7.1).

Proof of Claim 1. Since A is of type I, we know that the von Neumann algebra $\pi_\theta(A)''$ is also of type I. By the structure result [81, Theorem V.1.27] there exist a family of (possibly empty) standard measure spaces $\{(X_n, \mu_n)\}_{n\in\mathbb{N}\cup\{\aleph_0\}}$ such that

$$\pi_\theta(A)'' \simeq \prod_{n\in\mathbb{N}\cup\{\aleph_0\}} \mathrm{B}(\mathbb{C}^n)\bar\otimes \mathrm{L}^\infty(X_n,\mu_n),$$

where $\mathbb{C}^{\aleph_0} = \ell^2(\mathbb{N})$. Note that there do not appear bigger cardinals in the above decomposition since we assume that A is separable. Form a measure space $(X,\mu) = \bigsqcup_{n\in\mathbb{N}\cup\{\aleph_0\}}(X_n,\mu_n)$ with the standard measurable field of Hilbert spaces $(\mathsf{K}_x)_{x\in X}$ given by $\mathsf{K}_x = \mathbb{C}^n$ for $x \in X_n$. We have

$$\pi_\theta(A)'' \simeq \int_X^{\oplus} \mathrm{B}(\mathsf{K}_x)\,\mathrm{d}\mu(x).$$

Now, let Tr_x be the (non normalized) trace on $\mathrm{B}(\mathsf{K}_x)\,(x\in X)$ and $\eta = \int_X^{\oplus}\mathrm{Tr}_x\,\mathrm{d}\mu(x)$ the direct integral weight on $\int_X^{\oplus}\mathrm{B}(\mathsf{K}_x)\,\mathrm{d}\mu(x)$ (see Appendix 7.1). It is a n.s.f. weight. One can easily check that the GNS construction of η is given by

$$\mathsf{H}_\eta = \int_X^{\oplus} \mathrm{HS}(\mathsf{K}_x)\,\mathrm{d}\mu(x),$$

$$\pi_\eta\colon \int_X^{\oplus}\mathrm{B}(\mathsf{K}_x)\,\mathrm{d}\mu(x) \ni \int_X^{\oplus} T_x\,\mathrm{d}\mu(x) \mapsto \int_X^{\oplus} T_x\otimes \mathbb{1}_{\overline{\mathsf{K}_x}}\,\mathrm{d}\mu(x) \in \mathrm{B}\big(\int_X^{\oplus}\mathrm{HS}(\mathsf{K}_x)\,\mathrm{d}\mu(x)\big),$$

$$\Lambda_\eta\colon \mathfrak{N}_\eta \ni \int_X^{\oplus} T_x\,\mathrm{d}\mu(x) \mapsto \int_X^{\oplus} T_x\,\mathrm{d}\mu(x) \in \int_X^{\oplus}\mathrm{HS}(\mathsf{K}_x)\,\mathrm{d}\mu(x).$$

The corresponding modular conjugation J_η is given by

$$J_\eta = \int_X^{\oplus} J_x\,\mathrm{d}\mu(x), \qquad \text{where} \qquad J_x\colon\ \mathrm{HS}(\mathsf{K}_x) \ni T_x \mapsto T_x^* \in \mathrm{HS}(\mathsf{K}_x)\ (x\in X).$$

The weight η is tracial, hence [87, Proposition 5.2] implies an existence of a strictly positive, self-adjoint operator D affiliated with $\int_X^{\oplus}\mathrm{B}(\mathsf{K}_x)\,\mathrm{d}\mu(x)$ such that $\tilde\theta \simeq \eta_{D^{-2}}$ (see also Section 2.1). Since the standard representation of a von Neumann algebra is unique up to a unitary automorphism, there exists a canonical unitary operator $\mathcal{P}\colon \mathsf{H}_{\tilde\theta} = \mathsf{H}_\theta \to \mathsf{H}_\eta$ conjugating between both representations. To ease the notation we will identify H_θ and H_η. By [58, Theorem 1.8] the operator D is decomposable as an unbounded operator, i.e. we can write

$$D = \int_X^{\oplus} D_x \otimes \mathbb{1}_{\overline{\mathsf{H}}_x} \,\mathrm{d}\mu(x).$$

Let us introduce a subspace

$$\mathfrak{N}_{\tilde{\theta}}^0 = \{y \in \pi_\theta(A)'' \,|\, y \circ D^{-1} \text{ is closable and } yD^{-1} \in \mathfrak{N}_\eta\}.$$

By construction of $\eta_{D^{-2}}$, the subspace $\mathfrak{N}_{\tilde{\theta}}^0$ is a core for $\Lambda_{\tilde{\theta}}$ ([87, Section 1]). For $y \in \mathfrak{N}_{\tilde{\theta}}^0$ we have

$$\Lambda_{\tilde{\theta}}(y) = \Lambda_\eta(yD^{-1}) = \int_X^{\oplus} (yD^{-1})_x \,\mathrm{d}\mu(x). \tag{3.1}$$

Define approximate units

$$e_n^x = \tfrac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} D_x^{it} \,\mathrm{d}t \quad (n \in \mathbb{N},\, x \in X)$$

and

$$e_n = \int_X^{\oplus} e_n^x \,\mathrm{d}\mu(x) = \tfrac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} D^{it} \,\mathrm{d}t \quad (n \in \mathbb{N}).$$

The above integrals converge in σ-WOT. Let $y \in \mathfrak{N}_{\tilde{\theta}}$. The operator ye_n belongs to $\mathfrak{N}_{\tilde{\theta}}^0$ for each $n \in \mathbb{N}$. Indeed, since D is affiliated with $\pi_\theta(A)''$ we have $ye_n \in \pi_\theta(A)''$ and clearly $(ye_n) \circ D^{-1}$ is closable with closure $y(e_nD^{-1})$. This operator belongs to $\mathfrak{N}_\eta$ by [87, Lemma 3.2]. Using the equation (3.1) we can calculate $\Lambda_{\tilde{\theta}}(ye_n)$:

$$\Lambda_{\tilde{\theta}}(ye_n) = \Lambda_\eta(y(e_nD^{-1})) = \int_X^{\oplus} y_x(e_n^xD_x^{-1}) \,\mathrm{d}\mu(x).$$

Let us write $\Lambda_{\tilde{\theta}}(y) = \int_X^{\oplus} w_x \,\mathrm{d}\mu(x)$ for certain $w_x \in \mathrm{HS}(\mathsf{K}_x)$. By [87, Proposition 2.5] we have $J_{\tilde{\theta}} = J_\eta$. Since $e_n = e_n^*$ is invariant under $(\sigma_t^{\tilde{\theta}})_{t\in\mathbb{R}}$ ([87, Corollary 2.7]), we have

$$\Lambda_{\tilde{\theta}}(ye_n) = J_{\tilde{\theta}}\pi_\eta(e_n)J_{\tilde{\theta}}\Lambda_{\tilde{\theta}}(y) = \int_X^{\oplus} w_xe_n^x \,\mathrm{d}\mu(x)$$

which implies

$$w_xe_n^x = y_x(e_n^xD_x^{-1})$$

for all $n \in \mathbb{N}$ and almost all $x \in X$. Since $\|e_n^x\| \leq 1$ and $e_n^x \xrightarrow[n\to\infty]{\text{SOT}} \mathbb{1}_{\mathsf{K}_x}$, the above equation implies that the operator $y_x \circ D_x^{-1}$ is closable and $y_xD_x^{-1} = w_x$. This proves

$$\Lambda_{\tilde{\theta}}(y) = \int_X^{\oplus} y_xD_x^{-1} \,\mathrm{d}\mu(x)$$

for all $y \in \mathfrak{N}_{\tilde{\theta}}$. We are left with the last statement: $\mathcal{Z}(\pi_\theta(A)'')$ is transformed onto the algebra of diagonalisable operators. This result is clear since

$$\Big(\int_X^\oplus \mathrm{B}(\mathsf{K}_x) \otimes \mathbb{1}_{\overline{\mathsf{K}}_x} \,\mathrm{d}\mu(x)\Big)' = \int_X^\oplus \mathbb{1}_{\mathsf{K}_x} \otimes \mathrm{B}(\overline{\mathsf{K}}_x) \,\mathrm{d}\mu(x),$$

this proves Claim 1. □

Claim 2. X can be identified with a Borel subset of $\mathrm{Irr}(A)$ and we can extend μ by zero to a measure on the whole $\mathrm{Irr}(A)$. There exists a measurable field $(\pi_\sigma)_{\sigma\in\mathrm{Irr}(A)}$ of representations of A on $(\mathsf{K}_\sigma)_{\sigma\in\mathrm{Irr}(A)}$ such that $\pi_\sigma \in \sigma$ for almost all $\sigma \in \mathrm{Irr}(A)$ and

$$\pi_\theta \simeq \int_{\mathrm{Irr}(A)}^\oplus \pi_\sigma \otimes \mathbb{1}_{\overline{\mathsf{K}}_\sigma} \,\mathrm{d}\mu(\sigma).$$

Proof of Claim 2. We already know that $\pi_\theta(A)' \supseteq \mathcal{Z}(\pi_\theta(A)'') \simeq \mathrm{Diag}(\int_X^\oplus \mathrm{HS}(\mathsf{K}_x)\,\mathrm{d}\mu(x))$, hence by [81, Theorem IV.8.25] there exists a measurable field of representations $(\zeta_x)_{x\in X}$ on $(\mathrm{HS}(\mathsf{K}_x))_{x\in X}$ such that

$$\pi_\theta = \int_X^\oplus \zeta_x \,\mathrm{d}\mu(x).$$

Since $\pi_\theta(A)'' = \int_X^\oplus \mathrm{B}(\mathsf{K}_x) \otimes \mathbb{1}_{\overline{\mathsf{K}}_x} \,\mathrm{d}\mu(x)$, we have $\zeta_x = \pi_x \otimes \mathbb{1}_{\overline{\mathsf{K}}_x}$ for a measurable field of representations $(\pi_x)_{x\in X}$. Furthermore, for μ-almost all $x \in X$ we have $\pi_x(A)'' = \mathrm{B}(\mathsf{K}_x)$, hence almost all π_x are irreducible. Since the algebra of diagonalizable operators is included in $\int_X^\oplus \mathrm{B}(\mathsf{K}_x) \otimes \mathbb{1}_{\overline{\mathsf{K}}_x} \,\mathrm{d}\mu(x)$, [33, Lemma 8.4.1 c)] implies that π_x's are pairwise nonequivalent for x outside a null set. Then [33, Proposition 8.1.8] shows that the almost everywhere defined map $f\colon X \ni x \mapsto [\pi_x] \in \mathrm{Irr}(A)$ is almost everywhere equal to a Borel mapping (we will neglect writing classes from now on). Assume that we have cut out from X the neglibible part of "bad representations", so that f is everywhere defined and Borel. Because π_x are pairwise nonequivalent, f is injective. By [33, Appendix B 21] $f(X)$ is Borel and $f\colon X \to f(X)$ is a Borel isomorphism. Consequently, we can transport the measure μ and extend if (by 0) to a standard measure on $\mathrm{Irr}(A)$. □

Note that in the above proof we have used the property that $\mathrm{Irr}(A)$ is standard.

Claim 3. The operator $\mathcal{P}$ transforms ρ_θ onto $\int_{\mathrm{Irr}(A)}^\oplus \mathbb{1}_{\mathsf{K}_\sigma} \otimes \pi_\sigma^c \,\mathrm{d}\mu(\sigma)$.

Proof of Claim 3. For $y \in A$ we have

$$\begin{aligned}\mathcal{P}\rho_\theta(y)\mathcal{P}^* = \mathcal{P}J_\theta\pi_\theta(y)^*J_\theta\mathcal{P}^* &= \int_{\mathrm{Irr}(A)}^\oplus J_\sigma(\pi_\sigma(y)^* \otimes \mathbb{1}_{\overline{\mathsf{K}}_\sigma})J_\sigma \,\mathrm{d}\mu(\sigma)\\ &= \int_{\mathrm{Irr}(A)}^\oplus \mathbb{1}_{\mathsf{K}_\sigma} \otimes \pi_\sigma^c(y) \,\mathrm{d}\mu(\sigma). \quad \square\end{aligned}$$

Claim 4. The weight θ is tracial if and only if $D_\sigma \in \mathbb{R}_{>0}\mathbb{1}_{\mathsf{K}_\sigma}$ for almost all $\sigma \in \operatorname{Irr}(A)$.

Proof of Claim 4. By [56, Propositon 1.32] θ is tracial if and only $\tilde{\theta}$ is tracial. Indeed, one direction is trivial, assume that θ is tracial and $y \in \mathfrak{N}_{\tilde{\theta}}$. Then there exists a bounded net $(y_i)_{i\in\mathcal{I}}$ in $\mathfrak{N}_\theta$ such that $\pi_\theta(y_i) \xrightarrow[i\in\mathcal{I}]{\text{SOT}^*} y$ and $\Lambda_\theta(y_i) = \Lambda_{\tilde{\theta}}(\pi_\theta(y_i)) \xrightarrow[i\in\mathcal{I}]{} \Lambda_{\tilde{\theta}}(y)$. By closedness of $\Lambda_{\tilde{\theta}}$ and the fact that $J_{\tilde{\theta}}\nabla_{\tilde{\theta}}^{\frac{1}{2}}$ is an isometry on $\Lambda_\theta(A)$ we have $y^* \in \mathfrak{N}_{\tilde{\theta}}$ and $\tilde{\theta}(y^*y) = \tilde{\theta}(yy^*)$. Now the claim follows from $\tilde{\theta} \simeq \eta_{D^{-2}}$ and [87, Corollary 2.6]. □ □

Now we can prove an existence of the Plancherel measure.

Proof of Theorem 3.3. Since $\mathbb{G}$ is second countable, type I locally compact quantum group, $A = \mathrm{C}_0^u(\widehat{\mathbb{G}})$ is a separable, type I C*-algebra. By [55, Proposition 5.9, Definition 5.10] the weights $\widehat{\varphi}^u, \widehat{\psi}^u$ are lower semicontinuous, densely defined and KMS, hence we can use Theorem 3.4. Let us first use this theorem for the left Haar integral $\widehat{\varphi}^u$. In this way we get a measure μ, measurable field of representations $(\pi_\sigma)_{\sigma\in\operatorname{Irr}(\mathbb{G})}$ (on the canonical measurable field of Hilbert spaces), measurable field of operators $(D_\sigma)_{\sigma\in\operatorname{Irr}(\mathbb{G})}$ and a unitary operator $\mathcal{Q}_L\colon \mathrm{L}^2(\mathbb{G}) \to \int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} \operatorname{HS}(\mathsf{H}_\sigma)\,\mathrm{d}\mu(\sigma)$. Recall [55, Proposition 5.2] that $\widehat{\varphi}^u = \widehat{\varphi}\circ\Lambda_{\widehat{\mathbb{G}}}$ and the GNS construction for $\widehat{\varphi}^u$ is given by $(\mathrm{L}^2(\mathbb{G}), \Lambda_{\widehat{\mathbb{G}}}, \Lambda_{\widehat{\varphi}}\circ\Lambda_{\widehat{\mathbb{G}}})$. Observe that if $\alpha \in \mathrm{L}^1(\mathbb{G})$ is such that $\lambda(\alpha) \in \mathfrak{N}_{\widehat{\varphi}}$ then $(\alpha\otimes\mathrm{id})\mathbb{W} \in \mathfrak{N}_{\widehat{\varphi}^u}$ and

$$(\alpha\otimes\mathrm{id})U^{\pi_\sigma} = \pi_\sigma\big((\alpha\otimes\mathrm{id})\mathbb{W}\big).$$

This shows point 1a) of Theorem 3.3. Since $\Lambda_{\widehat{\varphi}}(\lambda(\mathrm{L}^1(\mathbb{G}))\cap\mathfrak{N}_{\widehat{\varphi}})$ is dense in $\mathrm{L}^2(\mathbb{G})$ (see Lemma 7.10), point 2a) is also clear. The second part of 3a) is a consequence of formula (2.11) and the definition of π_σ^c. The rest of the claim follows from analogous results in Theorem 3.4.
Now we perform a similar construction for $\widehat{\psi}^u$. This way we obtain $\mu^R, (\pi_\sigma^R)_{\sigma\in\operatorname{Irr}(\mathbb{G})}, (E_\sigma)_{\sigma\in\operatorname{Irr}(\mathbb{G})}$ and $\mathcal{Q}_R^0$. We can take $\mu^R = \mu$ and $\pi_\sigma^R = \pi_\sigma$ essentially because the GNS representations for $\widehat{\varphi}^u, \widehat{\psi}^u$ are two standard representations of the von Neumann algebra $\mathrm{L}^\infty(\widehat{\mathbb{G}})$, hence are unitarily equivalent [17, Theorem 1.6.3]. Define

$$\mathcal{Q}_R = \mathcal{Q}_R^0 \circ J_\varphi J_{\widehat{\varphi}},$$

it is straightforward to check that these objects satisfy properties listed in Theorem 3.3. From now on we will abuse the notation and neglect writing the measurable field of representations, e.g. we will write $\pi \in \operatorname{Irr}(\mathbb{G})$ instead of $\pi_\sigma \in \sigma \in \operatorname{Irr}(\mathbb{G})$. This should not cause any confusion. □

One can prove a (type of) uniqueness result for the Plancherel measure.

Proposition 3.5. *Let $\mathbb{G}$ be a second countable, type I locally compact quantum group and let $\mu, (\mathsf{H}_\pi)_{\pi\in\operatorname{Irr}(\mathbb{G})}, (D_\pi)_{\pi\in\operatorname{Irr}(\mathbb{G})}, \mathcal{Q}_L$ be the objects given by Theorem 3.3. Assume that we have objects of the same type $\mu', (\mathsf{H}'_{\pi'})_{\pi'\in\operatorname{Irr}(\mathbb{G})}, (D'_{\pi'})_{\pi'\in\operatorname{Irr}(\mathbb{G})}, \mathcal{Q}'_L$ (together with a measurable field of representations $\pi' \in \operatorname{Irr}(\mathbb{G})$ such that $\pi' \in [\pi'] \in \operatorname{Irr}(\mathbb{G})$ for μ'-almost all $\pi' \in \operatorname{Irr}(\mathbb{G})$). If*

1) the operator $\mathcal{Q}'_L$ satisfies

$$\mathcal{Q}'_L(\omega \otimes \mathrm{id})\mathrm{W} = \Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega \otimes \mathrm{id})U^{\pi'} \otimes \mathbb{1}_{\overline{\mathsf{H}}'_{\pi'}} \, \mathrm{d}\mu'(\pi')\Big)\mathcal{Q}'_L \quad (\omega \in \mathrm{L}^1(\mathbb{G})),$$

2) $\mathcal{Q}'_L$ transforms $\mathcal{Z}(\mathrm{L}^\infty(\widehat{\mathbb{G}}))$ onto diagonalisable operators,

then the measures μ, μ' are equivalent. If moreover

3) the operator $\mathcal{Q}'_L$ satisfies

$$\mathcal{Q}'_L(\omega \otimes \mathrm{id})\chi(\mathrm{V}) = \Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}'_\pi} \otimes \pi'^c((\omega \otimes \mathrm{id})\mathrm{W}) \, \mathrm{d}\mu'(\pi')\Big)\mathcal{Q}'_L \quad (\omega \in \mathrm{L}^1(\mathbb{G})),$$

4) we have the equality

$$\mathcal{Q}'_L \Lambda_{\widehat{\varphi}}(\lambda(\alpha)) = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\alpha \otimes \mathrm{id})(U^{\pi'}) D'^{\,-1}_{\pi'} \, \mathrm{d}\mu'(\pi')$$

for all $\lambda(\alpha) \in X$, where X is a subspace of $\mathfrak{N}_{\widehat{\varphi}}$ containing an approximate identity and invariant under $(\sigma_t^{\widehat{\varphi}})_{t \in \mathbb{R}}$,

then for μ-almost all $\pi \in \mathrm{Irr}(\mathbb{G})$ there exists a unitary intertwiner $T_\pi \colon \mathsf{H}_\pi \to \mathsf{H}'_\pi$ such that

$$D'_\pi = \sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\pi)}\, T_\pi D_\pi T_\pi^{-1}.$$

Furthermore, objects with prime satisfy all the properties of Theorem 3.3.
An analogous result holds for the objects associated with the right Haar integral $\widehat{\psi}$.

This result is again based on a more general result for C*-algebras with a chosen weight (see also a proof of [50, Theorem 3.3, 3.4]).

Lemma 3.6. *Let A be a separable C^*-algebra of type I and θ a lower semicontinuous densely defined KMS weight on A with a modular group $(\sigma_t^\theta)_{t \in \mathbb{R}}$. Let $\mu, (\pi_\sigma)_{\sigma \in \mathrm{Irr}(A)}, (\mathsf{K}_\sigma)_{\sigma \in \mathrm{Irr}(A)}, (D_\sigma)_{\sigma \in \mathrm{Irr}(A)}$ be given by Theorem 3.4. Assume that μ' is a standard measure on $\mathrm{Irr}(A)$, $(K'_\sigma)_{\sigma \in \mathrm{Irr}(A)}$ is a measurable family of Hilbert spaces, $(\pi'_\sigma)_{\sigma \in \mathrm{Irr}(A)}$ is a measurable family of representations such that $\pi'_\sigma \in \sigma$ for μ'-every σ. Assume moreover that there exists a unitary operator $\mathcal{P}' \colon \mathsf{H}_\theta \to \int_{\mathrm{Irr}(A)}^{\oplus} K'_\sigma \otimes \overline{K'_\sigma} \, \mathrm{d}\mu'(\sigma)$. If*

1) operator $\mathcal{P}'$ transforms π_θ onto $\int_{\mathrm{Irr}(A)}^{\oplus} (\pi'_\sigma \otimes 1) \, \mathrm{d}\mu'(\sigma)$,

2) operator $\mathcal{P}'$ transforms $\pi_\theta(A)'' \cap \pi_\theta(A)'$ onto the algebra of diagonalisable operators

then the measures μ, μ' are equivalent. If moreover there exists a measurable family of strictly positive self-adjoint operators $(D'_\sigma)_{\sigma \in \mathrm{Irr}(A)}$ and

3) operator $\mathcal{P}'$ transforms ρ_θ onto $\int^\oplus_{\mathrm{Irr}(A)}(\mathbb{1}\otimes\pi'^c_\sigma)\,\mathrm{d}\mu(\sigma)$,

4) we have the equality

$$\mathcal{P}'\Lambda_\theta(x)=\int^\oplus_{\mathrm{Irr}(A)}\pi'_\sigma(x)D'^{-1}_\sigma\,\mathrm{d}\mu'(\sigma)$$

for all x in a subspace $X\subseteq\mathfrak{N}_\theta$ containing a bounded approximate identity such that $\sigma^\theta_t(X)=X$ $(t\in\mathbb{R})$,

then for μ-almost all $\sigma\in\mathrm{Irr}(A)$ there exists a unitary intertwiner $T_\sigma\colon K_\sigma\to K'_\sigma$ such that

$$D'_\sigma=\sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}T_\sigma D_\sigma T_\sigma^{-1}.$$

Proof. Let us define a unitary operator

$$\mathcal{U}=\mathcal{P}'\circ\mathcal{P}^{-1}\colon\int^\oplus_{\mathrm{Irr}(A)}K_\sigma\otimes\overline{K_\sigma}\,\mathrm{d}\mu(\sigma)\to\int^\oplus_{\mathrm{Irr}(A)}K'_\sigma\otimes\overline{K'_\sigma}\,\mathrm{d}\mu'(\sigma).$$

It transforms diagonalisable operators onto diagonalisable operators. Consider the following representations of A:

$$\int^\oplus_{\mathrm{Irr}(A)}\pi_\sigma\otimes\mathbb{1}\,\mathrm{d}\mu(\sigma),\quad\int^\oplus_{\mathrm{Irr}(A)}\pi'_\sigma\otimes\mathbb{1}\,\mathrm{d}\mu'(\sigma).$$

We would like to use [33, Proposition 8.2.4]. In order to do that, we need to check that $\mathcal{U}$ is a morphism between these representations. Let $a\in A$. Thanks to properties of $\mathcal{P},\mathcal{P}'$ we have

$$\mathcal{U}\big(\int^\oplus_{\mathrm{Irr}(A)}\pi_\sigma\otimes\mathbb{1}\,\mathrm{d}\mu(\sigma)\big)(a)\mathcal{U}^{-1}=\mathcal{P}'\pi_\theta(a)\mathcal{P}'^{-1}=\big(\int^\oplus_{\mathrm{Irr}(A)}\pi'_\sigma\otimes\mathbb{1}\,\mathrm{d}\mu'(\sigma)\big)(a).$$

Now, [33, Proposition 8.2.4] gives us subsets N, $N'\subseteq\mathrm{Irr}(A)$ which are correspondingly of μ and μ'-measure 0, Borel isomorphism $\eta\colon\mathrm{Irr}(A)\setminus N\to\mathrm{Irr}(A)\setminus N'$ which maps μ onto a measure $\tilde{\mu}'$ equivalent to μ' and a family $(V(\sigma))_{\sigma\in\mathrm{Irr}(A)\setminus N}$ such that each $V(\sigma)\colon K_\sigma\otimes\overline{K_\sigma}\to K'_{\eta(\sigma)}\otimes\overline{K'_{\eta(\sigma)}}$ is a unitary map and a vector field $(\xi_\sigma)_{\sigma\in\mathrm{Irr}(A)\setminus N}$ is measurable with respect to $(K_\sigma\otimes\overline{K_\sigma})_{\sigma\in\mathrm{Irr}(A)\setminus N}$ if and only if $(V(\sigma)\xi_\sigma)_{\eta(\sigma)\in\mathrm{Irr}(A)\setminus N'}$ is measurable with respect to $(K'_{\eta(\sigma)}\otimes\overline{K'_{\eta(\sigma)}})_{\eta(\sigma)\in\mathrm{Irr}(A)\setminus N'}$. Such a family is called η-isomorphism [33, A 70]. For $\sigma\in\mathrm{Irr}(A)\setminus N$ operator $V(\sigma)$ is a unitary morphism between $\pi_\sigma\otimes\mathbb{1}$ and $\pi'_{\eta(\sigma)}\otimes\mathbb{1}$, moreover

$$\mathcal{U}=\big(\int^\oplus_{\mathrm{Irr}(A)}K'_\sigma\otimes\overline{K'_\sigma}\,\mathrm{d}\tilde{\mu}'(\sigma)\to\int^\oplus_{\mathrm{Irr}(A)}K'_\sigma\otimes\overline{K'_\sigma}\,\mathrm{d}\mu'(\sigma)\big)\circ\int^\oplus_{\mathrm{Irr}(A)}V(\sigma)\,\mathrm{d}\mu(\sigma).$$

Fix $\overline{\zeta}\in\overline{K_\sigma},\overline{\zeta}'\in\overline{K'_{\eta(\sigma)}}$ and define a bounded operator $S^\sigma_{\overline{\zeta}',\overline{\zeta}}\in\mathrm{B}(K_\sigma,K'_{\eta(\sigma)})$ via equality

$$\langle\xi'\,|\,S^\sigma_{\overline{\zeta}',\overline{\zeta}}\xi\rangle=\langle\xi'\otimes\overline{\zeta}'\,|\,V(\sigma)(\xi\otimes\overline{\zeta})\rangle\quad(\xi\in K_\sigma,\,\xi'\in K'_\sigma).$$

For $a \in A$ and arbitrary ξ, ξ' we have

$$\begin{aligned}\langle \xi' \,|\, S^{\sigma}_{\overline{\zeta}',\overline{\zeta}} \pi_\sigma(a)\xi\rangle &= \langle \xi' \otimes \overline{\zeta}' \,|\, V(\sigma)(\pi_\sigma(a)\xi \otimes \overline{\zeta})\rangle \\ = \langle \pi'_{\eta(\sigma)}(a^*)\xi' \otimes \overline{\zeta}' \,|\, V(\sigma)(\xi \otimes \overline{\zeta})\rangle &= \langle \pi'_{\eta(\sigma)}(a^*)\xi' \,|\, S^{\sigma}_{\overline{\zeta}',\overline{\zeta}}\xi\rangle = \langle \xi' \,|\, \pi'_{\eta(\sigma)}(a) S^{\sigma}_{\overline{\zeta}',\overline{\zeta}}\xi\rangle.\end{aligned}$$

This means that $S^{\sigma}_{\overline{\zeta}',\overline{\zeta}}$ is a morphism between π_σ and $\pi'_{\eta(\sigma)}$. It is clear that there exist $\overline{\zeta}$, $\overline{\zeta}'$ for which $S^{\sigma}_{\overline{\zeta}',\overline{\zeta}}$ is non-zero. Consequently, as there are no non-trivial morphisms between nonequivalent irreducible representations, η needs to be identity on $\mathrm{Irr}(A) \setminus (N \cup N')$. Therefore $\mu = \tilde{\mu}'$ on this set. This proves the first part of the lemma.
Assume now that $\mathcal{P}'$ transforms ρ_θ onto $\int^{\oplus}_{\mathrm{Irr}(A)} (\mathbb{1} \otimes {\pi'}^c_\sigma)\, \mathrm{d}\mu(\sigma)$, and we have a family $(D'_\sigma)_{\sigma \in \mathrm{Irr}(A)}$ which meets conditions stated in the lemma. Then $V(\sigma)$ is also a morphism between $\mathbb{1} \otimes \pi^c_\sigma$ and $\mathbb{1} \otimes {\pi'}^c_\sigma$. Thanks to the Schur's lemma we have

$$S^{\sigma}_{\overline{\zeta}',\overline{\zeta}} = q(\overline{\zeta}', \overline{\zeta}) T_\sigma$$

for a unitary intertwiner $T_\sigma \in \mathrm{B}(K_\sigma, K'_\sigma)$ and a bounded sesquilinear form q. We know how forms like this looks: there exists an operator $\tilde{T}_\sigma \in \mathrm{B}(\overline{K_\sigma}, \overline{K'_\sigma})$ such that

$$\langle \xi' \otimes \overline{\zeta}' \,|\, V(\sigma)\xi \otimes \overline{\zeta}\rangle = \langle \xi' \,|\, S^{\sigma}_{\overline{\zeta}',\overline{\zeta}}\xi\rangle = \langle \xi' \otimes \overline{\zeta}' \,|\, (T_\sigma \otimes \tilde{T}_\sigma)(\xi \otimes \overline{\zeta})\rangle.$$

Operator $\tilde{T}_\sigma$ is a morphism between π^c_σ and ${\pi'}^c_\sigma$. Indeed, take $a, b \in A$. Then we have

$$\begin{aligned}\langle \xi' \otimes \overline{\zeta}' \,|\, (T_\sigma \otimes \tilde{T}_\sigma)(\pi_\sigma(a)\xi \otimes \pi^c_\sigma(b)\overline{\zeta})\rangle &= \langle \xi' \otimes \overline{\zeta}' \,|\, V(\sigma)(\pi_\sigma(a)\xi \otimes \pi^c_\sigma(b)\overline{\zeta})\rangle \\ = \langle \xi' \otimes \overline{\zeta}' \,|\, (\pi'_\sigma(a) \otimes {\pi'}^c_\sigma(b))V(\sigma)(\xi \otimes \overline{\zeta})\rangle &= \langle \xi' \otimes \overline{\zeta}' \,|\, (\pi'_\sigma(a) \otimes {\pi'}^c_\sigma(b))(T_\sigma \otimes \tilde{T}_\sigma)(\xi \otimes \overline{\zeta})\rangle.\end{aligned}$$

Taking a to be an approximate identity shows that $\tilde{T}_\sigma$ is morphism between π^c_σ and ${\pi'}^c_\sigma$. The calculation

$$T^\top_\sigma {\pi'}^c_\sigma(a)\overline{\xi} = T^\top_\sigma \overline{\pi'_\sigma(a^*)\xi} = \overline{T^*_\sigma \pi'_\sigma(a^*)\xi} = \overline{\pi_\sigma(a^*)T^*_\sigma\xi} = \pi^c(a)T^\top_\sigma\overline{\xi} \quad (\overline{\xi} \in \overline{K'_\sigma},\ a \in A)$$

implies that $T^\top_\sigma$ is a unitary morphism ${\pi'}^c_\sigma \to \pi^c_\sigma$. Schur's lemma shows that $\tilde{T}_\sigma = z_\sigma(T^{-1}_\sigma)^\top$ for a certain $z_\sigma \in \mathbb{C}$. Since

$$1 = \|V(\sigma)\| = \|T_\sigma \otimes \tilde{T}_\sigma\| = \|\tilde{T}_\sigma\| = |z_\sigma|,$$

we know that $\tilde{T}_\sigma = z_\sigma(T^{-1}_\sigma)^\top$ is a unitary operator. Let us see how $V(\sigma)$ acts on $\mathrm{HS}(K_\sigma) = K_\sigma \otimes \overline{K_\sigma}$. For every $\xi \otimes \overline{\zeta} \in K_\sigma \otimes \overline{K_\sigma}$ we have

$$\begin{aligned}V(\sigma)(|\xi\rangle\langle\zeta|) &= V(\sigma)(\xi \otimes \overline{\zeta}) = (T_\sigma\xi) \otimes (z_\sigma(T^{-1}_\sigma)^\top\overline{\zeta}) = z_\sigma(T_\sigma\xi \otimes \overline{T_\sigma\zeta}) \\ &= z_\sigma|T_\sigma\xi\rangle\langle T_\sigma\zeta| = z_\sigma T_\sigma(|\xi\rangle\langle\zeta|)T^{-1}_\sigma\end{aligned}$$

Let us make use of our knowledge about operator $\mathcal{P}'$. For a in the subspace $X \subseteq \mathfrak{N}_\theta$ we have

$$\begin{aligned}\int^{\oplus}_{\mathrm{Irr}(A)} \pi'_\sigma(a){D'}^{-1}_\sigma \,\mathrm{d}\mu'(\sigma) &= \mathcal{U}\int^{\oplus}_{\mathrm{Irr}(A)} \pi_\sigma(a)D^{-1}_\sigma \,\mathrm{d}\mu(\sigma) \\ = \int^{\oplus}_{\mathrm{Irr}(A)} &\sqrt{\tfrac{\mathrm{d}\mu}{\mathrm{d}\mu'}(\sigma)}V(\sigma)(\pi_\sigma(a)D^{-1}_\sigma)\,\mathrm{d}\mu'(\sigma),\end{aligned}$$

which implies

$$\pi'_\sigma(a)D'^{-1}_\sigma = \sqrt{\tfrac{\mathrm{d}\mu}{\mathrm{d}\mu'}(\sigma)}V(\sigma)(\pi_\sigma(a)D^{-1}_\sigma) = z_\sigma\sqrt{\tfrac{\mathrm{d}\mu}{\mathrm{d}\mu'}(\sigma)}T_\sigma(\pi_\sigma(a)D^{-1}_\sigma)T^{-1}_\sigma$$

for almost all $\sigma \in \operatorname{Irr}(A)$. Taking the adjoint of both sides gives us (note that $\pi'_\sigma(a), \pi_\sigma(a)$ are bounded, see [78, Proposition 9.2])

$$D'^{-1}_\sigma\pi'_\sigma(a^*) = \overline{z_\sigma}\sqrt{\tfrac{\mathrm{d}\mu}{\mathrm{d}\mu'}(\sigma)}T_\sigma(D^{-1}_\sigma\pi_\sigma(a^*))T^{-1}_\sigma$$

and

$$D^{-1}_\sigma\pi_\sigma(a^*) = z_\sigma\sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}T^{-1}_\sigma(D'^{-1}_\sigma\pi'_\sigma(a^*))T_\sigma. \tag{3.2}$$

Observe that (3.2) is an equality of bounded operators. Equation (3.2) shows that operators D^{-1}_σ, $z_\sigma\sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}T^{-1}_\sigma D'^{-1}_\sigma T_\sigma$ are equal on the dense subspace

$$\mathscr{V} = \operatorname{span}\{\pi_\sigma(a^*)\xi \,|\, a \in X,\, \xi \in \mathsf{K}_\sigma\} \subseteq \mathsf{K}_\sigma$$

(in particular $\mathscr{V}$ is contained in the domain of D^{-1}_σ and $z_\sigma\sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}T^{-1}_\sigma D'^{-1}_\sigma T_\sigma$). Take any $\eta \in \mathscr{V}\setminus\{0\}$, then

$$0 < \langle \eta \,|\, D^{-1}_\sigma\eta\rangle = \big\langle \eta \big| z_\sigma\sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}T^{-1}_\sigma D'^{-1}_\sigma T_\sigma\eta\big\rangle = z_\sigma\sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}\big\langle\eta\big|T^{-1}_\sigma D'^{-1}_\sigma T_\sigma\eta\big\rangle,$$

hence $z_\sigma = 1$.
Now we need to use the assumption that θ is a KMS weight. Let $(\sigma^\theta_t)_{t\in\mathbb{R}}$ be a modular group for θ. Let us show that

$$\pi_\sigma(\sigma^\theta_t(a)) = D^{it}_\sigma\pi_\sigma(a)D^{-it}_\sigma \quad (a \in A, t \in \mathbb{R}) \tag{3.3}$$

for almost all $\sigma \in \operatorname{Irr}(A)$. Define $\rho^1_t, \rho^2_t \colon \pi_\theta(A) \to \pi_\theta(A)$ via

$$\rho^1_t(\pi_\theta(a)) = \pi_\theta(\sigma^\theta_t(a)), \quad \rho^2_t(\pi_\theta(a)) = \mathcal{P}^{-1}\Big(\int^\oplus_{\operatorname{Irr}(A)} D^{it}_\sigma\pi_\sigma(a)D^{-it}_\sigma \otimes \mathbb{1}_{\overline{\mathsf{K}_\sigma}}\,\mathrm{d}\mu(\sigma)\Big)\mathcal{P}$$

for all $t \in \mathbb{R}, a \in A$. It is clear that ρ^2_t is well defined. So is ρ^1_t. Indeed, by [53, Theorem 6.20] (see also Section 2.1) we have $\Lambda_\theta(\sigma^\theta_t(a')) = \nabla^{it}_{\tilde\theta}\Lambda_\theta(a')$ for $a' \in \mathfrak{N}_\theta$. Using this equation we can prove

$$\pi_\theta(\sigma^\theta_t(a)) = \sigma^{\tilde\theta}_t(\pi_\theta(a)) \tag{3.4}$$

for $t \in \mathbb{R}, a \in A$. Indeed, for $a' \in \mathfrak{N}_\theta$

$$\begin{aligned}\pi_\theta(\sigma^\theta_t(a))\Lambda_\theta(a') &= \Lambda_\theta(\sigma^\theta_t(a)a') = \nabla^{it}_{\tilde\theta}\Lambda_\theta(a\sigma^\theta_{-t}(a')) \\ &= \nabla^{it}_{\tilde\theta}\pi_\theta(a)\nabla^{-it}_{\tilde\theta}\nabla^{it}_{\tilde\theta}\Lambda_\theta(\sigma^\theta_{-t}(a')) = \sigma^{\tilde\theta}_t(\pi_\theta(a))\Lambda_\theta(a')\end{aligned}$$

and equation (3.4) follows. Consequently, if $a \in \ker(\pi_\theta)$ then $\sigma_t^\theta(a) \in \ker(\pi_\theta)$. Next,

$$
\begin{aligned}
&\langle \Lambda_\theta(b) \,|\, \rho_t^2(\pi_\theta(a))\Lambda_\theta(b')\rangle = \langle \mathcal{P}\Lambda_\theta(b) \,|\, \int_{\mathrm{Irr}(A)}^{\oplus} D_\sigma^{it}\pi_\sigma(a)D_\sigma^{-it} \otimes \mathbb{1}_{\overline{\mathsf{K}_\sigma}} \,\mathrm{d}\mu(\sigma)\mathcal{P}\Lambda_\theta(b')\rangle \\
&= \Big\langle \int_{\mathrm{Irr}(A)}^{\oplus} \pi_\sigma(b)D_\sigma^{-1}\,\mathrm{d}\mu(\sigma) \Big| \int_{\mathrm{Irr}(A)}^{\oplus} D_\sigma^{it}\pi_\sigma(a)D_\sigma^{-it} \otimes \mathbb{1}_{\overline{\mathsf{K}_\sigma}}\,\mathrm{d}\mu(\sigma) \int_{\mathrm{Irr}(A)}^{\oplus} \pi_\sigma(b')D_\sigma^{-1}\,\mathrm{d}\mu(\sigma)\Big\rangle \\
&= \Big\langle \int_{\mathrm{Irr}(A)}^{\oplus} \pi_\sigma(b)D_\sigma^{-1}\,\mathrm{d}\mu(\sigma) \Big| \int_{\mathrm{Irr}(A)}^{\oplus} D_\sigma^{it}\pi_\sigma(a)D_\sigma^{-it}\pi_\sigma(b')D_\sigma^{-1}\,\mathrm{d}\mu(\sigma)\Big\rangle \\
&= \langle \Lambda_{\tilde\theta}(\sigma_{-t}^{\tilde\theta}(\pi_\theta(b))) \,|\, \pi_\theta(a)\Lambda_{\tilde\theta}(\sigma_{-t}^{\tilde\theta}(\pi_\theta(b')))\rangle = \tilde\theta(\sigma_{-t}^{\tilde\theta}(\pi_\theta(b^*))\pi_\theta(a)\sigma_{-t}^{\tilde\theta}(\pi_\theta(b'))) \\
&= \tilde\theta(\pi_\theta(b^*)\sigma_t^{\tilde\theta}(\pi_\theta(a))\pi_\theta(b')) = \tilde\theta(\pi_\theta(b^*)\pi_\theta(\sigma_t^\theta(a))\pi_\theta(b')) = \theta(b^*\sigma_t^\theta(a)b) \\
&= \langle \Lambda_\theta(\pi_\theta(b)) \,|\, \pi_\theta(\sigma_t^\theta(a))\Lambda_\theta(b')\rangle = \langle \Lambda_\theta(\pi_\theta(b)) \,|\, \rho_t^1(\pi_\theta(a))\Lambda_\theta(b')\rangle
\end{aligned}
$$

for $b, b' \in \mathfrak{N}_\theta$, hence $\rho_t^1 = \rho_t^2$. It follows that

$$
\pi_\sigma(\sigma_t^\theta(a)) = D_\sigma^{it}\pi_\theta(a)D_\sigma^{-it} \quad (a \in A, t \in \mathbb{R}) \tag{3.5}
$$

for almost all $\sigma \in \mathrm{Irr}(A)$. Consequently $\mathscr{V}$ is invariant under $(D_\sigma^{it})_{t\in\mathbb{R}}$, hence [54, Corollary 1.22] implies that $\mathscr{V}$ is a core for D_σ^{-1}. It follows that

$$
D_\sigma^{-1} \subseteq \sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}T_\sigma^{-1}D_\sigma'^{-1}T_\sigma,
$$

and as self-adjoint operators do not admit proper self-adjoint extensions ([78, Section 9.2]), we arrive at

$$
D_\sigma^{-1} = \sqrt{\tfrac{\mathrm{d}\mu'}{\mathrm{d}\mu}(\sigma)}T_\sigma^{-1}D_\sigma'^{-1}T_\sigma
$$

for almost all $\sigma \in \mathrm{Irr}(A)$. □

3.2 Operators related to the modular theory of $\mathbb{G}$ and $\widehat{\mathbb{G}}$

Let assume for the rest of this and the next section that $\mathbb{G}$ is a second countable, type I locally compact quantum group. Theorem 3.3 gives us a Plancherel measure together with objects $\mathcal{Q}_L, \mathcal{Q}_R, (D_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}, (E_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$.

As advertised at the beginning of the Section 3 we will now express various objects related to the modular theory of $\mathbb{G}$ and $\widehat{\mathbb{G}}$ using the direct integral picture and operators $(D_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}, (E_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$. Let us start with a description of $J_{\widehat{\varphi}}, J_{\widehat{\psi}}, \mathrm{L}^\infty(\widehat{\mathbb{G}})$ and $\mathrm{L}^\infty(\widehat{\mathbb{G}})'$ on the level of direct integrals.

Proposition 3.7. *Define an antiunitary operator* $\Sigma = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} J_{\mathsf{H}_\pi}\,\mathrm{d}\mu(\pi)$, *where*

$$
J_{\mathsf{H}_\pi}\colon\ \mathrm{HS}(\mathsf{H}_\pi) \ni \xi \mapsto \xi^* \in \mathrm{HS}(\mathsf{H}_\pi) \quad (\pi \in \mathrm{Irr}(\mathbb{G})).
$$

We have

$$
\nu^{\frac{i}{4}}J_{\widehat{\psi}} = J_{\widehat{\varphi}} = \mathcal{Q}_L^*\Sigma\mathcal{Q}_L = \mathcal{Q}_R^*\Sigma\mathcal{Q}_R.
$$

Furthermore, the following equalities of von Neumann algebras hold:

$$\mathcal{Q}_L\,\mathrm{L}^\infty(\widehat{\mathbb{G}})\mathcal{Q}_L^* = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{B}(\mathsf{H}_\pi)\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi), \quad \mathcal{Q}_L\,\mathrm{L}^\infty(\widehat{\mathbb{G}})'\mathcal{Q}_L^* = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}_\pi}\otimes \mathrm{B}(\overline{\mathsf{H}_\pi})\,\mathrm{d}\mu(\pi)$$

$$\mathcal{Q}_R\,\mathrm{L}^\infty(\widehat{\mathbb{G}})\mathcal{Q}_R^* = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}_\pi}\otimes \mathrm{B}(\overline{\mathsf{H}_\pi})\,\mathrm{d}\mu(\pi), \quad \mathcal{Q}_R\,\mathrm{L}^\infty(\widehat{\mathbb{G}})'\mathcal{Q}_R^* = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{B}(\mathsf{H}_\pi)\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)$$

Proof. By Theorem 3.4 and [55, Proposition 5.2, Definition 5.10] we have the equality $\mathcal{Q}_L^*\Sigma\mathcal{Q}_L = J_{\widehat{\varphi}^u} = J_{\widehat{\varphi}}$.
Similarly, $\mathcal{Q}_{R,0}$ transforms $J_{\widehat{\psi}}$ to Σ: $J_{\widehat{\psi}} = \mathcal{Q}_{R,0}^*\Sigma\mathcal{Q}_{R,0}$. Operator $\mathcal{Q}_R$ was defined as $\mathcal{Q}_R = \mathcal{Q}_{R,0}J_\varphi J_{\widehat{\varphi}}$. Consequently, we get $J_{\widehat{\psi}} = J_\varphi J_{\widehat{\varphi}}\mathcal{Q}_R^*\Sigma\mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi$. Using the commutation relation $J_{\widehat{\varphi}}J_\varphi = \nu^{\frac{i}{4}} J_\varphi J_{\widehat{\varphi}}$ (see equation (2.14)) and formula $J_{\widehat{\psi}} = \nu^{-\frac{i}{4}} J_{\widehat{\varphi}}$ (equation (2.3), the scaling constant of $\widehat{\mathbb{G}}$ is $\hat{\nu} = \nu^{-1}$) we arrive at

$$\mathcal{Q}_R^*\Sigma\mathcal{Q}_R = J_{\widehat{\varphi}}J_\varphi(\nu^{-\frac{i}{4}}J_{\widehat{\varphi}})J_\varphi J_{\widehat{\varphi}} = \nu^{-\frac{i}{4}}\nu^{\frac{i}{4}}J_\varphi J_{\widehat{\varphi}} J_{\widehat{\varphi}} J_\varphi J_{\widehat{\varphi}} = J_{\widehat{\varphi}}.$$

The first two equalities in the claim are direct consequences of Theorem 3.4. The equalities involving $\mathcal{Q}_R$ can be proven using $\mathcal{Q}_R = \mathcal{Q}_{R,0}J_\varphi J_{\widehat{\varphi}}$ and $J_\varphi\,\mathrm{L}^\infty(\widehat{\mathbb{G}})J_\varphi = \mathrm{L}^\infty(\widehat{\mathbb{G}})$ (equation (2.10)). □

In the theory of compact quantum groups one often perform calculations on special elements $U^\alpha_{i,j}$ (called matrix coefficients) which form a linearly w*-dense subset inside $\mathrm{L}^\infty(\mathbb{G})$. We will now define an analog of these elements for type I, second countable locally compact quantum group $\mathbb{G}$. Elements of this form were already considered in [17].

Definition 3.8. For $\xi,\eta \in \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$ we define elements of $\mathrm{L}^\infty(\mathbb{G})$:

$$M^L_{\xi,\eta} = \int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id}\otimes\omega_{\xi_\pi,\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi), \quad M^R_{\xi,\eta} = \int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id}\otimes\omega_{\xi_\pi,\eta_\pi})(U^{\pi})\,\mathrm{d}\mu(\pi).$$

The above elements will be referred to as left (resp. right) matrix coefficients.

Note that the above (weak) integrals converge in σ-WOT and we have $(M^L_{\xi,\eta})^* = M^R_{\eta,\xi}$.

We will now recall some results obtained by Caspers and Koelink in [17, 18]. We remark that one needs to be careful when taking equations from these papers as there is a difference in convention: we prefer to use inner products linear on the right and functionals $\omega_{\xi,\eta}$ defined accordingly. That is why we choose to state explicitly used results with necessary changes.
Let us introduce two positive, self-adjoint operators on $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$:

$$D = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi\,\mathrm{d}\mu(\pi), \quad E = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi\,\mathrm{d}\mu(\pi).$$

(see [58] and Section 7.1 for the meaning of a direct integral of unbounded operators).
First, we can transport a left (resp. right) matrix coefficient via $\mathcal{Q}_L$ (resp. $\mathcal{Q}_R$). The following is a reformulation of [18, Lemma 3.7, Lemma 3.9].

Lemma 3.9.

1) *If* $\xi,\eta\in\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$, $\xi\in\mathrm{Dom}(D)$ *and the vector field* $(\eta_\pi\otimes\overline{D_\pi\xi_\pi})_{\pi\in\mathrm{Irr}(\mathbb{G})}$ *is square integrable, then* $M^L_{\xi,\eta}\in\mathfrak{N}_\varphi$ *and* $\mathcal{Q}_L\Lambda_\varphi(M^L_{\xi,\eta})=\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\eta_\pi\otimes\overline{D_\pi\xi_\pi}\,\mathrm{d}\mu(\pi)$.

2) *If* $\xi,\eta\in\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$, $\xi\in\mathrm{Dom}(E)$ *and the vector field* $(\eta_\pi\otimes\overline{E_\pi\xi_\pi})_{\pi\in\mathrm{Irr}(\mathbb{G})}$ *is square integrable, then* $M^R_{\xi,\eta}\in\mathfrak{N}_\psi$ *and* $\mathcal{Q}_R\Lambda_\psi(M^R_{\xi,\eta})=\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\eta_\pi\otimes\overline{E_\pi\xi_\pi}\,\mathrm{d}\mu(\pi)$.

Using the above result and the fact that $\mathcal{Q}_L,\mathcal{Q}_R$ are unitary, one can easily derive the following density results:

Lemma 3.10.

1) *The set* $\{\Lambda_\varphi(M^L_{\xi,\eta})\}$, *where* ξ,η *run over vectors in* $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$ *such that* $\xi\in\mathrm{Dom}(D)$ *and* $(\eta_\pi\otimes\overline{D_\pi\xi_\pi})_{\pi\in\mathrm{Irr}(\mathbb{G})}$ *is square integrable, is lineary dense in* $\mathrm{L}^2(\mathbb{G})$.

2) *The set* $\{\Lambda_\psi(M^R_{\xi,\eta})\}$, *where* ξ,η *run over vectors in* $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$ *such that* $\xi\in\mathrm{Dom}(E)$ *and* $(\eta_\pi\otimes\overline{E_\pi\xi_\pi})_{\pi\in\mathrm{Irr}(\mathbb{G})}$ *is square integrable, is lineary dense in* $\mathrm{L}^2(\mathbb{G})$.

Consider an antilinear map[22]

$$\Lambda_\psi(\mathfrak{N}_\psi\cap\mathfrak{N}_\varphi{}^*)\ni\Lambda_\psi(x)\mapsto\Lambda_\varphi(x^*)\in\mathrm{L}^2(\mathbb{G})\tag{3.6}$$

and define T' to be its closure. Let $T'=J'\nabla'^{\frac12}$ be the polar decomposition of T'. It is well known that J' is antiunitary and $\nabla'^{\frac12}$ is strictly positive and self-adjoint. In Corollary 3.19 we will describe these operators, for now let us recall how they look on the level of direct integrals.

Proposition 3.11. *We have* $\mathcal{Q}_LJ'\mathcal{Q}_R^*=\Sigma$ *and* $\mathcal{Q}_R\nabla'^{\frac12}\mathcal{Q}_R^*=\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}D_\pi\otimes(E_\pi^{-1})^{\top}\,\mathrm{d}\mu(\pi)$.

The above proposition is a combination of [18, Proposition 4.4, Proposition 4.5, Theorem 4.6]. We also need formulas expressing the action of modular automorphism groups on the matrix coefficients.

Proposition 3.12. *For each* $\xi,\eta\in\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathsf{H}_\pi\,\mathrm{d}\mu(\pi),t\in\mathbb{R}$ *the following holds:*

$$\begin{aligned}
\sigma^\psi_t(M^R_{\xi,\eta})&=\nu^{\frac12it^2}\delta^{it}\,M^R_{E^{2it}\xi,D^{2it}\eta}, &\quad \sigma^\varphi_t(M^R_{\xi,\eta})&=\nu^{\frac12it^2}\,M^R_{E^{2it}\xi,D^{2it}\eta}\,\delta^{it},\\
\sigma^\psi_t(M^L_{\xi,\eta})&=\nu^{-\frac12it^2}M^L_{D^{2it}\xi,E^{2it}\eta}\,\delta^{-it}, &\quad \sigma^\varphi_t(M^L_{\xi,\eta})&=\nu^{-\frac12it^2}\delta^{-it}\,M^L_{D^{2it}\xi,E^{2it}\eta}.
\end{aligned}$$

The formulas expressing the action of $\sigma^\varphi,\sigma^\psi$ on $M^R_{\xi,\eta}$ are stated in [17, Remark 2.2.11]. The other two follow by taking the adjoints. We note that they can be derived using the formula for ∇' (Proposition 3.11) and equation $\nu^{\frac12it^2}\delta^{it}=\nabla_\psi^{it}\,\nabla'^{-it}$ (see [82, Equations (29),

[22] This map appears during a construction of the Radon–Nikodym derivative between ψ and φ, see [82].

(30), page 112] and the proof of [93, Theorem 3.11]).

Our next goal is to describe the polar decomposition of operator $T' \colon \Lambda_\psi(x) \mapsto \Lambda_\varphi(x^*)$ using more standard operators on $\mathrm{L}^2(\mathbb{G})$. This result which is interesting on its own, will give us an tremendously useful relation between $\mathcal{Q}_L$ and $\mathcal{Q}_R$. Before we dive into technical details, let us see through a formal calcualtion what kind of result we should expect:

$$\begin{aligned} T'\Lambda_\psi(x) &= \Lambda_\varphi(x^*) = J_\varphi \nabla_\varphi^{\frac12}\Lambda_\varphi(x) = J_\varphi \nabla_\varphi^{\frac12} J_\varphi \sigma^\varphi_{i/2}(\delta^{-\frac12})^* J_\varphi \Lambda_\varphi(x\delta^{\frac12}) \\ &= \nabla_\varphi^{-\frac12}(\nu^{-\frac{i}{4}}\delta^{-\frac12})^* J_\varphi\Lambda_\psi(x) = (\nu^{\frac{i}{8}}J_\varphi)(J_\varphi \nu^{\frac{i}{8}}\nabla_\varphi^{-\frac12}\delta^{-\frac12}J_\varphi)\Lambda_\psi(x). \end{aligned} \tag{3.7}$$

We need to include the factor $\nu^{\frac{i}{8}}$ due to the following lemma:

Lemma 3.13. *For all $s,t\in\mathbb{R}$ operators $\nabla_\varphi^s\circ\delta^t$, $\delta^t\circ\nabla_\varphi^s$ are closable. We have the equality $\nu^{\frac{ist}{2}}\nabla_\varphi^s\delta^t = \nu^{-\frac{ist}{2}}\delta^t\nabla_\varphi^s$ of strictly positive, self-adjoint operators, moreover*

$$(\nu^{\frac{ist}{2}}\nabla_\varphi^s\delta^t)^{ir} = \nu^{-\frac{ist}{2}r^2}\nabla_\varphi^{isr}\delta^{itr} = \nu^{\frac{ist}{2}r^2}\delta^{itr}\nabla_\varphi^{isr} \quad (r\in\mathbb{R}).$$

The above result is a consequence of the commutation relation $\nabla_\varphi^{is}\delta^{it} = \nu^{ist}\delta^{it}\nabla_\varphi^{is}$ $(s,t\in\mathbb{R})$. Indeed, if $\nu=1$ then ∇_φ^s and δ^t strongly commute and the claim is clear. Otherwise operators $\nabla_\varphi^s,\delta^t$ satisfy the Zakrzewski relation (or the Weyl relation after passing to logarithms). Then Lemma 3.13 follows from [102, Example 3.1, Theorem 3.1]. The next lemma describes the action of the unbounded operator δ^t.

Lemma 3.14.

1) *Let $t\in\mathbb{R}, x\in\mathfrak{N}_\varphi$ be such that $x\circ\delta^t$ is closable and $x\delta^t\in\mathfrak{N}_\varphi$. Then $J_\varphi\Lambda_\varphi(x)\in\operatorname{Dom}(\delta^t)$ and $\nu^{\frac{it}{2}}J_\varphi\delta^tJ_\varphi\Lambda_\varphi(x) = \Lambda_\varphi(x\delta^t)$.*

2) *Let $t\in\mathbb{R}, x\in\mathfrak{N}_\psi$ be such that $x\circ\delta^t$ is closable and $x\delta^t\in\mathfrak{N}_\psi$. Then $J_\varphi\Lambda_\psi(x)\in\operatorname{Dom}(\delta^t)$ and $\nu^{\frac{it}{2}}J_\varphi\delta^tJ_\varphi\Lambda_\psi(x) = \Lambda_\psi(x\delta^t)$.*

Proof. We prove only the first assertion, the second one can be derived analogously. Take $x\in\mathfrak{N}_\varphi, t\in\mathbb{R}$ which satisfy conditions of the lemma and define

$$x_n = \sqrt{\tfrac{n}{\pi}}\int_{\mathbb{R}} e^{-np^2}x\delta^{ip}\,\mathrm{d}p \in \mathrm{L}^\infty(\mathbb{G}) \quad (n\in\mathbb{N})$$

(the above weak integral converges in σ-WOT). Operator $x_n\circ\delta^t$ is closable and we have

$$x_n\delta^t = \sqrt{\tfrac{n}{\pi}}\int_{\mathbb{R}} e^{-np^2}(x\delta^t)\delta^{ip}\,\mathrm{d}p = \sqrt{\tfrac{n}{\pi}}\int_{\mathbb{R}} e^{-n(p+it)^2}x\delta^{ip}\,\mathrm{d}p. \tag{3.8}$$

Clearly $x_n, x_n\delta^t\in\mathfrak{N}_\varphi$ and due to the Hille's theorem

$$\Lambda_\varphi(x_n) = \sqrt{\tfrac{n}{\pi}}\int_{\mathbb{R}} e^{-np^2}\Lambda_\varphi(x\delta^{ip})\,\mathrm{d}p = \sqrt{\tfrac{n}{\pi}}J_\varphi\int_{\mathbb{R}} e^{-np^2}\nu^{-\frac{p}{2}}\delta^{-ip}J_\varphi\Lambda_\varphi(x)\,\mathrm{d}p,$$

similarly thanks to the equation (3.8) we have

$$\Lambda_\varphi(x_n\delta^t) = \sqrt{\tfrac{n}{\pi}} J_\varphi \int_{\mathbb{R}} e^{-np^2}\nu^{-\frac{p}{2}}\delta^{-ip} J_\varphi\Lambda_\varphi(x\delta^t)\,\mathrm{d}p = \sqrt{\tfrac{n}{\pi}} J_\varphi \int_{\mathbb{R}} e^{-n(p-it)^2}\nu^{-\frac{p}{2}}\delta^{-ip} J_\varphi\Lambda_\varphi(x)\,\mathrm{d}p.$$

Consequently, $\Lambda_\varphi(x_n) \xrightarrow[n\to\infty]{} \Lambda_\varphi(x)$ and $\Lambda_\varphi(x_n\delta^t) \xrightarrow[n\to\infty]{} \Lambda_\varphi(x\delta^t)$. For each $r\in\mathbb{R}$ we have

$$\delta^{ir} J_\varphi\Lambda_\varphi(x_n) = \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-np^2}\nu^{-\frac{p}{2}}\delta^{-i(p-r)} J_\varphi\Lambda_\varphi(x)\,\mathrm{d}p = f_n(r),$$

where f_n is the entire function

$$f_n\colon \mathbb{C}\ni z\mapsto \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-n(p+z)^2}\nu^{-\frac{p+z}{2}}\delta^{-ip} J_\varphi\Lambda_\varphi(x)\,\mathrm{d}p \in \mathrm{L}^2(\mathbb{G}).$$

From the above follows that $J_\varphi\Lambda_\varphi(x_n)\in\mathrm{Dom}(\delta^z)$ for all $z\in\mathbb{C}$ and $\delta^z J_\varphi\Lambda_\varphi(x_n) = f_n(-iz)$ [78, Corollary 9.15]. Let us show that the sequence $(\delta^t J_\varphi\Lambda_\varphi(x_n))_{n\in\mathbb{N}}$ converges to $\nu^{\frac{it}{2}} J_\varphi\Lambda_\varphi(x\delta^t)$:

$$\begin{aligned}\delta^t J_\varphi\Lambda_\varphi(x_n) = f_n(-it) &= \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-n(p-it)^2}\nu^{-\frac{p-it}{2}}\delta^{-ip} J_\varphi\Lambda_\varphi(x)\,\mathrm{d}p \\ = \nu^{\frac{it}{2}}\sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-n(p-it)^2}\nu^{-\frac{p}{2}}\delta^{-ip} J_\varphi\Lambda_\varphi(x)\,\mathrm{d}p &= \nu^{\frac{it}{2}} J_\varphi\Lambda_\varphi(x_n\delta^t) \xrightarrow[n\to\infty]{} \nu^{\frac{it}{2}} J_\varphi\Lambda_\varphi(x\delta^t).\end{aligned}$$

Norm closedness of δ^t implies $J_\varphi\Lambda_\varphi(x)\in\mathrm{Dom}(\delta^t)$ and $\delta^t J_\varphi\Lambda_\varphi(x) = \nu^{\frac{it}{2}} J_\varphi\Lambda_\varphi(x\delta^t)$. □

We will now introduce a space $\mathcal{D}_0$ of sufficiently nice vectors on which calculation (3.7) is justified and which forms a core for the operators involved. First, define

$$\delta_{n,z} = \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-nt^2}\nu^{zt}\delta^{it}\,\mathrm{d}t \in \mathrm{L}^\infty(\mathbb{G}) \quad (n\in\mathbb{N}, z\in\mathbb{C}).$$

Note that for each $z\in\mathbb{C}$, the sequence $(\delta_{n,z})_{n\in\mathbb{N}}$ is bounded and converges to $\mathbb{1}$ in SOT. Next, for $x\in\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi{}^*\cap\mathfrak{N}_\psi\cap\mathfrak{N}_\psi{}^*, k\in\mathbb{N}, A=(A_1,A_2)\in\mathbb{C}^2$ define

$$x_{k,A} = \tfrac{k}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-k(t-A_1)^2-k(s-A_2)^2}\sigma^\varphi_t\circ\sigma^\psi_s(x)\,\mathrm{d}t\,\mathrm{d}s \in \mathrm{L}^\infty(\mathbb{G}).$$

Finally, define a subspace $\mathcal{D}_0$ via

$$\mathcal{D}_0 = \mathrm{span}\{\Lambda_\psi(\delta_{n,z}x_{k,A}\delta_{m,w})\,|\,x,x^*\in\mathfrak{N}_\varphi\cap\mathfrak{N}_\psi, n,m,k\in\mathbb{N}, A\in\mathbb{C}^2, z,w\in\mathbb{C}\}.$$

Lemma 3.15.

- *The subspace $\mathcal{D}_0$ is a core for $\nabla_\varphi^{-\frac{1}{2}}$. Moreover, for $\xi\in\mathrm{Dom}(\nabla_\varphi^{-\frac{1}{2}})$ we can find a sequence $(\xi_p)_{p\in\mathbb{N}}$ in*

$$\{\Lambda_\psi(x_{k,A}\delta_{m,w})\,|\,x,x^*\in\mathfrak{N}_\varphi\cap\mathfrak{N}_\psi, m,k\in\mathbb{N}, A\in\mathbb{C}^2, w\in\mathbb{C}\}$$

such that $\xi_p \xrightarrow[p\to\infty]{} \xi$ and $\nabla_\varphi^{-\frac{1}{2}}\xi_p \xrightarrow[p\to\infty]{} \nabla_\varphi^{-\frac{1}{2}}\xi$

- *Each element of $\mathcal{D}_0$ can be written as $\Lambda_\psi(x)$ for some $x \in \mathrm{L}^\infty(\mathbb{G})$ such that $x, x^* \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi \cap \bigcap_{z\in\mathbb{C}} \mathrm{Dom}(\sigma_z^\psi)$. Moreover, $\sigma_z^\psi(x) \in \mathfrak{N}_\psi$ and $\Lambda_\psi(x^*), \Lambda_\psi(\sigma_z^\psi(x)) \in \mathcal{D}_0$. Additionally, $\Lambda_\psi(x) \in \bigcap_{z\in\mathbb{C}} \mathrm{Dom}(\nabla_\psi^z)$ and $\nabla_\psi^{iz}\Lambda_\psi(x) = \Lambda_\psi(\sigma_z^\psi(x))$.*
- *For all $z, w \in \mathbb{C}, \Lambda_\psi(x) \in \mathcal{D}_0$ the operator $\delta^z \circ x \circ \delta^w$ is closable and its closure belongs to $\mathfrak{N}_\psi \cap \mathfrak{N}_\varphi$.*
- *We have $J_\varphi \mathcal{D}_0 = \mathcal{D}_0$.*

A proof of the above lemma requires only standard reasoning, hence will be skipped. In the next two lemmas we prove properties of $\mathcal{D}_0$ which allow us to derive the polar decomposition of T'.

Lemma 3.16. *The subspace $\mathcal{D}_0$ is a core for $\nu^{-\frac{i}{4}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi$. We have*

$$\nu^{-\frac{i}{4}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi \Lambda_\psi(x) = \Lambda_\varphi(x)$$

for all $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi$ such that $x \circ \delta^{-\frac{1}{2}}$ is closable and $x\delta^{-\frac{1}{2}} \in \mathfrak{N}_\psi$. Moreover, the operator

$$(J_\varphi \nabla_\varphi^{\frac{1}{2}}) \circ (\nu^{-\frac{i}{4}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi) = \nu^{\frac{i}{4}} (\nabla_\varphi^{-\frac{1}{2}} \circ \delta^{-\frac{1}{2}}) J_\varphi$$

is closable and $\mathcal{D}_0$ is a core for its closure $\nu^{\frac{i}{4}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} J_\varphi$.

Proof. It is clear that $\mathrm{span} \bigcup_{n\in\mathbb{N}} \delta_{n,0} \mathrm{L}^2(\mathbb{G})$ is a core for $\delta^{-\frac{1}{2}}$. Take $\xi = \delta_{n,0}\eta \in \mathrm{Dom}(\delta^{-\frac{1}{2}})$ for some $n \in \mathbb{N}$ and let $(\eta_p)_{p\in\mathbb{N}}$ be a sequence of vectors of the form $\Lambda_\psi(x_{k,A}\delta_{m,w})$ (see the first point of the Lemma 3.15) converging to η. We have $\delta_{n,0}\eta_p \in \mathcal{D}_0$,

$$\|\xi - \delta_{n,0}\eta_p\| \le \|\eta - \eta_p\| \xrightarrow[p\to\infty]{} 0 \quad \text{and} \quad \|\delta^{-\frac{1}{2}}\xi - \delta^{-\frac{1}{2}}\delta_{n,0}\eta_p\| \le \|\delta^{-\frac{1}{2}}\delta_{n,0}\| \|\eta - \eta_p\| \xrightarrow[p\to\infty]{} 0,$$

which shows that $\mathcal{D}_0$ is a core for $\delta^{-\frac{1}{2}}$. Since $\mathcal{D}_0$ is invariant under J_φ, it is also a core for $\nu^{-\frac{i}{4}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi$.
Take $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi$ such that $x \circ \delta^{-\frac{1}{2}}$ is closable and $x\delta^{-\frac{1}{2}} \in \mathfrak{N}_\psi$. Lemma 3.14 gives us $J_\varphi \Lambda_\psi(x) \in \mathrm{Dom}(\delta^{-\frac{1}{2}})$ and $\nu^{-\frac{i}{4}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi \Lambda_\psi(x) = \Lambda_\psi(x\delta^{-\frac{1}{2}}) = \Lambda_\varphi(x)$.
Equality from the claim $(J_\varphi \nabla_\varphi^{\frac{1}{2}}) \circ (\nu^{-\frac{i}{4}} J_\varphi \delta^{-\frac{1}{2}} J_\varphi) = \nu^{\frac{i}{4}} (\nabla_\varphi^{-\frac{1}{2}} \circ \delta^{-\frac{1}{2}}) J_\varphi$ is a straightforward consequence of the relation $J_\varphi \nabla_\varphi^{\frac{1}{2}} = \nabla_\varphi^{-\frac{1}{2}} J_\varphi$.
To deduce the last assertion let us observe that Lemma 3.13 gives us an equality $\nu^{\frac{i}{8}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} = \nu^{-\frac{i}{8}} \delta^{-\frac{1}{2}} \nabla_\varphi^{-\frac{1}{2}}$. It follows that the closure of $\nu^{-\frac{i}{4}} \delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}}$ is $\nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}}$. Take ξ in $\mathrm{Dom}(\nu^{-\frac{i}{4}} \delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}})$. For each $n \in \mathbb{N}$ we have $\delta_{n,0}\xi \in \mathrm{Dom}(\nu^{-\frac{i}{4}} \delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}})$,

$$\delta_{n,0}\xi \xrightarrow[n\to\infty]{} \xi \tag{3.9}$$

and

$$\begin{aligned} \nu^{-\frac{i}{4}} \delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}} \, (\delta_{n,0}\xi) &= \nu^{-\frac{i}{4}} \sigma_{i/2}^\varphi(\delta_{n,0}) \delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}}(\xi) \\ &= \nu^{-\frac{i}{4}} \delta_{n,-1/2} \delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}}(\xi) \xrightarrow[n\to\infty]{} \nu^{-\frac{i}{4}} \delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}}(\xi). \end{aligned} \tag{3.10}$$

As previously, since $\mathcal{D}_0$ is invariant for J_φ, it is enough to check that $\mathcal{D}_0$ is a core for $\nabla_\varphi^{-\frac{1}{2}}\delta^{-\frac{1}{2}}$. Take $\xi \in \mathrm{Dom}(\nabla_\varphi^{-\frac{1}{2}}\delta^{-\frac{1}{2}})$. The above reasoning and equations (3.9), (3.10) show that it is enough to take vector of the form $\xi = \delta_{n,0}\eta$ for $\eta \in \mathrm{Dom}(\nu^{-\frac{i}{4}}\delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}})$ and some $n \in \mathbb{N}$. Let $(\eta_p)_{p\in\mathbb{N}}$ be a sequence of vectors of the form $\Lambda_\psi(x_{k,A,B}\delta_{m,w})$ such that $\eta_p \xrightarrow[p\to\infty]{} \eta$ and $\nabla_\varphi^{-\frac{1}{2}}\eta_p \xrightarrow[p\to\infty]{} \nabla_\varphi^{-\frac{1}{2}}\eta$. We have $\delta_{n,0}\eta_p \in \mathcal{D}_0$, $\delta_{n,0}\eta_p \xrightarrow[p\to\infty]{} \delta_{n,0}\eta = \xi$ and

$$\begin{aligned}&\|\nu^{-\frac{i}{4}}\delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}}(\delta_{n,0}\eta - \delta_{n,0}\eta_p)\| = \|\delta_{n,-1/2}\delta^{-\frac{1}{2}} \circ \nabla_\varphi^{-\frac{1}{2}}(\eta - \eta_p)\| \\ &\le \|\delta_{n,-1/2}\delta^{-\frac{1}{2}}\|\|\nabla_\varphi^{-\frac{1}{2}}(\eta - \eta_p)\| \xrightarrow[p\to\infty]{} 0.\end{aligned}$$

□

Lemma 3.17. *The subspace $\mathcal{D}_0$ is a core for T'.*

Proof. Take $x \in \mathfrak{N}_\psi \cap \mathfrak{N}_\varphi{}^*$ and define x_n as $x_n = \frac{n}{\pi}\int_\mathbb{R}\int_\mathbb{R} e^{-n(r^2+p^2)}\delta^{ip}x\delta^{ir}\,\mathrm{d}r\,\mathrm{d}p \quad (n \in \mathbb{N})$. We have $x_n, x_n^* \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi$. Next, define $x_{n,n} = \frac{n}{\pi}\int_\mathbb{R}\int_\mathbb{R} e^{-n(t^2+s^2)}\sigma_t^\varphi \circ \sigma_s^\psi(x_n)\,\mathrm{d}s\,\mathrm{d}t$. We have $\delta_{n,0}x_{n,n}\delta_{n,0} \in \mathfrak{N}_\psi \cap \mathfrak{N}_\varphi{}^*, \Lambda_\psi(\delta_{n,0}x_{n,n}\delta_{n,0}) \in \mathcal{D}_0$, $\Lambda_\psi(\delta_{n,0}x_{n,n}\delta_{n,0}) \xrightarrow[n\to\infty]{} \Lambda_\psi(x)$ and

$$T'\Lambda_\psi(\delta_{n,0}x_{n,n}\delta_{n,0}) = \Lambda_\varphi(\delta_{n,0}x_{n,n}^*\delta_{n,0}) \xrightarrow[n\to\infty]{} \Lambda_\varphi(x^*) = T'\Lambda_\varphi(x).$$

□

Now we can derive the main results of this section.

Proposition 3.18. *We have $(J_\varphi\nabla_\varphi^{\frac{1}{2}}) \circ (\nu^{-\frac{i}{4}}J_\varphi\delta^{-\frac{1}{2}}J_\varphi) = \nu^{\frac{i}{4}}(\nabla_\varphi^{-\frac{1}{2}} \circ \delta^{-\frac{1}{2}})J_\varphi$ and after closure*

$$\nu^{\frac{i}{4}}\,\nabla_\varphi^{-\frac{1}{2}}\delta^{-\frac{1}{2}}\,J_\varphi = T'.$$

Proof. The first equality is a consequence of the equation $J_\varphi\nabla_\varphi^{\frac{1}{2}} = \nabla_\varphi^{\frac{1}{2}}J_\varphi$. Take $\Lambda_\psi(x) \in \mathcal{D}_0$. Lemmas 3.16, 3.15 justify the following calculation:

$$(J_\varphi\nabla_\varphi^{\frac{1}{2}}) \circ (\nu^{-\frac{i}{4}}J_\varphi\delta^{-\frac{1}{2}}J_\varphi)\,\Lambda_\psi(x) = J_\varphi\nabla_\varphi^{\frac{1}{2}}\Lambda_\varphi(x) = \Lambda_\varphi(x^*) = T'\Lambda_\psi(x).$$

In lemmas 3.17, 3.16 we have shown that $\mathcal{D}_0$ is a core for T' and $\nu^{\frac{i}{4}}\nabla_\varphi^{-\frac{1}{2}}\delta^{-\frac{1}{2}}J_\varphi$, which shows $T' = \nu^{\frac{i}{4}}\nabla_\varphi^{-\frac{1}{2}}\delta^{-\frac{1}{2}}J_\varphi$. □

The above result has a number of interesting corollaries.

Corollary 3.19. The polar decomposition of T' is $T' = (\nu^{\frac{i}{8}}J_\varphi)\,(J_\varphi\nu^{\frac{i}{8}}\nabla_\varphi^{-\frac{1}{2}}\delta^{-\frac{1}{2}}J_\varphi)$. Moreover, we have

$$(J_\varphi\nu^{\frac{i}{8}}\nabla_\varphi^{-\frac{1}{2}}\delta^{-\frac{1}{2}}J_\varphi)^{it} = \nu^{\frac{i}{8}t^2}J_\varphi\nabla_\varphi^{\frac{it}{2}}\delta^{\frac{it}{2}}J_\varphi \quad (t \in \mathbb{R}). \tag{3.11}$$

Proof. The first equality follows directly from Proposition 3.18. Let us justify that it is indeed the polar decomposition of T'. First, it is clear that $\nu^{\frac{i}{8}} J_\varphi$ is antiunitary. Next, Lemma 3.13 implies that $\nu^{\frac{i}{8}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}}$ is self-adjoint and strictly positive. Consequently, the operator $J_\varphi \nu^{\frac{i}{8}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} J_\varphi$ has the same properties. Uniqueness of the polar decomposition gives us the first claim. The second formula follows from Lemma 3.13:

$$
\begin{aligned}
&(J_\varphi \nu^{\frac{i}{8}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}} J_\varphi)^{it} = f(\nu^{\frac{i}{8}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}})^{it} = f((\nu^{\frac{i}{8}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}})^{it}) \\
&= J_\varphi (\nu^{\frac{i}{8}} \nabla_\varphi^{-\frac{1}{2}} \delta^{-\frac{1}{2}})^{-it} J_\varphi = J_\varphi \nu^{-\frac{i}{8}t^2} \nabla_\varphi^{\frac{it}{2}} \delta^{\frac{it}{2}} J_\varphi,
\end{aligned}
$$

where $f\colon a \mapsto J_\varphi a^* J_\varphi$. □

Now we combine our polar decomposition of T' with the result of Caspers, Koelink (Proposition 3.11), Proposition 3.7 and commutation relation $J_{\widehat{\varphi}} J_\varphi = \nu^{\frac{i}{4}} J_\varphi J_{\widehat{\varphi}}$ (equation (2.14)).

Corollary 3.20. We have $\mathcal{Q}_L \nu^{\frac{i}{8}} J_\varphi \mathcal{Q}_R^* = \Sigma$ and $\mathcal{Q}_R^* \mathcal{Q}_L = \mathcal{Q}_L^* \mathcal{Q}_R = \nu^{-\frac{i}{8}} J_{\widehat{\varphi}} J_\varphi$.

Formula $\mathcal{Q}_R^* \mathcal{Q}_L = \nu^{-\frac{i}{8}} J_{\widehat{\varphi}} J_\varphi$ is of great importance and will be used many times throught the paper.
Now we can prove several results in which we express operators related to $\mathbb{G}$ and $\widehat{\mathbb{G}}$ on the level of direct integrals. The first result of this type comes from the polar decomposition of T'.

Proposition 3.21. *For all $t \in \mathbb{R}$ we have*

$$
\begin{aligned}
\nabla_\psi^{it} \delta^{-it} &= J_\varphi \nabla_\varphi^{it} \delta^{it} J_\varphi = \nu^{-\frac{i}{2}t^2} \mathcal{Q}_R^* \Big(\int_{\mathrm{Irr}(\mathbb{G})}^\oplus D_\pi^{2it} \otimes (E_\pi^{-2it})^\top \, \mathrm{d}\mu(\pi) \Big) \mathcal{Q}_R, \\
J_\varphi \nabla_\psi^{it} \delta^{-it} J_\varphi &= \nabla_\varphi^{it} \delta^{it} = \nu^{\frac{i}{2}t^2} \mathcal{Q}_L^* \Big(\int_{\mathrm{Irr}(\mathbb{G})}^\oplus E_\pi^{2it} \otimes (D_\pi^{-2it})^\top \, \mathrm{d}\mu(\pi) \Big) \mathcal{Q}_L, \\
\nabla_\varphi^{-it} \delta^{-it} &= J_\varphi \nabla_\psi^{-it} \delta^{it} J_\varphi = \nu^{\frac{i}{2}t^2} \mathcal{Q}_R^* \Big(\int_{\mathrm{Irr}(\mathbb{G})}^\oplus E_\pi^{2it} \otimes (D_\pi^{-2it})^\top \, \mathrm{d}\mu(\pi) \Big) \mathcal{Q}_R, \\
J_\varphi \nabla_\varphi^{-it} \delta^{-it} J_\varphi &= \nabla_\psi^{-it} \delta^{it} = \nu^{-\frac{i}{2}t^2} \mathcal{Q}_L^* \Big(\int_{\mathrm{Irr}(\mathbb{G})}^\oplus D_\pi^{2it} \otimes (E_\pi^{-2it})^\top \, \mathrm{d}\mu(\pi) \Big) \mathcal{Q}_L.
\end{aligned}
$$

Proof. First, observe that we have $\nabla_\psi^{it} = J_{\widehat{\varphi}} \nabla_\varphi^{-it} J_{\widehat{\varphi}} = \delta^{it} (J_\varphi \delta^{it} J_\varphi) \nabla_\varphi^{it}$ (see [93, Theorem 5.18] and equation (2.14)). It follows that

$$
\nabla_\psi^{it} \delta^{-it} = \nu^{-it^2} \delta^{-it} \nabla_\psi^{it} = \nu^{-it^2} J_\varphi \delta^{it} \nabla_\varphi^{it} J_\varphi = J_\varphi \nabla_\varphi^{it} \delta^{it} J_\varphi,
$$

and the first equation in each row easily follows. The formula expressing $J_\varphi \nabla_\varphi^{it} \delta^{it} J_\varphi$ via direct integral of operators follows from equation (3.11) combined with Proposition 3.11.

The second equation can be found using already derived relation $\mathcal{Q}_L^*\mathcal{Q}_R=\nu^{-\frac{i}{8}}J_{\widehat{\varphi}}J_\varphi$:

$$\begin{aligned}J_\varphi\nabla_\varphi^{it}\delta^{it}J_\varphi&=\nu^{-\frac{i}{2}t^2}\nu^{\frac{i}{8}}J_\varphi J_{\widehat{\varphi}}\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}D_\pi^{2it}\otimes(E_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L\nu^{-\frac{i}{8}}J_{\widehat{\varphi}}J_\varphi\\&=\nu^{-\frac{i}{2}t^2}J_\varphi\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}E_\pi^{2it}\otimes(D_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_LJ_\varphi,\end{aligned}$$

which implies $\nabla_\varphi^{it}\delta^{it}=\nu^{\frac{i}{2}t^2}\mathcal{Q}_L^*(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}E_\pi^{2it}\otimes(D_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi))\mathcal{Q}_L$. The last two equations come from applying the operation $J_{\widehat{\varphi}}\cdot J_{\widehat{\varphi}}$ to both sides of the already derived formulas. □

Let us now derive an interesting corollary of these results.

Corollary 3.22. There exists a unique measurable function $f\colon\mathrm{Irr}(\mathbb{G})\to\mathbb{R}_{>0}$ such that

$$\begin{aligned}J_\varphi\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}D_\pi^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_RJ_\varphi&=\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}f(\pi)^{it}E_\pi^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R,\\J_\varphi\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{1}_{\mathsf{H}_\pi}\otimes(E_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_RJ_\varphi&=\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}f(\pi)^{it}\mathbb{1}_{\mathsf{H}_\pi}\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R,\\J_\varphi\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{1}_{\mathsf{H}_\pi}\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_LJ_\varphi&=\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}f(\pi)^{it}\mathbb{1}_{\mathsf{H}_\pi}\otimes(E_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L,\\J_\varphi\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}E_\pi^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_LJ_\varphi&=\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}f(\pi)^{it}D_\pi^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L\end{aligned}$$

for all $t\in\mathbb{R}$.

We note that the function f might depend on the choice of the measure μ.

Proof. Fix $t\in\mathbb{R}$. The first and the third row in Proposition 3.21 imply

$$J_\varphi\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}D_\pi^{2it}\otimes(E_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_RJ_\varphi=\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}E_\pi^{-2it}\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R.$$

Since $J_\varphi\,\mathrm{L}^\infty(\widehat{\mathbb{G}})J_\varphi=\mathrm{L}^\infty(\widehat{\mathbb{G}})$, $J_\varphi\mathrm{L}^\infty(\widehat{\mathbb{G}})'J_\varphi=\mathrm{L}^\infty(\widehat{\mathbb{G}})'$ and the center of $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{1}_{\mathsf{H}_\pi}\otimes\mathrm{B}(\overline{\mathsf{H}_\pi})\,\mathrm{d}\mu(\pi)$ is $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{C}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\pi)}\,\mathrm{d}\mu(\pi)$, Proposition 3.7 implies that there exists a measurable function $f_t\colon\mathrm{Irr}(\mathbb{G})\to\mathbb{T}$ such that

$$\begin{aligned}&J_\varphi\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}D_\pi^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_RJ_\varphi\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}E_\pi^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R\\&=J_\varphi\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{1}_{\mathsf{H}_\pi}\otimes(E_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_RJ_\varphi\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{1}_{\mathsf{H}_\pi}\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R\\&=\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\overline{f_t(\pi)}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\pi)}\,\mathrm{d}\mu(\pi).\end{aligned}$$

The above equations imply

$$J_\varphi \mathcal{Q}_R^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus D_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_R J_\varphi = \mathcal{Q}_R^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus f_t(\pi)E_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R \tag{3.12}$$

and

$$J_\varphi \mathcal{Q}_R^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus \mathbb{1}_{\mathsf{H}_\pi}\otimes (E_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_R J_\varphi = \mathcal{Q}_R^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus f_t(\pi)\mathbb{1}_{\mathsf{H}_\pi}\otimes (D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R. \tag{3.13}$$

Equation (3.12) together with relation $\mathcal{Q}_L^*\mathcal{Q}_R=\nu^{-\frac{i}{8}}J_{\widehat{\varphi}}J_\varphi$ (Corollary 3.20) gives us

$$\begin{aligned}&J_\varphi J_\varphi J_{\widehat{\varphi}}\mathcal{Q}_L^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus D_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_L J_{\widehat{\varphi}}J_\varphi J_\varphi\\ &= J_\varphi J_{\widehat{\varphi}}\mathcal{Q}_L^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus f_t(\pi)E_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L J_{\widehat{\varphi}}J_\varphi,\end{aligned}$$

hence also thanks to $\mathcal{Q}_L J_{\widehat{\varphi}}\mathcal{Q}_L^*=\Sigma$ (see Proposition 3.7)

$$\mathcal{Q}_L^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus \mathbb{1}_{\mathsf{H}_\pi}\otimes (D_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi)\big)^*\mathcal{Q}_L = J_\varphi\mathcal{Q}_L^*\big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus \overline{f_t(\pi)}\mathbb{1}_{\mathsf{H}_\pi}\otimes (E_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L J_\varphi.$$

The last equation can be derived from equation (3.13) in a similar manner. Clearly we have $f_t(\pi)=f(\pi)^{it}$ for a measurable function $f\colon \operatorname{Irr}(\mathbb{G})\to\mathbb{R}_{>0}$. ☐

In the last part of this section, we will transport operators $\nabla_{\widehat{\varphi}}^{it},\nabla_{\widehat{\psi}}^{it},\hat{\delta}^{it}\,(t\in\mathbb{R})$ to $\int_{\operatorname{Irr}(\mathbb{G})}^\oplus \operatorname{HS}(\mathsf{H}_\pi)\,\mathrm{d}\mu(\pi)$. We start with a formula expressing the action of $(\tau_t)_{t\in\mathbb{R}}$ on matrix coefficients introduced in Definition 3.8. Recall that $\hat{\delta}_u$ is the modular element of $\widehat{\mathbb{G}}$ living in the universal level, i.e. it is a positive self-adjoint operators affiliated with $\mathrm{C}_0^u(\widehat{\mathbb{G}})$.

Lemma 3.23. *For $\xi,\eta\in\int_{\operatorname{Irr}(\mathbb{G})}^\oplus \mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$ and $t\in\mathbb{R}$ we have*

$$\begin{aligned}\tau_t(M^L_{\xi,\eta}) &= \nu^{-\frac12 it^2}\delta^{-it}\int_{\operatorname{Irr}(\mathbb{G})}(\mathrm{id}\otimes\omega_{D_\pi^{2it}\pi(\hat{\delta}_u^{-it})\xi_\pi,E_\pi^{2it}\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)\\ &= \nu^{-\frac12 it^2}\int_{\operatorname{Irr}(\mathbb{G})}(\mathrm{id}\otimes\omega_{D_\pi^{-2it}\xi_\pi,E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)\delta^{it},\\ \tau_t(M^R_{\xi,\eta}) &= \nu^{\frac12 it^2}\int_{\operatorname{Irr}(\mathbb{G})}(\mathrm{id}\otimes\omega_{E_\pi^{2it}\xi_\pi,D_\pi^{2it}\pi(\hat{\delta}_u^{-it})\eta_\pi})(U^{\pi})\,\mathrm{d}\mu(\pi)\delta^{it}\\ &= \nu^{\frac12 it^2}\delta^{-it}\int_{\operatorname{Irr}(\mathbb{G})}(\mathrm{id}\otimes\omega_{E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\xi_\pi,D_\pi^{-2it}\eta_\pi})(U^{\pi})\,\mathrm{d}\mu(\pi).\end{aligned}$$

Later on in Theorem 3.25 we will get simpler expressions for these actions (once we find out what $\pi(\hat{\delta}_u^{it})$ is).

Proof. The proof is based on several facts from the theory of locally compact quantum groups. First of all, we know that $\hat{\delta}^{it} = P^{-it}\nabla_{\psi}^{-it}$ (equation (2.14)). Next, [93, Lemma 5.14] gives us

$$(\sigma_t^{\varphi} \otimes \mathrm{id})\mathrm{W} = (\mathbb{1} \otimes P^{-it})\mathrm{W}(\mathbb{1} \otimes \nabla_{\psi}^{-it}), \quad (\sigma_t^{\psi} \otimes \mathrm{id})\mathrm{W} = (\mathbb{1} \otimes \nabla_{\psi}^{-it})\mathrm{W}(\mathbb{1} \otimes P^{-it}),$$

and $(\tau_t \otimes \mathrm{id})\mathrm{W} = (\mathrm{id} \otimes \hat{\tau}_{-t})\mathrm{W}$. We note also that $\hat{\delta}^{it} \in \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}}))$, $\hat{\delta}_u^{it} \in \mathrm{M}(\mathrm{C}_0^u(\widehat{\mathbb{G}}))$ and $\Lambda_{\widehat{\mathbb{G}}}(\hat{\delta}_u^{it}) = \hat{\delta}^{it}$ ([55]). Fix $t \in \mathbb{R}$, a representation $\pi \in \mathrm{Irr}(\mathbb{G})$ which factorises through $\mathrm{C}_0(\widehat{\mathbb{G}})$ (i.e. $\pi = \pi' \circ \Lambda_{\widehat{\mathbb{G}}}$ for a representation $\pi'\colon \mathrm{C}_0(\widehat{\mathbb{G}}) \to \mathrm{B}(\mathsf{H}_\pi)$) and arbitrary vectors $\xi_\pi, \eta_\pi \in \mathsf{H}_\pi$. We have

$$\begin{aligned}
&\tau_t((\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi})(U^{\pi *})) = (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi})(\tau_t \otimes \mathrm{id})(\mathrm{id} \otimes \pi)(\mathbb{W}^*)\\
&= (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi')(\tau_t \otimes \mathrm{id})(\mathrm{W}^*) = (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi')(\mathrm{id} \otimes \hat{\tau}_{-t})(\mathrm{W}^*)\\
&= (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi')((\mathbb{1} \otimes P^{-it})(\mathrm{W}^*)(\mathbb{1} \otimes P^{it})).
\end{aligned}$$

Now we write the above expression in two different ways: we have

$$\begin{aligned}
&\tau_t((\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi})(U^{\pi *})) = (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi')((\mathbb{1} \otimes P^{-it}\nabla_{\psi}^{-it})(\mathbb{1} \otimes \nabla_{\psi}^{it})(\mathrm{W}^*)(\mathbb{1} \otimes P^{it}))\\
&= (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi')((\mathbb{1} \otimes \hat{\delta}^{it})\,(\sigma_t^{\varphi} \otimes \mathrm{id})(\mathrm{W}^*)) = \sigma_t^{\varphi}((\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi)((\mathbb{1} \otimes \hat{\delta}_u^{it})\,(\mathbb{W}^*)))\\
&= \sigma_t^{\varphi}((\mathrm{id} \otimes \omega_{\pi(\hat{\delta}_u^{-it})\xi_\pi,\eta_\pi})(U^{\pi *}))
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\tau_t((\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi})(U^{\pi *})) = (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi')((\mathbb{1} \otimes P^{-it})(\mathrm{W}^*)(\mathbb{1} \otimes \nabla_{\psi}^{-it})(\mathbb{1} \otimes \nabla_{\psi}^{it}P^{it}))\\
&= (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi')((\sigma_{-t}^{\psi} \otimes \mathrm{id})(\mathrm{W}^*)\,(\mathbb{1} \otimes \hat{\delta}^{-it}))\\
&= (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi} \circ \pi)((\sigma_{-t}^{\psi} \otimes \mathrm{id})(\mathbb{W}^*)\,(\mathbb{1} \otimes \hat{\delta}_u^{-it}))\\
&= (\mathrm{id} \otimes \omega_{\xi_\pi,\pi(\hat{\delta}_u^{-it})\eta_\pi})((\sigma_{-t}^{\psi} \otimes \mathrm{id})(U^{\pi *})) = \sigma_{-t}^{\psi}((\mathrm{id} \otimes \omega_{\xi_\pi,\pi(\hat{\delta}_u^{-it})\eta_\pi})(U^{\pi *})).
\end{aligned} \tag{3.15}$$

Let now ξ, η be vectors in $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathsf{H}_\pi \,\mathrm{d}\mu(\pi)$. Then fields $(\pi(\hat{\delta}_u^{-it})\xi_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})}$, $(\pi(\hat{\delta}_u^{-it})\eta_\pi)_{\pi \in \mathrm{Irr}(\mathbb{G})}$ are also measurable and square integrable. Using equations (3.14), (3.15) and Proposition 3.12 we arrive at

$$\begin{aligned}
&\tau_t(M_{\xi,\eta}^L) = \tau_t(\int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)) = \int_{\mathrm{Irr}(\mathbb{G})} \tau_t((\mathrm{id} \otimes \omega_{\xi_\pi,\eta_\pi})(U^{\pi *}))\,\mathrm{d}\mu(\pi)\\
&= \int_{\mathrm{Irr}(\mathbb{G})} \sigma_t^{\varphi}((\mathrm{id} \otimes \omega_{\pi(\hat{\delta}_u^{-it})\xi_\pi,\eta_\pi})(U^{\pi *}))\,\mathrm{d}\mu(\pi) = \sigma_t^{\varphi}(\int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id} \otimes \omega_{\pi(\hat{\delta}_u^{-it})\xi_\pi,\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi))\\
&= \nu^{-\frac{1}{2}it^2}\delta^{-it} \int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id} \otimes \omega_{D_\pi^{2it}\pi(\hat{\delta}_u^{-it})\xi_\pi,E_\pi^{2it}\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)
\end{aligned}$$

and

$$\tau_t(M^L_{\xi,\eta}) = \tau_t\big(\int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id}\otimes\omega_{\xi_\pi,\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)\big) = \int_{\mathrm{Irr}(\mathbb{G})} \tau_t((\mathrm{id}\otimes\omega_{\xi_\pi,\eta_\pi})(U^{\pi *}))\,\mathrm{d}\mu(\pi)$$
$$= \int_{\mathrm{Irr}(\mathbb{G})} \sigma^{\psi}_{-t}((\mathrm{id}\otimes\omega_{\xi_\pi,\pi(\hat\delta_u^{-it})\eta_\pi})(U^{\pi *}))\,\mathrm{d}\mu(\pi) = \sigma^{\psi}_{-t}\big(\int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id}\otimes\omega_{\xi_\pi,\pi(\hat\delta_u^{-it})\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)\big)$$
$$= \nu^{-\frac{1}{2}it^2}\int_{\mathrm{Irr}(\mathbb{G})} (\mathrm{id}\otimes\omega_{D_\pi^{-2it}\xi_\pi,E_\pi^{-2it}\pi(\hat\delta_u^{-it})\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)\delta^{it}.$$

The second pair of equations follow by applying the adjoint. □

Now we are ready to obtain the main results of this section. Even though we will prove them together, they are of different nature, hence we prefer to state them separately. First, we have a couple of equations expressing important operators on the level of direct integrals.

Theorem 3.24. *For every $t\in\mathbb{R}$ we have*

$$\begin{aligned}\nabla_{\hat\psi}^{-it} = \delta^{it}P^{it} &= \mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{2it}\otimes(E_\pi^{-2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L\\ &= \mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{-2it}\otimes(D_\pi^{2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R,\\ \nabla_{\hat\varphi}^{it} = J_\varphi\delta^{it}P^{it}J_\varphi &= \mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{-2it}\otimes(D_\pi^{2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L\\ &= \mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{2it}\otimes(E_\pi^{-2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R.\end{aligned}$$

Next, we show that the modular element for $\widehat{\mathbb{G}}$ can be expressed using operators $(D_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})},(E_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$.

Theorem 3.25. *For all $t\in\mathbb{R}$ we have*

$$\begin{aligned}\hat\delta^{it} &= \nu^{-\frac{i}{2}t^2}\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{2it}E_\pi^{-2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L\\ &= \nu^{-\frac{i}{2}t^2}\mathcal{Q}_R^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}_\pi}\otimes(D_\pi^{-2it}E_\pi^{2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_R.\end{aligned}$$

Moreover, $\pi(\hat\delta_u^{it}) = \nu^{\frac{it^2}{2}}E_\pi^{-2it}D_\pi^{2it}$ and $\nu^{ist}D_\pi^{2is}E_\pi^{2it} = E_\pi^{2it}D_\pi^{2is}$ for all $s,t\in\mathbb{R}$ and almost all $\pi\in\mathrm{Irr}(\mathbb{G})$. We also get better expressions for the action of $(\tau_t)_{t\in\mathbb{R}}$ on matrix coefficients:

$$\begin{aligned}\tau_t(M^L_{\xi,\eta}) &= \delta^{-it}M^L_{E^{2it}\xi,E^{2it}\eta} = M^L_{D^{-2it}\xi,D^{-2it}\eta}\delta^{it},\\ \tau_t(M^R_{\xi,\eta}) &= M^R_{E^{2it}\xi,E^{2it}\eta}\,\delta^{it} = \delta^{-it}\,M^R_{D^{-2it}\xi,D^{-2it}\eta}\end{aligned}$$

for all $t\in\mathbb{R}$ and $\xi,\eta\in\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathsf{H}_\pi\,\mathrm{d}\mu(\pi)$.

Proof. Let ξ, η be vector fields satisfying conditions from the first point of Lemma 3.9. Note that vector fields $(D_\pi^{-2it}\xi_\pi)_{\pi\in\operatorname{Irr}(\mathbb{G})}, (E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\eta_\pi)_{\pi\in\operatorname{Irr}(\mathbb{G})}$ also satisfy conditions of this lemma. Using the second equation from Lemma 3.23 we get:

$$\begin{aligned}
\mathcal{Q}_L P^{it}\Lambda_\varphi(M^L_{\xi,\eta}) &= \nu^{\frac{t}{2}}\mathcal{Q}_L\Lambda_\varphi(\tau_t(M^L_{\xi,\eta}))\\
&= \nu^{\frac{t-it^2}{2}}\mathcal{Q}_L\Lambda_\varphi\Big(\int_{\operatorname{Irr}(\mathbb{G})}(\mathrm{id}\otimes\omega_{D_\pi^{-2it}\xi_\pi, E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)\,\delta^{it}\Big)\\
&= \nu^{\frac{t-it^2}{2}}\mathcal{Q}_L J_\varphi\sigma^\varphi_{i/2}(\delta^{it})^* J_\varphi\Lambda_\varphi\Big(\int_{\operatorname{Irr}(\mathbb{G})}(\mathrm{id}\otimes\omega_{D_\pi^{-2it}\xi_\pi, E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\eta_\pi})(U^{\pi *})\,\mathrm{d}\mu(\pi)\Big)\\
&= \nu^{\frac{-it^2}{2}}\mathcal{Q}_L J_\varphi\delta^{-it}J_\varphi\mathcal{Q}_L^*\int^\oplus_{\operatorname{Irr}(\mathbb{G})} E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\eta_\pi\otimes\overline{D_\pi D_\pi^{-2it}\xi_\pi}\,\mathrm{d}\mu(\pi)\\
&= \nu^{\frac{-it^2}{2}}\mathcal{Q}_L J_\varphi\delta^{-it}J_\varphi\mathcal{Q}_L^*\Big(\int^\oplus_{\operatorname{Irr}(\mathbb{G})} E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L\Lambda_\varphi(M^L_{\xi,\eta}).
\end{aligned}$$

Since the set of $\Lambda_\varphi(M^L_{\xi,\eta})$ with ξ,η as above is lineary dense in $\mathrm{L}^2(\mathbb{G})$ (Lemma 3.10), we get

$$J_\varphi\delta^{it}J_\varphi P^{it} = \mathcal{Q}_L^*\Big(\int^\oplus_{\operatorname{Irr}(\mathbb{G})}(\nu^{-\frac{it^2}{2}}E_\pi^{-2it}\pi(\hat{\delta}_u^{-it}))\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L. \tag{3.16}$$

Since $(J_\varphi\delta^{it}J_\varphi P^{it})_{t\in\mathbb{R}}$, $((D_\pi^{2it})^\top)_{t\in\mathbb{R}}$ are strongly continuous groups (see equation (3.7)), the same is true for $(\nu^{-\frac{it^2}{2}}E_\pi^{-2it}\pi(\hat{\delta}_u^{-it}))_{t\in\mathbb{R}}$. Using relations gathered in equation (3.7) one easily checks that $J_{\widehat{\varphi}}$ commutes with $J_\varphi\delta^{it}J_\varphi P^{it}$. Since $J_{\widehat{\varphi}} = \mathcal{Q}_L^*\Sigma\mathcal{Q}_L$, Lemma 7.8 implies

$$\nu^{-\frac{it^2}{2}}E_\pi^{-2it}\pi(\hat{\delta}_u^{-it}) = D_\pi^{-2it} \quad\Rightarrow\quad \pi(\hat{\delta}_u^{it}) = \nu^{\frac{it^2}{2}}E_\pi^{-2it}D_\pi^{2it} \quad (\pi\in\operatorname{Irr}(\mathbb{G}), t\in\mathbb{R}) \tag{3.17}$$

Let us choose $s,t\in\mathbb{R}$ and use the fact that $(\pi(\hat{\delta}_u^{ip}))_{p\in\mathbb{R}}$ is a group: we have

$$\nu^{\frac{i(t+s)^2}{2}}E_\pi^{-2i(t+s)}D_\pi^{2i(t+s)} = \pi(\hat{\delta}_u^{i(t+s)}) = \pi(\hat{\delta}_u^{it})\pi(\hat{\delta}_u^{is}) = \nu^{\frac{it^2}{2}}E_\pi^{-2it}D_\pi^{2it}\nu^{\frac{is^2}{2}}E_\pi^{-2is}D_\pi^{2is},$$

and formula $\nu^{ist}E_\pi^{-2is}D_\pi^{2it} = D_\pi^{2it}E_\pi^{-2is}$ easily follows. Equations expressing the action of $(\tau_t)_{t\in\mathbb{R}}$ on matrix coefficients follows from the equation $\pi(\hat{\delta}_u^{it}) = \nu^{\frac{i}{2}t^2}E_\pi^{-2it}D_\pi^{2it}$, commutation relation between E_π^{it} and D_π^{is} and Lemma 3.23. Let us now plug in the above results to equation (3.16):

$$\begin{aligned}
J_\varphi\delta^{it}J_\varphi P^{it} &= \nu^{-\frac{it^2}{2}}\mathcal{Q}_L^*\Big(\int^\oplus_{\operatorname{Irr}(\mathbb{G})}E_\pi^{-2it}\pi(\hat{\delta}_u^{-it})\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L\\
&= \nu^{-\frac{it^2}{2}}\mathcal{Q}_L^*\Big(\int^\oplus_{\operatorname{Irr}(\mathbb{G})}E_\pi^{-2it}\nu^{\frac{it^2}{2}}E_\pi^{2it}D_\pi^{-2it}\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L\\
&= \mathcal{Q}_L^*\Big(\int^\oplus_{\operatorname{Irr}(\mathbb{G})}D_\pi^{-2it}\otimes(D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L,
\end{aligned}\tag{3.18}$$

which is the third equation of Theorem 3.24. If we use formula $\mathcal{Q}_R^*\mathcal{Q}_L = \nu^{-\frac{i}{8}}J_{\widehat{\varphi}}J_\varphi$, we readily get the second equation. Now we can derive the first pair of equations of Theorem

3.25. Since for all $t \in \mathbb{R}$ we have $\nabla_{\psi}^{it} = \hat{\delta}^{-it} P^{-it}$ and $J_{\varphi}\hat{\delta}^{it} = \hat{\delta}^{it} J_{\varphi}$, it follows that $\hat{\delta}^{it} = J_{\varphi}\hat{\delta}^{it} J_{\varphi} = (J_{\varphi}P^{-it}\delta^{-it}J_{\varphi})(J_{\varphi}\delta^{it}\nabla_{\psi}^{-it}J_{\varphi})$, which we can express using equation (3.18) and Proposition 3.21:

$$\begin{aligned}\mathcal{Q}_L\hat{\delta}^{it}\mathcal{Q}_L^* &= \Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{2it} \otimes (D_{\pi}^{-2it})^{\top} \,\mathrm{d}\mu(\pi)\Big)\nu^{-\frac{i}{2}t^2}\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} E_{\pi}^{-2it} \otimes (D_{\pi}^{2it})^{\top} \,\mathrm{d}\mu(\pi)\Big)\\ &= \nu^{-\frac{i}{2}t^2}\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{2it}E_{\pi}^{-2it} \otimes \mathbb{1}_{\overline{\mathsf{H}_{\pi}}} \,\mathrm{d}\mu(\pi).\end{aligned}$$

On the other hand, we also have $\hat{\delta}^{it} = (\nabla_{\psi}^{-it}\delta^{it})(\delta^{-it}P^{-it})$, hence

$$\mathcal{Q}_R\hat{\delta}^{it}\mathcal{Q}_R^* = \nu^{-\frac{i}{2}t^2}\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{-2it} \otimes (E_{\pi}^{2it})^{\top} \,\mathrm{d}\mu(\pi)\Big)\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{2it} \otimes (D_{\pi}^{-2it})^{\top} \,\mathrm{d}\mu(\pi)\Big),$$

which implies the second equation for $\hat{\delta}^{it}$ and ends the proof of Theorem 3.25. In order to finish the proof of Theorem 3.24 we have to derive a lemma concerning the function f introduced in Corollary 3.22.

Lemma 3.26. *For all $t \in \mathbb{R}$ we have*

$$\begin{aligned}J_{\varphi}\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}\,\mathbb{1}_{\mathrm{HS}(\mathsf{H}_{\pi})}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_L J_{\varphi} &= \mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{-it}\,\mathbb{1}_{\mathrm{HS}(\mathsf{H}_{\pi})}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L,\\ J_{\varphi}\mathcal{Q}_R^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}\,\mathbb{1}_{\mathrm{HS}(\mathsf{H}_{\pi})}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_R J_{\varphi} &= \mathcal{Q}_R^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{-it}\,\mathbb{1}_{\mathrm{HS}(\mathsf{H}_{\pi})}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_R.\end{aligned}$$

Proof of Lemma 3.26. Recall that $J_{\varphi}\hat{\delta}^{it}J_{\varphi} = \hat{\delta}^{it}$, hence

$$\nu^{\frac{i}{2}t^2}J_{\varphi}\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{2it}E_{\pi}^{-2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_{\pi}}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L J_{\varphi} = \nu^{-\frac{i}{2}t^2}\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{2it}E_{\pi}^{-2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_{\pi}}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L.$$

Using the above relation, the fourth equation of Corollary 3.22 and the commutation relation $\nu^{ist}D_{\pi}^{2is}E_{\pi}^{2it} = E_{\pi}^{2it}D_{\pi}^{2is}$ we get

$$\begin{aligned}&J_{\varphi}\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_{\pi}}}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_L J_{\varphi}\\ &= J_{\varphi}\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{-2it}E_{\pi}^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_{\pi}}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L J_{\varphi}\,J_{\varphi}\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} E_{\pi}^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_{\pi}}}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_L J_{\varphi}\\ &= \nu^{-it^2}\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} D_{\pi}^{-2it}E_{\pi}^{2it}\otimes\mathbb{1}_{\mathsf{H}_{\pi}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L\,\mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}D_{\pi}^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_{\pi}}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L\\ &= \mathcal{Q}_L^*\Big(\int_{\operatorname{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}E_{\pi}^{2it}\otimes\mathbb{1}_{\overline{\mathsf{H}_{\pi}}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L,\end{aligned}$$

consequently

$$\mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L = J_\varphi\Big(J_\varphi\mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_L J_\varphi\Big)^* J_\varphi$$

$$= J_\varphi\mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it} D_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_L J_\varphi$$

$$= \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it} E_\pi^{2it}\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L J_\varphi \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\pi)}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_L J_\varphi$$

and

$$J_\varphi\mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\pi)}\,\mathrm{d}\mu(\pi)\Big)^*\mathcal{Q}_L J_\varphi = \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{-it}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\pi)}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L.$$

The second equation can be proved analogously or using equation $\mathcal{Q}_R^*\mathcal{Q}_L = \nu^{-\frac{i}{8}} J_{\widehat{\varphi}} J_\varphi$. □

Using the above lemma and Corollary 3.22 we can derive the first equation of Theorem 3.24 from the third one:

$$\delta^{it}P^{it} = J_\varphi J_\varphi \delta^{it}P^{it} J_\varphi J_\varphi = J_\varphi \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{-2it}\otimes (D_\pi^{2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L J_\varphi$$

$$= J_\varphi\mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\pi)}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L J_\varphi J_\varphi \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{-it} D_\pi^{-2it}\otimes (D_\pi^{2it})^\top \mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L J_\varphi$$

$$= \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{it}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\pi)}\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} f(\pi)^{-it} E_\pi^{2it}\otimes (E_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L$$

$$= \mathcal{Q}_L^*\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{2it}\otimes (E_\pi^{-2it})^\top\,\mathrm{d}\mu(\pi)\Big)\mathcal{Q}_L.$$

Now, the last equation of Theorem 3.24 follows as usual from the formula relating $\mathcal{Q}_L$ and $\mathcal{Q}_R$ (Corollary 3.20). This concludes the proof of Theorem 3.24 and Theorem 3.25. □

The commutation relation $\nu^{ist} D_\pi^{2is} E_\pi^{2it} = E_\pi^{2it} D_\pi^{2is}$ $(t, s \in \mathbb{R})$ derived in the previous proposition has the following consequence.

Corollary 3.27. If $\nu \neq 1$ then for almost all $\pi \in \mathrm{Irr}(\mathbb{G})$, operators D_π, E_π have empty point spectrum. In particular, if $\nu \neq 1$ then the set of finite dimensional irreducible representations is of measure zero.

3.3 Special classes of type I locally compact quantum groups

When $\mathbb{G}$ is a compact quantum group, then a number of conditions related to the modular theory of $\mathbb{G}, \widehat{\mathbb{G}}$ turn out to be equivalent. For example, $\widehat{\mathbb{G}}$ is unimodular if and only if the Haar integrals on $\widehat{\mathbb{G}}$ are tracial, which happens if and only if the Haar integral on $\mathbb{G}$ is

tracial [64]. In fact, these properties are governed by the family of operators $(\rho_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}$ (see Section 2.3). In the main result of this section (Theorem 3.34) we will obtain similar, though more complicated, characterisation for type I, second countable locally compact quantum groups.
We start with presenting some results concerning modular theory of a general locally compact quantum group $\mathbb{G}$ and its dual. These results are probably well known to the experts.

Lemma 3.28. *The following conditions are equivalent:*

1) $P^{it} \in \mathrm{L}^\infty(\mathbb{G})'$ *for all* $t \in \mathbb{R}$,

2) *the scaling group of* $\mathbb{G}$ *is trivial,*

3) $P^{it} = \mathbb{1}$ *for all* $t \in \mathbb{R}$.

Proof. Implications 1) $\Leftrightarrow$ 2) $\Leftarrow$ 3) follow from the equation $\tau_t(x) = P^{it}xP^{-it}$ $(x \in \mathrm{L}^\infty(\mathbb{G}),$ $t \in \mathbb{R})$. For all $x \in \mathfrak{N}_\varphi$ and $t \in \mathbb{R}$ we have $P^{it}\Lambda_\varphi(x) = \nu^{\frac{t}{2}}\Lambda_\varphi(\tau_t(x))$, hence 2) implies $P^{it} = \nu^{\frac{t}{2}}\mathbb{1}$. Taking the norm of both sides gives us $1 = \nu^{\frac{t}{2}}$ hence $\nu = 1$. $\square$

Lemma 3.29.

1) *The Haar integrals on* $\mathbb{G}$ *are tracial if, and only if* $P = \hat{\delta} = \mathbb{1}$.

2) $\widehat{\mathbb{G}}$ *is unimodular if, and only if* $\nabla_\varphi^{it} = \nabla_\psi^{-it}$ $(t \in \mathbb{R})$.

Proof. We will use formulas gathered in equation (2.14). Equality $\nabla_\psi^{it} = \hat{\delta}^{-it}P^{-it}$ $(t \in \mathbb{R})$ shows that $P = \hat{\delta} = \mathbb{1}$ implies $\nabla_\psi^{it} = \mathbb{1}$ and the traciality of ψ. Then φ is tracial because $\nabla_\varphi^{it} = J_{\widehat{\varphi}}\nabla_\psi^{-it}J_{\widehat{\varphi}}$. Let us prove the converse implication. If $\nabla_\psi^{it} = \mathbb{1}$ then $P^{it} = \hat{\delta}^{-it} \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$ for all $t \in \mathbb{R}$. Since P^{it} commutes with $J_{\widehat{\varphi}}$, we have $P^{it} = J_{\widehat{\varphi}}P^{it}J_{\widehat{\varphi}} \in \mathrm{L}^\infty(\widehat{\mathbb{G}})'$ and by the previous lemma $P^{it} = \mathbb{1} = \hat{\delta}^{-it}$.
If $\widehat{\mathbb{G}}$ is unimodular, then we have $J_{\widehat{\varphi}}\nabla_\varphi^{-it}J_{\widehat{\varphi}} = \nabla_\psi^{it} = P^{-it}$ for all $t \in \mathbb{R}$. Since P^{-it} commutes with $J_{\widehat{\varphi}}$, it follows that $\nabla_\psi^{it} = \nabla_\varphi^{-it}$. On the other hand, if $\nabla_\psi^{it} = \nabla_\varphi^{-it}$ for all $t \in \mathbb{R}$, then

$$\hat{\delta}^{-it}P^{-it} = \nabla_\psi^{it} = \nabla_\varphi^{-it} = J_{\widehat{\varphi}}\nabla_\psi^{it}J_{\widehat{\varphi}} = J_{\widehat{\varphi}}\hat{\delta}^{-it}P^{-it}J_{\widehat{\varphi}} = J_{\widehat{\varphi}}\hat{\delta}^{-it}J_{\widehat{\varphi}}P^{-it}$$

and we get $\hat{\delta}^{it} = J_{\widehat{\varphi}}\hat{\delta}^{it}J_{\widehat{\varphi}}$. This in particular means that $\hat{\delta}^{it} \in \mathcal{Z}(\mathrm{L}^\infty(\widehat{\mathbb{G}}))$ and [82, Proposition 1.23] implies $\hat{\delta}^{it} = J_{\widehat{\varphi}}\hat{\delta}^{-it}J_{\widehat{\varphi}}$, unimodularity of $\widehat{\mathbb{G}}$ follows. $\square$

Lemma 3.30. *For all* $t, s \in \mathbb{R}$, *if* $\sigma_t^\varphi = \sigma_s^\psi$ *then* $\nabla_\varphi^{it} = \nabla_\psi^{is}$. *If* $(s,t) \neq (0,0)$ *then also* $\nu = 1$.

Proof. For all $x \in \mathfrak{N}_\varphi$ we have $\nabla_\psi^{-is}\nabla_\varphi^{it}\Lambda_\varphi(x) = \nu^{\frac{1}{2}s}\Lambda_\varphi(\sigma_{-s}^\psi(\sigma_t^\varphi(x))) = \nu^{\frac{1}{2}s}\Lambda_\varphi(x)$ (see [93, Remark 5.2 ii)]), hence $\nabla_\psi^{-is}\nabla_\varphi^{it} = \nu^{\frac{1}{2}s}\mathbb{1}$. Taking the norm of both sides implies $\nu^{\frac{1}{2}s} = 1$ and proves the first claim. If $s \neq 0$ then we get $\nu = 1$, if $s = 0$ and $(s,t) \neq (0,0)$ then $t \neq 0$ and we get $\nabla_\varphi^{it} = \mathbb{1}$. Formula $\nabla_\varphi^{it}\Lambda_\psi(y) = \nu^{\frac{t}{2}}\Lambda_\psi(\sigma_t^\varphi(y)) = \nu^{\frac{t}{2}}\Lambda_\psi(y)$ $(y \in \mathfrak{N}_\psi)$ implies $\nu = 1$. $\square$

Lemma 3.31. *For all $t \in \mathbb{R}$, $\delta^{it} \in \mathcal{Z}(\mathrm{L}^\infty(\mathbb{G}))$ if and only of $\sigma_t^\varphi = \sigma_t^\psi$. If these conditions hold, then $\nu^{it} = 1$.*

Proof. The first part of the result is a consequence of the formula $\sigma_t^\psi(x) = \delta^{it}\sigma_t^\varphi(x)\delta^{-it}$ ($x \in \mathrm{L}^\infty(\mathbb{G})$) – see equation (2.4). For the second part, observe that when $\delta^{it} \in \mathcal{Z}(\mathrm{L}^\infty(\mathbb{G}))$, we have $\delta^{it} = \sigma_s^\varphi(\delta^{it}) = \nu^{ist}\delta^{it}$ ($s \in \mathbb{R}$) by equation (2.9). It follows that $\nu^{ist} = 1$ for all $s \in \mathbb{R}$, hence $\nu^{it} = 1$. □

For unimodular quantum groups we obtain another useful piece of information (cf. equations (2.20), (2.21)):

Proposition 3.32. *If $\mathbb{G}$ is unimodular, then*

$$\sigma_t^{\widehat{\varphi}}(x) = \hat{\tau}_t(x) = \hat{\delta}^{-\frac{it}{2}} x \hat{\delta}^{\frac{it}{2}}, \quad \sigma_t^{\widehat{\psi}}(x) = \hat{\delta}^{\frac{it}{2}} x \hat{\delta}^{-\frac{it}{2}}$$

and

$$\Delta_{\widehat{\mathbb{G}}}(\sigma_t^{\widehat{\varphi}}(x)) = (\sigma_t^{\widehat{\varphi}} \otimes \sigma_t^{\widehat{\varphi}})\Delta_{\widehat{\mathbb{G}}}(x), \quad \Delta_{\widehat{\mathbb{G}}}(\sigma_t^{\widehat{\psi}}(x)) = (\sigma_t^{\widehat{\psi}} \otimes \sigma_t^{\widehat{\psi}})\Delta_{\widehat{\mathbb{G}}}(x)$$

for all $t \in \mathbb{R}, x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$.

Proof. This proposition is a straightforward consequence of $P^{-2it} = \delta^{it}(J_\varphi \delta^{it} J_\varphi)\hat{\delta}^{it}(J_{\widehat{\varphi}}\hat{\delta}^{it}J_{\widehat{\varphi}})$, $\nabla_{\widehat{\varphi}}^{it} = J_\varphi \delta^{it} P^{it} J_\varphi$ (equation (2.14)) and $\Delta_{\widehat{\mathbb{G}}}(\hat{\delta}^{it}) = \hat{\delta}^{it} \otimes \hat{\delta}^{it}$ (Section 2.2). □

Proposition 3.33. *The Haar integrals on $\mathbb{G}$ and $\widehat{\mathbb{G}}$ are tracial if, and only if $\mathbb{G}$ and $\widehat{\mathbb{G}}$ are unimodular.*

We remark that in [49] this result was stated as a corollary of Theorem 3.34 hence only for type I, second countable locally compact quantum groups. However, we have realised that this assumption is superfluous.

Proof. Implication "⇒" is an easy corollary of Lemma 3.29. Assume that $\mathbb{G}$ and $\widehat{\mathbb{G}}$ are unimodular. Using formulas from equation (2.14) we arrive at

$$P^{-it} = \nabla_\psi^{it} = J_{\widehat{\varphi}} \nabla_\varphi^{-it} J_{\widehat{\varphi}} = J_{\widehat{\varphi}} \nabla_\psi^{-it} J_{\widehat{\varphi}} = J_{\widehat{\varphi}} P^{it} J_{\widehat{\varphi}} = P^{it},$$

hence $P = \delta = \hat{\delta} = \mathbb{1}$ and Lemma 3.29 gives us the claim. □

The next theorem is the main result of this section. It presents a web of connections between various properties of a type I, second countable locally compact quantum group (and its dual).

Theorem 3.34. *Let $\mathbb{G}$ be a second countable, type I locally compact quantum group. Consider the following conditions:*

1) $D_\pi^{it} \in \mathbb{C}\mathbb{1}_{\mathsf{H}_\pi}$ for all $t \in \mathbb{R}$ and almost all $\pi \in \mathrm{Irr}(\mathbb{G})$,

2) $E_\pi^{it} \in \mathbb{C}\mathbb{1}_{\mathsf{H}_\pi}$ for all $t \in \mathbb{R}$ and almost all $\pi \in \mathrm{Irr}(\mathbb{G})$, *3) the Haar integrals on $\widehat{\mathbb{G}}$ are tracial (left $\Leftrightarrow$ right $\Leftrightarrow$ both),*

4) $\hat{\delta}^{it} \in \mathcal{Z}(\mathrm{L}^\infty(\widehat{\mathbb{G}}))$ for all $t \in \mathbb{R}$,

5) $\mathbb{G}$ is unimodular,

6) $E_\pi^{it} D_\pi^{-it} \in \mathbb{C}\mathbb{1}_{\mathsf{H}_\pi}$ for all $t \in \mathbb{R}$ and almost all π,

7) $E_\pi^{it} = D_\pi^{it}$ for all $t \in \mathbb{R}$ and almost all $\pi \in \operatorname{Irr}(\mathbb{G})$,

8) $\widehat{\mathbb{G}}$ is unimodular,

9) $E_\pi^{it} D_\pi^{it} \in \mathbb{C}\mathbb{1}_{\mathsf{H}_\pi}$ for all $t \in \mathbb{R}$ and almost all $\pi \in \operatorname{Irr}(\mathbb{G})$,

Then the following implications hold:

$$\begin{tikzcd}
1) \arrow[r, Leftrightarrow] \arrow[dd, Rightarrow] & 2) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & 3) & & \\
& 5) \arrow[r, Leftrightarrow] & 9) & 7) \arrow[r, Leftrightarrow] & 8) \arrow[d, Rightarrow] \\
6) \arrow[rrrr, Leftrightarrow] & & & & 4)
\end{tikzcd}$$

Moreover, each of the above conditions implies $\nu = 1$.

Proof. First, let us note that φ is tracial if and only ψ is tracial: it is a consequence of the equation $\nabla_\psi^{it} = J_{\widehat{\varphi}} \nabla_\varphi^{-it} J_{\widehat{\varphi}}\,(t \in \mathbb{R})$. An analogous result holds for $\widehat{\varphi}$ and $\widehat{\psi}$. Equivalence 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) is a part of the Desmedt's theorem, one can also deduce this from formulas for $\nabla_{\widehat{\varphi}}, \nabla_{\widehat{\psi}}$ – see Theorem 3.24. Equivalence 6) $\Leftrightarrow$ 4) follows from the formula for $\hat{\delta}^{it}$ in Theorem 3.25 and $\mathcal{Q}_L \mathrm{L}^\infty(\widehat{\mathbb{G}}) \mathcal{Q}_L^* = \int_{\operatorname{Irr}(\mathbb{G})}^\oplus \mathrm{B}(\mathsf{H}_\pi) \otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}} \,\mathrm{d}\mu(\pi)$ (see Proposition 3.7). Equivalence 7) $\Leftrightarrow$ 8) is a straightforward consequence of Theorem 3.25.
Assume 5), i.e. that $\mathbb{G}$ is unimodular and let us derive 9). Fix $t \in \mathbb{R}$. Theorem 3.24 gives us

$$P^{it} = \mathcal{Q}_L^* \Big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus E_\pi^{2it} \otimes (E_\pi^{-2it})^\top \,\mathrm{d}\mu(\pi) \Big) \mathcal{Q}_L = \mathcal{Q}_L^* \Big(\int_{\operatorname{Irr}(\mathbb{G})}^\oplus D_\pi^{-2it} \otimes (D_\pi^{2it})^\top \,\mathrm{d}\mu(\pi) \Big) \mathcal{Q}_L,$$

which implies $E_\pi^{2it} \otimes (E_\pi^{-2it})^\top = D_\pi^{-2it} \otimes (D_\pi^{2it})^\top \,(\pi \in \operatorname{Irr}(\mathbb{G}))$. Consequently, $D_\pi^{2it} E_\pi^{2it} S = S D_\pi^{2it} E_\pi^{2it}$ for all $S \in \mathrm{HS}(\mathsf{H}_\pi)$. This means that $D_\pi^{2it} E_\pi^{2it} = \lambda_t \mathbb{1}_{\mathsf{H}_\pi}$ for some $\lambda_t \in \mathbb{C}$ and we arrive at the point 9). On the other hand, point 9) implies that there exists $\lambda_{t,\pi} \in \mathbb{T}$ such that $E_\pi^{it} = \lambda_{t,\pi} D_\pi^{-it}$. It follows that $\nu = 1$ and $\lambda_{-t,\pi} = {\lambda_{t,\pi}}^{-1}$, moreover the first and the third row of Theorem 3.24 imply $\delta^{it} = J_\varphi \delta^{it} J_\varphi$. This in particular means that δ^{it} belongs to the center of $\mathrm{L}^\infty(\mathbb{G})$ – we have $\delta^{it} = J_\varphi (\delta^{it})^* J_\varphi$ [82, Proposition 1.23]. These two equations together imply $\delta = \mathbb{1}$.
The remaining implications are trivial. Let us now argue why all of the above conditions

imply $\nu = 1$. Clearly we only need to justify this for 6) and 5). If $E_\pi^{it} D_\pi^{-it} \in \mathbb{C}\mathbb{1}_{\mathsf{H}_\pi}$ then $\nu^{ist} D_\pi^{2is} E_\pi^{2it} = E_\pi^{2it} D_\pi^{2is}$ forces $\nu = 1$. If $\delta^{it} \in \mathcal{Z}(\mathrm{L}^\infty(\mathbb{G}))$ then $\nu^{it^2} \delta^{it} = \sigma_t^\varphi(\delta^{it}) = \delta^{it}$ for all $t \in \mathbb{R}$ ([82, Proposition 1.23]), hence also in this case $\nu = 1$. □

Let us now show how certain classes of quantum groups fit into the above diagram.

Proposition 3.35. *Let $\mathbb{G}$ be a type I, second countable locally compact quantum group.*

- *If $\mathbb{G}$ is classical and non-unimodular, then it satisfies* 8) *and does not satisfy* 5).
- *If $\widehat{\mathbb{G}}$ is classical and non-unimodular, then $\mathbb{G}$ satisfies* 3) *and does not satisfy* 8).
- *If $\mathbb{G}$ is compact and not of Kac type, then it satisfies* 5) *and does not satisfy* 4).
- *If $\mathbb{G}$ is discrete and non-unimodular, then it satisfies* 8) *and does not satisfy* 5).

The numbering in the above proposition corresponds to the numbering introduced in Theorem 3.34. Clearly each of the above classes is non-empty: examples are given by the classical $ax+b$ group, its dual, the $\mathrm{SU}_q(2)$ group and its dual (see Example 3.6).

In the next four sections we will describe some examples of type I locally compact quantum groups. This description is taken (with minor changes) from [50] and [49].

3.4 Example: compact quantum groups

Let $\mathbb{G}$ be a compact quantum group with countably many classes of irreducible representations (recall that we have introduced compact and discrete quantum groups as well as their basic properties in Section 2.3). We equip $\mathrm{Irr}(\mathbb{G})$ with the discrete measurable structure and declare all vector fields on $\mathrm{Irr}(\mathbb{G})$ to be measurable.
Define positive invertible operators $D_\alpha, E_\alpha \in \mathrm{B}(\mathsf{H}_\alpha)$ and a measure μ on $\mathrm{Irr}(\mathbb{G})$ via

$$D_\alpha = \rho_\alpha^{\frac{1}{2}},\; E_\alpha = \rho_\alpha^{-\frac{1}{2}},\; \mu(\{\alpha\}) = \dim_q(\alpha) \quad (\alpha \in \mathrm{Irr}(\mathbb{G})).$$

Next, define operators $\mathcal{Q}_L, \mathcal{Q}_R$ via

$$\mathcal{Q}_L\colon\; \mathrm{L}^2(\mathbb{G}) \ni \Lambda_{\widehat{\varphi}}\big((T_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\big) \mapsto \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} T_\alpha\, \rho_\alpha^{-\frac{1}{2}}\,\mathrm{d}\mu(\alpha) \in \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{HS}(\mathsf{H}_\alpha)\,\mathrm{d}\mu(\alpha),$$
$$\mathcal{Q}_R\colon\; \mathrm{L}^2(\mathbb{G}) \ni J_{\widehat{\varphi}} J_h \Lambda_{\widehat{\psi}}\big((T_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\big) \mapsto \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} T_\alpha\, \rho_\alpha^{\frac{1}{2}}\,\mathrm{d}\mu(\alpha) \in \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{HS}(\mathsf{H}_\alpha)\,\mathrm{d}\mu(\alpha),$$

where $(T_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}$ belongs respectively: to $\mathfrak{N}_{\widehat{\varphi}}$ in the case of $\mathcal{Q}_L$ and $\mathfrak{N}_{\widehat{\psi}}$ in the case of $\mathcal{Q}_R$.

Proposition 3.36. *The objects*

$$\mathcal{Q}_L,\; \mathcal{Q}_R,\; \mu,\; (D_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})},\; (E_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}$$

satisfy all the conditions of Theorem 3.3.

In order to prove this result, we will use Proposition 3.5. First, let us check that $\mathcal{Q}_L$ is a well defined isometry:

$$\begin{aligned}\Big\|\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} T_\alpha \mathsf{\rho}_\alpha{}^{-\frac{1}{2}}\,\mathrm{d}\mu(\alpha)\Big\|^2 &= \int_{\mathrm{Irr}(\mathbb{G})} \big\|T_\alpha \mathsf{\rho}_\alpha{}^{-\frac{1}{2}}\big\|_{\mathrm{HS}}^2\,\mathrm{d}\mu(\alpha)\\ &= \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \dim_q(\alpha)\,\mathrm{Tr}(\mathsf{\rho}_\alpha{}^{-1} T_\alpha{}^* T_\alpha) = \|\Lambda_{\widehat{\varphi}}((T_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})})\|^2.\end{aligned}$$

It is clear that the image of $\mathcal{Q}_L$ is dense, hence $\mathcal{Q}_L$ is a unitary operator. An analogous argument shows that $\mathcal{Q}_R$ also is unitary. For $\omega\in\mathrm{L}^1(\mathbb{G})$ such that $\lambda(\omega)=\big((\omega\otimes\mathrm{id})U^\alpha\big)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\in\mathfrak{N}_{\widehat{\varphi}}$ (see equation (2.22)) we have

$$\mathcal{Q}_L\Lambda_{\widehat{\varphi}}(\lambda(\omega)) = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})(U^\alpha)\mathsf{\rho}_\alpha{}^{-\frac{1}{2}}\,\mathrm{d}\mu(\alpha).$$

Similarly, for $\omega\in\mathrm{L}^1(\mathbb{G})$ such that $\lambda(\omega)\in\mathfrak{N}_{\widehat{\psi}}$ we have

$$\mathcal{Q}_R J_{\widehat{\varphi}} J_h \Lambda_{\widehat{\psi}}(\lambda(\omega)) = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})(U^\alpha)\,\mathsf{\rho}_\alpha^{\frac{1}{2}}\,\mathrm{d}\mu(\alpha),$$

which proves point 4) of Proposition 3.5 (note that $X=\{\lambda(\omega)\,|\,\omega\in\mathrm{L}^1(\mathbb{G})\}\cap\mathfrak{N}_{\widehat{\psi}}$ satisfies assumptions of Proposition 3.5, see [93, Lemma 5.14]). Take $x=(x_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\in\mathfrak{N}_{\widehat{\varphi}}$ and $\omega\in\mathrm{L}^1(\mathbb{G})$. We have

$$\mathcal{Q}_L((\omega\otimes\mathrm{id})\mathrm{W})\Lambda_{\widehat{\varphi}}(x) = \mathcal{Q}_L\Lambda_{\widehat{\varphi}}(((\omega\otimes\mathrm{id})\mathrm{W})x) = \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})(U^\alpha)x_\alpha\mathsf{\rho}_\alpha{}^{-\frac{1}{2}}\,\mathrm{d}\mu(\alpha),$$

on the other hand

$$\begin{aligned}&\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})(U^\alpha)\otimes\mathbb{1}_{\overline{\mathsf{H}_\alpha}}\,\mathrm{d}\mu(\alpha)\Big)\mathcal{Q}_L\Lambda_{\widehat{\varphi}}(x)\\ &= \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} ((\omega\otimes\mathrm{id})(U^\alpha)\otimes\mathbb{1}_{\overline{\mathsf{H}_\alpha}})x_\alpha\mathsf{\rho}_\alpha{}^{-\frac{1}{2}}\,\mathrm{d}\mu(\alpha)\\ &= \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})(U^\alpha)x_\alpha\mathsf{\rho}_\alpha{}^{-\frac{1}{2}}\,\mathrm{d}\mu(\alpha).\end{aligned}$$

The last equality follows from the isomorphism $\mathrm{HS}(\mathsf{H}_\alpha)=\mathsf{H}_\alpha\otimes\overline{\mathsf{H}_\alpha}$. The above calculation proves the commutation rule

$$\mathcal{Q}_L(\omega\otimes\mathrm{id})\mathrm{W} = \Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})(U^\alpha)\otimes\mathbb{1}_{\overline{\mathsf{H}_\alpha}}\,\mathrm{d}\mu(\alpha)\Big)\mathcal{Q}_L \quad (\omega\in\mathrm{L}^1(\mathbb{G})).$$

Let us introduce a dense *-subalgebra in $\mathrm{c}_0(\widehat{\mathbb{G}})$:

$$\mathrm{c}_{00}(\widehat{\mathbb{G}}) = \overset{alg}{\bigoplus_{\alpha\in\mathrm{Irr}(\mathbb{G})}} \mathrm{B}(\mathsf{H}_\alpha).$$

In order to show the second commutation rule, we need the following lemma:

Lemma 3.37. *The subspace $c_{00}(\widehat{\mathbb{G}})$ is a σ-SOT $\times \|\cdot\|$ core for $\Lambda_{\widehat{\varphi}}$.*

Above (and everywhere else) we treat $\ell^\infty(\widehat{\mathbb{G}})$ as a subalgebra of $\mathrm{B}(\mathrm{L}^2(\mathbb{G}))$, not as a subalgebra of $\mathrm{B}(\bigoplus_{\alpha\in\mathrm{Irr}(\mathbb{G})}\mathsf{H}_\alpha)$.

Proof. Let $T=(T_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\in\mathfrak{N}_{\widehat{\varphi}}$, that is

$$\widehat{\varphi}(T^*T)=\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\dim_q(\alpha)\operatorname{Tr}(T_\alpha^*T_\alpha\rho_\alpha^{-1})<+\infty. \tag{3.19}$$

Let $\{X_n\,|\,n\in\mathbb{N}\}$ be any increasing family of finite subsets of $\mathrm{Irr}(\mathbb{G})$ such that $\bigcup_{n\in\mathbb{N}}X_n=\mathrm{Irr}(\mathbb{G})$. Let $(T^n)_{n\in\mathbb{N}}$ be a sequence of elements of $c_{00}(\widehat{\mathbb{G}})$ given by $T^n_\alpha=1_{X_n}(\alpha)T_\alpha$. It is clear that $T^n\in\mathfrak{N}_{\widehat{\varphi}}$ for each $n\in\mathbb{N}$. An easy calculation using (3.19) shows $\Lambda_{\widehat{\varphi}}(T^n)\xrightarrow[n\to\infty]{}\Lambda_{\widehat{\varphi}}(T)$. Furthermore, we have $T^n\xrightarrow[n\to\infty]{\sigma\text{-SOT}}T$. Indeed: as the sequence $(T^n)_{n\in\mathbb{N}}$ is bounded, it is enough to check convergence in SOT and on vectors from a dense subspace $\{\Lambda_{\widehat{\varphi}}(S)\,|\,S\in\mathfrak{N}_{\widehat{\varphi}}\cap\mathrm{Dom}(\sigma^{\widehat{\varphi}}_{i/2})\}$. For such a $S\in\mathfrak{N}_{\widehat{\varphi}}\cap\mathrm{Dom}(\sigma^{\widehat{\varphi}}_{i/2})$ we have

$$\begin{aligned}&\|T\Lambda_{\widehat{\varphi}}(S)-T^n\Lambda_{\widehat{\varphi}}(S)\|=\|\Lambda_{\widehat{\varphi}}((T-T^n)S)\|=\|J_{\widehat{\varphi}}\sigma^{\widehat{\varphi}}_{i/2}(S)^*J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(T-T^n)\|\\&\le\|\sigma^{\widehat{\varphi}}_{i/2}(S)\|\|\Lambda_{\widehat{\varphi}}(T-T^n)\|\xrightarrow[n\to\infty]{}0,\end{aligned}$$

which proves the claim. ☐

Let us now check the second commutation rule, i.e.

$$\mathcal{Q}_L(\omega\otimes\mathrm{id})\chi(\mathrm{V})=\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{1}_{\mathsf{H}_\alpha}\otimes\alpha^c((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu(\alpha)\Big)\mathcal{Q}_L\quad(\omega\in\mathrm{L}^1(\mathbb{G})).$$

Take any $T=(T_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\in c_{00}(\widehat{\mathbb{G}})$ and $\omega\in\mathrm{L}^1(\mathbb{G})$ such that $\lambda(\omega)\in c_{00}(\widehat{\mathbb{G}})$. Let us note that the unbounded operators $\hat{\delta}^a=\bigoplus_{\alpha\in\mathrm{Irr}(\mathbb{G})}\rho_\alpha^{2a}$ and $\nabla^a_{\widehat{\varphi}}$ $(a\in\mathbb{R})$ have the subspace $\Lambda_{\widehat{\varphi}}(c_{00}(\widehat{\mathbb{G}}))$ in their domain, and moreover this subspace is preserved by them. Indeed, it is clear for $\hat{\delta}^a$, and we know that $\nabla^a_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(e^\alpha_{i,j})=\Lambda_{\widehat{\varphi}}(\sigma^{\widehat{\varphi}}_{-ia}(e^\alpha_{i,j}))=\frac{(\rho_\alpha)^a_j}{(\rho_\alpha)^a_i}\Lambda_{\widehat{\varphi}}(e^\alpha_{i,j})$. Recall equation (2.11)

$$\chi(\mathrm{V})=(J_{\widehat{\varphi}}\otimes J_{\widehat{\varphi}})\mathrm{W}^*(J_{\widehat{\varphi}}\otimes J_{\widehat{\varphi}}),$$

hence using equation (2.12) we arrive at

$$(\omega\otimes\mathrm{id})\chi(\mathrm{V})=J_{\widehat{\varphi}}((\omega\circ R\otimes\mathrm{id})\mathrm{W})^*J_{\widehat{\varphi}}=J_{\widehat{\varphi}}\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^*J_{\widehat{\varphi}} \tag{3.20}$$

and

$$\mathcal{Q}_L(\omega\otimes\mathrm{id})\chi(\mathrm{V})\Lambda_{\widehat{\varphi}}(T)=\mathcal{Q}_LJ_{\widehat{\varphi}}\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^*J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(T).$$

On the other hand

$$
\begin{aligned}
&\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}_\alpha}\otimes\alpha^c((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu(\alpha)\Big)\mathcal{Q}_L\Lambda_{\widehat{\varphi}}(T)\\
&=\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\big(\mathbb{1}_{\mathsf{H}_\alpha}\otimes\alpha^c((\omega\otimes\mathrm{id})\mathrm{W})\big)\Big(\sum_{j=1}^{\dim(\alpha)}|\zeta_j^\alpha\rangle\langle(T_\alpha\rho_\alpha^{-\frac12})^*\zeta_j^\alpha|\,\mathrm{d}\mu(\alpha)\\
&=\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\sum_{j=1}^{\dim(\alpha)}\zeta_j^\alpha\otimes\overline{\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^*\,(T_\alpha\rho_\alpha^{-\frac12})^*\zeta_j^\alpha}\,\mathrm{d}\mu(\alpha)\\
&=\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}T_\alpha\rho_\alpha^{-\frac12}\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu(\alpha)\\
&=\mathcal{Q}_L\Lambda_{\widehat{\varphi}}((T_\alpha\rho_\alpha^{-\frac12}\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})\rho_\alpha^{\frac12})_{\alpha\in\mathrm{Irr}(\mathbb{G})})\\
&=\mathcal{Q}_L J_{\widehat{\varphi}}\nabla_{\widehat{\varphi}}^{\frac12}(\rho_\alpha^{-\frac12}\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})\rho_\alpha^{\frac12})^*_{\alpha\in\mathrm{Irr}(\mathbb{G})}J_{\widehat{\varphi}}\nabla_{\widehat{\varphi}}^{\frac12}\Lambda_{\widehat{\varphi}}(T)\\
&=\mathcal{Q}_L J_{\widehat{\varphi}}\nabla_{\widehat{\varphi}}^{\frac12}(\rho_\alpha^{\frac12}\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^*\rho_\alpha^{-\frac12})_{\alpha\in\mathrm{Irr}(\mathbb{G})}\nabla_{\widehat{\varphi}}^{-\frac12}J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(T),
\end{aligned}
$$

where $\{\zeta_j^\alpha\,|\,j\in\{1,\dots,\dim(\alpha)\}\}$ is any orthonormal basis in H_α. Notice that by Lemma 3.13 the operator $\nabla_{\widehat{\varphi}}^{2a}\circ\hat{\delta}^a$ is closable for all $a\in\mathbb{R}$. Furthermore, on $\Lambda_{\widehat{\varphi}}(\mathrm{c}_{00}(\widehat{\mathbb{G}}))$ it commutes with operators from $\ell^\infty(\widehat{\mathbb{G}})$. Indeed[23], by equation (2.14) and Lemma 3.29 we have

$$
\nabla_{\widehat{\varphi}}^{-it}=\nabla_{\widehat{\psi}}^{it}=\delta^{-it}P^{-it}=P^{-it},\quad P^{-2it}=\delta^{it}(J_h\delta^{it}J_h)\hat{\delta}^{it}(J_{\widehat{\varphi}}\hat{\delta}^{it}J_{\widehat{\varphi}})=\hat{\delta}^{it}(J_{\widehat{\varphi}}\hat{\delta}^{it}J_{\widehat{\varphi}})
$$

hence

$$
\nabla_{\widehat{\varphi}}^{2it}\hat{\delta}^{it}=P^{2it}\hat{\delta}^{it}=\hat{\delta}^{-it}(J_{\widehat{\varphi}}\hat{\delta}^{-it}J_{\widehat{\varphi}})\hat{\delta}^{it}=J_{\widehat{\varphi}}\hat{\delta}^{-it}J_{\widehat{\varphi}}\in\ell^\infty(\widehat{\mathbb{G}})'
$$

for all $t\in\mathbb{R}$ and $\nabla_{\widehat{\varphi}}^{2a}\hat{\delta}^a$ is affiliated with $\ell^\infty(\widehat{\mathbb{G}})'$ ([78, Exercise E.9.25]). Consequently

$$
\begin{aligned}
&\mathcal{Q}_L J_{\widehat{\varphi}}\nabla_{\widehat{\varphi}}^{\frac12}(\rho_\alpha^{\frac12}\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^*\rho_\alpha^{-\frac12})_{\alpha\in\mathrm{Irr}(\mathbb{G})}\nabla_{\widehat{\varphi}}^{-\frac12}J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(T)\\
&=\mathcal{Q}_L J_{\widehat{\varphi}}(\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^*)_{\alpha\in\mathrm{Irr}(\mathbb{G})}(\nabla_{\widehat{\varphi}}^{\frac12}\hat{\delta}^{\frac14})(\hat{\delta}^{-\frac14}\nabla_{\widehat{\varphi}}^{-\frac12})J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(T)\\
&=\mathcal{Q}_L J_{\widehat{\varphi}}(\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^*)_{\alpha\in\mathrm{Irr}(\mathbb{G})}J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(T),
\end{aligned}
$$

and the second commutation relation holds.
Assume that $x=(x_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}$ is an element of $\ell^\infty(\widehat{\mathbb{G}})\cap\ell^\infty(\widehat{\mathbb{G}})'$. Triviality of the center of $\mathrm{B}(\mathsf{H}_\alpha)$ impliess that $x_\alpha\in\mathbb{C}\mathbb{1}_\alpha$ for each $\alpha\in\mathrm{Irr}(\mathbb{G})$. Operator x is mapped via $\mathcal{Q}_L$ to $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}x_\alpha\,\mathrm{d}\mu(\alpha)$, which is a diagonalisable operator. On the other hand, any diagonalisable operator $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}y_\alpha\,\mathrm{d}\mu(\alpha)$ ($y_\alpha\in\mathbb{C}\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\alpha)}$) is an image of $(y_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\in\ell^\infty(\widehat{\mathbb{G}})\cap\ell^\infty(\widehat{\mathbb{G}})'$. This proves that we have identified objects that are given by the left version of Theorem 3.3.
Let us now check that $\mathcal{Q}_R$ and $E_\alpha=\rho_\alpha{}^{-\frac12}$ satisfy conditions from Proposition 3.5: we

[23] A more direct proof is also possible, see [50].

only need to check the commutation rules, since the rest is clear. Now we need to use the formula $\nabla_{\widehat{\psi}}^{a}\Lambda_{\widehat{\psi}}(e^{\alpha}_{i,j}) = \frac{(\rho_\alpha)^a_i}{(\rho_\alpha)^a_j}\Lambda_{\widehat{\psi}}(e^{\alpha}_{i,j})$. Take $\omega \in \mathrm{L}^1(\mathbb{G})$ and $T \in \mathfrak{N}_{\widehat{\psi}}$ as before. We have

$$\begin{aligned}
&\mathcal{Q}_R J_{\widehat{\varphi}} J_h (\omega\otimes\mathrm{id})\mathrm{W}\Lambda_{\widehat{\psi}}(T) = \mathcal{Q}_R J_{\widehat{\varphi}} J_h \Lambda_{\widehat{\psi}}(((\omega\otimes\mathrm{id})\mathrm{W})\,T)\\
&= \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})U^{\alpha} T_\alpha \rho_\alpha^{\frac12}\,\mathrm{d}\mu(\alpha) = \big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} (\omega\otimes\mathrm{id})U^{\alpha}\otimes \mathbb{1}_{\overline{\mathsf{H}_\alpha}}\,\mathrm{d}\mu(\alpha)\big)\mathcal{Q}_R J_{\widehat{\varphi}} J_h \Lambda_{\widehat{\psi}}(T),
\end{aligned}$$

which shows the first commutation rule. Let us now prove the second one using this time the result that $\nabla_{\widehat{\psi}}^{-2a}\hat{\delta}^{a}$ is affiliated with $\ell^\infty(\widehat{\mathbb{G}})'$ (recall also that $\hat{\nu}=1$, hence $J_{\widehat{\varphi}} = J_{\widehat{\psi}}$):

$$\begin{aligned}
&\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathbb{1}_{\mathsf{H}_\alpha}\otimes \alpha^{c}((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu(\alpha)\big)\mathcal{Q}_R J_{\widehat{\varphi}} J_h \Lambda_{\widehat{\psi}}(T)\\
&= \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} T_\alpha \rho_\alpha^{\frac12}\,\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu(\alpha)\\
&= \mathcal{Q}_R J_{\widehat{\varphi}} J_h \Lambda_{\widehat{\psi}}\big((T_\alpha \rho_\alpha^{\frac12}\,\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})\rho_\alpha^{-\frac12})_{\alpha\in\mathrm{Irr}(\mathbb{G})}\big)\\
&= \mathcal{Q}_R J_{\widehat{\varphi}} J_h J_{\widehat{\varphi}} \nabla_{\widehat{\psi}}^{\frac12}\big(\rho_\alpha^{\frac12}\,\alpha\circ\hat{R}((\omega\otimes\mathrm{id})\mathrm{W})\rho_\alpha^{-\frac12}\big)^{*}_{\alpha\in\mathrm{Irr}(\mathbb{G})} J_{\widehat{\varphi}} \nabla_{\widehat{\psi}}^{\frac12}\Lambda_{\widehat{\psi}}(T)\\
&= \mathcal{Q}_R J_{\widehat{\varphi}} J_h J_{\widehat{\varphi}} \hat{R}((\omega\otimes\mathrm{id})\mathrm{W})^{*} J_{\widehat{\varphi}} \Lambda_{\widehat{\psi}}(T) = \mathcal{Q}_R J_{\widehat{\varphi}} J_h (\omega\otimes\mathrm{id})\chi(\mathrm{V})\Lambda_{\widehat{\psi}}(T),
\end{aligned}$$

which concludes the proof of Proposition 3.36.

Remark. Note that one gets a general Plancherel measure by taking any positive measure on $\mathrm{Irr}(\mathbb{G})$ with full support. Indeed, let $c\colon \mathrm{Irr}(\mathbb{G}) \to \mathbb{R}_{>0}$ be an arbitrary function. Define measure $\mu^c\colon \{\alpha\} \mapsto c(\alpha)$. It is equivalent to the above measure $\mu = \mu^{\dim_q}$ and we have

$$\frac{\mathrm{d}\mu^c}{\mathrm{d}\mu} = \frac{c}{\dim_q}.$$

With this choice of a Plancherel measure we can relate the following Duflo-Moore operators:

$$\begin{aligned}
D_\alpha &= \sqrt{\frac{\mathrm{d}\mu^c}{\mathrm{d}\mu}(\alpha)}\rho_\alpha^{\frac12} = c(\alpha)^{\frac12}\dim_q(\alpha)^{-\frac12}\rho_\alpha^{\frac12},\\
E_\alpha &= \sqrt{\frac{\mathrm{d}\mu^c}{\mathrm{d}\mu}(\alpha)}\rho_\alpha^{-\frac12} = c(\alpha)^{\frac12}\dim_q(\alpha)^{-\frac12}\rho_\alpha^{-\frac12}.
\end{aligned}$$

3.5 Example: quantum groups dual to classical

Assume now that $\widehat{\mathbb{G}}$ is a classical locally compact group which is second countable (for preliminary results see Section 2.2.1). We have equality of C*-algebras $\mathrm{C}_0^u(\widehat{\mathbb{G}}) = \mathrm{C}_0(\widehat{\mathbb{G}})$, its spectrum can be identified with $\mathrm{Irr}(\mathbb{G}) = \widehat{\mathbb{G}}$ as a topological space, and every point $\zeta \in \widehat{\mathbb{G}}$ corresponds to the one dimensional representation of $\mathrm{C}_0(\widehat{\mathbb{G}})$ given by evaluation at ζ. We

will abuse the notation and identify (as sets) H_ζ and $\mathrm{B}(\mathsf{H}_\zeta)$ with $\mathbb{C}$ for each $\zeta \in \widehat{\mathbb{G}}$.
Take any $p \in \mathbb{R}$. Define a measure $\mu_p = \hat{\delta}^p \mu_L = \hat{\delta}^{p-1}\mu_R$, the structure of measurable field of Hilbert spaces $(\mathbb{C})_{\zeta\in\widehat{\mathbb{G}}}$ given by measurable functions on $\widehat{\mathbb{G}}$, positive operators $D_\zeta = \hat{\delta}(\zeta)^{\frac{p}{2}}, E_\zeta = \hat{\delta}(\zeta)^{\frac{p-1}{2}}$ ($\zeta \in \widehat{\mathbb{G}}$) and operators $\mathcal{Q}_L, \mathcal{Q}_R$ given by

$$\mathcal{Q}_L \colon \mathrm{L}^2(\mathbb{G}) \ni \Lambda_{\widehat{\varphi}}(f) \mapsto \int_{\widehat{\mathbb{G}}}^{\oplus} f(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}} \,\mathrm{d}\mu_p(\zeta) \in \int_{\widehat{\mathbb{G}}}^{\oplus} \mathrm{HS}(\mathsf{H}_\zeta)\,\mathrm{d}\mu_p(\zeta),$$
$$\mathcal{Q}_R \colon \mathrm{L}^2(\mathbb{G}) \ni J_{\widehat{\varphi}}J_\varphi\Lambda_{\widehat{\psi}}(f) \mapsto \int_{\widehat{\mathbb{G}}}^{\oplus} f(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}} \,\mathrm{d}\mu_p(\zeta) \in \int_{\widehat{\mathbb{G}}}^{\oplus} \mathrm{HS}(\mathsf{H}_\zeta)\,\mathrm{d}\mu_p(\zeta).$$

Operators $\mathcal{Q}_L, \mathcal{Q}_R$ are at first only densely defined: f belongs respectively to $\mathfrak{N}_{\widehat{\varphi}}$ and $\mathfrak{N}_{\widehat{\psi}}$.

Proposition 3.38. *For each $p \in \mathbb{R}$ the objects*

$$\mathcal{Q}_L,\ \mathcal{Q}_R,\ \mu_p,\ (D_\zeta)_{\zeta\in\widehat{\mathbb{G}}},\ (E_\zeta)_{\zeta\in\widehat{\mathbb{G}}}$$

satisfy all the conditions of Theorem 3.3.

From this proposition follows that a general Plancherel measure is given by $g\mu_L$ for a strictly positive function g. We restrict our attention to the case $g = \hat{\delta}^p$ because this choice simplifies our calculations. Furthermore, this family of measures includes a measure invariant under conjugation (when $p = \frac{1}{2}$, see [50, Section 13.2]) and the natural choices of left and right invariant Haar measures μ_L, μ_R.

First, let us check that the densely defined operators $\mathcal{Q}_L, \mathcal{Q}_R$ are isometric:

$$\Big\|\int_{\widehat{\mathbb{G}}}^{\oplus} f(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}} \,\mathrm{d}\mu_p(\zeta)\Big\|^2 = \int_{\widehat{\mathbb{G}}} |f(\zeta)|^2\hat{\delta}(\zeta)^{-p}\hat{\delta}(\zeta)^p \,\mathrm{d}\mu_L(\zeta) = \|\Lambda_{\widehat{\varphi}}(f)\|^2,$$
$$\Big\|\int_{\widehat{\mathbb{G}}}^{\oplus} f(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}} \,\mathrm{d}\mu_p(\zeta)\Big\|^2 = \int_{\widehat{\mathbb{G}}} |f(\zeta)|^2\hat{\delta}(\zeta)^{-p+1}\hat{\delta}(\zeta)^{p-1} \,\mathrm{d}\mu_R(\zeta) = \|\Lambda_{\widehat{\psi}}(f)\|^2.$$

It follows that they extend to the whole $\mathrm{L}^2(\mathbb{G})$. It is clear that they have dense image, therefore are unitary. As before, to prove Proposition 3.38 we will use Proposition 3.5. We have

$$\begin{aligned}\mathcal{Q}_L(\Lambda_{\widehat{\varphi}}(\lambda(\alpha))) &= \int_{\widehat{\mathbb{G}}}^{\oplus} ((\alpha\otimes\mathrm{id})\mathrm{W})(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}} \,\mathrm{d}\mu_p(\zeta)\\ &= \int_{\widehat{\mathbb{G}}}^{\oplus} (\alpha\otimes\mathrm{id})(U^\zeta)D_\zeta^{-1} \,\mathrm{d}\mu_p(\zeta)\end{aligned}$$

for $\alpha \in \mathrm{L}^1(\mathbb{G})$ such that $\lambda(\alpha) \in \mathfrak{N}_{\widehat{\varphi}}$. Similarly,

$$\begin{aligned}\mathcal{Q}_R J_{\widehat{\varphi}}J_\varphi(\Lambda_{\widehat{\psi}}(\lambda(\alpha))) &= \int_{\widehat{\mathbb{G}}}^{\oplus} ((\alpha\otimes\mathrm{id})\mathrm{W})(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}} \,\mathrm{d}\mu_p(\zeta)\\ &= \int_{\widehat{\mathbb{G}}}^{\oplus} (\alpha\otimes\mathrm{id})(U^\zeta)E_\zeta^{-1} \,\mathrm{d}\mu_p(\zeta)\end{aligned}$$

for $\alpha \in \mathrm{L}^1(\mathbb{G})$ such that $\lambda(\alpha) \in \mathfrak{N}_{\widehat{\psi}}$. Consequently, point 4) holds. Now, for $f \in \mathfrak{N}_{\widehat{\varphi}}$ and $\omega \in \mathrm{L}^1(\mathbb{G})$ we have

$$\begin{aligned}
&\mathcal{Q}_L(\omega\otimes\mathrm{id})\mathrm{W}\Lambda_{\widehat{\varphi}}(f)\\
=&\mathcal{Q}_L\Lambda_{\widehat{\varphi}}((\omega\otimes\mathrm{id})(\mathrm{W})f)\\
=&\int_{\widehat{\mathbb{G}}}^{\oplus}((\omega\otimes\mathrm{id})\mathrm{W})(\zeta)f(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}}\,\mathrm{d}\mu_p(\zeta)\\
=&\Big(\int_{\widehat{\mathbb{G}}}^{\oplus}((\omega\otimes\mathrm{id})\mathrm{W})(\zeta)\otimes\mathbb{1}_{\overline{\mathsf{H}_\zeta}}\,\mathrm{d}\mu_p(\zeta)\Big)\mathcal{Q}_L\Lambda_{\widehat{\varphi}}(f)\\
=&\Big(\int_{\widehat{\mathbb{G}}}^{\oplus}(\omega\otimes\mathrm{id})(U^\zeta)\otimes\mathbb{1}_{\overline{\mathsf{H}_\zeta}}\,\mathrm{d}\mu_p(\zeta)\Big)\mathcal{Q}_L\Lambda_{\widehat{\varphi}}(f),
\end{aligned}$$

which gives us the first commutation relation. The Haar integrals on $\widehat{\mathbb{G}}$ are tracial, hence the operator $J_{\widehat{\varphi}}$ acts as follows: $J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(f)=\Lambda_{\widehat{\varphi}}(f^*)\,(f\in\mathfrak{N}_{\widehat{\varphi}})$. Consequently for each $x\in\mathrm{L}^\infty(\widehat{\mathbb{G}}), f,g\in\mathfrak{N}_{\widehat{\varphi}}$ the following holds

$$\langle\Lambda_{\widehat{\varphi}}(g)\,|\,J_{\widehat{\varphi}}x^*J_{\widehat{\varphi}}\Lambda_{\widehat{\varphi}}(f)\rangle=\langle\Lambda_{\widehat{\varphi}}(g)\,|\,\Lambda_{\widehat{\varphi}}(fx)\rangle=\widehat{\varphi}(g^*fx)=\widehat{\varphi}(g^*xf)=\langle\Lambda_{\widehat{\varphi}}(g)\,|\,x\Lambda_{\widehat{\varphi}}(f)\rangle.$$

It follows that $J_{\widehat{\varphi}}x^*J_{\widehat{\varphi}}=x\,(x\in\mathrm{L}^\infty(\widehat{\mathbb{G}}))$ and

$$(\omega\otimes\mathrm{id})\chi(\mathrm{V})=R^{\widehat{\mathbb{G}}}((\omega\otimes\mathrm{id})\mathrm{W})\quad(\omega\in\mathrm{L}^1(\mathbb{G})).$$

Clearly we have $R^{\widehat{\mathbb{G}}}(x)(\zeta)=x(\zeta^{-1})\,(x\in\mathrm{L}^\infty(\widehat{\mathbb{G}}),\zeta\in\widehat{\mathbb{G}})$, therefore

$$\begin{aligned}
&\mathcal{Q}_L(\omega\otimes\mathrm{id})\chi(\mathrm{V})\Lambda_{\widehat{\varphi}}(f)\\
=&\int_{\widehat{\mathbb{G}}}^{\oplus}((\omega\otimes\mathrm{id})\chi(\mathrm{V})f)(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}}\,\mathrm{d}\mu_p(\zeta)\\
=&\int_{\widehat{\mathbb{G}}}^{\oplus}(\omega\otimes\mathrm{id})\mathrm{W}(\zeta^{-1})f(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}}\,\mathrm{d}\mu_p(\zeta)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\Big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus}\mathbb{1}_{\mathsf{H}_\zeta}\otimes\zeta^c((\omega\otimes\mathrm{id})\mathrm{W})\,\mathrm{d}\mu_p(\zeta)\Big)\mathcal{Q}_L\Lambda_{\widehat{\varphi}}(f)\\
=&\Big(\int_{\widehat{\mathbb{G}}}^{\oplus}\mathbb{1}_{\mathsf{H}_\zeta}\otimes((\omega\otimes\mathrm{id})\mathrm{W})(\zeta^{-1})^\top\,\mathrm{d}\mu_p(\zeta)\Big)\mathcal{Q}_L\Lambda_{\widehat{\varphi}}(f)\\
=&\int_{\widehat{\mathbb{G}}}^{\oplus}\big(\mathbb{1}_{\mathsf{H}_\pi}\otimes((\omega\otimes\mathrm{id})\mathrm{W})(\zeta^{-1})^\top\big)(f(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}})\,\mathrm{d}\mu_p(\zeta)\\
=&\int_{\widehat{\mathbb{G}}}^{\oplus}((\omega\otimes\mathrm{id})\mathrm{W})(\zeta^{-1})f(\zeta)\hat{\delta}(\zeta)^{-\frac{p}{2}}\,\mathrm{d}\mu_p(\zeta)
\end{aligned}$$

for $\omega\in\mathrm{L}^1(\mathbb{G}), f\in\mathfrak{N}_{\widehat{\varphi}}$, which ends the proof of commutation relations for $\mathcal{Q}_L$.
We have $\mathrm{L}^\infty(\widehat{\mathbb{G}})\cap\mathrm{L}^\infty(\widehat{\mathbb{G}})'=\mathrm{L}^\infty(\widehat{\mathbb{G}})$ and it is clear that operator $\mathcal{Q}_L$ maps a function

$x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$ to the operator $\int_{\widehat{\mathbb{G}}}^\oplus x(\zeta)\,\mathrm{d}\mu_p(\zeta)$. Note that for each $x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$ and $f \in \mathfrak{N}_{\widehat{\psi}}$ we have

$$\begin{aligned}
&\mathcal{Q}_R x \mathcal{Q}_R^* \int_{\widehat{\mathbb{G}}}^\oplus f(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}}\,\mathrm{d}\mu_p(\zeta) = \mathcal{Q}_R x J_{\widehat{\varphi}} J_\varphi \Lambda_{\widehat{\psi}}(f) \\
&= \mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi J_\varphi x^* J_\varphi \Lambda_{\widehat{\psi}}(f) = \mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi R^{\widehat{\mathbb{G}}}(x)\Lambda_{\widehat{\psi}}(f) = \mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi \Lambda_{\widehat{\psi}}(R^{\widehat{\mathbb{G}}}(x) f) \\
&= \int_{\mathrm{Irr}(\mathbb{G})}^\oplus R^{\widehat{\mathbb{G}}}(x)(\zeta) f(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}}\,\mathrm{d}\mu_p(\zeta),
\end{aligned}$$

therefore $\mathcal{Q}_R x \mathcal{Q}_R^* = \int_{\widehat{\mathbb{G}}}^\oplus R^{\widehat{\mathbb{G}}}(x)(\zeta)\,\mathrm{d}\mu_p(\zeta)$. Consequently, $\mathcal{Q}_L, \mathcal{Q}_R$ transform $\mathcal{Z}(\mathrm{L}^\infty(\widehat{\mathbb{G}}))$ onto the algebra of diagonalisable operators. We are left to show the commutation relations for $\mathcal{Q}_R$. Take any $\omega \in \mathrm{L}^1(\mathbb{G})$ and $f \in \mathfrak{N}_{\widehat{\psi}}$. We have

$$\begin{aligned}
&\mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi (\omega \otimes \mathrm{id})\mathrm{W}\Lambda_{\widehat{\psi}}(f) = \int_{\widehat{\mathbb{G}}}^\oplus ((\omega \otimes \mathrm{id})\mathrm{W})(\zeta) f(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}}\,\mathrm{d}\mu_p(\zeta) \\
&= \int_{\widehat{\mathbb{G}}}^\oplus (\omega \otimes \mathrm{id})(U^\zeta) f(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}}\,\mathrm{d}\mu_p(\zeta) = \big(\int_{\widehat{\mathbb{G}}}^\oplus (\omega \otimes \mathrm{id})(U^\zeta) \otimes \mathbb{1}_{\mathsf{H}_\zeta}\,\mathrm{d}\mu_p(\zeta)\big)\mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi \Lambda_{\widehat{\psi}}(f)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi (\omega \otimes \mathrm{id})\chi(\mathrm{V})\Lambda_{\widehat{\psi}}(f) = \mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi R^{\widehat{\mathbb{G}}}((\omega \otimes \mathrm{id})\mathrm{W})\Lambda_{\widehat{\psi}}(f) \\
&= \int_{\widehat{\mathbb{G}}}^\oplus R^{\widehat{\mathbb{G}}}((\omega \otimes \mathrm{id})\mathrm{W})(\zeta) f(\zeta)\hat{\delta}(\zeta)^{-\frac{p-1}{2}}\,\mathrm{d}\mu_p(\zeta) \\
&= \big(\int_{\widehat{\mathbb{G}}}^\oplus \mathbb{1}_{\mathsf{H}_\zeta} \otimes \zeta^c((\omega \otimes \mathrm{id})\mathrm{W})\,\mathrm{d}\mu_p(\zeta)\big)\mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi \Lambda_{\widehat{\psi}}(f).
\end{aligned}$$

This concludes the proof of Proposition 3.38.

3.6 Example: $\widehat{\mathrm{SU}_q(2)}$

Fix a real number $q \in]-1,1[\setminus \{0\}$. Let $\mathbb{G}$ be the quantum group $\mathrm{SU}_q(2)$ (see Section 2.3.1) and let $\mathbb{\Gamma}$ be the dual discrete quantum group $\mathbb{\Gamma} = \widehat{\mathrm{SU}_q(2)}$. To avoid confusion, in this section we will decorate objects related to $\mathrm{SU}_q(2)$ (resp. $\widehat{\mathrm{SU}_q(2)}$) with $\mathbb{G}$ (resp. $\mathbb{\Gamma}$). We have already said that $\mathbb{G}$ is coamenable, consequently $\mathrm{C}(\mathrm{SU}_q(2)) = \mathrm{C}^u(\mathrm{SU}_q(2))$. This C^*-algebra is separable and type I (see [99, Theorem A2.3]) hence $\mathbb{\Gamma}$ is an interesting example of a second countable, type I discrete quantum group[24]. We will describe the Plancherel measure for this group and show how various operators related to $\mathbb{\Gamma}$ act on the level of direct integrals. Let us start with describing the measurable space $\mathrm{Irr}(\mathbb{\Gamma})$ (i.e. the spectrum of $\mathrm{C}(\mathrm{SU}_q(2))$). The following result is a reformulation of [90, Theorem 3.2] (see also [44, Section 3.2]):

[24]In this section $\mathbb{\Gamma}$ is the "main" group and $\mathbb{G}$ is the "dual" one.

Proposition 3.39. *Measurable space* $\operatorname{Irr}(\mathbb{\Gamma})$ *can be identified with the disjoint union of two circles* $\mathbb{T}\sqcup\mathbb{T}=\{\psi^{1,\rho}\,|\,\rho\in\mathbb{T}\}\cup\{\psi^{2,\lambda}\,|\,\lambda\in\mathbb{T}\}$. *Representations* $\psi^{1,\rho}$ *are one dimensional and given by*

$$\psi^{1,\rho}(\alpha)=\rho,\quad \psi^{1,\rho}(\alpha^*)=\overline{\rho},\quad \psi^{2,\rho}(\gamma)=0,\quad \psi^{2,\rho}(\gamma^*)=0\quad(\rho\in\mathbb{T}).$$

Representations $\psi^{2,\lambda}$ *act on a separable Hilbert space* $\mathsf{H}_\lambda=\ell^2(\mathbb{Z}_+)$ *with an orthonormal basis* $\{\phi_k\,|\,k\in\mathbb{Z}_+\}$ *via*

$$\begin{aligned}&\psi^{2,\lambda}(\alpha)\phi_k=\sqrt{1-q^{2k}}\phi_{k-1}, && \psi^{2,\lambda}(\alpha^*)\phi_k=\sqrt{1-q^{2(k+1)}}\phi_{k+1},\\ &\psi^{2,\lambda}(\gamma)\phi_k=\lambda q^k\phi_k, && \psi^{2,\lambda}(\gamma^*)\phi_k=\overline{\lambda}q^k\phi_k, \qquad (\lambda\in\mathbb{T},k\in\mathbb{Z}_+),\end{aligned}$$

with the convention $\phi_{-n}=0\,(n\in\mathbb{N})$.

In what follows, φ,ψ are the Haar integrals on $\mathbb{\Gamma}=\widehat{\mathrm{SU}_q(2)}$ and h is the Haar integral on $\mathbb{G}=\mathrm{SU}_q(2)$.
In the next proposition we calculate the Plancherel measure of $\mathbb{\Gamma}$, the unitary operator $\mathcal{Q}_L$ and operators $(D_\pi)_{\pi\in\operatorname{Irr}(\mathbb{\Gamma})}$. Then, as $\mathbb{G}$ is unimodular, we have $E_\pi=D_\pi,(\pi\in\operatorname{Irr}(\mathbb{\Gamma}))$ and $\mathcal{Q}_R=\mathcal{Q}_L J_\varphi J_h$ (see Remark 3.1).

Proposition 3.40. *The Plancherel measure of* $\mathbb{\Gamma}$ *equals* 0 *on* $\{\psi^{1,\rho}\,|\,\rho\in\mathbb{T}\}$ *and the normalized Lebesgue measure on the second circle* $\{\psi^{2,\lambda}\,|\,\lambda\in\mathbb{T}\}$. *Consequently, we will identify* $\operatorname{Irr}(\mathbb{\Gamma})$ *with* $\mathbb{T}$. *Operators* $\{D_\lambda\,|\,\lambda\in\mathbb{T}\}$ *are given by*

$$D_\lambda=(1-q^2)^{-\frac{1}{2}}\operatorname{Diag}(1,|q|^{-1},|q|^{-2},\dots)\quad(\lambda\in\mathbb{T})$$

with respect to the basis $\{\phi_k\,|\,k\in\mathbb{Z}_+\}$. *Operator* $\mathcal{Q}_L$ *is given by*

$$\mathcal{Q}_L\colon\ \mathrm{L}^2(\mathbb{G})\ni\Lambda_h(a)\mapsto\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(a)D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)\in\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)\quad(a\in\mathrm{C}(\mathrm{SU}_q(2))).$$

Proof. Define μ to be the normalized Lebesgue measure on the second circle of $\operatorname{Irr}(\mathbb{\Gamma})=\mathbb{T}\sqcup\mathbb{T}$ and let $\mathcal{Q}_L$ be the operator given by the above formula. In order to show that these objects are the ones given by Desmedt's theorem, we will use Proposition 3.5. Let us start with showing that $\mathcal{Q}_L$ is well defined and unitary. First, it is clear that for $a\in\mathrm{C}(\mathrm{SU}_q(2))$ the field of operators $(\psi^{2,\lambda}(a)D_\lambda^{-1})_{\lambda\in\mathbb{T}}$ is measurable and square integrable. Consequently, we can introduce a densely defined linear map $\mathcal{Q}_L\colon\Lambda_h(a)\mapsto\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(a)D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)$. Since $\|\mathcal{Q}_L\Lambda_h(a)\|\le\|a\|\,(a\in\mathrm{C}(\mathrm{SU}_q(2)))$, the linear map $\mathcal{Q}_L\circ\Lambda_h$ is bounded. Let us now show that $\mathcal{Q}_L$ is isometry, i.e. $\langle\mathcal{Q}_L\Lambda_h(a')\,|\,\mathcal{Q}_L\Lambda_h(a)\rangle=\langle\Lambda_h(a')\,|\,\Lambda_h(a)\rangle$ for all $a,a'\in\mathrm{C}(\mathrm{SU}_q(2))$. Since

$$\begin{aligned}\langle\mathcal{Q}_L\Lambda_h(a')\,|\,\mathcal{Q}_L\Lambda_h(a)\rangle&=\Big\langle\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(a')D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)\Big|\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(a)D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)\Big\rangle\\ =\Big\langle\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(\mathbb{1})D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)\Big|&\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(a'^*a)D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)\Big\rangle=\langle\mathcal{Q}_L\Lambda_h(\mathbb{1})\,|\,\mathcal{Q}_L\Lambda_h(a'^*a)\rangle\end{aligned}$$

and $\langle \Lambda_h(a') \,|\, \Lambda_h(a)\rangle = \langle \Lambda_h(\mathbb{1}) \,|\, \Lambda_h(a'^* a)\rangle$, it is enough to consider the case $a' = \mathbb{1}$. Next, as maps $\mathcal{Q}_L \circ \Lambda_h, \Lambda_h$ are bounded and linear, it is enough to consider a in a basis of $\mathrm{Pol}(\mathrm{SU}_q(2))$, $\{\alpha^l \gamma^n \gamma^{*m}, \alpha^{*l'} \gamma^n \gamma^{*m} \,|\, l, n, m \in \mathbb{Z}_+, l' \in \mathbb{N}\}$ (see [99, Theorem 1.2]).
In order to calculate $\langle \Lambda_h(\mathbb{1}) \,|\, \Lambda_h(a)\rangle$ we need to introduce a faithful representation $\pi_0\colon \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{B}(\ell^2(\mathbb{Z}_+ \times \mathbb{Z}))$ defined in [99]. One can express the Haar integral h as

$$h(a) = (1-q^2) \sum_{k=0}^{\infty} q^{2k} \langle \phi_{k,0} \,|\, \pi_0(a)\phi_{k,0}\rangle \quad (a \in \mathrm{C}(\mathrm{SU}_q(2))),$$

where $\{\phi_{k,p} \,|\, (k,p) \in \mathbb{Z}_+ \times \mathbb{Z}\}$ is the standard basis of $\ell^2(\mathbb{Z}_+ \times \mathbb{Z})$. Now, for $l, n, m \in \mathbb{Z}_+$ we have

$$\langle \Lambda_h(\mathbb{1}) \,|\, \Lambda_h(\alpha^l \gamma^n \gamma^{*m})\rangle = h(\alpha^l \gamma^n \gamma^{*m}) = \delta_{l,0}(1-q^2) \sum_{k=0}^{\infty} q^{2k} \delta_{n,m} q^{(n+m)k} = \delta_{l,0}\delta_{n,m} \tfrac{1-q^2}{1-q^{2(1+n)}}$$

and similarly $\langle \Lambda_h(\mathbb{1}) \,|\, \Lambda_h(\alpha^{*l} \gamma^n \gamma^{*m})\rangle = \delta_{l,0}\delta_{n,m} \tfrac{1-q^2}{1-q^{2(1+n)}}$. On the other hand

$$\langle \mathcal{Q}_L \Lambda_h(\mathbb{1}) \,|\, \mathcal{Q}_L \Lambda_h(\alpha^l \gamma^n \gamma^{*m})\rangle = \Big\langle \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} D_\lambda^{-1} \, \mathrm{d}\mu(\lambda) \Big| \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \psi^{2,\lambda}(\alpha^l \gamma^n \gamma^{*m}) D_\lambda^{-1} \, \mathrm{d}\mu(\lambda)\Big\rangle$$

$$= \delta_{l,0}(1-q^2) \int_{\mathrm{Irr}(\mathbb{\Gamma})} \sum_{k=0}^{\infty} \langle \phi_k \,|\, \lambda^{n-m} q^{(n+m)k} q^{2k} \phi_k\rangle \, \mathrm{d}\mu(\lambda)$$

$$= \delta_{l,0}\delta_{n,m}(1-q^2) \sum_{k=0}^{\infty} q^{(n+m)k} q^{2k} = \delta_{l,0}\delta_{n,m} \tfrac{1-q^2}{1-q^{2(1+n)}}.$$

In an analogous manner we check $\langle \mathcal{Q}_L \Lambda_h(\mathbb{1}) \,|\, \mathcal{Q}_L \Lambda_h(\alpha^{*l} \gamma^n \gamma^{*m})\rangle = \delta_{l,0}\delta_{n,m} \tfrac{1-q^2}{1-q^{2(1+n)}}$. This shows that $\mathcal{Q}_L$ is isometry and consequently extends to the whole of $\mathrm{L}^2(\mathbb{G})$. Let us now argue that $\mathcal{Q}_L$ is surjective. Fix $\lambda \in \mathbb{T}$, $k, l \in \mathbb{Z}_+$. We have $\psi^{2,\lambda}(\gamma\gamma^*)\phi_k = q^{2k}\phi_k$, hence $\psi^{2,\lambda}(1_{\{q^{2l}\}}(\gamma\gamma^*))\phi_k = \delta_{k,l}\phi_k$ (note that operator $1_{\{q^{2l}\}}(\gamma\gamma^*)$ belongs to $\mathrm{C}(\mathrm{SU}_q(2))$ because q^{2l} is an isolated point in the spectrum of $\gamma\gamma^*$). Next, for $n \in \mathbb{Z}_+$ the following holds

$$\psi^{2,\lambda}(\alpha^n 1_{q^{2l}}(\gamma\gamma^*))\phi_k = \delta_{k,l}\Big(\prod_{a=0}^{n-1} (1-q^{2(k-a)})^{\frac{1}{2}}\Big)\phi_{k-n} = \delta_{k,l}\Big(\prod_{a=0}^{n-1} (1-q^{2(k-a)})^{\frac{1}{2}}\Big)\phi_{l-n}$$

which (together with a similar reasoning for α^*) implies that for all $l, n \in \mathbb{Z}_+$ there exists an operator $E_{n,l} \in \mathrm{C}(\mathrm{SU}_q(2))$ such that $\psi^{2,\lambda}(E_{n,l})\phi_k = \delta_{l,k}\phi_n$ ($k \in \mathbb{Z}_+, \lambda \in \mathbb{T}$). Next, for $m \in \mathbb{Z}_+$ we have

$$\psi^{2,\lambda}(q^{-lm} E_{n,l}\gamma^m)\phi_k = \delta_{l,k}\lambda^m \phi_n, \quad \psi^{2,\lambda}(q^{-lm} E_{n,l}\gamma^{*m})\phi_k = \delta_{l,k}\lambda^{-m}\phi_n \quad (k \in \mathbb{Z}_+, \lambda \in \mathbb{T})$$

and consequently for any polynomial function P in $\lambda, \overline{\lambda}$ and $n, l \in \mathbb{Z}_+$ an operator $\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} P(\lambda)\psi^{2,\lambda}(E_{n,l}) \, \mathrm{d}\mu(\lambda)$ belongs to the image of $\mathcal{Q}_L$. From the density of such polynomials in $\mathrm{L}^2(\mathbb{T})$ it follows that for all $f \in \mathrm{L}^2(\mathbb{T})$

$$\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} f(\lambda)\psi^{2,\lambda}(E_{n,l}) \, \mathrm{d}\mu(\lambda) \in \mathcal{Q}_L(\mathrm{L}^2(\mathbb{G})). \tag{3.21}$$

We have an isomorphism (given by choice of bases) $\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda) \simeq \mathrm{L}^2(\mathbb{T})\otimes\mathrm{HS}(\ell^2(\mathbb{Z}_+))$, hence it is clear that operators as in (3.21) span a dense subspace in $\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)$, and consequently $\mathcal{Q}_L$ is unitary. Let us now check the first commutation relation of Proposition 3.5. We have

$$\mathcal{Q}_L\lambda^{\mathbb{\Gamma}}(\omega)\mathcal{Q}_L^*(\mathcal{Q}_L\Lambda_h(a)) = \mathcal{Q}_L\Lambda_h(\lambda^{\mathbb{\Gamma}}(\omega)a) = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \psi^{2,\lambda}(\lambda^{\mathbb{\Gamma}}(\omega)a)D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)$$
$$= \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \psi^{2,\lambda}(\lambda^{\mathbb{\Gamma}}(\omega))\psi^{2,\lambda}(a)D_\lambda^{-1}\,\mathrm{d}\mu(\lambda) = \big(\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \psi^{2,\lambda}(\lambda^{\mathbb{\Gamma}}(\omega))\otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda)\big)\mathcal{Q}_L\Lambda_h(a),$$

for all $\omega\in\ell^1(\mathbb{\Gamma}), a\in\mathrm{C}(\mathrm{SU}_q(2))$ where $\lambda^{\mathbb{\Gamma}}(\omega)=(\omega\otimes\mathrm{id})\mathrm{W}^{\mathbb{\Gamma}}$, hence

$$\mathcal{Q}_L\lambda^{\mathbb{\Gamma}}(\omega)\mathcal{Q}_L^* = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \psi^{2,\lambda}(\lambda^{\mathbb{\Gamma}}(\omega))\otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda) \quad (\omega\in\ell^1(\mathbb{\Gamma})). \tag{3.22}$$

In order to show the second commutation relation, let us show that $\mathcal{Q}_L$ transports J_h to the direct integral of adjoints. For $a\in\mathrm{Pol}(\mathrm{SU}_q(2))$ we have

$$\mathcal{Q}_L J_h\Lambda_h(a) = \mathcal{Q}_L\Lambda_h(\sigma^h_{-i/2}(a^*)) = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \psi^{2,\lambda}(\sigma^h_{-i/2}(a^*))D_\lambda^{-1}\,\mathrm{d}\mu(\lambda).$$

Next, observe that $\psi^{2,\lambda}(\sigma^h_t(a)) = D_\lambda^{-2it}\psi^{2,\lambda}(a)D_\lambda^{2it}$ for all $\lambda\in\mathbb{T}, t\in\mathbb{R}, a\in\mathrm{Pol}(\mathrm{SU}_q(2))$. Indeed, we have $\sigma^h_t(\alpha)=|q|^{-2it}\alpha, \sigma^h_t(\gamma)=\gamma\ (t\in\mathbb{R})$ and consequently

$$\psi^{2,\lambda}(\sigma^h_t(\gamma)) = \psi^{2,\lambda}(\gamma) = D_\lambda^{-2it}\psi^{2,\lambda}(\gamma)D_\lambda^{2it} \quad (t\in\mathbb{R})$$

and similarly for all $k\in\mathbb{Z}_+, t\in\mathbb{R}$

$$D_\lambda^{-2it}\psi^{2,\lambda}(\alpha)D_\lambda^{2it}\phi_k = (1-q^{2k})^{\frac{1}{2}}|q|^{-2ikt}|q|^{2i(k-1)t}\phi_{k-1} = |q|^{-2it}\psi^{2,\lambda}(\alpha)\phi_k = \psi^{2,\lambda}(\sigma^h_t(\alpha))\phi_k.$$

It follows that for all $a\in\mathrm{Pol}(\mathrm{SU}_q(2))$

$$\mathcal{Q}_L J_h\Lambda_h(a) = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} D_\lambda^{-1}\psi^{2,\lambda}(a^*)D_\lambda D_\lambda^{-1}\,\mathrm{d}\mu(\lambda) = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} (\psi^{2,\lambda}(a)D_\lambda^{-1})^*\,\mathrm{d}\mu(\lambda),$$

hence $\mathcal{Q}_L J_h\mathcal{Q}_L^*$ equals $\Sigma = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} J_{\mathsf{H}_\lambda}\,\mathrm{d}\mu(\lambda)$. Now we can show the second commutation relation. Recall that formula $\chi(\mathrm{V}^{\mathbb{\Gamma}}) = (J_h\otimes J_h)(\mathrm{W}^{\mathbb{\Gamma}})^*(J_h\otimes J_h)$ implies that for all $\omega\in\ell^1(\mathbb{\Gamma})$ we have $(\omega\otimes\mathrm{id})\chi(\mathrm{V}^{\mathbb{\Gamma}}) = J_hR^{\mathbb{G}}((\omega\otimes\mathrm{id})\mathrm{W}^{\mathbb{\Gamma}})^*J_h$ (equation (3.20)) and consequently

$$\mathcal{Q}_L(\omega\otimes\mathrm{id})\chi(\mathrm{V}^{\mathbb{\Gamma}})\mathcal{Q}_L^* = \mathcal{Q}_L J_h\mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} (\psi^{2,\lambda}\circ R^{\mathbb{G}})(\lambda^{\mathbb{\Gamma}}(\omega))\otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda)\big)^*\mathcal{Q}_L J_h\mathcal{Q}_L^*$$
$$= \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathbb{1}_{\mathsf{H}_\lambda}\otimes(\psi^{2,\lambda}\circ R^{\mathbb{G}})(\lambda^{\mathbb{\Gamma}}(\omega))^{\mathsf{T}}\,\mathrm{d}\mu(\lambda) = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathbb{1}_{\mathsf{H}_\lambda}\otimes(\psi^{2,\lambda})^c(\lambda^{\mathbb{\Gamma}}(\omega))\,\mathrm{d}\mu(\lambda),$$

which is the second commutation relation. We are left to show

$$\mathcal{Q}_L(\mathrm{L}^\infty(\mathbb{G})\cap \mathrm{L}^\infty(\mathbb{G})')\mathcal{Q}_L^* = \operatorname{Diag}(\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)),$$

let us first argue that

$$\mathcal{Q}_L\,\mathrm{L}^\infty(\mathbb{G})\mathcal{Q}_L^* = \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathrm{B}(\mathsf{H}_\lambda)\otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda)). \tag{3.23}$$

Inclusion $\subseteq$ follows from the commutation relation (3.22). On the other hand, equation (3.22) and a reasoning similar to the one showing that $\mathcal{Q}_L$ is unitary, implies that for any polynomial P in $\lambda,\overline{\lambda}$ and $n,l\in\mathbb{Z}_+$ we have

$$\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} P(\lambda)\psi^{2,\lambda}(E_{n,l})\otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda)\in \mathcal{Q}_L\,\mathrm{L}^\infty(\mathbb{G})\mathcal{Q}_L^*.$$

The σ-WOT density of polynomials in $\mathrm{L}^\infty(\mathbb{T})$ and the isomorphism $\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus}\mathrm{B}(\mathsf{H}_\lambda)\otimes\mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda)\simeq \mathrm{L}^\infty(\mathbb{T})\bar{\otimes}\,\mathrm{B}(\ell^2(\mathbb{Z}_+))$ gives us (3.23). Consequently

$$\begin{aligned}\mathcal{Q}_L(\mathrm{L}^\infty(\mathbb{G})\cap \mathrm{L}^\infty(\mathbb{G})')\mathcal{Q}_L^* &= \big(\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathrm{B}(\mathsf{H}_\lambda)\otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda)\big)\cap\big(\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathbb{1}_{\mathsf{H}_\lambda}\otimes \mathrm{B}(\overline{\mathsf{H}_\lambda})\,\mathrm{d}\mu(\lambda)\big)\\ &= \operatorname{Diag}(\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)).\end{aligned}$$

□

In the next proposition we find the action of the operator P^{it} on the level of direct integrals.

Proposition 3.41. *For each $t\in\mathbb{R}$, operator $\mathcal{Q}_L P^{it}\mathcal{Q}_L^*$ acts on $\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus}\mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)$ as follows:*

$$\mathcal{Q}_L P^{it}\mathcal{Q}_L^*\colon \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} T_\lambda\,\mathrm{d}\mu(\lambda)\mapsto \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} T_{\lambda\,|q|^{2it}}\,\mathrm{d}\mu(\lambda).$$

Note that the above result implies that $\mathcal{Q}_L P^{it}\mathcal{Q}_L^*$ is not decomposable.

Proof. Let $\tilde{P}^{it}$ be the operator in the claim, i.e. $\tilde{P}^{it}\colon \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} T_\lambda\,\mathrm{d}\mu(\lambda)\mapsto\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus} T_{\lambda|q|^{2it}}\,\mathrm{d}\mu(\lambda)$. Clearly it is well defined and bounded. Recall that the scaling group of $\mathbb{G}=\mathrm{SU}_q(2)$ acts as follows

$$\tau_t^{\mathbb{G}}(\alpha)=\alpha,\quad \tau_t^{\mathbb{G}}(\alpha^*)=\alpha^*,\quad \tau_t^{\mathbb{G}}(\gamma)=|q|^{2it}\gamma,\quad \tau_t^{\mathbb{G}}(\gamma^*)=|q|^{-2it}\gamma^*\quad (t\in\mathbb{R}).$$

and P^{it} satisfies $P^{it}\Lambda_h(a)=\Lambda_h(\tau^{\mathbb{G}}_t(a))$ for all $t\in\mathbb{R}, a\in \mathrm{C}(\mathbb{G})$. Fix $l,k,n,m\in\mathbb{Z}_+,\lambda\in\mathbb{T}$ and the corresponding operator $\alpha^l\gamma^n\gamma^{*m}$ in the basis of $\operatorname{Pol}(\mathbb{G})$. We have

$$
\begin{aligned}
\psi^{2,\lambda}(\alpha^l\gamma^n\gamma^{*m})\phi_k &= (\prod_{a=0}^{l-1}(1-q^{2(k-a)})^{\frac12})\lambda^{n-m}q^{k(n+m)}\phi_{k-l}\\
&= |q|^{-2it(n-m)}(\prod_{a=0}^{l-1}(1-q^{2(k-a)})^{\frac12})(\lambda\,|q|^{2it})^{n-m}q^{k(n+m)}\phi_{k-l}\\
&= |q|^{-2it(n-m)}\psi^{2,\lambda|q|^{2it}}(\alpha^l\gamma^n\gamma^{*m})\phi_k,
\end{aligned}
$$

(recall that we use the convention $\phi_{-p}=0$ for $p\in\mathbb{N}$) and consequently

$$
\begin{aligned}
&\mathcal{Q}_L P^{it}\Lambda_h(\alpha^l\gamma^n\gamma^{*m}) = |q|^{2it(n-m)}\mathcal{Q}_L\Lambda_h(\alpha^l\gamma^n\gamma^{*m})\\
&= \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda|q|^{2it}}(\alpha^l\gamma^n\gamma^{*m})D_\lambda^{-1}\,\mathrm{d}\mu(\lambda) = \tilde{P}^{it}\mathcal{Q}_L\Lambda_h(\alpha^l\gamma^n\gamma^{*m}).
\end{aligned}
$$

In a similar manner we check $\mathcal{Q}_L P^{it}\Lambda_h(\alpha^{*l}\gamma^n\gamma^{*m})=\tilde{P}^{it}\mathcal{Q}_L\Lambda_h(\alpha^{*l}\gamma^n\gamma^{*m})$. The claim follows because $\Lambda_h(\operatorname{Pol}(\mathbb{G}))$ is dense in $\mathrm{L}^2(\mathbb{G})$. $\square$

The last result of this section describes the action of an operator $\mathcal{Q}_L J_\varphi\mathcal{Q}_L^*$.

Proposition 3.42. *Operator $\mathcal{Q}_L J_\varphi Q_L^*$ acts on $\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus}\mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)$ as follows:*

$$
\mathcal{Q}_L J_\varphi\mathcal{Q}_L^*\colon \int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus}T_\lambda\,\mathrm{d}\mu(\lambda)\mapsto\int_{\mathrm{Irr}(\mathbb{\Gamma})}^{\oplus}j_\lambda T_{-\operatorname{sgn}(q)\lambda}j_\lambda\,\mathrm{d}\mu(\lambda),
$$

where j_λ is the antilinear operator on $\mathsf{H}_\lambda=\ell(\mathbb{Z}_+)$ given by $j_\lambda\phi_k=\phi_k$ $(\lambda\in\mathbb{T},k\in\mathbb{Z}_+)$.

Note that this result implies that operator $\mathcal{Q}_L J_\varphi\mathcal{Q}_L^*$ is not decomposable if $q>0$.

Proof. Using the formula $R^{\mathbb{G}}=S^{\mathbb{G}}\tau^{\mathbb{G}}_{i/2}$ and [99, Equation 1.14] one easily checks that

$$
R^{\mathbb{G}}(\alpha)=\alpha^*,\quad R^{\mathbb{G}}(\alpha^*)=\alpha,\quad R^{\mathbb{G}}(\gamma)=-\operatorname{sgn}(q)\gamma,\quad R^{\mathbb{G}}(\gamma^*)=-\operatorname{sgn}(q)\gamma^*.
$$

On the other hand we have $R^{\mathbb{G}}(a)=J_\varphi a^* J_\varphi$ for all $a\in\mathrm{C}(\mathrm{SU}_q(2))$, hence

$$
J_\varphi\alpha=\alpha J_\varphi,\quad J_\varphi\alpha^*=\alpha^*J_\varphi,\quad J_\varphi\gamma=-\operatorname{sgn}(q)\gamma^*J_\varphi,\quad J_\varphi\gamma^*=-\operatorname{sgn}(q)\gamma J_\varphi.
$$

Denote by $\tilde{J}_\varphi$ the operator from the claim and fix $\lambda\in\mathbb{T},k,n,m,l\in\mathbb{Z}_+$. We have

$$
\begin{aligned}
\psi^{2,\lambda}(\alpha^l\gamma^m\gamma^{*n})\phi_k &= \lambda^{m-n}q^{(m+n)k}(\prod_{a=0}^{l-1}(1-q^{2(k-a)})^{\frac12})\phi_{k-l}\\
&= (-\operatorname{sgn}(q))^{m+n}(-\operatorname{sgn}(q)\lambda)^{m-n}q^{(m+n)k}(\prod_{a=0}^{l-1}(1-q^{2(k-a)})^{\frac12})\phi_{k-l}\\
&= (-\operatorname{sgn}(q))^{m+n}j_\lambda\psi^{2,-\operatorname{sgn}(q)\lambda}(\alpha^l\gamma^n\gamma^{*m})j_\lambda\phi_k,
\end{aligned}
$$

consequently

$$
\begin{aligned}
&\mathcal{Q}_L J_\varphi \Lambda_h(\alpha^l\gamma^n\gamma^{*m}) = \mathcal{Q}_L \alpha^l(-\operatorname{sgn}(q))^n\gamma^{*n}(-\operatorname{sgn}(q))^m\gamma^m J_\varphi\Lambda_h(\mathbb{1})\\
&= (-\operatorname{sgn}(q))^{n+m}\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(\alpha^l\gamma^m\gamma^{*n})D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)\\
&= \int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus} j_\lambda\psi^{2,-\operatorname{sgn}(q)\lambda}(\alpha^l\gamma^n\gamma^{*m})j_\lambda D_\lambda^{-1}\,\mathrm{d}\mu(\lambda)\\
&= \tilde{J}_\varphi\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\psi^{2,\lambda}(\alpha^l\gamma^n\gamma^{*m})D_\lambda^{-1}\,\mathrm{d}\mu(\lambda) = \tilde{J}_\varphi\mathcal{Q}_L\Lambda_h(\alpha^l\gamma^n\gamma^{*m}).
\end{aligned}
$$

Equation $\mathcal{Q}_L J_\varphi\Lambda_h(\alpha^{*l}\gamma^n\gamma^{*m}) = \tilde{J}_\varphi\mathcal{Q}_L\Lambda_h(\alpha^{*l}\gamma^n\gamma^{*m})$ can be checked similarly. □

Remark. In propositions 3.41, 3.42 we have expressed operators $P^{it}\,(t\in\mathbb{R})$ and J_φ on $\int_{\operatorname{Irr}(\mathbb{\Gamma})}^{\oplus}\operatorname{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)$. Theorem 3.24 and Proposition 3.21 allow us to do the same for $\delta_{\mathbb{\Gamma}}^{it}, \nabla_\varphi^{it}, \nabla_\psi^{it}\,(t\in\mathbb{R})$ – operators obtained in this way are not decomposable.

3.7 Example: quantum $az+b$ group

In this section we will describe some aspects of the theory of the quantum $az+b$ group. We begin by introducing a complex number q and an abelian group $\Gamma_q\subseteq\mathbb{C}^\times$. We will consider three cases:

1) $q=e^{\frac{2\pi i}{N}}$ for a natural number $N\in 2\mathbb{N}\setminus\{2\}$ and $\Gamma_q=\{q^k r\,|\,k\in\mathbb{Z}, r\in\mathbb{R}_{>0}\}$,
2) q is a real number in $]0,1[$ and $\Gamma_q=\{q^{i\theta+k}\,|\,\theta\in\mathbb{R},k\in\mathbb{Z}\}$,
3) $q=e^{\frac{1}{\rho}}$, where $\operatorname{Re}(\rho)<0, \operatorname{Im}(\rho)=\frac{N}{2\pi}$ and $N\in 2\mathbb{Z}\setminus\{0\}$. In this case $\Gamma_q=\{e^{\frac{k+it}{\rho}}\,|\,k\in\mathbb{Z},t\in\mathbb{R}\}$.

It will be more convenient for us to work in the dual picure[25]: let $\widehat{\mathbb{G}}$ be the quantum $az+b$ group associated with the parameter q. We refer the reader to papers [103, 73, 104] for construction of these groups, here we will recall only necessary properties.
We treat all three cases simultaneously. The group Γ_q has closure given by $\overline{\Gamma}_q=\Gamma_q\cup\{0\}$ and is selfdual. This duality is implemented by a certain bicharacter $\chi\colon\Gamma_q\times\Gamma_q\to\mathbb{T}$. We choose a Haar measure on Γ_q in such a way that the Fourier transform $\mathcal{F}(f)(\gamma)=\int_{\Gamma_q}\chi(\gamma,\gamma')f(\gamma')\,\mathrm{d}\mu(\gamma')$ is a unitary operator on $\mathrm{L}^2(\Gamma_q)$. Next, the group Γ_q acts on $\mathrm{C}_0(\overline{\Gamma}_q)$ by translations: $\sigma_\gamma(f)(\gamma')=f(\gamma\gamma')\,(f\in\mathrm{C}_0(\overline{\Gamma}_q),\gamma\in\Gamma_q,\gamma'\in\overline{\Gamma}_q)$. Let $\mathrm{C}_0(\overline{\Gamma}_q)\rtimes_\sigma\Gamma_q\subseteq\mathrm{B}(\mathrm{L}^2(\Gamma_q))$ be the associated crossed product C^*-algebra (note that since Γ_q is abelian, the reduced crossed product is universal). The C^*-algebra $\mathrm{C}_0(\widehat{\mathbb{G}})$ is isomorphic to the crossed product $\mathrm{C}_0(\overline{\Gamma}_q)\rtimes_\sigma\Gamma_q$. Furthermore, it is known that $\widehat{\mathbb{G}}$ is coamenable. Indeed,

[25] In fact, $\mathbb{G}$ is isomorphic to the quantum group opposite to quantum $az+b$.

it was pointed in [73, 74]. It follows from an easy observation that the universal property of $\mathrm{C}_0(\overline{\Gamma}_q) \rtimes_\sigma \Gamma_q$ together with the trivial representation of Γ_q and the character $\mathrm{C}_0(\overline{\Gamma}_q) \ni f \mapsto f(0) \in \mathbb{C}$ give rise to a character of $\mathrm{C}_0(\overline{\Gamma}_q) \rtimes_\sigma \Gamma_q \simeq \mathrm{C}_0(\widehat{\mathbb{G}})$. Then [8, Theorem 3.1] implies that $\widehat{\mathbb{G}}$ is coamenable.

One easily checks that the quotient space $\overline{\Gamma}_q/\Gamma_q$ consists of two points and is not antidiscrete. Consequently, [97, Proposition 7.30] implies that $\mathbb{G}$ is second countable and type I. Using [97, Theorem 8.39] one can describe the spectrum of $\mathrm{C}_0(\widehat{\mathbb{G}}) \simeq \mathrm{C}_0(\overline{\Gamma}_q) \rtimes_\sigma \Gamma_q$: there is a family of one dimensional representations indexed by $\widehat{\Gamma}_q$ and one faithful irreducible infinite dimensional representation given by the inclusion into $\mathrm{B}(\mathrm{L}^2(\Gamma_q))$. Denote this representation by π.

Proposition 3.43. *The Plancherel measure of $\mathbb{G}$ equals the Dirac measure at π, a representation corresponding to the inclusion $\pi\colon \mathrm{C}_0(\widehat{\mathbb{G}}) \xrightarrow{\simeq} \mathrm{C}_0(\overline{\Gamma}_q) \rtimes_\sigma \Gamma_q \hookrightarrow \mathrm{B}(\mathrm{L}^2(\Gamma_q))$. Consequently we have $\mathcal{Q}_L, \mathcal{Q}_R\colon \mathrm{L}^2(\mathbb{G}) \to \mathrm{HS}(\mathrm{L}^2(\Gamma_q))$.*

Proof. It is observed in [104] that we have $\widehat{\psi} \circ \tau_t^{\widehat{\mathbb{G}}} = |q^{-4it}|\widehat{\psi}$ for all $t \in \mathbb{R}$, hence the scaling constant of $\mathbb{G}$ equals $\nu = \hat{\nu}^{-1} = |q^{-4i}|$. In the first and the third case q is not real and it follows that ν is non-trivial. Corollary 3.27 implies that the set of one dimensional representations is of measure zero, and the claim follows[26]. Let us now consider the second case, i.e. $q \in \,]0,1[$. It is argued in [92, Section 5, Proposition A.3] that the von Neumann algebra $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is isomorphic to the von Neumann algebra M associated with a pair (a,b) of admissible normal operators (see [92, Definition 5.1]). Moreover, up to an isomorphism M does not depend on the choice of (a,b), in particular we can take a pair (a,b) introduced in [92, Proposition 5.2]. In this case one easily sees that the resulting von Neumann algebra equals the whole $\mathrm{B}(\ell^2(\mathbb{Z}))$. In particular it is a factor, hence Proposition 3.7 implies that the Plancherel measure of $\mathbb{G}$ must be the Dirac measure at π. □

Now we turn to the problem of identifying operators D_π, E_π. To simplify the notation, we will call these operators respectively D and E. Let us start with introducing two normal (unbounded) operators on $\mathrm{L}^2(\Gamma_q)$: a and b. Operator b acts by multiplication: $(bf)(\gamma) = \gamma f(\gamma)$ $(f \in \mathrm{Dom}(b), \gamma \in \Gamma_q)$ and has the obvious domain. The second operator a can be defined as $a = \mathcal{F} b \mathcal{F}^*$.
Note that there exists an isomorphism of von Neumann algebras $\Phi_R\colon \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{B}(\mathrm{L}^2(\Gamma_q))$ induced by $\mathcal{Q}_R J_{\widehat{\varphi}} J_\varphi$, such that $\Phi_R(x) = \pi(x)$ for $x \in \mathrm{C}_0(\widehat{\mathbb{G}})$ (see Theorem 3.3 and Proposition 3.7). Under this isomorphism, the right Haar integral $\widehat{\psi}$ is transformed to $\mathrm{Tr}(E^{-1} \cdot E^{-1})$ – it follows from the construction of the Plancherel measure in Theorem 3.4. On the other hand, we have $\widehat{\psi}(x) = \mathrm{Tr}(|b|\pi(x)|b|)$ for all $x \in \mathrm{C}_0(\widehat{\mathbb{G}})^+$ ([104, Theorem 3.1]). This means that the weights $\mathrm{Tr}(E^{-1} \cdot E^{-1})$, $\mathrm{Tr}(|b| \cdot |b|)$ are equal on $\Phi_R(\mathrm{C}_0(\widehat{\mathbb{G}}))$. Let θ be the restriction of these weights to $\Phi_R(\mathrm{C}_0(\widehat{\mathbb{G}}))$. The modular automorphism group of $\mathrm{Tr}(E^{-1} \cdot E^{-1})$ is

[26] We remark that it was already observed in [92] that in the first case, $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is isomorphic to the algebra of bounded operators on a separable Hilbert space.

given by $\sigma_t^{\operatorname{Tr}_{E^{-1}}}(A)=E^{-2it}AE^{2it}$, similarly $\sigma_t^{\operatorname{Tr}_{|b|}}(A)=|b|^{2it}A|b|^{-2it}$ $(A\in \mathrm{B}(\mathrm{L}^2(\Gamma_q)),t\in\mathbb{R})$. Next, the weight θ satisfies the KMS condition for both groups $(\sigma_t^{\operatorname{Tr}_{E^{-1}}}|_{\Phi_R(\mathrm{C}_0(\widehat{\mathbb{G}}))})_{t\in\mathbb{R}}$ and $(\sigma_t^{\operatorname{Tr}_{|b|}}|_{\Phi_R(\mathrm{C}_0(\widehat{\mathbb{G}}))})_{t\in\mathbb{R}}$ and as this weight is faithful, [53, Corollary 6.35] implies $E^{-2it}AE^{2it}=|b|^{2it}A|b|^{-2it}$ for all $A\in\Phi_R(\mathrm{C}_0(\widehat{\mathbb{G}})),t\in\mathbb{R}$. By the σ-WOT density of $\Phi_R(\mathrm{C}_0(\widehat{\mathbb{G}}))$ in $\mathrm{B}(\mathrm{L}^2(\Gamma_q))$ we get $E=c|b|^{-1}$ for some $c>0$. Equality $\operatorname{Tr}(E^{-1}\cdot E^{-1})=\operatorname{Tr}(|b|\cdot|b|)$ on $\Phi_R(\mathrm{C}_0(\widehat{\mathbb{G}}))$ forces $c=1$ and consequently $E=|b|^{-1}$.
The next step is to identify the operator D. Observe that Lemma 3.26 implies $f(\pi)=1$, where f is the function from Corollary 3.22. Recall ([73, Section 6.2], [103, Equation 3.18]) that operator $a^{-1}\circ b$ is closable and its closure $a^{-1}b$ is normal. Moreover, we have $R^{\widehat{\mathbb{G}}}(\pi^{-1}(b))=\pi^{-1}(-qa^{-1}b)$. If we combine this information together with Corollary 3.22 and the equality $E=|b|^{-1}$ we arrive at

$$\begin{aligned}
&\mathcal{Q}_L^*(D^{2it}\otimes\mathbb{1}_{\overline{\mathrm{L}^2(\Gamma_q)}})\mathcal{Q}_L=R^{\widehat{\mathbb{G}}}(\mathcal{Q}_L^*(E^{2it}\otimes\mathbb{1}_{\overline{\mathrm{L}^2(\Gamma_q)}})\mathcal{Q}_L)=R^{\widehat{\mathbb{G}}}(\mathcal{Q}_L^*(|b|^{-2it}\otimes\mathbb{1}_{\overline{\mathrm{L}^2(\Gamma_q)}})\mathcal{Q}_L)\\
&=\mathcal{Q}_L^*(|-qa^{-1}b|^{-2it}\otimes\mathbb{1}_{\overline{\mathrm{L}^2(\Gamma_q)}})\mathcal{Q}_L=\mathcal{Q}_L^*(|qa^{-1}b|^{-2it}\otimes\mathbb{1}_{\overline{\mathrm{L}^2(\Gamma_q)}})\mathcal{Q}_L,
\end{aligned}$$

which implies $D=|qa^{-1}b|^{-1}$.

Proposition 3.44. *We have $D=|qa^{-1}b|^{-1}$ and $E=|b|^{-1}$.*

4 The quantum disc

Choose $0 \leq q < 1$ and let $\mathcal{T}_q$ be the universal unital C*-algebra generated by an element s_q satisfying

$$s_q^* s_q - q s_q s_q^* = (1-q)\mathbb{1}. \tag{4.1}$$

This C*-algebra was considered (under the name $\mathrm{C}_{0,q}(\overline{U})$) in [48]. Let us recall an argument ([48, Proposition IV.1]) which shows that such an C*-algebra exists.

Lemma 4.1. *Let* H *be a Hilbert space and* $t \in \mathrm{B}(\mathsf{H})$ *an operator satisfying* $t^*t - qtt^* = (1-q)\mathbb{1}$. *Then* $\|t\| = 1$.

Proof. We have

$$t^*t = qtt^* + (1-q)\mathbb{1},$$

hence

$$\|t\|^2 = \|t^*t\| = \|qtt^* + (1-q)\mathbb{1}\| = q\|tt^*\| + (1-q) = q\|t\|^2 + (1-q)$$

and as $q \neq 1$, the claim follows. □

The above lemma shows that s_q is a contraction in $\mathcal{T}_q$. If we formally set $q = 1$ in (4.1), the resulting $\mathcal{T}_1$ would have to be a universal C*-algebra generated by a normal (bounded) operator – such an algebra does not exists. However, we can define such an algebra if we put an additional condition (superfluous in the case $0 \leq q < 1$) that the generator is a contraction. Due to the spectral theorem, such obtained C*-algebra is isomorphic to the C*-algebra of continuous functions on the closed unit disc, $\mathrm{C}(\mathbb{D})$. Consequently, we can think of algebras $\mathcal{T}_q$ as the algebras of continuous functions on quantum discs.

Our next aim is to realise $\mathcal{T}_q$ as algebras of operators on $\ell^2(\mathbb{Z}_+)$. To this end, let us introduce a weighted shift

$$S_q \colon e_n \mapsto \sqrt{1-q^{n+1}}\, e_{n+1} \quad (n \in \mathbb{Z}_+),$$

where $\{e_n\}_{n\in\mathbb{Z}_+}$ is the standard basis of $\ell^2(\mathbb{Z}_+)$. In particular, $S = S_0 \in \mathrm{B}(\ell^2(\mathbb{Z}_+))$ is the unilateral shift. One easily checks that

$$S_q^* \colon e_n \mapsto \sqrt{1-q^n}\, e_{n-1} \quad (n \in \mathbb{N}), \qquad S_q^* \colon e_0 \mapsto 0$$

and

$$S_q^* S_q - q S_q S_q^* = (1-q)\mathbb{1}.$$

It follows that there exists a $\star$-epimorphism $\rho_q \colon \mathcal{T}_q \to \mathrm{C}^*(S_q)$ defined by $\rho_q(s_q) = S_q$.

Proposition 4.2. *Each* ρ_q *is an* $\star$*-isomorphism and we have* $\mathrm{C}^*(S_q) = \mathrm{C}^*(S)$.

Proof. ρ_0 is a $\star$-isomorphism by Coburn's theorem (see e.g. [27, Theorem V 2.2]). Assume therefore that $q > 0$. Let us first prove the second claim. Since $\mathbb{1} - SS^* \in \mathrm{C}^*(S)$ is the projection onto $\mathbb{C}e_0$, one easily checks that the C^*-algebra of compact operators is contained in $\mathrm{C}^*(S)$. Next, consider compact operator $\mathbf{q} \in \mathrm{B}(\ell^2(\mathbb{Z}_+))$ given by $\mathbf{q}e_n = q^n e_n$ $(n \in \mathbb{Z}_+)$. Clearly $S_q = \sqrt{\mathbb{1} - \mathbf{q}}S$, hence $S_q \in \mathrm{C}^*(S)$ and $\mathrm{C}^*(S_q) \subseteq \mathrm{C}^*(S)$. To prove the converse inclusion observe that $S_q^* S_q = \mathbb{1} - q\mathbf{q} \in \mathrm{C}^*(S_q)$, hence $\mathbf{q} \in \mathrm{C}^*(S_q)$ and

$$S = (\mathbb{1} - \mathbf{q})^{-\frac{1}{2}} S_q = S_q (\mathbb{1} - q\mathbf{q})^{-\frac{1}{2}} \in \mathrm{C}^*(S_q). \tag{4.2}$$

Let us now prove that ρ_q is a $\star$-isomorphism. To this end we will find the inverse map $\mathrm{C}^*(S) = \mathrm{C}^*(S_q) \to \mathcal{T}_q$ using the universal property of $\mathrm{C}^*(S)$. The idea is to construct a "s operator" instide $\mathcal{T}_q$ using formula (4.2). Let us first show that

$$\mathrm{Sp}(s_q s_q^*) \subseteq \{0, 1-q, 1-q^2, \dots\} \cup \{1\}. \tag{4.3}$$

Recall $s_q^* s_q - q s_q s_q^* = (1-q)\mathbb{1}$, hence

$$\begin{aligned} &\mathrm{Sp}(s_q s_q^*) \setminus \{0\} = \mathrm{Sp}(s_q^* s_q) \setminus \{0\} \\ &= \mathrm{Sp}(q s_q s_q^* + (1-q)\mathbb{1}) \setminus \{0\} = (q\,\mathrm{Sp}(s_q s_q^*) + (1-q)) \setminus \{0\}. \end{aligned} \tag{4.4}$$

Assume by contradiction that we have a number $\lambda = \lambda_1 \in \mathrm{Sp}(s_q s_q^*) \setminus (\{0, 1-q, 1-q^2, \dots\} \cup \{1\})$, then using (4.4) we can define

$$\lambda_2 = q^{-1}(\lambda_1 - (1-q)) = q^{-1}\lambda_1 + (1-q^{-1}) \in \mathrm{Sp}(s_q s_q^*) \setminus (\{0, 1-q, 1-q^2, \dots\} \cup \{1\}).$$

Indeed, clearly $\lambda_2 \neq 0$ and if $\lambda_2 = 1$, then

$$q = \lambda_1 - (1-q) \quad \Rightarrow \quad \lambda_1 = 1,$$

a contradiction. Similarly, if $\lambda_2 = 1 - q^n$ for some $n \in \mathbb{N}$ then

$$1 - q^n = q^{-1}(\lambda_1 - (1-q))$$

and

$$\begin{aligned} \lambda_1 &= (1-q) + q(1-q^n) = (1-q)(1 + q(1 + q + \dots + q^{n-1})) \\ &= (1-q)(1 + q + \dots + q^n) = 1 - q^{n+1}, \end{aligned}$$

which again gives us a contradiciton. Consequently, we can inductively define numbers $\lambda_k \in \mathrm{Sp}(s_q s_q^*)$ $(k \in \mathbb{N})$ such that

$$\lambda_{k+1} = q^{-1}\lambda_k + (1 - q^{-1}) \quad (k \in \mathbb{N}).$$

It follows that

$$\lambda_k = q^{-k+1}\lambda_1 + (1-q^{-1}) \sum_{n=0}^{k-2} q^{-n} = q^{-k+1}\lambda_1 + (1-q^{-1}) \tfrac{1-q^{-k+1}}{1-q^{-1}} = 1 + (\lambda_1 - 1) q^{-k+1}$$

for $k \geq 2$. As $0 < q < 1$ and $\lambda_1 \neq 1$, the sequence $(\lambda_k)_{k\in\mathbb{N}}$ diverges to $+\infty$ and we arrive at a contradiction. We have showed equation (4.3). It follows that also

$$\mathrm{Sp}(s_q^* s_q) \subseteq \{0, 1-q, 1-q^2, \dots\} \cup \{1\}.$$

However, if $0 \in \mathrm{Sp}(s_q^* s_q) = q\,\mathrm{Sp}(s_q s_q^*) + (1-q)$ then $-q^{-1}(1-q) \in \mathrm{Sp}(s_q s_q^*)$, a contradiction. This shows that in fact

$$\mathrm{Sp}(s_q^* s_q) \subseteq \{1-q, 1-q^2, \dots\} \cup \{1\}.$$

Recall that $S_q^* S_q = \mathbb{1} - q\mathbf{q}$, hence $q^{-1}(\mathbb{1} - s_q^* s_q)$ plays the role of $\mathbf{q}$ inside $\mathcal{T}_q$. Consequently, let us define $\mathfrak{s} = s_q(s_q^* s_q)^{-\frac{1}{2}} \in \mathcal{T}_q$. One easily sees that $\mathfrak{s}$ is an isometry, hence by the universal property of $\mathrm{C}^*(S)$ there exists a $\star$-homomorphism

$$\mathrm{C}^*(S_q) = \mathrm{C}^*(S) \ni S \mapsto \mathfrak{s} \in \mathcal{T}_q.$$

Clearly this map is the inverse to ρ_q. □

The above proposition tells us that all the quantum discs underlying $\mathcal{T}_q$ $(0 \leq q < 1)$ are homeomorphic. Due to this reason, henceforth we will only consider $\mathcal{T} = \mathcal{T}_0$ which is usually called the Toeplitz algebra. To ease the notation we will write $\mathrm{C}^*(S) = \mathcal{T}$.

The main question we will answer in this section is whether there exists a compact quantum group structure on $\mathcal{T}$, or in other words, whether the quantum disc is a quantum group. This question was posed by Piotr M. Sołtan in his paper [75]. Let us first note that in the classical setting an analogous question has negative answer: there is no compact group structure on the unit disc $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. To see this observe that $\mathbb{D}$ is not homogenous – e.g. there is no homeomorphism of $\mathbb{D}$ taking 0 to 1 – which shows that a structure of a topological group on $\mathbb{D}$ cannot exist[27]. The same result holds for the quantum disc:

Theorem 4.3. *There is no compact quantum group $\mathbb{G}$ with $\mathrm{C}(\mathbb{G}) \simeq \mathcal{T}$.*

Note that we do not assume that $\mathrm{C}(\mathbb{G})$ is universal or reduced.

We've obtained this result together with Piotr M. Sołtan, which resulted in a publication [51]. The proof underwent some modifications. During the revision process an anonymous referee suggested us some changes which made the argument shorter but also more relying on the structure of $\mathcal{T}$ and less on the theory of type I quantum groups. This is why we will present here its former version. Let us mention also that afterwards, together with Alexandru Chirvasitu, we were able to generalise this result to C*-algebras with discrete CCR ideal (see [21] for the definition of a discrete CCR ideal and a precise result).

[27] Let us mention that a similar question for spheres S^n $(n \in \mathbb{N})$ is much more subtle, see [1].

The rest of this section will be devoted to the proof of Theorem 4.3. Assume by contradiction that there exists a compact quantum group $\mathbb{G}$ with $\mathrm{C}(\mathbb{G})$ isomorphic to $\mathcal{T}$. Let us denote the dual discrete quantum group by $\mathbb{\Gamma} = \widehat{\mathbb{G}}$. It will be more convenient for us to work with the concrete C*-algebra of operators $\mathrm{C}^*(S) \subseteq \mathrm{B}(\ell^2(\mathbb{Z}_+))$ generated by the unilateral shift. Let

$$\pi_\bullet \colon \mathrm{C}(\mathbb{G}) \to \mathcal{T} = \mathrm{C}^*(S) \subseteq \mathrm{B}(\mathsf{H}_\bullet)$$

be the coresponding isomorphism, where $\mathsf{H}_\bullet = \ell^2(\mathbb{Z}_+)$.
To proceed we need to recall some properties of the Toeplitz algebra. First, by the universal property of $\mathcal{T}$, there exists a $\star$-epimorphism $\rho\colon \mathcal{T} \to \mathrm{C}(\mathbb{T})$ (called the symbol map) given by $S \mapsto z$, where $z \in \mathrm{C}(\mathbb{T})$ is the identity function on $\mathbb{T}$. The kernel of ρ equals $\mathcal{K} \subseteq \mathrm{B}(\ell^2(\mathbb{Z}_+))$, the algebra of compact operators, hence we have a short exact sequence (see e.g. [27, Theorem V.I.5] or [10, Example II.8.3.2 (v)])

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\rho} \mathrm{C}(\mathbb{T}) \to 0. \tag{4.5}$$

Furthermore, $\mathcal{K}$ is an essential ideal in $\mathcal{T}$ [22, Theorem 1]. It follows that $\mathcal{T}$ is postliminal, hence of type I ([33, Theorem 9.1]). The spectrum of $\mathcal{T}$, $\mathrm{Irr}(\mathbb{\Gamma}) = \mathrm{Irr}(\mathcal{T})$ is equal to $\{\bullet\} \cup \mathbb{T}$, where $\bullet$ corresponds to the representation $\pi_\bullet$ and each $\lambda \in \mathbb{T}$ is associated with the character

$$\rho_\lambda \colon \mathcal{T} \xrightarrow{\rho} \mathrm{C}(\mathbb{T}) \xrightarrow{\mathrm{ev}_\lambda} \mathbb{C},$$

where ev_λ is the evaluation at $\lambda \in \mathbb{T}$. It is not difficult to check that the (Mackey Borel) measurable structure on $\mathrm{Irr}(\mathbb{\Gamma})$ is the "obvious one": $\{\bullet\}$ is measurable and the measurable structure on $\mathbb{T}$ is the Borel measurable structure corresponding to the standard topology (see [33, Section 3.8.1]). Then because $\mathcal{T}$ is of type I, by [33, Proposition 4.6.1] this measurable structure equals the measurable structure induced by topology.

After recalling these results about the Toeplitz algebra, we can prove some preliminary results concerning $\mathbb{G}$.

Claim 1. The Haar integral on $\mathbb{G}$ is faithful, i.e. $\mathrm{C}(\mathbb{G}) = \mathrm{C}^r(\mathbb{G})$.
Proof. Denote the Haar integral by h. Its kernel $\ker(h) = \{a \in \mathrm{C}(\mathbb{G}) \,|\, h(a^*a) = 0\}$ is an ideal ([98, Page 656], see also [59, Proposition 7.9]), hence if it is non-trivial, we have $\mathcal{K} \subseteq \ker(h)$. But then the reduced C*-algebra of continuous functions on $\mathbb{G}$, $\mathrm{C}^r(\mathbb{G})$, which is the quotient $\mathrm{C}^r(\mathbb{G}) = \mathrm{C}(\mathbb{G})/\ker(h)$ (see [7, Section 2]) would be commutative. However, the quotient map is injective on $\mathrm{Pol}(\mathbb{G}) \subseteq \mathrm{C}(\mathbb{G})$ which forces $\ker(h) = \{0\}$. □

Claim 2. $\mathbb{G}$ is coamenable, in particular $\mathrm{C}(\mathbb{G}) = \mathrm{C}^u(\mathbb{G})$.
Proof. This claim follows from [7, Corollary 2.9] – a compact quantum group $\mathbb{H}$ is coamenable if and only if there exists a character on the reduced C*-algebra of continuous functions on $\mathbb{H}$. Since $\mathrm{C}(\mathbb{G}) \simeq \mathcal{T}$, there exists a character on $\mathrm{C}(\mathbb{G})$. □

Claim 3. $\mathbb{G}$ is not of Kac type.
Proof. If $\mathbb{G}$ were of Kac type, then its Haar integral h would give a faithful tracial state

on $\mathcal{T}$. However, the operator $\mathbb{1} - SS^* \in \mathcal{T}$ is non-zero, positive and it is annihilated by any tracial state. $\square$

In the remainder of the proof we will use theory of non-Kac type quantum groups together with theory of type I quantum groups to arrive at a contradiction.
Let us denote by $\tilde{\mathbb{G}}$ the group of characters on $\mathrm{C}(\mathbb{G})$, with the group operation given by the convolution $\tilde{\mathbb{G}} \times \tilde{\mathbb{G}} \ni (\phi, \phi') \mapsto \phi \star \phi' = (\phi \otimes \phi') \circ \Delta \in \tilde{\mathbb{G}}$. $\tilde{\mathbb{G}}$ is equipped with the w*-topology of $\mathrm{C}(\mathbb{G})^*$ – this makes $\tilde{\mathbb{G}}$ into a compact Hausdorff group ([7, Theorem 3.5]). Since $\mathrm{C}(\mathbb{G}) \simeq \mathcal{T}$, we have $\tilde{\mathbb{G}} = \{\rho_\lambda \circ \pi_\bullet \,|\, \lambda \in \mathbb{T}\}$. Recall that in Section 2.3 we have introduced a family of functionals $\{f_z\}_{z\in\mathbb{C}}$ on $\mathrm{Pol}(\mathbb{G})$. As $\{f_{it}\}_{t\in\mathbb{R}}$ are $\star$-preserving ([64, Proposition 1.7.2 (ii)]), they extend to characters on $\mathrm{C}(\mathbb{G})$ ([7, Theorem 3.3]). Let us denote by F_W the set $\{f_{it}\}_{t\in\mathbb{R}}$. It is easy to see that F_W is a subgroup of $\tilde{\mathbb{G}}$ (see [64, Proposition 1.7.2 (iii)]), furthermore the map $\mathbb{R} \ni t \mapsto f_{it} \in \tilde{\mathbb{G}}$ is continuous. As $\mathbb{R}$ is connected it follows that F_W is a non-trivial connected subgroup of $\tilde{\mathbb{G}}$ which is homeomorphic to a circle – consequently we have $F_W = \tilde{\mathbb{G}}$:

Lemma 4.4. *The group of characters on* $\mathrm{C}(\mathbb{G})$ *equals* $\{\rho_\lambda \circ \pi_\bullet \,|\, \lambda \in \mathbb{T}\} = \{f_{it} \,|\, t \in \mathbb{R}\}$.

For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ let us choose a basis in H_α which diagonalises operator $\uprho_\alpha$. Denote the corresponding eigenvalues by $\uprho_{\alpha,i}$ $(i \in \{1, \dots, \dim(\alpha)\})$. Equation (2.18) implies

$$f_{it}(U^\alpha_{i,j}) = \delta_{i,j} \uprho^{it}_{\alpha,i} \quad (\alpha \in \mathrm{Irr}(\mathbb{G}),\, i, j \in \{1, \dots, \dim(\alpha)\}).$$

In next two propositions which are based on Lemma 4.4 we connect properties coming from two pictures – quantum group $\mathrm{C}(\mathbb{G})$ and operator algebra $\mathcal{T} \subseteq \mathrm{B}(\ell^2(\mathbb{Z}_+))$.

Proposition 4.5. *An operator* $A \in \mathcal{T}$ *is compact if and only if* $f_{it}(\pi_\bullet^{-1}(A)) = 0$ *for all* $t \in \mathbb{R}$.

Proof. The short exact sequence (4.5) implies that A is compact if and only if $\rho_\lambda(A) = 0$ for all $\lambda \in \mathbb{T}$. As $\{\rho_\lambda \,|\, \lambda \in \mathbb{T}\} = \{f_{it} \circ \pi_\bullet^{-1} \,|\, t \in \mathbb{R}\}$ we get the claim. $\square$

Proposition 4.6. *For any* $\alpha \in \mathrm{Irr}(\mathbb{G})$ *and* $i, j \in \{1, \dots, \dim(\alpha)\}$ *the operator* $\pi_\bullet(U^\alpha_{i,j})$ *is compact if and only if* $i \neq j$. *Moreover,* $\pi_\bullet(U^\alpha_{i,i})$ *is a Fredholm operator.*

Proof. The first part of the proposition follows immediately from Proposition 4.5. For the second part note that from the unitarity of U^α:

$$\sum_{j=1}^{n_\alpha} U^\alpha_{i,j} {U^\alpha_{i,j}}^* = \mathbb{1}$$

and from the fact that $\pi_\bullet(U^\alpha_{i,j})$ are compact for $i \neq j$ it follows that $\rho\big(\pi_\bullet(U^\alpha_{i,i})\big)$ is unitary, so the operator $\pi_\bullet(U^\alpha_{i,i})$ is Fredholm. $\square$

The observation that $\pi_\bullet(U^\alpha_{i,i})$ is Fredholm, will be a crucial ingredient in our proof. The next step in our reasoning is to treat $\mathbb{\Gamma} = \widehat{\mathbb{G}}$ as a second countable (discrete) type I quantum group. Theorem 3.3 gives us a measure μ on $\mathrm{Irr}(\mathbb{\Gamma}) = \{\bullet\} \cup \mathbb{T}$, unitary operator $\mathcal{Q}_L$ and a measurable field of strictly positive, self-adjoint operators $D_\bullet, D_\lambda (\lambda \in \mathbb{T})$. Since $\mathrm{L}^\infty(\mathbb{G})$ is non-commutative, Proposition 3.7 implies that subset $\{\bullet\}$ has positive measure (recall that representations coresponding to $\mathbb{T}$ are one dimensional), hence after rescalling we may assume that $\mu(\{\bullet\}) = 1$. Consequently we will write

$$
\begin{aligned}
\mathcal{Q}_L(\mathrm{L}^2(\mathbb{G})) &= \mathrm{HS}(\mathsf{H}_\bullet) \oplus \int_{\mathbb{T}}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda)\, \mathrm{d}\mu(\lambda) \\
\mathcal{Q}_L \mathrm{L}^\infty(\mathbb{G}) \mathcal{Q}_L^* &= (\mathrm{B}(\mathsf{H}_\bullet) \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}}) \oplus \int_{\mathbb{T}}^{\oplus} \mathrm{B}(\mathsf{H}_\lambda) \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\, \mathrm{d}\mu(\lambda)
\end{aligned}
\tag{4.6}
$$

and work with the above decomposition. It will be useful to introduce the following notation: write M for $\mathcal{Q}_L \mathrm{L}^\infty(\mathbb{G}) \mathcal{Q}_L^*$ and $\mathrm{M}_1, \mathrm{M}_2$ for the two summands in the above decomposition, so that $\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}_2$. Next, let us denote by $\mathbb{1}_i \in \mathrm{M}_i \,(i \in \{1,2\})$ the units, so that $\mathbb{1}_1 + \mathbb{1}_2 = \mathbb{1} \in \mathrm{M}$. The next lemma holds for general type I, second countable locally compact quantum groups.

Lemma 4.7. *Let $\mathbb{H}$ be a type I, second countable locally compact quantum group with Plancherel measure $\mu_{\mathbb{H}}$, corresponding unitary operator $\mathcal{Q}_L^{\mathbb{H}}$ and measurable field of representations $(\pi_x)_{x \in \mathrm{Irr}(\mathbb{H})}$. Then for any $a \in \mathrm{C}_0^u(\widehat{\mathbb{H}})$ we have*

$$
\mathcal{Q}_L^{\mathbb{H}} \Lambda_{\widehat{\mathbb{H}}}(a) \mathcal{Q}_L^{\mathbb{H}*} = \int_{\mathrm{Irr}(\mathbb{H})}^{\oplus} (\pi_x(a) \otimes \mathbb{1}_{\overline{\mathsf{H}_x}})\, \mathrm{d}\mu_{\mathbb{H}}(x), \tag{4.7}
$$

where $\Lambda_{\widehat{\mathbb{H}}} \colon \mathrm{C}_0^u(\widehat{\mathbb{H}}) \to \mathrm{C}_0(\widehat{\mathbb{H}})$ is the reducing morphism.

Proof. Given $a \in \mathrm{C}_0^u(\widehat{\mathbb{H}})$, element $\mathcal{Q}_L^{\mathbb{H}} \Lambda_{\widehat{\mathbb{H}}}(a) \mathcal{Q}_L^{\mathbb{H}*}$ can be written as

$$
\mathcal{Q}_L^{\mathbb{H}} \Lambda_{\widehat{\mathbb{H}}}(a) \mathcal{Q}_L^{\mathbb{H}*} = \int_{\mathrm{Irr}(\mathbb{H})}^{\oplus} (a_x \otimes \mathbb{1}_{\overline{\mathsf{H}_x}})\, \mathrm{d}\mu_{\mathbb{H}}(x).
$$

for some measurable field $(a_x)_{x \in \mathrm{Irr}(\mathbb{H})}$. By Desmedt's result (Theorem 3.3) for any $\omega \in \mathrm{L}^1(\mathbb{H})$ we have

$$
\mathcal{Q}_L^{\mathbb{H}}((\omega \otimes \mathrm{id})(\mathrm{W})) \mathcal{Q}_L^{\mathbb{H}*} = \int_{\mathrm{Irr}(\mathbb{H})}^{\oplus} (\omega \otimes \mathrm{id})(U^{\pi_x}) \otimes \mathbb{1}_{\overline{\mathsf{H}_x}}\, \mathrm{d}\mu_{\mathbb{H}}(x),
$$

where U^{π_x} is the unitary representation of $\mathbb{H}$ corresponding to π_x. Thus

$$
\mathcal{Q}_L^{\mathbb{H}} \Lambda_{\widehat{\mathbb{H}}}(\lambda^u(\omega)) \mathcal{Q}_L^{\mathbb{H}*} = \int_{\mathrm{Irr}(\mathbb{H})}^{\oplus} \pi_x(\lambda^u(\omega)) \otimes \mathbb{1}_{\overline{\mathsf{H}_x}}\, \mathrm{d}\mu_{\mathbb{H}}(x).
$$

Both sides of the above equation are continuous with respect to $\lambda^u(\omega)$ (for the right hand side we can use [34, Section 2.3, Proposition 4] because the range of λ^u is dense in $\mathrm{C}_0^u(\widehat{\mathbb{H}})$) and (4.7) follows. □

Lemma 4.8. *Let β be an automorphism of M. Then β preserves the decomposition $M = M_1 \oplus M_2$. In particular $\beta(\mathbb{1}_1) = \mathbb{1}_1$ and $\beta(\mathbb{1}_2) = \mathbb{1}_2$.*

Proof. Let E_1 and E_2 be the projections of M onto the two summands M_1 and M_2 (so that $E_i(x) = \mathbb{1}_i x$ for any $x \in M$). The map $M_1 \ni y \mapsto E_2\big(\beta(y)\big) \in M_2$ is a normal $*$-homomorphism $M_1 \to M_2$ which must be zero because M_1 is a factor and M_2 is commutative. It follows that $\beta(M_1) \subseteq M_1$ and since this is also true for the automorphism β^{-1}, we have $\beta^{-1}(M_1) \subseteq M_1$ and acting with β on both sides gives $M_1 \subseteq \beta(M_1)$. It follows that β restricts to an automorphisms of M_1, so it must preserve $\mathbb{1}_1$. Clearly if there were $z \in M_2$ such that $E_1\big(\beta(z)\big) \neq 0$ then $z = \beta^{-1}(\beta(z)) = \beta^{-1}\big(E_1(\beta(z)) + E_2(\beta(z))\big) = \beta^{-1}\big(E_1(\beta(z))\big) + \beta^{-1}\big(E_2(\beta(z))\big)$ which is a contradiction because β^{-1} is injective and preserves M_1, so $\beta^{-1}\big(E_1(\beta(z))\big) = 0$ and hence $E_1(\beta(z)) = 0$. It follows that β preserves M_2 and consequently $\beta(\mathbb{1}_2) = \mathbb{1}_2$. $\square$

One way to use Lemma 4.8 is to apply it to the scaling automorphisms of $\mathrm{L}^\infty(\mathbb{G})$ transferred to M via the unitary $\mathcal{Q}_L$:

$$\beta_t(a) = \mathcal{Q}_L \tau_t\big(\mathcal{Q}_L^* a \mathcal{Q}_L\big)\mathcal{Q}_L^*, \quad (a \in \mathrm{M},\ t \in \mathbb{R}).$$

It follows that the one-parameter group $(\beta_t)_{t\in\mathbb{R}}$ restricts to a one-parameter group of automorphisms of M_1. Thus we obtain a one-parameter group of automorphisms $(\alpha_t)_{t\in\mathbb{R}}$ of $\mathrm{B}(\mathsf{H}_\bullet)$ defined by

$$\alpha_t(x) \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}} = \mathcal{Q}_L \tau_t(\mathcal{Q}_L^*(x \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}})\mathcal{Q}_L)\mathcal{Q}_L^* \quad (x \in \mathrm{B}(\mathsf{H}_\bullet),\ t \in \mathbb{R})$$

and thus by [46, Theorem 4.13] (see also [47]) there exists a strongly continuous one-parameter group of unitary operators $(A_t)_{t\in\mathbb{R}}$ on $\mathsf{H}_\bullet$ such that

$$\alpha_t(x) \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}} = A_t x A_{-t} \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}}, \quad (x \in \mathrm{B}(\mathsf{H}_\bullet),\ t \in \mathbb{R}).$$

The one-parameter group $(P^{it})_{t\in\mathbb{R}}$ also induces automorphisms of $\mathrm{L}^\infty(\mathbb{G})' \subseteq \mathrm{B}(\mathrm{L}^2(\mathbb{G}))$ which can be transferred to automorphisms of M' by the unitary $\mathcal{Q}_L$. Clearly $M' = {M_1}' \oplus {M_2}'$ and by a process analogous to the one presented for M we obtain a σ-weakly continuous one-parameter group of automorphisms of ${M_1}' = \mathbb{1}_{\mathsf{H}_\bullet} \otimes \mathrm{B}(\overline{\mathsf{H}_\bullet})$ which yields a group $(\alpha'_t)_{t\in\mathbb{R}}$ of automorphisms of $\mathrm{B}(\overline{\mathsf{H}_\bullet})$:

$$\mathbb{1}_{\mathsf{H}_\bullet} \otimes \alpha'_t(y) = \mathcal{Q}_L P^{it}\big(\mathcal{Q}_L^*(\mathbb{1}_{\mathsf{H}_\bullet} \otimes y)\mathcal{Q}_L\big)P^{-it}\mathcal{Q}_L^* \quad (y \in \mathrm{B}(\overline{\mathsf{H}_\bullet}),\ t \in \mathbb{R}).$$

It follows that there is a strongly continuous one-parameter group of unitary operators $(B_t)_{t\in\mathbb{R}}$ on $\mathsf{H}_\bullet$ such that

$$\mathbb{1}_{\mathsf{H}_\bullet} \otimes \alpha'_t(y) = \mathbb{1}_{\mathsf{H}_\bullet} \otimes B_t^\top y B_{-t}^\top, \quad (y \in \mathrm{B}(\overline{\mathsf{H}_\bullet}),\ t \in \mathbb{R})$$

(for future notational convenience we choose to consider the group $(B_t)_{t\in\mathbb{R}}$ on $\mathsf{H}_\bullet$ and work with the transposed operators on $\overline{\mathsf{H}_\bullet}$).

Clearly the group $(\beta_t)_{t\in\mathbb{R}}$ is implemented by the unitary operators $(\mathcal{Q}_L P^{it}\mathcal{Q}_L^*)_{t\in\mathbb{R}}$ and since the group preserves the projections $\mathbb{1}_1$ and $\mathbb{1}_2$, these operators are block-diagonal in the decomposition (4.6). It follows that for any $x\in \mathrm{B}(\mathsf{H}_\bullet)$ and $y\in \mathrm{B}(\overline{\mathsf{H}_\bullet})$ we have

$$(\mathbb{1}_1\mathcal{Q}_L P^{it}\mathcal{Q}_L^*)(x\otimes y)(\mathbb{1}_1\mathcal{Q}_L P^{-it}\mathcal{Q}_L^*)=\alpha_t(x)\otimes\alpha'_t(y)=(A_t\otimes B_t^\top)(x\otimes y)(A_{-t}\otimes B_{-t}^\top)$$

for all $t\in\mathbb{R}$. This implies

$$\mathbb{1}_1\mathcal{Q}_L P^{it}\mathcal{Q}_L^*=\lambda_t(A_t\otimes B_t^\top),\qquad t\in\mathbb{R}$$

for some complex numbers $(\lambda_t)_{t\in\mathbb{R}}$ of absolute value 1 depending continuously on t. Moreover, since the two one-parameter groups $(\mathcal{Q}_L P^{it}\mathcal{Q}_L^*)_{t\in\mathbb{R}}$ and $(A_t\otimes B_t^\top)_{t\in\mathbb{R}}$ obviously commute, $t\mapsto\lambda_t$ is also a homomorphism, so defining $\widetilde{A}_t=\lambda_t A_t$ we obtain a strongly continuous one-parameter group of unitaries such that

$$\mathbb{1}_1\mathcal{Q}_L P^{it}\mathcal{Q}_L^*=\widetilde{A}_t\otimes B_t^\top\quad(t\in\mathbb{R}).$$

Proposition 4.9. *With the notation introduced above we have $\widetilde{A}_t=B_{-t}$ for all $t\in\mathbb{R}$.*

Proof. We will use the fact that for all t we have $P^{it}J_{\mathbb{G}}=J_{\mathbb{G}}P^{it}$ (equation (2.14)). Moreover, by Proposition 3.7 the operator

$$\mathcal{Q}_L J_{\mathbb{G}}\mathcal{Q}_L^*\colon\ \mathrm{HS}(\mathsf{H}_\bullet)\oplus\int_{\mathbb{T}}^{\oplus}\mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)\to\mathrm{HS}(\mathsf{H}_\bullet)\oplus\int_{\mathbb{T}}^{\oplus}\mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda)$$

acts as

$$(\xi\otimes\overline{\eta})\oplus\int_{\mathbb{T}}^{\oplus}(\xi_\lambda\otimes\overline{\eta_\lambda})\,\mathrm{d}\mu(\lambda)\mapsto(\eta\otimes\overline{\xi})\oplus\int_{\mathbb{T}}^{\oplus}(\eta_\lambda\otimes\overline{\xi_\lambda})\,\mathrm{d}\mu(\lambda).$$

In particular $\mathcal{Q}_L J_{\mathbb{G}}\mathcal{Q}_L^*$ is block-diagonal and

$$\mathbb{1}_1\mathcal{Q}_L J_{\mathbb{G}}\mathcal{Q}_L^*\colon\ \mathrm{HS}(\mathsf{H}_\bullet)\ni(\xi\otimes\overline{\eta})\mapsto(\eta\otimes\overline{\xi})\in\mathrm{HS}(\mathsf{H}_\bullet).$$

Therefore

$$\begin{aligned}\mathbb{1}_1\mathcal{Q}_L J_{\mathbb{G}}\mathcal{Q}_L^*\mathbb{1}_1(\widetilde{A}_t\otimes B_t^\top)&=\mathbb{1}_1\mathcal{Q}_L J_{\mathbb{G}}P^{it}\mathcal{Q}_L^*\mathbb{1}_1\\&=\mathbb{1}_1\mathcal{Q}_L P^{it}J_{\mathbb{G}}\mathcal{Q}_L^*\mathbb{1}_1=(\widetilde{A}_t\otimes B_t^\top)\mathbb{1}_1\mathcal{Q}_L J_{\mathbb{G}}\mathcal{Q}_L^*\mathbb{1}_1\end{aligned}$$

for all $t\in\mathbb{R}$. The claim follows from Lemma 7.8. □

Corollary 4.10. For each $t\in\mathbb{R}$ the restriction of $\mathcal{Q}_L P^{it}\mathcal{Q}_L^*$ to $\mathrm{HS}(\mathsf{H}_\bullet)$ is equal to $B_{-t}\otimes B_t^\top$.

In what follows we let B be the unique strictly positive, self-adjoint operator on $\mathsf{H}_\bullet$ satisfying $B_t=B^{it}$ for all t.

Recall that using operators $D_\bullet,(D_\lambda)_{\lambda\in\mathbb{T}}$ we can express the Haar integral on $\widehat{\mathbb{\Gamma}}=\mathbb{G}$ (see Theorem 3.3): in particular we have

$$1=h(\mathbb{1})=\mathrm{Tr}(D_\bullet^{-2})+\int_{\mathbb{T}}\mathrm{Tr}(D_\lambda^{-2})\,\mathrm{d}\mu(\lambda).$$

This shows that $D_\bullet^{-1}$ is a Hilbert-Schmidt operator, in particular it is compact hence $D_\bullet^{-1} \in \mathcal{T} \subseteq \mathrm{B}(\mathsf{H}_\bullet)$. The eigenvalues of $D_\bullet^{-1}$ are of finite multiplicity, they form a countable subset of $]0, +\infty[$ and we have the norm convergent series

$$D_\bullet^{-1} = \sum_{q \in \mathrm{Sp}(D_\bullet^{-1})} q 1_{\{q\}}(D_\bullet^{-1}).$$

Proposition 4.11. *The operators B and $D_\bullet$ strongly commute.*

Proof. From the properties of the Plancherel measure we get

$$\mathcal{Q}_L \Lambda_h(\mathbb{1}) = D_\bullet^{-1} \oplus \int_{\mathbb{T}}^{\oplus} D_\lambda^{-1} \, \mathrm{d}\mu(\lambda) \in \mathrm{HS}(\mathsf{H}_\bullet) \oplus \int_{\mathbb{T}}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda) \, \mathrm{d}\mu(\lambda).$$

Now we fix $t \in \mathbb{R}$ and note that since $\tau_t(\mathbb{1}) = \mathbb{1}$, we have

$$\begin{aligned} D_\bullet^{-1} \oplus \int_{\mathbb{T}}^{\oplus} D_\lambda^{-1} \, \mathrm{d}\mu(\lambda) &= \mathcal{Q}_L \Lambda_h(\mathbb{1}) = \mathcal{Q}_L \Lambda_h(\tau_t(\mathbb{1})) \\ &= \mathcal{Q}_L P^{it} \Lambda_h(\mathbb{1}) = B_{-t} D_\bullet^{-1} B_t \oplus \big(\mathbb{1}_2(\mathcal{Q}_L P^{it} \mathcal{Q}_L^*)\big)\Big(\int_{\mathbb{T}}^{\oplus} D_\lambda^{-1} \, \mathrm{d}\mu(\lambda)\Big), \end{aligned}$$

which implies that $B_t D_\bullet^{-1} = D_\bullet^{-1} B_t$. ☐

Corollary 4.12. The operator B preserves the decomposition of $\mathsf{H}_\bullet$ into eigenspaces of $D_\bullet^{-1}$:

$$\mathsf{H}_\bullet = \bigoplus_{q \in \mathrm{Sp}(D_\bullet^{-1})} 1_{\{q\}}(D_\bullet^{-1}) \mathsf{H}_\bullet$$

so that

$$B = \bigoplus_{q \in \mathrm{Sp}(D_\bullet^{-1})} 1_{\{q\}}(D_\bullet^{-1}) \, B \, 1_{\{q\}}(D_\bullet^{-1}).$$

Now we will prove three lemmas relating the structure of the compact quantum group $\mathbb{G}$ to the decomposition of $\mathsf{H}_\bullet$ into eigenspaces of $D_\bullet^{-1}$.

Lemma 4.13. *For $a \in \mathcal{T}$ and $t \in \mathbb{R}$ we have*

$$\begin{aligned} \pi_\bullet\big(\sigma_t^h(\pi_\bullet^{-1}(a))\big) &= D_\bullet^{-2it} a D_\bullet^{2it}, \\ \pi_\bullet\big(\tau_t(\pi_\bullet^{-1}(a))\big) &= B_{-t} a B_t. \end{aligned}$$

Proof. By Theorem 3.24 the modular operator for h transported via the unitary $\mathcal{Q}_L$ acts as follows:

$$\mathcal{Q}_L \nabla_h^{it} \mathcal{Q}_L^* = \big(D_\bullet^{-2it} \otimes (D_\bullet^{2it})^\top\big) \oplus \int_{\mathbb{T}}^{\oplus} \big(D_\lambda^{-2it} \otimes (D_\lambda^{2it})^\top\big) \, \mathrm{d}\mu(\lambda).$$

Take $a \in \mathrm{C}(\mathbb{G})$. Denoting $\pi_\lambda = \rho_\lambda \circ \pi_\bullet$ we conclude using Lemma 4.7 that

$$\mathcal{Q}_L a \mathcal{Q}_L^* = \bigl(\pi_\bullet(a) \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}}\bigr) \oplus \int_{\mathbb{T}}^{\oplus} \bigl(\pi_\lambda(a) \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\bigr)\, \mathrm{d}\mu(\lambda),$$

so

$$\begin{aligned}&\bigl(\pi_\bullet\bigl(\sigma_t^h(a)\bigr) \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}}\bigr) \oplus \int_{\mathbb{T}}^{\oplus} \bigl(\pi_\lambda\bigl(\sigma_t^h(a)\bigr) \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\bigr)\, \mathrm{d}\mu(\lambda)\\ &= \mathcal{Q}_L \sigma_t^h(a) \mathcal{Q}_L^* = \mathcal{Q}_L \nabla_h^{it} a \nabla_h^{-it} \mathcal{Q}_L^*\\ &= \bigl(D_\bullet^{-2it} \pi_\bullet(a) D_\bullet^{2it} \otimes \mathbb{1}_{\overline{\mathsf{H}_\bullet}}\bigr) \oplus \int_{\mathbb{T}}^{\oplus} \bigl(\pi_\lambda(a) \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\bigr)\, \mathrm{d}\mu(\lambda)\end{aligned}$$

(representations $\{\pi_\lambda\}_{\lambda\in\mathbb{T}}$ are one-dimensional) and consequently

$$\pi_\bullet\bigl(\sigma_t^h(\pi_\bullet^{-1}(a))\bigr) = D_\bullet^{-2it} a D_\bullet^{2it} \quad (a \in \mathcal{T},\ t \in \mathbb{R}).$$

The second part of the lemma is proved analogously using Corollary 4.12. □

Lemma 4.14. *For any $\alpha \in \mathrm{Irr}(\mathbb{G})$, $i \in \{1, \dots, \dim(\alpha)\}$ and $q \in \mathrm{Sp}(D_\bullet^{-1})$ the operator $\pi_\bullet(U_{i,i}^\alpha)$ shifts the eigenspaces of $D_\bullet^{-1}$ as follows:*

$$\pi_\bullet(U_{i,i}^\alpha) 1_{\{q\}}(D_\bullet^{-1}) \mathsf{H}_\bullet \subseteq 1_{\{q\rho_{\alpha,i}\}}(D_\bullet^{-1}) \mathsf{H}_\bullet.$$

Proof. For $t \in \mathbb{R}$ and $\xi \in 1_{\{q\}}(D_\bullet^{-1})\mathsf{H}_\bullet$, by Lemma 4.13 we have

$$D_\bullet^{it} \pi_\bullet(U_{i,i}^\alpha)\xi = D_\bullet^{it} \pi_\bullet(U_{i,i}^\alpha) D_\bullet^{-it} D_\bullet^{it}\xi = \pi_\bullet\bigl(\sigma_{-t/2}^h(U_{i,i}^\alpha)\bigr) q^{-it}\xi = q^{-it} \rho_{\alpha,i}^{-it} \pi_\bullet(U_{i,i}^\alpha)\xi$$

(cf. Section 2.3) which means that $\pi_\bullet(U_{i,i}^\alpha)\xi \in 1_{\{q\rho_{\alpha,i}\}}(D_\bullet^{-1})\mathsf{H}_\bullet$. □

Clearly for any $q \in \mathrm{Sp}(D_\bullet^{-1})$ the operator $1_{\{q\}}(D_\bullet^{-1})\, B\, 1_{\{q\}}(D_\bullet^{-1})$ acting on $1_{\{q\}}(D_\bullet^{-1})\mathsf{H}_\bullet$ is bounded and positive. We let Δ_q denote its spectrum:

$$\Delta_q = \mathrm{Sp}\bigl(1_{\{q\}}(D_\bullet^{-1})\, B\, 1_{\{q\}}(D_\bullet^{-1})\bigr).$$

Lemma 4.15. *For any $\alpha \in \mathrm{Irr}(\mathbb{G})$, $i \in \{1, \dots, \dim(\alpha)\}$, $q \in \mathrm{Sp}(D_\bullet^{-1})$ and $c \in \Delta_q$ we have*

$$\pi_\bullet(U_{i,i}^\alpha) 1_{\{c\}}\bigl(1_{\{q\}}(D_\bullet^{-1})\, B\, 1_{\{q\}}(D_\bullet^{-1})\bigr)\mathsf{H}_\bullet \subseteq 1_{\{c\}}\bigl(1_{\{q\rho_{\alpha,i}\}}(D_\bullet^{-1})\, B\, 1_{\{q\rho_{\alpha,i}\}}(D_\bullet^{-1})\bigr)\mathsf{H}_\bullet. \tag{4.8}$$

Proof. Fix $t \in \mathbb{R}$. Since $U_{i,i}^\alpha$ is invariant under the scaling group (Section 2.3), from Lemma 4.13 we know that $B_t \pi_\bullet(U_{i,i}^\alpha) B_{-t} = \pi_\bullet\bigl(\tau_{-t}(U_{i,i}^\alpha)\bigr) = \pi_\bullet(U_{i,i}^\alpha)$. Therefore if

$$\xi \in 1_{\{c\}}\bigl(1_{\{q\}}(D_\bullet^{-1})\, B\, 1_{\{q\}}(D_\bullet^{-1})\bigr)\mathsf{H}_\bullet$$

then

$$B\pi_\bullet(U_{i,i}^\alpha)\xi = B\pi_\bullet(U_{i,i}^\alpha) B^{-1} B\xi = c\pi_\bullet(U_{i,i}^\alpha)\xi$$

and (4.8) follows (note that there are no domain issues because we are restricting to finite-dimensional eigenspaces of $D_\bullet^{-1}$ for the eigenvalues q and $q\rho_{\alpha,i}$). □

Theorem 4.16. *The set*

$$\bigcup_{q\in\mathrm{Sp}(D_\bullet^{-1})} \Delta_q \tag{4.9}$$

is finite.

Proof. First let us choose $\alpha \in \mathrm{Irr}(\mathbb{G})$ and $i \in \{1,\dots,\dim(\alpha)\}$ such that $\rho_{\alpha,i} > 1$ (this is possible because $\mathbb{G}$ is not of Kac type and $\mathrm{Tr}(\rho_\alpha) = \mathrm{Tr}(\rho_\alpha^{-1})$).

We have the decomposition of $\mathsf{H}_\bullet$ into eigenspaces of the positive compact operator $D_\bullet^{-1}$:

$$\mathsf{H}_\bullet = \bigoplus_{q\in\mathrm{Sp}(D_\bullet^{-1})} 1_{\{q\}}(D_\bullet^{-1})\mathsf{H}_\bullet.$$

Consider $\eta \in \ker \pi_\bullet(U^\alpha_{i,i})$ with decomposition

$$\eta = \sum_{q\in\mathrm{Sp}(D_\bullet^{-1})} \eta_q,$$

where $\eta_q \in 1_{\{q\}}(D_\bullet^{-1})\mathsf{H}_\bullet$. Now

$$0 = \pi_\bullet(U^\alpha_{i,i})\eta = \sum_{q\in\mathrm{Sp}(D_\bullet^{-1})} \pi_\bullet(U^\alpha_{i,i})\eta_q$$

and by Lemma 4.14 each summand is orthogonal to the remaining ones. It follows that $\pi_\bullet(U^\alpha_{i,i})\eta_q = 0$ for all q and consequently

$$\ker \pi_\bullet(U^\alpha_{i,i}) = \bigoplus_{q\in\mathrm{Sp}(D_\bullet^{-1})} \ker \pi_\bullet(U^\alpha_{i,i}) \cap 1_{\{q\}}(D_\bullet^{-1})\mathsf{H}_\bullet. \tag{4.10}$$

As $\pi_\bullet(U^\alpha_{i,i})$ is a Fredholm operator, its kernel is finite-dimensional. In particular (since the summands on the right hand side of (4.10) are pairwise orthogonal) there exists q_0 in $\mathrm{Sp}(D_\bullet^{-1})$ such that $\pi_\bullet(U^\alpha_{i,i})$ is injective on $1_{\{q\}}(D_\bullet^{-1})\mathsf{H}_\bullet$ for all $q \in \mathrm{Sp}(D_\bullet^{-1})$ such that $q < q_0$.

Clearly, since there are only finitely many eigenvalues of $D_\bullet^{-1}$ greater than q_0 and each is of finite multiplicity, the set

$$\bigcup_{\substack{q\in\mathrm{Sp}(D_\bullet^{-1})\\ q\geq q_0}} \Delta_q \tag{4.11}$$

is finite. Therefore, if (4.9) is infinite, there exists $\tilde{c} > 0$ such that

$$\tilde{c} \in \Delta_{\tilde{q}} = \mathrm{Sp}\big(1_{\{\tilde{q}\}}(D_\bullet^{-1})\, B\, 1_{\{\tilde{q}\}}(D_\bullet^{-1})\big)$$

for some $\tilde{q} < q_0$ and $\tilde{c}$ does not belong to (4.11).

Consider now a unit vector $\xi \in 1_{\{\tilde{c}\}}(1_{\tilde{q}}(D_\bullet^{-1})\, B\, 1_{\{\tilde{q}\}}(D_\bullet^{-1}))\mathsf{H}_\bullet$. For $k \in \mathbb{Z}_+$ we have

$$\pi_\bullet(U^\alpha_{i,i})^k\xi \in 1_{\{\tilde{c}\}}\big(1_{\{\tilde{q}\rho^k_{\alpha,i}\}}(D_\bullet^{-1})\, B\, 1_{\{\tilde{q}\rho^k_{\alpha,i}\}}(D_\bullet^{-1})\big)\mathsf{H}_\bullet.$$

As k increases $\tilde{q}\rho_{\alpha,i}^{k}$ tends to infinity, so we let $\tilde{k} = \max\bigl\{k \in \mathbb{Z}_+ \bigm| \tilde{q}\rho_{\alpha,i}^{k} < q_0\bigr\}$.

We have

$$\begin{aligned}\pi_\bullet(U_{i,i}^{\alpha})\bigl(\pi_\bullet(U_{i,i}^{\alpha})^{\tilde{k}}\xi\bigr) &= \pi_\bullet(U_{i,i}^{\alpha})^{\tilde{k}+1}\xi \\ &\in 1_{\{\tilde{c}\}}\bigl(1_{\{\tilde{q}\rho_{\alpha,i}^{(\tilde{k}+1)}\}}(D_\bullet^{-1})\,B\,1_{\{\tilde{q}\rho_{\alpha,i}^{(\tilde{k}+1)}\}}(D_\bullet^{-1})\bigr)\mathsf{H}_\bullet.\end{aligned}$$

But the latter subspace is $\{0\}$ because $\tilde{q}\rho_{\alpha,i}^{(\tilde{k}+1)} \geq q_0$ and $\tilde{c}$ is not in the spectrum of $1_{\{q\}}(D_\bullet^{-1})\,B\,1_{\{q\}}(D_\bullet^{-1})$ for $q \geq q_0$. This contradicts injectivity of $\pi_\bullet(U_{i,i}^{\alpha})$ on $1_{\{\tilde{q}\rho_{\alpha,i}^{\tilde{k}}\}}(D_\bullet^{-1})\mathsf{H}_\bullet$, since

$$\pi_\bullet(U_{i,i}^{\alpha})^{\tilde{k}}\xi \in 1_{\{\tilde{c}\}}\bigl(1_{\{\tilde{q}\rho_{\alpha,i}^{\tilde{k}}\}}(D_\bullet^{-1})\,B\,1_{\{\tilde{q}\rho_{\alpha,i}^{\tilde{k}}\}}(D_\bullet^{-1})\bigr)\mathsf{H}_\bullet \subseteq 1_{\{\tilde{q}\rho_{\alpha,i}^{\tilde{k}}\}}(D_\bullet^{-1})\mathsf{H}_\bullet$$

because B preserves the decomposition of $\mathsf{H}_\bullet$ into eigenspaces of $D_\bullet^{-1}$ (Corollary 4.12). ☐

Corollary 4.17. The quantum group $\mathbb{G}$ is of Kac type.

Proof. Recall from Corollary 4.12 that

$$\begin{aligned}B &= \bigoplus_{q\in\mathrm{Sp}(D_\bullet^{-1})} 1_{\{q\}}(D_\bullet^{-1})\,B\,1_{\{q\}}(D_\bullet^{-1}) \\ &= \bigoplus_{q\in\mathrm{Sp}(D_\bullet^{-1})}\bigoplus_{c\in\Delta_q} c\,1_{\{c\}}\bigl(1_{\{q\}}(D_\bullet^{-1})\,B\,1_{\{q\}}(D_\bullet^{-1})\bigr).\end{aligned}$$

Therefore Theorem 4.16 implies that B and B^{-1} are bounded with

$$\|B\| = \sup_{q\in\mathrm{Sp}(D_\bullet^{-1})}\sup_{c\in\Delta_q} c < +\infty \quad\text{and}\quad \|B^{-1}\| = \sup_{q\in\mathrm{Sp}(D_\bullet^{-1})}\sup_{c\in\Delta_q} c^{-1} < +\infty.$$

By Lemma 4.13 for any $a \in \mathrm{C}(\mathbb{G})$

$$\pi_\bullet(\tau_t(a)) = B_{-t}\pi_\bullet(a)B_t = B^{-it}\pi_\bullet(a)B^{it} \quad (t \in \mathbb{R}),$$

so for $a \in \mathrm{Pol}(\mathbb{G})$ the holomorphic continuation of the function $\mathbb{R} \ni t \mapsto \pi_\bullet(\tau_t(a))$ to $t = -i$ is given by $\pi_\bullet(\tau_{-i}(a)) = B^{-1}\pi_\bullet(a)B$ and hence

$$\bigl\|\pi_\bullet(\tau_{-i}(a))\bigr\| \leq \|B^{-1}\|\|a\|\|B\|.$$

Thus for any $\alpha \in \mathrm{Irr}(\mathbb{G})$ and $i, j \in \{1, \ldots, \dim(\alpha)\}$

$$\rho_{\alpha,i}\rho_{\alpha,j}^{-1}\|U_{i,j}^{\alpha}\| = \|\rho_{\alpha,i}\rho_{\alpha,j}^{-1}U_{i,j}^{\alpha}\| = \bigl\|\tau_{-i}(U_{i,j}^{\alpha})\bigr\| \leq \|B^{-1}\|\|U_{i,j}^{\alpha}\|\|B\|,$$

so that

$$\rho_{\alpha,i}\rho_{\alpha,j}^{-1} \leq \|B^{-1}\|\|B\|$$

which implies that $\mathbb{G}$ is a compact quantum of Kac type (cf. [64, Remarks after Example 1.7.10]). ☐

As we already mentioned in the introduction, the assumption that there is a compact quantum group $\mathbb{G}$ such that the C*-algebra $\mathrm{C}(\mathbb{G})$ is isomorphic to the C*-algebra of continuous functions on the quantum disc, $\mathcal{T}$, leads to the contradiction between the relatively easy conclusion that $\mathbb{G}$ cannot be of Kac type (Claim 3) and the conclusion of Corollary 4.17 that $\mathbb{G}$ is of Kac type. It follows that no such compact quantum group exists.

5 The von Neumann algebra of class functions

In this section we will study a certain structural property of the algebra of L^∞ functions on a compact quantum group (mostly not of Kac type). These results were obtained together with Mateusz Wasilewski and resulted in a preprint [52]. Let us first start with a motivation.
Let F_n be the free group with $n \geq 2$ generators $g_1, \dots, g_n$ and let $\mathrm{L}(F_n) = \lambda(F_n)''$ be the corresponding group von Neumann algebra. Inside $\mathrm{L}(F_n)$ one finds the so called radial subalgebra $\mathscr{R}$, the von Neumann algebra generated by the operator $(\lambda_{g_1}+\lambda_{g_1}^*)+\cdots+(\lambda_{g_n}+\lambda_{g_n}^*)$. Its name stems from the property that if we (informally) write $x = \sum_{w\in F_n} f(w)\lambda_w \in \mathrm{L}(F_n)$ then $x \in \mathscr{R}$ if and only if f is a radial function on F_n, i.e. $f(w)$ depends only on the length $|w|$ of w. The radial algebra was intensively studied – the result most important for is [71] where Pytlik proved that $\mathscr{R}$ is maximal abelian (MASA) in $\mathrm{L}(F_n)$. Later $\mathscr{R}$ was proved to be singular [72] and even maximal injective [16].
One may look at the element $(\lambda_{g_1}+\lambda_{g_1}^*)+\cdots+(\lambda_{g_n}+\lambda_{g_n}^*)$ from a different perspetive. It is the character of a fundamental representation of the compact quantum group $\widehat{F_n}$, dual to F_n. Consequently, the radial algebra $\mathscr{R}$ is the von Neumann algebra generated by this character. It is therefore natural to wonder whether similar properties holds for other discrete (or by duality, compact) quantum groups. This question was studied in particular in the case of the (Kac type) free orthogonal quantum group O_N^+ (see Example 2.3.2). In [42] Freslon and Vergnioux showed that $\mathscr{C}_{O_N^+}$, the von Neumann algebra generated by the character of the fundamental representation, is a singular MASA in $\mathrm{L}^\infty(O_N^+)$. Observe that now $\mathscr{C}_{O_N^+}$ has also a different description – it coincides with the von Neumann algebra generated by all characters of irreducible representations. Let us take this description as a general definition of $\mathscr{C}_{\mathbb{G}}$ for a compact quantum group $\mathbb{G}$[28].

Definition 5.1. For a compact quantum group $\mathbb{G}$ we define the von Neumann algebra of class functions $\mathscr{C}_{\mathbb{G}} = \{\chi_\alpha \,|\, \alpha \in \mathrm{Irr}(\mathbb{G})\}''$.

We choose to call $\mathscr{C}_{\mathbb{G}}$ “the von Neumann algebra of class functions” because the two coincide for classical compact groups.

Lemma 5.2. *Let G be a compact group. Then $\{\chi_\alpha \,|\, \alpha \in \mathrm{Irr}(G)\}''$ equals the von Neumann algebra of bounded measurable class functions, i.e. the set of $f \in \mathrm{L}^\infty(G)$ satisfying $f(hgh^{-1}) = f(g)\,(g,h\in G)$.*

Proof. Since every character χ_α is a class function, one of the inclusions is clear. Let μ_G be the Haar measure on G and $\mathbb{E}\colon f \mapsto \int_G f(h\cdot h^{-1})\,\mathrm{d}\mu_G(h)$ $(f\in\mathrm{L}^\infty(G))$ the normal conditional expectation onto the von Neumann algebra of bounded measurable class fuctions. As matrix coefficients of irreducible representations span a w*-dense subspace in $\mathrm{L}^\infty(G)$, by Lemma 7.8 it is enough to show that $\mathbb{E}(u^\alpha_{\xi,\eta}) = 0$ for all $\alpha \in \mathrm{Irr}(G)$ and orthogonal

[28] Note however that for $\mathbb{G} = \widehat{F_n}$ we do not have equality of $\mathscr{R}$ and $\mathscr{C}_{\mathbb{G}}$. In fact, $\mathscr{C}_{\mathbb{G}} = \mathrm{L}^\infty(\mathbb{G})$ holds for all abelian compact quantum groups.

vectors $\xi, \eta \in \mathsf{H}_\alpha$. It is a consequence of the orthogonality relations:

$$\int_G |\mathbb{E}(u^\alpha_{\xi,\eta})|^2 \,\mathrm{d}\mu_G = \int_G \int_G \Big(\int_G \overline{\langle \xi \,|\, \alpha(hgh^{-1})\eta \rangle} \langle \xi \,|\, \alpha(h' g h'^{-1})\eta \rangle \,\mathrm{d}\mu_G(g) \Big) \,\mathrm{d}\mu_G(h) \,\mathrm{d}\mu_G(h')$$
$$= \tfrac{1}{\dim(\alpha)} \int_G \int_G \overline{\langle \alpha(h^{-1})\xi \,|\, \alpha(h'^{-1})\xi \rangle} \langle \alpha(h^{-1})\eta \,|\, \alpha(h'^{-1})\eta \rangle \,\mathrm{d}\mu_G(h) \,\mathrm{d}\mu_G(h')$$
$$= \tfrac{1}{\dim(\alpha)^2} \int_G \overline{\langle \alpha(h^{-1})\xi \,|\, \alpha(h^{-1})\eta \rangle} \langle \xi \,|\, \eta \rangle \,\mathrm{d}\mu_G(h) = 0.$$

□

Let us mention that in [2, Theorem 3.7] Alaghmandan and Crann obtained an analogous result for compact quantum groups: $\mathscr{C}_{\mathbb{G}} = \{x \in \mathrm{L}^\infty(\mathbb{G}) \,|\, \Delta(x) = \Delta^{op}(x)\}$.

The question that motivated our work was whether $\mathscr{C}_{O_F^+}$ is MASA in $\mathrm{L}^\infty(O_F^+)$ for non-Kac type quantum groups O_F^+. The main tool that was used in solving this riddle was the notion of quasi-split inclusions of von Neumann algebras.

5.1 Quasi-split inclusions

In this subsection we assume that $\mathrm{B} \subseteq \mathrm{M}$ are von Neumann algebras with the same unit, separable preduals and φ is a normal faithful state on M.

Definition 5.3.

1) The inclusion $\mathrm{B} \subseteq \mathrm{M}$ is *split*, if there is a type I factor F such that $\mathrm{B} \subseteq \mathrm{F} \subseteq \mathrm{M}$.

2) The inclusion $\mathrm{B} \subseteq \mathrm{M}$ is *quasi-split*, if the map

$$\mathrm{B} \otimes_{alg} \mathrm{M}^{op} \ni b \otimes y^{op} \mapsto b J_\varphi y^* J_\varphi \in \mathrm{B}(\mathsf{H}_\varphi)$$

extends to a normal $\star$-homomorphism $\mathrm{B} \bar{\otimes} \mathrm{M}^{op} \to \mathrm{B} \vee \mathrm{M}' \subseteq \mathrm{B}(\mathsf{H}_\varphi)$.

(Quasi)-split inclusions of von Neumann algebras were extensively studied: let us mention papers [15, 25, 35] and later works [9, 37]. We will present here results which are mainly taken from [9].

Let us start with a remark that if $\mathrm{B} \subseteq \mathrm{M}$ is quasi-split and B, M are factors or one of them is a type III algebra then the inclusion is in fact split [25, Corollary 1].

We are interested in this condition mainly because it is, in some sense, opposite to being an inclusion of a MASA. We will present here two results in this spirit, the first one assumes that the "big algebra" is of type III.

Proposition 5.4 ([9, Corollary 3.11])**.** *If* $\mathrm{B} \subseteq \mathrm{M}$ *is a quasi-split inclusion and* M *is a type III von Neumann algebra then* $\mathrm{B}' \cap \mathrm{M}$ *is also of type III. In particular,* B *is not a MASA in* M.

Proof. By assumption, the map $\mathrm{B}\otimes_{alg}\mathrm{M}^{op}\ni b\otimes y^{op}\mapsto bJ_\varphi y^*J_\varphi\in\mathrm{B}\vee\mathrm{M}'$ extends to a normal $\star$-epimorphism, hence there exists a central projection $0\neq p\in\mathcal{Z}(\mathrm{B}\,\bar{\otimes}\,\mathrm{M}^{op})$ such that $p(\mathrm{B}\,\bar{\otimes}\,\mathrm{M}^{op})\simeq\mathrm{B}\vee\mathrm{M}'$. As M is of type III, so is $\mathrm{B}\,\bar{\otimes}\,\mathrm{M}^{op}$ and $p(B\bar{\otimes}\,\mathrm{M}^{op})$ (see [81, Theorem V.2.30] and [78, Exercise E.4.18]). It follows that the isomorphism $p(\mathrm{B}\,\bar{\otimes}\,\mathrm{M}^{op})\simeq\mathrm{B}\vee\mathrm{M}'$ is spatial ([78, Corollary 8.13]), hence the respective commutants are also isomorphic. It follows by [81, Corollary 2.24] that $\mathrm{B}'\cap\mathrm{M}$ is a von Neumann algebra of type III. □

The next proposition gives us a structural result on M under the assumption that B is a MASA in M and $\mathrm{B}\subseteq\mathrm{M}$ is quasi-split. The benefit of this result is that we do not need to assume anything on M.

Proposition 5.5 ([9, Remark 3.10 (2)])**.** *If* $\mathrm{B}\subseteq\mathrm{M}$ *is a quasi-split inclusion and* B *is a MASA in* M*, then* M *is isomorphic to a direct product of type I factors.*

Proof. Let us first argue that the abelian von Neumann algebra $\mathcal{Z}(\mathrm{M})$ is purely atomic. Using the $\star$-homomorphism provided by the quasi-split inclusion $\mathrm{B}\subseteq\mathrm{M}$, we deduce that $\mathcal{Z}(\mathrm{M})\subseteq\mathrm{M}$ is quasi-split with associated $\star$-homomorphism $\eta\colon\mathcal{Z}(\mathrm{M})\bar{\otimes}\,\mathrm{M}^{op}\to\mathcal{Z}(\mathrm{M})\vee\mathrm{M}'$. If we compress $\eta\restriction_{\mathcal{Z}(\mathrm{M})\bar{\otimes}\mathcal{Z}(\mathrm{M})}$ with the Jones projection $e_{\mathcal{Z}(\mathrm{M})}$ associated to $\mathcal{Z}(\mathrm{M})$, we get a $\star$-epimorphism $\mathcal{Z}(\mathrm{M})\bar{\otimes}\mathcal{Z}(\mathrm{M})\to\mathcal{Z}(\mathrm{M})\subseteq\mathrm{B}(e_{\mathcal{Z}(\mathrm{M})}\mathsf{H}_\varphi)$, i.e. $\mathcal{Z}(\mathrm{M})\subseteq\mathcal{Z}(\mathrm{M})$ is also quasi-split. It follows that $\mathcal{Z}(\mathrm{M})$ is purely atomic. Indeed, otherwise there is a von Neumann subalgebra in $\mathcal{Z}(\mathrm{M})$ isomorphic to $\mathrm{L}^\infty([0,1])$. After further restriction of η we see that the multiplication map extends to a normal $\star$-homomorphism $\mathrm{L}^\infty([0,1])\bar{\otimes}\,\mathrm{L}^\infty([0,1])\to\mathrm{L}^\infty([0,1])$. One can easily see that it has to act via $F\mapsto F\circ\delta$, where $\delta(x)=(x,x)\,(x\in[0,1])$, at least for continuous functions $F\in\mathrm{C}([0,1]\times[0,1])$. However, it is not difficult to find a sequence of functions $(F_n)_{n\in\mathbb{N}}$ in $\mathrm{C}([0,1]\times[0,1])$ which converge to $\mathbb{1}$ in w*, but are zero on the diagonal: $F_n(x,x)=0\,(n\in\mathbb{N},x\in[0,1])$ – this gives us a contradiction. Consequently M is a direct product of factors.
Take now a non-zero central projection $p_0\in\mathcal{Z}(\mathrm{M})$. It is easy to see that the inclusion $p_0\,\mathrm{B}\subseteq p_0\,\mathrm{M}$ is also quasi-split, hence by the previous discussion it is enough to deduce that M is of type I assuming it is a factor.
Since B is a MASA in M, $\mathrm{B}\vee\mathrm{M}'=(\mathrm{B}'\cap\mathrm{M})'=\mathrm{B}'$ is of type I (see [81, Corollary V.2.24]). Reasoning from the proof of Proposition 5.4 gives us a central projection $p\in\mathcal{Z}(\mathrm{M})$ such that $p(\mathrm{B}\,\bar{\otimes}\,\mathrm{M}^{op})\simeq\mathrm{B}\vee\mathrm{M}'$. But M is a factor, hence $p=\mathbb{1}$ and $\mathrm{B}\,\bar{\otimes}\,\mathrm{M}^{op}$ (and consequently M) is of type I. □

We would like to present now a useful criterion for proving that a given inclusion is quasi-split, which is a variant of Proposition 2.3 from [15]. Observe first, that if a von Neumann algebra M with a faithful normal state φ is represented on a Hilbert space H_φ then we have an inclusion $\Phi_2:\mathrm{M}\to\mathsf{H}_\varphi$ given by $x\mapsto\nabla_\varphi^{\frac{1}{4}}\Lambda_\varphi(x)$.

We will also need the notion of a nuclear map between two Banach spaces X and Y. A map $T:X\to Y$ is called nuclear if there are sequences $(y_n)_{n\in\mathbb{N}}\subseteq Y$ and $(x_n^*)_{n\in\mathbb{N}}\subseteq X^*$ such that $T(x)=\sum_{n\in\mathbb{N}}x_n^*(x)y_n\,(x\in X)$ and $\sum_{n\in\mathbb{N}}\|x_n^*\|\|y_n\|<\infty$.

Proposition 5.6 ([9, Proposition 3.7])**.** *If the map* $\Phi_2\restriction_{\mathrm{B}}:\mathrm{B}\to\mathsf{H}_\varphi$ *is nuclear then the inclusion* $\mathrm{B}\subseteq\mathrm{M}$ *is quasi-split.*

Let $\mathbb{G}$ be a second countable compact quantum group. Using the above criterion we will derive a useful condition (Theorem 5.9) which says the the inclusion $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$ is quasi-split provided $\mathbb{G}$ is "sufficiently non-Kac". Before doing that we need some preparation regarding the action of the modular group on characters of unitary representations.

Our aim is to show that the map $\Phi_2\restriction_{\mathscr{C}_{\mathbb{G}}} : \mathscr{C}_{\mathbb{G}} \to \mathrm{L}^2(\mathbb{G})$ given by $x \mapsto \nabla_h^{\frac{1}{4}}\Lambda_h(x)$ is nuclear. Note that $\mathscr{C}_{\mathbb{G}}$ is the closed linear span of the characters of (finite dimensional) unitary representations and these characters are analytic elements for the modular group. Therefore $\nabla_h^{\frac{1}{4}}\Lambda_h(\chi) = \Lambda_h(\sigma^h_{-\frac{i}{4}}(\chi))$ holds for every character χ; we first have to understand the action of the modular group on the characters.

Recall that (see [64, Page 30] or Section 2.3) for any (unitary, finite dimensional) representation U on H_U and $z \in \mathbb{C}$ we have $(\sigma^h_z \otimes \mathbb{1}_U)(U) = (\rho_U^{iz} \otimes \mathbb{1}_U)U(\rho_U^{iz} \otimes \mathbb{1}_U)$. If we choose an orthonormal basis of H_U in which ρ_U is diagonal then we can write more concretely that $\sigma^h_z(U_{k,l}) = \rho_{U,k}^{iz}\rho_{U,l}^{iz}U_{k,l}$. Therefore for the character $\chi_U = \sum_{k=1}^{\dim(U)} U_{k,k}$ we have $\sigma^h_z(\chi_U) = \sum_{k=1}^{\dim(U)} \rho_{U,k}^{2iz}U_{k,k}$. We will now compute the L^2-norm of this element.

Lemma 5.7. *We have* $\|\sigma^h_{a+ib}(\chi_U)\|_2^2 = \frac{\mathrm{Tr}(\rho_U^{-4b-1})}{\mathrm{Tr}(\rho_U)}$ *for all* $a, b \in \mathbb{R}$.

Proof. Recall that by definition $\|x\|_2^2 = h(x^*x)$, so in our case we get

$$\|\sigma^h_{a+ib}(\chi_U)\|_2^2 = \sum_{k,l=1}^{\dim(U)} \rho_{U,k}^{-2ia-2b}\rho_{U,l}^{2ia-2b}h(U_{k,k}^*U_{l,l}).$$

Using the orthogonality relations (see [64, Theorem 1.4.3] or Section 3.4) we get $h(U_{k,k}^*U_{l,l}) = \delta_{k,l}\frac{\rho_{U,k}^{-1}}{\dim_q(U)}$. Therefore we obtain

$$\|\sigma^h_{a+ib}(\chi_U)\|_2^2 = \sum_{k=1}^{\dim(U)} \rho_{U,k}^{-4b}\frac{\rho_{U,k}^{-1}}{\dim_q(U)} = \frac{\mathrm{Tr}(\rho_U^{-4b-1})}{\dim_q(U)}.$$

To finish the proof we just have to recall that $\dim_q(U) = \mathrm{Tr}(\rho_U)$. □

Corollary 5.8. We have $\|\chi_U\|_2 = 1$ and $\|\sigma^h_{-\frac{i}{4}}(\chi_U)\|_2^2 = \frac{\dim(U)}{\dim_q(U)}$.

Relation between the quantum and the usual dimension will be crucial for proving that the inclusion of the algebra of class functions is quasi-split.

Theorem 5.9. *Let* $\mathbb{G}$ *be a compact quantum group. Suppose that* $\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}(\frac{\dim(\alpha)}{\dim_q(\alpha)})^{\frac{1}{2}} < \infty$. *Then the inclusion* $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$ *is quasi-split.*

Proof. We want to show that the map $\Phi_2\restriction_{\mathscr{C}_{\mathbb{G}}} : \mathscr{C}_{\mathbb{G}} \to \mathrm{L}^2(\mathbb{G})$ is nuclear. Note that it is a composition of two maps: the inclusion $\Lambda_h : \mathscr{C}_{\mathbb{G}} \to \Lambda_h(\mathscr{C}_{\mathbb{G}})$ and $\nabla_h^{\frac{1}{4}} : \Lambda_h(\mathscr{C}_{\mathbb{G}}) \to \mathrm{L}^2(\mathbb{G})$.

We will first show that $\nabla_h^{\frac{1}{4}}$ extends to a contraction from $\mathrm{L}^2(\mathscr{C}_{\mathbb{G}}) = \overline{\Lambda_h(\mathscr{C}_{\mathbb{G}})}$ to $\mathrm{L}^2(\mathbb{G})$. Note that the (images of) characters of irreducible representations are linearly dense in

$\mathrm{L}^2(\mathscr{C}_{\mathbb{G}})$ and they form an orthonormal set. Let $x = \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} c_\alpha \chi_\alpha$ be a finite sum of characters of irreducible representations. Note that $\|\Lambda_h(x^*)\|^2 = \|\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \overline{c_\alpha}\Lambda_h(\chi_{\overline{\alpha}})\|^2 = \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} |c_\alpha|^2 = \|\Lambda_h(x)\|^2$. It follows that

$$\begin{aligned}\|\nabla_h^{\frac{1}{4}}\Lambda_h(x)\|^2 &= \langle \Lambda_h(x) \,|\, \nabla_h^{\frac{1}{2}}\Lambda_h(x)\rangle = \langle J_h\nabla_h^{\frac{1}{2}}\Lambda_h(x) \,|\, J_h\Lambda_h(x)\rangle\\ &= \langle \Lambda_h(x^*) \,|\, J_h\Lambda_h(x)\rangle \leq \|\Lambda_h(x^*)\|\,\|\Lambda_h(x)\| = \|\Lambda_h(x)\|^2,\end{aligned}$$

so $\nabla_h^{\frac{1}{4}}$ extends to a contraction from $\mathrm{L}^2(\mathscr{C}_{\mathbb{G}})$ to $\mathrm{L}^2(\mathbb{G})$.
We will now show that $\Phi_2\restriction_{\mathscr{C}_{\mathbb{G}}}\colon \mathscr{C}_{\mathbb{G}} \to \mathrm{L}^2(\mathbb{G})$ is a nuclear map. Take $x \in \mathscr{C}_{\mathbb{G}}$. As $\Lambda_h(x) \in \mathrm{L}^2(\mathscr{C}_{\mathbb{G}})$, we can write $\Lambda_h(x) = \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\langle \Lambda_h(\chi_\alpha) \,|\, \Lambda_h(x)\rangle \Lambda_h(\chi_\alpha)$. Since $\nabla_h^{\frac{1}{4}}\restriction_{\mathrm{L}^2(\mathscr{C}_{\mathbb{G}})}\colon \mathrm{L}^2(\mathscr{C}_{\mathbb{G}}) \to \mathrm{L}^2(\mathbb{G})$ is bounded, we have

$$\Phi_2(x) = \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \langle \Lambda_h(\chi_\alpha) \,|\, \Lambda_h(x)\rangle \nabla_h^{\frac{1}{4}}\Lambda_h(\chi_\alpha).$$

If we define functionals $\omega_\alpha\colon \mathscr{C}_{\mathbb{G}} \ni y \mapsto \langle \Lambda_h(\chi_\alpha) \,|\, \Lambda_h(y)\rangle \in \mathbb{C}$ then it suffices to check that

$$\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \|\omega_\alpha\|\,\|\nabla_h^{\frac{1}{4}}\Lambda_h(\chi_\alpha)\| < \infty.$$

We already know that $\|\nabla_h^{\frac{1}{4}}\Lambda_h(\chi_\alpha)\| = (\frac{\dim(\alpha)}{\dim_q(\alpha)})^{\frac{1}{2}}$ and it is clear that $\|\omega_\alpha\| \leqslant \|\chi_\alpha\|_2 = 1$, hence

$$\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \|\omega_\alpha\|\,\|\nabla_h^{\frac{1}{4}}\Lambda_h(\chi_\alpha)\| \leq \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \big(\tfrac{\dim(\alpha)}{\dim_q(\alpha)}\big)^{\frac{1}{2}} < \infty.$$

By Proposition 5.6 the inclusion $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$ is quasi-split. □

This result is already enough to prove that in many cases the radial subalgebra in $\mathrm{L}^\infty(O_F^+)$ is not a MASA; it follows from [89, Theorem 7.1] that $\mathrm{L}^\infty(O_F^+)$ is often a type III factor and we can use Proposition 5.4. We will be able to generalize this result (see Corollary 5.22).

5.2 Relative commutant of $\mathscr{C}_{\mathbb{G}}$ and inner scaling automorphisms

We will be interested in the relative commutant $\mathscr{C}_{\mathbb{G}}' \cap \mathrm{L}^\infty(\mathbb{G})$. If $\mathscr{C}_{\mathbb{G}}$ is commutative, as is the case for example for the free orthogonal quantum groups, then the condition $\mathscr{C}_{\mathbb{G}}' \cap \mathrm{L}^\infty(\mathbb{G}) \subseteq \mathscr{C}_{\mathbb{G}}$ precisely means that $\mathscr{C}_{\mathbb{G}}$ is a MASA in $\mathrm{L}^\infty(\mathbb{G})$.
Our strategy for proving that $\mathscr{C}_{\mathbb{G}}$ cannot be a MASA in many cases will be the following. We will show that if $\mathscr{C}_{\mathbb{G}}$ were a MASA then $\mathrm{L}^\infty(\mathbb{G})$ would have to be a factor. Moreover, if the inclusion $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$ were quasi-split then it would have to be a type I factor. We will then use properties of the scaling automorphisms to exclude this case. Let us now move on to more precise statements.

Proposition 5.10. *Let $\mathbb{G}$ be a compact quantum group such that $\mathscr{C}_{\mathbb{G}}' \cap \mathrm{L}^\infty(\mathbb{G}) \subseteq \mathscr{C}_{\mathbb{G}}$. Then*

$$\mathcal{Z}(\mathrm{L}^\infty(\mathbb{G})) \subseteq \overline{\operatorname{span}}^{\mathrm{w}^*}\{\chi_\alpha \,|\, \alpha \in \operatorname{Irr}(\mathbb{G})\colon \rho_\alpha = \mathbb{1}_\alpha\} = \mathscr{C}_{\mathbb{G}} \cap \mathrm{L}^\infty(\mathbb{G})^\sigma.$$

In particular, if $\rho_\alpha \neq \mathbb{1}_\alpha$ for all non-trivial irreducible representation of $\mathbb{G}$ then $\mathrm{L}^\infty(\mathbb{G})$ is a factor.

Proof. We first argue that $\overline{\operatorname{span}}^{\mathrm{w}^*}\{\chi_\alpha \,|\, \alpha \in \operatorname{Irr}(\mathbb{G})\colon \rho_\alpha = \mathbb{1}_\alpha\} = \mathscr{C}_{\mathbb{G}} \cap \mathrm{L}^\infty(\mathbb{G})^\sigma$. Clearly the left-hand side (denoted from now on by A) is contained in the right-hand side. If $x \in \mathscr{C}_{\mathbb{G}} \cap \mathrm{L}^\infty(\mathbb{G})^\sigma$ then we can write $\Lambda_h(x) = \sum_{\alpha\in\operatorname{Irr}(\mathbb{G})} c_\alpha \Lambda_h(\chi_\alpha)$. As x belongs to the centralizer of the Haar integral, we have $\Lambda_h(x) = \Lambda_h(\sigma_t^h(x)) = \sum_{\alpha\in\operatorname{Irr}(\mathbb{G})} c_\alpha \Lambda_h(\sigma_t^h(\chi_\alpha))$. It follows from the orthogonality relations that the elements $\sigma_t(\chi_\alpha^h)$ and χ_β are orthogonal unless $\alpha=\beta$. It follows that $\sigma_t^h(\chi_\alpha) = \chi_\alpha$ or $c_\alpha = 0$. The condition $\sigma_t^h(\chi_\alpha) = \chi_\alpha$ implies that $\rho_\alpha = \mathbb{1}_\alpha$, so we proved that any element $x \in \mathscr{C}_{\mathbb{G}} \cap \mathrm{L}^\infty(\mathbb{G})^\sigma$ satisfies $\Lambda_h(x) \in \overline{\Lambda_h(A)}$. Since A is contained in the centralizer, there exists a normal, state-preserving conditional expectation onto it ([82, Theorem 4.2]), and it is easy to conclude that it implies $x \in A$. Alternatively one can invoke Lemma 5.18 because the Haar integral on the algebra A is tracial.

To finish the proof, note that the condition $\mathscr{C}_{\mathbb{G}}' \cap \mathrm{L}^\infty(\mathbb{G}) \subseteq \mathscr{C}_{\mathbb{G}}$ implies that $\mathcal{Z}(\mathrm{L}^\infty(\mathbb{G})) \subseteq \mathscr{C}_{\mathbb{G}}$. Moreover the center is always contained in the centralizer ([78, Corollary 2.10.13]), so we obtain $\mathcal{Z}(\mathrm{L}^\infty(\mathbb{G})) \subseteq \mathscr{C}_{\mathbb{G}} \cap \mathrm{L}^\infty(\mathbb{G})^\sigma$. □

The next technical ingredient, featuring the scaling group, is the following proposition.

Proposition 5.11. *Let $\mathbb{G}$ be a compact quantum group and let $t \in \mathbb{R}$. Suppose that $\mathscr{C}_{\mathbb{G}}' \cap \mathrm{L}^\infty(\mathbb{G}) \subseteq \mathscr{C}_{\mathbb{G}}$, the scaling automorphism τ_t is inner and is implemented by $v \in \mathrm{L}^\infty(\mathbb{G})$. Then $v \in \overline{\operatorname{span}}^{\mathrm{w}^*}\{\chi_\alpha \,|\, \alpha \in \operatorname{Irr}(\mathbb{G})\colon \rho_\alpha = \mathbb{1}_\alpha\}$.*

Proof. The scaling group acts trivially on characters, so we have $\chi_\alpha = \tau_t(\chi_\alpha) = v\chi_\alpha v^*$ for any $\alpha \in \operatorname{Irr}(\mathbb{G})$. Therefore $v \in \mathscr{C}_{\mathbb{G}}' \cap \mathrm{L}^\infty(\mathbb{G}) \subseteq \mathscr{C}_{\mathbb{G}}$. Because of that we can write $\Lambda_h(v) = \sum_{\alpha\in\operatorname{Irr}(\mathbb{G})} c_\alpha \Lambda_h(\chi_\alpha)$.

We will now use the fact that the scaling group and the modular group commute, so for any $x \in \mathrm{L}^\infty(\mathbb{G}), s \in \mathbb{R}$ we have

$$vxv^* = \tau_t(x) = \sigma_s^h \tau_t(\sigma_{-s}^h(x)) = \sigma_s^h(v\sigma_{-s}^h(x)v^*) = \sigma_s^h(v)x\sigma_s^h(v^*).$$

It follows that $\sigma_s^h(v^*)vx = x\sigma_s^h(v^*)v$, hence $\sigma_s^h(v^*)v \in \mathcal{Z}(\mathrm{L}^\infty(\mathbb{G}))$. We can write $\sigma_s^h(v) = vw_s$ for some $w_s \in \mathcal{Z}(\mathrm{L}^\infty(\mathbb{G}))$. As $\mathcal{Z}(\mathrm{L}^\infty(\mathbb{G})) \subseteq \mathscr{C}_{\mathbb{G}}$, we have $vw_s \in \mathscr{C}_{\mathbb{G}}$, so $\Lambda_h(vw_s) = \sum_{\alpha\in\operatorname{Irr}(\mathbb{G})} d_{s,\alpha}\Lambda_h(\chi_\alpha)$. On the other hand $\Lambda_h(v) = \sum_\alpha c_\alpha \Lambda_h(\chi_\alpha)$, so

$$\Lambda_h(\sigma_s^h(v)) = \sum_{\alpha\in\operatorname{Irr}(\mathbb{G})} c_\alpha \Lambda_h(\sigma_s^h(\chi_\alpha)).$$

By orthogonality relations this implies that $c_\alpha \sigma_s^h(\chi_\alpha) = d_{s,\alpha}\chi_\alpha$. If $\rho_\alpha \neq \mathbb{1}_\alpha$ we can once again infer that $c_\alpha = 0$, hence $\Lambda_h(v) = \sum_{\alpha:\rho_\alpha = \mathbb{1}_\alpha} c_\alpha \Lambda_h(\chi_\alpha)$ and exactly as in the proof of the previous proposition we conclude that $v \in \overline{\operatorname{span}}^{\mathrm{w}^*}\{\chi_\alpha \,|\, \alpha \in \operatorname{Irr}(\mathbb{G})\colon \rho_\alpha = \mathbb{1}_\alpha\}$. □

Now we can prove our main result of this section.

Theorem 5.12. *Suppose that* $\mathbb{G}$ *is a non-trivial second countable compact quantum group such that the inclusion* $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$ *is quasi-split and* $\rho_\alpha \neq \mathbb{1}_\alpha$ *for any non-trivial* $\alpha \in \mathrm{Irr}(\mathbb{G})$. *Then* $\mathscr{C}_{\mathbb{G}}$ *is not a MASA in* $\mathrm{L}^\infty(\mathbb{G})$.

Proof. Suppose that $\mathscr{C}_{\mathbb{G}}$ is a MASA in $\mathrm{L}^\infty(\mathbb{G})$. It follows from Proposition 5.5 that $\mathrm{L}^\infty(\mathbb{G})$ is a direct sum of type I factors. Moreover, it follows from Proposition 5.10 that $\mathrm{L}^\infty(\mathbb{G})$ is a factor, so $\mathrm{L}^\infty(\mathbb{G}) \simeq \mathrm{B}(\ell^2)$.

By our assumptions there exists an irreducible representation α with $\rho_\alpha \neq \mathbb{1}_\alpha$, thus $\mathbb{G}$ is not of Kac type – there exists a non-trivial scaling automorphism τ_t. All automorphisms of $\mathrm{B}(\ell^2)$ are inner, hence there exists $v \in \mathrm{L}^\infty(\mathbb{G})$ implementing it. By Proposition 5.11 $v \in \overline{\operatorname{span}}^{\mathrm{w}^*}\{\chi_\alpha \,|\, \alpha \in \mathrm{Irr}(\mathbb{G})\colon \rho_\alpha = \mathbb{1}_\alpha\} = \mathbb{C}\mathbb{1}$. But that means that τ_t is a trivial automorphism and this leads us to the desired contradiction. □

We will provide examples to which this result can be applied in Subsection 5.5 (see Corollary 5.22).

5.3 Properties of $\mathrm{SU}_q(2)$

Fix $q \in \,]-1,1[\, \setminus \{0\}$. In this subsection we establish a number of properties of $\mathscr{C}_{\mathrm{SU}_q(2)}$: we show that it is not a MASA in $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ (Proposition 5.14), however it is a MASA in $\mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau$, the fixed point subalgebra of the scaling group (Proposition 5.15). This property will be used in the next subsection, where we construct new compact quantum group out of $\mathrm{SU}_q(2)$ and $\mathbb{Q}$, using a bicrossed product construction.

Recall that we have defined the quantum group $\mathrm{SU}_q(2)$ in Section 2.3.1 and given its description from the dual perspective in Section 3.6. In particular, we can identify $\mathrm{Irr}(\widehat{\mathrm{SU}_q(2)})$ with the circle $\mathbb{T}$ and the Plancherel measure μ with the normalised Lebesgue measure, consequently the unitary operator $\mathcal{Q}_L$ gives us a unitary isomorphism

$$\mathrm{L}^\infty(\mathrm{SU}_q(2)) \simeq \int_{\mathbb{T}}^{\oplus} \mathrm{B}(\mathsf{H}_\lambda) \otimes \mathbb{1}_{\overline{\mathsf{H}}_\lambda} \,\mathrm{d}\mu(\lambda). \tag{5.1}$$

Each $\lambda \in \mathbb{T}$ corresponds to an irreducible representation $\psi^{2,\lambda}$ of $\mathrm{C}(\mathrm{SU}_q(2))$ (see Proposition 3.39). To ease the notation we will write $\psi^\lambda = \psi^{2,\lambda}\,(\lambda \in \mathbb{T})$.

The next lemma says that the von Neumann algebra generated by the real part of a weighted shift is MASA in $\mathrm{B}(\ell^2(\mathbb{Z}_+))$.

Lemma 5.13. *Let* $S \in \mathrm{B}(\ell^2(\mathbb{Z}_+))$ *be the shift operator given by* $S\phi_k = \phi_{k-1}\,(k \in \mathbb{Z}_+)$, *and let* $\mathrm{M}_f \in \mathrm{B}(\ell^2(\mathbb{Z}_+))$ *be the multiplication operator associated with a function* $f \in \ell^\infty(\mathbb{Z}_+)$. *If* $f(\mathbb{N}) \subseteq \mathbb{R}_{>0}$ *then the von Neumann algebra* $\mathscr{B}$ *generated by* $T = S\,\mathrm{M}_f + \mathrm{M}_f\,S^*$ *is maximal abelian in* $\mathrm{B}(\ell^2(\mathbb{Z}_+))$.

Proof. According to [68, Theorem 4.7.7], the claim follows once we show that there exists a cyclic vector for $\mathscr{B} = \{T\}''$. We claim that ϕ_0 is such a vector. Indeed, it is clear that ϕ_0 belongs to $V = \overline{\mathscr{B}\phi_0}$. Next, assume that $\phi_0, \dots, \phi_n \in V$ for some $n \in \mathbb{Z}_+$. Then $\phi_{n+1} = \frac{1}{f(n+1)} T\phi_n - \frac{f(n)}{f(n+1)}\phi_{n-1}$ belongs to V and consequently $V = \mathrm{B}(\ell^2(\mathbb{Z}_+))$. ☐

One easily sees that $\mathscr{C}_{\mathrm{SU}_q(2)}$ is abelian and $\mathscr{C}_{\mathrm{SU}_q(2)} = \{\alpha + \alpha^*\}''$. Indeed, $\alpha + \alpha^*$ is the character of the fundamental representation and the fusion rules of $\mathrm{SU}_q(2)$ (equation (2.23)) imply that $\chi(U^n) \in \{\alpha + \alpha^*\}''$ for all $U^n \in \mathrm{Irr}(\mathrm{SU}_q(2))$. Consequently,

$$\mathcal{Q}_L \mathscr{C}_{\mathrm{SU}_q(2)} \mathcal{Q}_L^* = \Big\{ \int_{\mathbb{T}}^{\oplus} T \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}} \, \mathrm{d}\mu(\lambda) \,|\, T \in \psi^1(\mathscr{C}_{\mathrm{SU}_q(2)}) \Big\} \simeq \psi^1(\mathscr{C}_{\mathrm{SU}_q(2)}) \otimes \mathbb{1}_{\mathrm{L}^\infty(\mathbb{T})} \tag{5.2}$$

(observe that $\psi^\lambda(\alpha + \alpha^*) = \psi^1(\alpha + \alpha^*)$ for all $\lambda \in \mathbb{T}$). Since $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ is not a factor[29], Proposition 5.10 implies that $\mathscr{C}_{\mathrm{SU}_q(2)}$ is not a MASA in $\mathrm{L}^\infty(\mathrm{SU}_q(2))$. The next result describes its relative commutant.

Proposition 5.14. *The relative commutant of $\mathscr{C}_{\mathrm{SU}_q(2)}$ is given by*

$$\begin{aligned}\mathcal{Q}_L(\mathscr{C}'_{\mathrm{SU}_q(2)} \cap \mathrm{L}^\infty(\mathrm{SU}_q(2)))\mathcal{Q}_L^* &= \Big\{ \int_{\mathbb{T}}^{\oplus} T_\lambda \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}} \, \mathrm{d}\mu(\lambda) \,|\, \forall_{\lambda \in \mathbb{T}} \, T_\lambda \in \psi^1(\mathscr{C}_{\mathrm{SU}_q(2)}) \Big\} \\ &\simeq \psi^1(\mathscr{C}_{\mathrm{SU}_q(2)}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{T}).\end{aligned}$$

Proof. Inclusion $\supseteq$ clearly follows from equation (5.2), assume that T belongs to the subalgebra $\mathcal{Q}_L(\mathscr{C}'_{\mathrm{SU}_q(2)} \cap \mathrm{L}^\infty(\mathrm{SU}_q(2)))\mathcal{Q}_L^*$. Using (5.1) we can write $T = \int_{\mathbb{T}}^{\oplus} T_\lambda \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}} \, \mathrm{d}\mu(\lambda)$ for some $T_\lambda \in \mathrm{B}(\mathsf{H}_\lambda)$. Our assumption forces $T_\lambda \in \psi^1(\mathscr{C}_{\mathrm{SU}_q(2)})'$ for almost all $\lambda \in \mathbb{T}$. From the definition of ψ^1 we see that $\psi^1(\alpha)$ is a weighted shift and Lemma 5.13 applies – $\psi^1(\mathscr{C}_{\mathrm{SU}_q(2)})$ is a MASA in $\mathrm{B}(\ell^2(\mathbb{Z}_+))$, hence $T_\lambda \in \psi^1(\mathscr{C}_{\mathrm{SU}_q(2)})$ and the claim follows. ☐

Despite the above negative result, we can nonetheless prove that $\mathscr{C}_{\mathrm{SU}_q(2)}$ is MASA in the smaller von Neumann algebra of fixed points for the scaling group. We denote this algebra by $\mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau$.

Proposition 5.15. *The algebra of class functions $\mathscr{C}_{\mathrm{SU}_q(2)}$ is MASA in $\mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau$, i.e.*

$$\mathscr{C}'_{\mathrm{SU}_q(2)} \cap \mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau = \mathscr{C}_{\mathrm{SU}_q(2)}.$$

Proof. Observe first that since $\mathscr{C}_{\mathrm{SU}_q(2)}$ is generated by characters, we have $\mathscr{C}_{\mathrm{SU}_q(2)} \subseteq \mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau$. Take $T = \int_{\mathbb{T}}^{\oplus} T_\lambda \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}} \, \mathrm{d}\mu(\lambda)$ in $\mathcal{Q}_L(\mathscr{C}'_{\mathrm{SU}_q(2)} \cap \mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau)\mathcal{Q}_L^*$. Proposition 5.14 implies that $T_\lambda \in \psi^1(\mathscr{C}_{\mathrm{SU}_q(2)})$ for almost all $\lambda \in \mathbb{T}$. Recall that P is the operator implementing the scaling group for $\mathrm{SU}_q(2)$ and its dual (see Section 2.2). We know how

[29] From (5.1) we see that $\mathrm{L}^\infty(\mathrm{SU}_q(2)) \simeq \mathrm{B}(\ell^2(\mathbb{Z}_+)) \bar{\otimes} \mathrm{L}^\infty(\mathbb{T})$, hence the center of $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ is isomorphic to $\mathrm{L}^\infty(\mathbb{T})$.

to express this operator on the level of direct integrals (Proposition 3.41): for all $t \in \mathbb{R}$ we have

$$\mathcal{Q}_L P^{it} \mathcal{Q}_L^* \colon \int_{\mathbb{T}}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda) \ni \int_{\mathbb{T}}^{\oplus} \xi_\lambda\,\mathrm{d}\mu(\lambda) \mapsto \int_{\mathbb{T}}^{\oplus} \xi_{\lambda|q|^{2it}}\,\mathrm{d}\mu(\lambda) \in \int_{\mathbb{T}}^{\oplus} \mathrm{HS}(\mathsf{H}_\lambda)\,\mathrm{d}\mu(\lambda), \tag{5.3}$$

and since T is invariant under the (transported) scaling group we have

$$\int_{\mathbb{T}}^{\oplus} T_\lambda \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda) = T = (\mathcal{Q}_L P^{it} \mathcal{Q}_L^*) T (\mathcal{Q}_L P^{-it} \mathcal{Q}_L^*) = \int_{\mathbb{T}}^{\oplus} T_{\lambda|q|^{2it}} \otimes \mathbb{1}_{\overline{\mathsf{H}_\lambda}}\,\mathrm{d}\mu(\lambda). \tag{5.4}$$

It follows that $\mathbb{T} \ni \lambda \mapsto T_\lambda \in \mathrm{B}(\ell^2(\mathbb{Z}_+))$ is constant almost everywhere. Indeed, for $\kappa \in \mathbb{T}$ denote by $f_\kappa \in \mathrm{B}(\ell^2(\mathbb{Z}_+))\bar{\otimes}\,\mathrm{L}^\infty(\mathbb{T})$ the function $\lambda \mapsto T_{\lambda\kappa}$. Equation (5.4) implies $f_1 = f_\kappa$ ($\kappa \in \mathbb{T}$). For all $\theta \in \mathrm{L}^1(\mathbb{T}), \omega \in \mathrm{B}(\ell^2(\mathbb{Z}_+))_*$ we get[30]

$$\begin{aligned}
(\omega \otimes \theta) f_1 &= (\omega \otimes \theta)\big(\int_{\mathbb{T}} f_\kappa\,\mathrm{d}\mu(\kappa)\big) = \int_{\mathbb{T}} (\omega \otimes \theta)(f_\kappa)\,\mathrm{d}\mu(\kappa)\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \theta(\lambda)\omega(f_\kappa(\lambda))\,\mathrm{d}\mu(\lambda)\,\mathrm{d}\mu(\kappa) = \int_{\mathbb{T}} \theta(\lambda) \int_{\mathbb{T}} \omega(T_{\lambda\kappa})\,\mathrm{d}\mu(\kappa)\,\mathrm{d}\mu(\lambda)\\
&= \big(\int_{\mathbb{T}} \theta(\lambda)\,\mathrm{d}\mu(\lambda)\big) \int_{\mathbb{T}} \omega(T_\kappa)\,\mathrm{d}\mu(\kappa) = (\omega \otimes \theta)\big(\int_{\mathbb{T}} T_\kappa\,\mathrm{d}\mu(\kappa) \otimes \mathbb{1}_{\mathrm{L}^\infty(\mathbb{T})}\big),
\end{aligned}$$

hence $f_1 = \int_{\mathbb{T}} T_\kappa\,\mathrm{d}\mu(\kappa) \otimes \mathbb{1}_{\mathrm{L}^\infty(\mathbb{T})}$. Consequently, T belongs to $\mathcal{Q}_L \mathscr{C}_{\mathrm{SU}_q(2)} \mathcal{Q}_L^*$ (see equation (5.2)). □

5.4 Certain bicrossed product construction

In this subsection we present a construction of a class of compact quantum groups $\mathbb{H}$ given by a bicrossed product of a compact quantum group $\mathbb{G}$ and the additive group of rational numbers $\mathbb{Q}$ (in this subsection we equip $\mathbb{Q}$ with the discrete topology), where $\mathbb{Q}$ acts on $\mathrm{L}^\infty(\mathbb{G})$ using the scaling grup of $\mathbb{G}$. Our construction is a slight variation of a construction presented in [26, Section 4.1] – the main difference is that we replace $\mathbb{R}$ with a discrete group $\mathbb{Q}$ in order to get a compact quantum group as the bicrossed product. The principal reason why we are interested in this family of quantum groups is the fact that they admit non-trivial inner scaling automorphisms – a property that appeared in Proposition 5.11 (see Lemma 5.16. Observe also that equation (5.3) implies that non-trivial scaling automorphisms of $\mathrm{SU}_q(2)$ are never inner). Another reason is that these bicrossed products provide examples of compact quantum groups $\mathbb{H}$ with $\mathrm{L}^\infty(\mathbb{H})$ being the injective factor of type II_∞.

Later on we will specify to $\mathbb{G} = \mathrm{SU}_q(2)$, for now let $\mathbb{G}$ be an arbitrary compact quantum group. Fix a non-zero number $\nu \in \mathbb{R} \setminus \{0\}$ and denote by ρ the normal $\star$-homomorphism $\ell^\infty(\mathbb{Q})\bar{\otimes}\,\mathrm{L}^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{Q})\bar{\otimes}\,\mathrm{L}^\infty(\mathbb{G})$ given by

$$\rho(F)(\gamma) = \tau^{\mathbb{G}}_{\nu\gamma}(F(\gamma)) \quad (\gamma \in \mathbb{Q}, F \in \ell^\infty(\mathbb{Q})\bar{\otimes}\,\mathrm{L}^\infty(\mathbb{G}))$$

[30]Integrals of $\mathrm{B}(\ell^2(\mathbb{Z}_+))$ or $\mathrm{L}^\infty(\mathbb{T})\bar{\otimes}\,\mathrm{B}(\ell^2(\mathbb{Z}_+))$–valued functions are understood in the sense of Pettis, where the von Neumann algebras are equipped with the w*–topology.

(we identify $\ell^\infty(\mathbb{Q})\bar{\otimes}\,\mathrm{L}^\infty(\mathbb{G})$ with $\ell^\infty(\mathbb{Q},\mathrm{L}^\infty(\mathbb{G}))$). Now, a pair $(\mathbb{Q},\mathbb{G})$ together with ρ forms a matched pair with trivial cocycles ([88, Definition 2.1]) let $\mathbb{H}=\mathbb{Q}\bowtie\mathbb{G}$ be the resulting bicrossed product quantum group. For the details of this construction and its properties we refer the reader to [88, 95] (see also [38]), here we will recall only some of its aspects. Using the notation of [88], $\alpha\colon \mathrm{L}^\infty(\mathbb{G})\ni x\mapsto \rho(\mathbb{1}\otimes x)\in \ell^\infty(\mathbb{Q})\bar{\otimes}\,\mathrm{L}^\infty(\mathbb{G})$ is given by $\alpha(x)(\gamma)=\tau^{\mathbb{G}}_{\nu\gamma}(x)$ and $\beta\colon \ell^\infty(\mathbb{Q})\ni f\mapsto \rho(f\otimes\mathbb{1})\in \ell^\infty(\mathbb{Q})\bar{\otimes}\,\mathrm{L}^\infty(\mathbb{G})$ is the trivial action $\beta(f)=f\otimes\mathbb{1}$. Furthermore, we have

$$\mathrm{L}^\infty(\mathbb{H})=\mathbb{Q}\ltimes_\alpha \mathrm{L}^\infty(\mathbb{G})=\{\alpha(x),u_\gamma\,|\,x\in\mathrm{L}^\infty(\mathbb{G}),\gamma\in\mathbb{Q}\}''$$

(where $\mathbb{Q}\ni\gamma\mapsto\lambda_\gamma\in\mathrm{B}(\ell^2(\mathbb{Q}))$ is the left regular representation, $u_\gamma=\lambda_\gamma\otimes\mathbb{1}$) and

$$\ell^\infty(\widehat{\mathbb{H}})=\ell^\infty(\mathbb{Q})\bar{\otimes}\ell^\infty(\widehat{\mathbb{G}}).$$

These von Neumann algebras are represented on the Hilbert space

$$\mathrm{L}^2(\mathbb{H})=\ell^2(\mathbb{Q})\otimes\mathrm{L}^2(\mathbb{G}).$$

The Haar integral on $\mathbb{H}$ is a state, hence $\mathbb{H}$ is compact (see [88, Definition 2.7]). In fact, the GNS map for $h_{\mathbb{H}}$ is given by

$$\Lambda_{h_{\mathbb{H}}}(u_\gamma\alpha(x))=\Lambda_{h_{\widehat{\mathbb{Q}}}}(\lambda_\gamma)\otimes\Lambda_{h_{\mathbb{G}}}(x)\quad(x\in\mathrm{L}^\infty(\mathbb{G}),\gamma\in\mathbb{Q}).\tag{5.5}$$

We can also identify the (left) Haar integral on $\widehat{\mathbb{H}}$ – it is equal to $\varphi_{\mathbb{Q}}\otimes\varphi_{\widehat{\mathbb{G}}}$ (where $\varphi_{\mathbb{Q}},\varphi_{\widehat{\mathbb{G}}}$ are the left Haar integrals on $\mathbb{Q}$ and $\widehat{\mathbb{G}}$), hence

$$\nabla_{\varphi_{\widehat{\mathbb{H}}}}=\mathbb{1}\otimes\nabla_{\varphi_{\widehat{\mathbb{G}}}}$$

(it is a combination of Proposition 2.9, Theorem 2.13 and Proposition 2.16 in [88]). Since the equality $\nabla^{it}_{\varphi_{\widehat{\mathbb{G}}}}=P^{it}_{\mathbb{G}}$ holds for any unimodular locally compact quantum group (equation (2.14)), we arrive at

$$P^{it}_{\mathbb{H}}=\nabla^{it}_{\varphi_{\widehat{\mathbb{H}}}}=\mathbb{1}\otimes\nabla^{it}_{\varphi_{\widehat{\mathbb{G}}}}=\mathbb{1}\otimes P^{it}_{\mathbb{G}}\quad(t\in\mathbb{R}).\tag{5.6}$$

It is a well known property of crossed products that automorphisms with which $\mathbb{Q}$ acts on $\mathrm{L}^\infty(\mathbb{G})$ become inner after the inclusion of $\mathrm{L}^\infty(\mathbb{G})$ into $\mathbb{Q}\ltimes_\alpha\mathrm{L}^\infty(\mathbb{G})$:

$$\alpha(\tau^{\mathbb{G}}_{\nu\gamma}(x))=u_\gamma\alpha(x)u_\gamma^*\quad(x\in\mathrm{L}^\infty(\mathbb{G}),\gamma\in\mathbb{Q}).\tag{5.7}$$

Let us now record a simple result concerning the scaling group of $\mathbb{H}$.

Lemma 5.16.

- *We have $\tau^{\mathbb{H}}_t(\alpha(x))=\alpha(\tau^{\mathbb{G}}_t(x))$ and $\tau^{\mathbb{H}}_t(u_\gamma)=u_\gamma$ for all $t\in\mathbb{R},x\in\mathrm{L}^\infty(\mathbb{G}),\gamma\in\mathbb{Q}$.*
- *For every $t\in\mathbb{R}$, the scaling automorphism $\tau^{\mathbb{H}}_t$ is trivial if and only if $\tau^{\mathbb{G}}_t$ is trivial. If $\gamma\in\mathbb{Q}$, then $\tau^{\mathbb{H}}_{\nu\gamma}$ is inner.*

Proof. The first part is a direct consequence of equations (5.5), (5.6):

$$\Lambda_{h_{\mathbb{H}}}(\tau_t^{\mathbb{H}}(\alpha(x))) = (\mathbb{1}\otimes P_{\mathbb{G}}^{it})(\Lambda_{h_{\widehat{\mathbb{Q}}}}(\mathbb{1})\otimes\Lambda_{h_{\mathbb{G}}}(x)) = \Lambda_{h_{\widehat{\mathbb{Q}}}}(\mathbb{1})\otimes\Lambda_{h_{\mathbb{G}}}(\tau_t^{\mathbb{G}}(x)) = \Lambda_{h_{\mathbb{H}}}(\alpha(\tau_t^{\mathbb{G}}(x)))$$

and

$$\Lambda_{h_{\mathbb{H}}}(\tau_t^{\mathbb{H}}(u_\gamma)) = (\mathbb{1}\otimes P_{\mathbb{G}}^{it})(\Lambda_{h_{\widehat{\mathbb{Q}}}}(\lambda_\gamma)\otimes\Lambda_{h_{\mathbb{G}}}(\mathbb{1})) = \Lambda_{h_{\widehat{\mathbb{Q}}}}(\lambda_\gamma)\otimes\Lambda_{h_{\mathbb{G}}}(\mathbb{1}) = \Lambda_{h_{\mathbb{H}}}(u_\gamma).$$

Since $h_{\mathbb{H}}$ is faithful on $\mathrm{L}^\infty(\mathbb{H})$ we get the first claim. As α is a monomorphism, $\tau_t^{\mathbb{H}}$ is trivial if and only so is $\tau_t^{\mathbb{G}}$. The last claim follows from equation (5.7). □

Let us end these general considerations with an observation that

$$u_\gamma\in\mathscr{C}_{\mathbb{H}}\quad(\gamma\in\mathbb{Q})\quad\text{and}\quad\alpha(\mathscr{C}_{\mathbb{G}})\subseteq\mathscr{C}_{\mathbb{H}}.$$

Indeed, it is a consequence of [95, Theorem 3.7].

Fix $q\in\,]-1,1[\setminus\{0\}$. From now on we consider the special case $\mathbb{G}=\mathrm{SU}_q(2)$ – accordingly $\mathbb{H}$ is given by $\mathbb{H}=\mathbb{Q}\bowtie\mathrm{SU}_q(2)$. Note that this quantum group depends on two parameters: q and ν and is not of Kac type. Using Proposition 5.15 ($\mathscr{C}_{\mathrm{SU}_q(2)}$ is MASA in $\mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau$) we are able to deduce the following interesting property of $\mathbb{H}$:

Proposition 5.17. *Let $\mathbb{H}=\mathbb{Q}\bowtie\mathrm{SU}_q(2)$. The von Neumann algebra $\mathscr{C}_{\mathbb{H}}$ is MASA in $\mathrm{L}^\infty(\mathbb{H})$.*

Proof. First, it is clear that $\mathscr{C}_{\mathbb{H}}$ is commutative. Indeed, since $\mathscr{C}_{\mathrm{SU}_q(2)}$ is commutative, commutativity of $\mathscr{C}_{\mathbb{H}}$ follows from [95, Theorem 3.7]. Take now $T\in\mathscr{C}_{\mathbb{H}}'\cap\mathrm{L}^\infty(\mathbb{H})$ – we want to show $T\in\mathscr{C}_{\mathbb{H}}$. Let $\mathbb{E}\colon\,\mathrm{L}^\infty(\mathbb{H})=\mathbb{Q}\ltimes_\alpha\mathrm{L}^\infty(\mathrm{SU}_q(2))\to\mathrm{L}^\infty(\mathrm{SU}_q(2))$ be the canonical faithful normal conditional expectation satisfying $\mathbb{E}(u_\gamma\alpha(x))=\delta_{\gamma,0}x$ for $\gamma\in\mathbb{Q},x\in\mathrm{L}^\infty(\mathrm{SU}_q(2))$. Define operators

$$T_\gamma=\mathbb{E}(u_\gamma^*T)\in\mathrm{L}^\infty(\mathrm{SU}_q(2))\quad(\gamma\in\mathbb{Q}).$$

Clearly we have

$$\langle\xi\,|\,T_\gamma\eta\rangle=\langle\xi\,|\,\mathbb{E}(u_\gamma^*T)\eta\rangle=\langle\delta_0\otimes\xi\,|\,(u_\gamma^*T)(\delta_0\otimes\eta)\rangle$$

for all $\xi,\eta\in\mathrm{L}^2(\mathrm{SU}_q(2))$. Fix $\gamma\in\mathbb{Q}$. Using the fact that $T\in\mathscr{C}_{\mathbb{H}}'\cap\mathrm{L}^\infty(\mathbb{H})$ we will now show $T_\gamma\in\mathscr{C}_{\mathrm{SU}_q(2)}'$. Since for any $y\in\mathscr{C}_{\mathrm{SU}_q(2)}$ operator $\alpha(y)$ belongs to $\mathscr{C}_{\mathbb{H}}$, we get

$$\begin{aligned}\langle\xi\,|\,T_\gamma y\eta\rangle&=\langle\delta_\gamma\otimes\xi\,|\,T(\delta_0\otimes y\eta)\rangle=\langle\delta_\gamma\otimes\xi\,|\,T\alpha(y)(\delta_0\otimes\eta)\rangle\\&=\langle\alpha(y^*)(\delta_\gamma\otimes\xi)\,|\,T(\delta_0\otimes\eta)\rangle=\langle\delta_0\otimes y^*\xi\,|\,(u_\gamma^*T)(\delta_0\otimes\eta)\rangle=\langle y^*\xi\,|\,T_\gamma\eta\rangle=\langle\xi\,|\,yT_\gamma\eta\rangle\end{aligned}$$

for all vectors $\xi,\eta\in\mathrm{L}^2(\mathrm{SU}_q(2))$ and consequently $T_\gamma\in\mathscr{C}_{\mathrm{SU}_q(2)}'$.
Take $\gamma'\in\mathbb{Q}$. Observe that Lemma 5.16 together with equation (5.7) implies that $\tau_{\nu\gamma'}^{\mathbb{H}}$ is implemeneted by $u_{\gamma'}\in\mathscr{C}_{\mathbb{H}}$. Using equation (5.6) we calculate

$$\begin{aligned}\langle\xi\,|\,\tau_{\nu\gamma'}^{\mathrm{SU}_q(2)}(T_\gamma)\eta\rangle&=\langle\delta_0\otimes P_{\mathrm{SU}_q(2)}^{-i\nu\gamma'}\xi\,|\,(u_\gamma^*T)(\delta_0\otimes P_{\mathrm{SU}_q(2)}^{-i\nu\gamma'}\eta)\rangle\\&=\langle\delta_\gamma\otimes\xi\,|\,P_{\mathbb{H}}^{i\nu\gamma'}TP_{\mathbb{H}}^{-i\nu\gamma'}(\delta_0\otimes\eta)\rangle=\langle\delta_\gamma\otimes\xi\,|\,\tau_{\nu\gamma'}^{\mathbb{H}}(T)(\delta_0\otimes\eta)\rangle\\&=\langle\delta_\gamma\otimes\xi\,|\,u_{\gamma'}Tu_{\gamma'}^*(\delta_0\otimes\eta)\rangle=\langle\delta_\gamma\otimes\xi\,|\,T(\delta_0\otimes\eta)\rangle=\langle\xi\,|\,T_\gamma\eta\rangle\end{aligned}$$

and as before we arrive at $\tau^{\mathrm{SU}_q(2)}_{\nu\gamma'}(T_\gamma) = T_\gamma$. Density of $\nu\mathbb{Q}$ in $\mathbb{R}$ implies $T_\gamma \in \mathrm{L}^\infty(\mathrm{SU}_q(2))^\tau$. These two properties of T_γ imply that $T_\gamma \in \mathscr{C}_{\mathrm{SU}_q(2)}$ (Proposition 5.15) and consequently $\alpha(T_\gamma) \in \mathscr{C}_{\mathbb{H}}$.

Formally we have $T = \sum_{\gamma\in\mathbb{Q}} u_\gamma \alpha(T_\gamma)$. However, this series does not need to converge in the w*-topology (see [62]), which is why we will argue on the L^2-level. Let us first prove that

$$\Lambda_{h_{\mathbb{H}}}(T) = \sum_{\gamma\in\mathbb{Q}} \Lambda_{h_{\mathbb{H}}}(u_\gamma \alpha(T_\gamma)) = \sum_{\gamma\in\mathbb{Q}} \Lambda_{h_{\widehat{\mathbb{Q}}}}(\lambda_\gamma) \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(T_\gamma). \tag{5.8}$$

Since $\{\delta_\gamma\}_{\gamma\in\mathbb{Q}}$ forms an orthonormal basis in $\ell^2(\mathbb{Q})$, we can write $\Lambda_{h_{\mathbb{H}}}(T) = \sum_{\gamma\in\mathbb{Q}} \delta_\gamma \otimes \tilde{T}_\gamma$ for some $\tilde{T}_\gamma \in \mathrm{L}^2(\mathrm{SU}_q(2))$. Then

$$\begin{aligned}\langle \xi \,|\, \tilde{T}_\gamma \rangle &= \langle \delta_\gamma \otimes \xi \,|\, \sum_{\gamma'\in\mathbb{Q}} \delta_{\gamma'} \otimes \tilde{T}_{\gamma'} \rangle = \langle \delta_\gamma \otimes \xi \,|\, \Lambda_{h_{\mathbb{H}}}(T) \rangle \\ &= \langle \delta_\gamma \otimes \xi \,|\, T(\delta_0 \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(\mathbb{1})) \rangle = \langle \xi \,|\, \Lambda_{h_{\mathrm{SU}_q(2)}}(T_\gamma) \rangle\end{aligned}$$

for all $\gamma \in \mathbb{Q}, \xi \in \mathrm{L}^2(\mathrm{SU}_q(2))$ which proves (5.8). Recall that $h_{\mathbb{H}}$ is tracial on $\mathscr{C}_{\mathbb{H}}$, hence the claim follows from equation (5.8) and the following lemma. $\square$

Lemma 5.18. *Let* M *be a von Neumann algebra with a fixed faithful normal state* ω. *Assume that* $\mathrm{N} \subseteq \mathrm{M}$ *is a von Neumann subalgebra such that* $\omega\restriction_{\mathrm{N}}$ *is tracial. If* $x \in \mathrm{M}$ *and* $\Lambda_\omega(x) \in \overline{\Lambda_\omega(N)}$ *then* $x \in \mathrm{N}$.

This lemma is well-known to experts but we were not able to locate a precise reference, so we decided to add a proof for completeness.

Proof. We will show that x commutes with every $y \in \mathrm{N}'$. Take $a, b \in \mathrm{M}$ that are analytic with respect to $(\sigma^\omega_t)_{t\in\mathbb{R}}$ and fix a net $(\Lambda_\omega(x_i))_{i\in I}$ ($x_i \in \mathrm{N}$) which converges to $\Lambda_\omega(x)$. Observe that since $\omega\restriction_{\mathrm{N}}$ is tracial, $J_\omega \nabla_\omega^{\frac{1}{2}}$ is an isometry on $\Lambda_\omega(\mathrm{N})$. As $J_\omega \nabla_\omega^{\frac{1}{2}}$ is closed, it follows that $\lim_{i\in I} \Lambda_\omega(x_i^*) = \Lambda_\omega(x^*)$. Consequently

$$\begin{aligned}&\langle \Lambda_\omega(a) \,|\, yx\Lambda_\omega(b) \rangle = \langle \Lambda_\omega(a) \,|\, yJ_\omega \sigma^\omega_{i/2}(b)^* J_\omega \Lambda_\omega(x) \rangle = \lim_{i\in I} \langle \Lambda_\omega(a) \,|\, yJ_\omega \sigma^\omega_{i/2}(b)^* J_\omega \Lambda_\omega(x_i) \rangle \\ &= \lim_{i\in I} \langle \Lambda_\omega(a) \,|\, yx_i \Lambda_\omega(b) \rangle = \lim_{i\in I} \langle J_\omega \sigma^\omega_{i/2}(a)^* J_\omega \Lambda_\omega(x_i^*) \,|\, y\Lambda_\omega(b) \rangle \\ &= \langle J_\omega \sigma^\omega_{i/2}(a)^* J_\omega \Lambda_\omega(x^*) \,|\, y\Lambda_\omega(b) \rangle = \langle x^* \Lambda_\omega(a) \,|\, y\Lambda_\omega(b) \rangle = \langle \Lambda_\omega(a) \,|\, xy\Lambda_\omega(b) \rangle.\end{aligned}$$

A standard density argument implies $x \in \mathrm{N}'' = \mathrm{N}$. $\square$

Remark. In the proof of Proposition 5.17, we argued on the L^2-level that $\alpha(T_\gamma) \in \mathscr{C}_{\mathbb{H}}$ ($\gamma \in \mathbb{Q}$) implies that $T \in \mathscr{C}_{\mathbb{H}}$. Alternatively, we could use a Fejér-type theorem for crossed products and arrive at the same conclusion (see e.g. [24, Theorem 4.10] for a general result).

In the penultimate result we prove about $\mathbb{H} = \mathbb{Q} \bowtie \mathrm{SU}_q(2)$ we study its von Neumann algebra of bounded functions. In particular, we show that for some values of ν, q, it is a factor of type II_∞ – we are not aware of another example of a compact quantum group in the literature with this property.

Proposition 5.19.

- *$\mathscr{Z}(\mathrm{L}^\infty(\mathbb{H}))$ is equal to $\{u_\gamma \,|\, \gamma \in \mathbb{Q} \cap \frac{\pi}{\nu \log(|q|)} \mathbb{Z}\}''$. In particular, it is trivial if $\nu \log(|q|) \notin \pi\mathbb{Q}$ and isomorphic to $\mathrm{L}^\infty(\mathbb{T})$ otherwise.*
- *Let $t \in \mathbb{R}$. The scaling automorphism $\tau_t^{\mathbb{H}}$ is trivial if and only if $t \in \frac{\pi}{\log(|q|)}\mathbb{Z}$. It is inner if and only if $t \in \nu\mathbb{Q} + \frac{\pi}{\log(|q|)}\mathbb{Z}$.*
- *$\mathbb{H}$ is coamenable and consequently $\mathrm{L}^\infty(\mathbb{H})$ is injective.*
- *If $\nu \log(|q|) \notin \pi\mathbb{Q}$ then $\mathrm{L}^\infty(\mathbb{H})$ is a factor of type II_∞.*

Proof. Observe first that for all $t \in \mathbb{R}$, the scaling automorphism $\tau_t^{\mathbb{H}}$ is trivial if and only $\tau_t^{\mathrm{SU}_q(2)}$ is trivial (Lemma 5.16) which happens if and only if $t \in \frac{\pi}{\log(|q|)}\mathbb{Z}$ (equation (5.3)). Take $x \in \mathscr{Z}(\mathrm{L}^\infty(\mathbb{H}))$. Since $\mathscr{C}_{\mathbb{H}}$ is MASA in $\mathrm{L}^\infty(\mathbb{H})$, we know that

$$x \in \overline{\operatorname{span}\{\chi_\beta \,|\, \beta \in \operatorname{Irr}(\mathbb{H}) : \rho_\beta = \mathbb{1}_\beta\}}^{\,\mathrm{SOT}} = \overline{\operatorname{span}\{u_\gamma \,|\, \gamma \in \mathbb{Q}\}}^{\,\mathrm{SOT}}$$

(Proposition 5.10). Write

$$x = \mathrm{SOT} - \lim_{i \in I} \sum_{\gamma \in \mathbb{Q}} C_\gamma^i u_\gamma, \quad \Lambda_{h_{\mathbb{H}}}(x) = \sum_{\gamma \in \mathbb{Q}} C_\gamma \Lambda_{h_{\mathbb{H}}}(u_\gamma)$$

for some $C_\gamma, C_\gamma^i \in \mathbb{C}$, where $\sum_{\gamma \in \mathbb{Q}} C_\gamma^i u_\gamma$ belongs to $\operatorname{span}\{u_\gamma \,|\, \gamma \in \mathbb{Q}\}$ for each $i \in I$. Take now $y \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$. Since $x \in \mathscr{Z}(\mathrm{L}^\infty(\mathbb{H}))$, we have

$$\begin{aligned}\sum_{\gamma \in \mathbb{Q}} C_\gamma \delta_\gamma \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(\tau_{\nu\gamma}^{\mathrm{SU}_q(2)}(y)) &= \alpha(y)\big(\sum_{\gamma \in \mathbb{Q}} C_\gamma \Lambda_{h_{\mathbb{H}}}(u_\gamma)\big) = \Lambda_{h_{\mathbb{H}}}(\alpha(y)x) = \Lambda_{h_{\mathbb{H}}}(x\alpha(y))\\ &= x(\delta_0 \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(y)) = \lim_{i \in I} \sum_{\gamma \in \mathbb{Q}} C_\gamma^i (\delta_\gamma \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(y)),\end{aligned}$$

which implies

$$C_\gamma \tau_{\nu\gamma}^{\mathrm{SU}_q(2)}(y) = \lim_{i \in I} C_\gamma^i \, y \quad (\gamma \in \mathbb{Q}).$$

As this equation holds for every $y \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$, we must have $C_\gamma = 0$ whenever $\tau_{\nu\gamma}^{\mathrm{SU}_q(2)}$ is non-trivial, i.e. for $\nu\gamma \notin \frac{\pi}{\log(|q|)}\mathbb{Z}$. Lemma 5.18 gives us

$$\mathscr{Z}(\mathrm{L}^\infty(\mathbb{H})) \subseteq \{u_\gamma \,|\, \gamma \in \mathbb{Q} \cap \tfrac{\pi}{\nu \log(|q|)}\mathbb{Z}\}''.$$

The inclusion $\supseteq$ is clear, hence we have identified the center of $\mathrm{L}^\infty(\mathbb{H})$. If $\nu \log(|q|) \notin \pi\mathbb{Q}$ then clearly $\mathbb{Q} \cap \frac{\pi}{\nu \log(|q|)}\mathbb{Z} = \{0\}$ and $\mathrm{L}^\infty(\mathbb{H})$ is a factor. Otherwise $\mathbb{Q} \cap \frac{\pi}{\nu \log(|q|)}\mathbb{Z}$ is a subgroup of $\mathbb{Q}$ isomorphic to $\mathbb{Z}$ and $\{u_\gamma \,|\, \gamma \in \mathbb{Q} \cap \frac{\pi}{\nu \log(|q|)}\mathbb{Z}\}''$ is therefore isomorphic to $\mathrm{L}(\mathbb{Z}) \simeq \mathrm{L}^\infty(\mathbb{T})$ [45, Theorem A]. This proves the first point[31].

Take now $t \in \mathbb{R}$. If $t = \nu\gamma + \frac{\pi}{\log(|q|)} s \in \nu\mathbb{Q} + \frac{\pi}{\log(|q|)}\mathbb{Z}$ then $\tau^{\mathbb{H}}_t = \tau^{\mathbb{H}}_{\nu\gamma}\tau^{\mathbb{H}}_{\pi s/\log(|q|)} = \tau^{\mathbb{H}}_{\nu\gamma}$ is inner by Lemma 5.16. Assume that $t \notin \nu\mathbb{Q} + \frac{\pi}{\log|q|}\mathbb{Z}$ and $\tau^{\mathbb{H}}_t = \mathrm{Ad}_v$ for some unitary $v \in \mathrm{L}^\infty(\mathbb{H})$. Proposition 5.11 implies that $v \in \{u_\gamma \,|\, \gamma \in \mathbb{Q}\}''$, hence we can write

$$\Lambda_{h_{\mathbb{H}}}(v) = \sum_{\gamma\in\mathbb{Q}} D_\gamma \Lambda_{h_{\mathbb{H}}}(u_\gamma) = \sum_{\gamma\in\mathbb{Q}} D_\gamma(\delta_\gamma \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(\mathbb{1})) \tag{5.9}$$

for some $D_\gamma \in \mathbb{C}$. Since v is unitary, we have $\sum_{\gamma\in\mathbb{Q}} |D_\gamma|^2 = 1$. Let $f \in \mathrm{L}^\infty(\mathbb{T})$ be the characteristic function of the arc $\{e^{i\theta} \,|\, \theta \in [0,\pi]\} \subseteq \mathbb{T}$ and $F = \mathcal{Q}_L^*(\int_{\mathbb{T}}^{\oplus} f(\lambda)\mathbb{1}_{\mathrm{HS}(\mathsf{H}_\lambda)}\,\mathrm{d}\mu(\lambda))\mathcal{Q}_L \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$. Equation (5.9) together with Lemma 5.16 gives us

$$\begin{aligned}&\sum_{\gamma\in\mathbb{Q}} D_\gamma(\delta_\gamma \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(\tau^{\mathrm{SU}_q(2)}_{\nu\gamma}(F))) = \alpha(F)\big(\sum_{\gamma\in\mathbb{Q}} D_\gamma(\delta_\gamma \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(\mathbb{1}))\big) = \Lambda_{h_{\mathbb{H}}}(\alpha(F)v)\\ &= v\Lambda_{h_{\mathbb{H}}}(\tau^{\mathbb{H}}_{-t}(\alpha(F))) = v\big(\delta_0 \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(\tau^{\mathrm{SU}_q(2)}_{-t}(F))\big).\end{aligned}$$

Since $v \in \{\lambda_\gamma \otimes \mathbb{1} \,|\, \gamma \in \mathbb{Q}\}''$, the last vector belongs to $\overline{\mathrm{span}}\{\delta_\gamma \otimes \Lambda_{h_{\mathrm{SU}_q(2)}}(\tau^{\mathrm{SU}_q(2)}_{-t}(F)) \,|\, \gamma \in \mathbb{Q}\}$. It follows that there exists $\gamma \in \mathbb{Q}$ such that

$$\tau^{\mathrm{SU}_q(2)}_{\nu\gamma}(F) = c\tau^{\mathrm{SU}_q(2)}_{-t}(F)$$

for some $c \in \mathbb{C}$. Each scaling automorphism acts by a rotation (equation (5.3)), hence $c = 1$ and $\tau^{\mathrm{SU}_q(2)}_{t+\nu\gamma}(F) = F$. However, $\tau^{\mathrm{SU}_q(2)}_{t+\nu\gamma}$ is a non-trivial rotation. Indeed, otherwise $t + \nu\gamma \in \frac{\pi}{\log(|q|)}\mathbb{Z}$ and we assumed that it is not the case. It follows that f is equal to its proper rotation, a contradiction. This ends the proof of the second bullet point.

The compact quantum group $\mathbb{H} = \mathbb{Q} \bowtie \mathrm{SU}_q(2)$ is coamenable because $\mathbb{Q}$ is amenable and $\mathrm{SU}_q(2)$ is coamenable [32, Theorem 15]. It follows that $\mathrm{L}^\infty(\mathbb{H})$ is injective [8, Theorem 3.3] (see also Section 6). Alternatively, to obtain injectivity of $\mathrm{L}^\infty(\mathbb{H})$ one can also use the fact that a crossed product of an injective von Neumann algebra by an action of an amenable group is injective [83, Theorem 3.16].

Assume $\nu\log(|q|) \notin \pi\mathbb{Q}$. We already know that $\mathrm{L}^\infty(\mathbb{H}) = \mathbb{Q} \ltimes_\alpha \mathrm{L}^\infty(\mathrm{SU}_q(2))$ is a factor. Since the n.s.f. tracial weight on $\mathrm{L}^\infty(\mathrm{SU}_q(2)) \simeq \mathrm{B}(\ell^2(\mathbb{Z}_+))\bar{\otimes}\mathrm{L}^\infty(\mathbb{T})$ given by $\mathrm{Tr}\otimes h_{\mathbb{T}}$ is invariant under the action of $\mathbb{Q}$, it gives rise to a n.s.f. tracial weight on $\mathrm{L}^\infty(\mathbb{H})$ ([82, Theorem 1.17]) and consequently $\mathrm{L}^\infty(\mathbb{H})$ is not of type III. It follows from the proof of [101, Theorem 1.3] that if there were a faithful normal tracial state on $\mathrm{L}^\infty(\mathbb{H})$, then $\mathbb{H}$ would be of Kac type. As this is not the case, $\mathrm{L}^\infty(\mathbb{H})$ cannot be of type II_1; we are left with two cases, I_∞ and II_∞. Clearly $|\nu\mathbb{Q} + \frac{\pi}{\log(|q|)}\mathbb{Z}| = \aleph_0 < |\mathbb{R}|$ hence there exists a scaling

[31] We could also argue that $\mathrm{L}^\infty(\mathbb{H})$ is a factor if $\nu\log(|q|) \notin \pi\mathbb{Q}$ using [69, Theorem 7.11.11].

automorphism $\tau_t^{\mathbb{H}}$ which is not inner. It is well known that all automorphisms of $\mathrm{B}(\ell^2(\mathbb{Z}_+))$ are inner ([10, II.5.5.14]), hence $\mathrm{L}^\infty(\mathbb{H})$ has to be of type II_∞.
Let us also give an alternative proof of the result that $\mathrm{L}^\infty(\mathbb{H})$ is not of type I_∞. Let

$$\mathbb{E}\colon \mathrm{L}^\infty(\mathbb{H}) = \mathbb{Q} \ltimes_\alpha \mathrm{L}^\infty(\mathrm{SU}_q(2)) \to \mathrm{L}^\infty(\mathrm{SU}_q(2))$$

be the canonical faithful normal conditional expectation. Assume by contradiction that $\mathrm{L}^\infty(\mathbb{H})$ is of type I_∞. Then it is purely atomic and it follows that $\mathrm{L}^\infty(\mathrm{SU}_q(2)) \simeq \mathrm{B}(\ell^2(\mathbb{Z}_+))\bar{\otimes}\mathrm{L}^\infty(\mathbb{T})$ is purely atomic as well ([10, Theorem IV.2.2.4]), which gives us a contradiction. $\square$

As a corollary, we can show that our family of bicrossed products contains uncountably many different isomorphism classes of quantum groups. To formulate this result, let us denote by $\mathbb{H}_{\nu,q}$ the bicrossed product $\mathbb{Q} \bowtie \mathrm{SU}_q(2)$ constructed using the parameter ν.

Corollary 5.20. Let $\nu, \nu' \in \mathbb{R}\setminus\{0\}, q, q' \in]-1,1[\setminus\{0\}$. If $\mathbb{H}_{\nu,q}$ and $\mathbb{H}_{\nu',q'}$ are isomorphic, then $|q| = |q'|$ and $\nu\mathbb{Q}+\frac{\pi}{\log(|q|)}\mathbb{Z} = \nu'\mathbb{Q}+\frac{\pi}{\log(|q|)}\mathbb{Z}$. In particular, for each $q \in]-1,1[\setminus\{0\}$ the family $\{\mathbb{H}_{\nu,q} \,|\, \nu \in \mathbb{R}\setminus\{0\}\}$ consists of $\mathfrak{c}$ isomorphism classes of compact quantum groups.

Proof. Let $\phi\colon \mathrm{C}(\mathbb{H}_{\nu,q}) \to \mathrm{C}(\mathbb{H}_{\nu',q'})$ be a Hopf $\star$-isomorphism implementing the isomorphism between $\mathbb{H}_{\nu,q}$ and $\mathbb{H}_{\nu',q'}$ (recall that $\mathbb{H}_{\nu,q}$ is coamenable). Since ϕ intertwines scaling groups ([63, Proposition 3.15]) it follows that for each $t \in \mathbb{R}$, $\tau_t^{\mathbb{H}_{\nu,q}}$ is trivial if and only if $\tau_t^{\mathbb{H}_{\nu',q'}}$ is trivial and consequently Proposition 5.19 implies $\frac{\pi}{\log(|q|)}\mathbb{Z} = \frac{\pi}{\log(|q'|)}\mathbb{Z} \Rightarrow |q| = |q'|$. Next, since inner scaling automorphisms of $\mathbb{H}_{\nu,q}$ are implemented by elements of $\mathrm{C}(\mathbb{H}_{\nu,q})$ (similarly for $\mathbb{H}_{\nu',q'}$) it follows from the same proposition that $\nu\mathbb{Q} + \frac{\pi}{\log(|q|)}\mathbb{Z} = \nu'\mathbb{Q} + \frac{\pi}{\log(|q|)}\mathbb{Z}$. The last claim is a consequence of $\dim_{\mathbb{Q}}(\mathbb{R}/(\mathbb{Q}\frac{\pi}{\log(|q|)})) = \mathfrak{c}$. $\square$

Remark. We have constructed a compact quantum group $\mathbb{H}$ such that $\mathrm{L}^\infty(\mathbb{H})$ is the injective type II_∞ factor (with separable predual). Clearly there exists a compact quantum group with $\mathrm{L}^\infty(\mathbb{H})$ being the type II_1 factor (with separable predual) – one simply has to take an amenable ICC group Γ, e.g. S_∞ and define $\mathbb{H} = \widehat{\Gamma}$. It is not difficult to observe that $\mathrm{L}^\infty(\mathbb{H})$ can be isomorphic to M_n ($n \in \mathbb{N}$) only for $n = 1$. Indeed, one line of reasoning would be as follows: since $\dim(\mathrm{M}_n) < +\infty$, $\mathbb{H}$ would have to be coamenable. But then $\mathrm{C}(\mathbb{H}) \simeq \mathrm{M}_n$ is universal and has a character, which forces $n = 1$. It is an interesting question, which is to our knowledge open, whether there exists a compact quantum group $\mathbb{H}$ with $\mathrm{L}^\infty(\mathbb{H})$ isomorphic to $\mathrm{B}(\ell^2)$ or a type III_λ ($\lambda \in [0,1]$) injective factor.

5.5 Examples with commutative $\mathscr{C}_{\mathbb{G}}$

In this subsection we will prove that the condition $\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\sqrt{\frac{\dim(\alpha)}{\dim_q(\alpha)}} < +\infty$ from Theorem 5.9 holds for a fairly general class of non-Kac type compact quantum groups. More precisely, in this subsection we consider any compact quantum group $\mathbb{G}$ with the following properties:

1) there exists an irreducible fundamental representation U with $\dim_q(U) > \dim(U)$ and $\overline{U} \simeq U$,

2) irreducible representations of $\mathbb{G}$ are labeled by $\mathbb{Z}_+$, so that $\mathrm{Irr}(\mathbb{G}) = \{U^n\}_{n\in\mathbb{Z}_+}$, where $U^1 = U$ and U^0 is the trivial representation,

3) the fusion rules are given by $U^1 \mathbin{\text{\textcircled{\scriptsize T}}} U^n \simeq \bigoplus_{k=0}^{n+1} C(k,n)U^k$, with $C(n+1,n) \geq 1$ and $\sup_{n\in\mathbb{Z}_+} C(n+1,n) < +\infty$.

Let us mention two classes of compact quantum groups that fit into the above description:

- Non-Kac type free orthogonal quantum group $\mathbb{G} = O_F^+$ satisfies the above conditions with U being the standard fundamental representation (see Section 2.3.2).
- Let (B,ψ) be a finite dimensional C*-algebra with a non-tracial δ-form. The non-Kac type quantum automorphism group $\mathbb{G}_{Aut}(B,\psi)$ also satisfies the above conditions (see Section 2.3.4).

To keep the notation lighter, let us write $\dim(n) = \dim(U^n)$ and $\dim_q(n) = \dim_q(U^n)$ for all $n \in \mathbb{Z}_+$. Using our assumptions on the representation theory of $\mathbb{G}$ we can show that $(\frac{\dim(n)}{\dim_q(n)})_{n\in\mathbb{Z}_+}$ decays at an exponential rate.

Lemma 5.21.

- *We have* $\overline{U^n} \simeq U^n$ *and* $U^n \mathbin{\text{\textcircled{\scriptsize T}}} U^m \simeq U^m \mathbin{\text{\textcircled{\scriptsize T}}} U^n$ *for all* $n, m \in \mathbb{Z}_+$.
- *There exists* $d > 0, c > 1$ *such that* $\frac{\dim(n)}{\dim_q(n)} \leq \frac{d}{c^n}$ *for all* $n \in \mathbb{Z}_+$.

Proof. Observe that we have $(U^1)^{\text{\textcircled{\scriptsize T}} n} \simeq \bigoplus_{k=0}^{n} c'_{k,n} U^k$ ($n \in \mathbb{N}$) for some $c'_{k,n} \in \mathbb{Z}_+$. As $\overline{U^1} \simeq U^1$, it follows inductively that $\overline{U^n} \simeq U^n$ for all $n \in \mathbb{Z}_+$. Equivalence $U^n \mathbin{\text{\textcircled{\scriptsize T}}} U^m \simeq U^m \mathbin{\text{\textcircled{\scriptsize T}}} U^n$ can now be justified with the following calculations

$$U^n \mathbin{\text{\textcircled{\scriptsize T}}} U^m \simeq \overline{U^n \mathbin{\text{\textcircled{\scriptsize T}}} U^m} \simeq \overline{U^m} \mathbin{\text{\textcircled{\scriptsize T}}} \overline{U^n} \simeq U^m \mathbin{\text{\textcircled{\scriptsize T}}} U^n.$$

To prove the second bullet point, let us introduce positive numbers $A_n \geq 1$ via $\dim_q(n) = A_n \dim(n)$ ($n \in \mathbb{Z}_+$). Clearly $A_0 = 1$ and we assume that $A_1 > 1$. The fusion rule $U^1 \mathbin{\text{\textcircled{\scriptsize T}}} U^n \simeq \bigoplus_{k=0}^{n+1} C(k,n)U^k$ implies

$$A_1 A_n \dim(1)\dim(n) = \dim_q(U^1 \mathbin{\text{\textcircled{\scriptsize T}}} U^n) = \dim_q\Big(\bigoplus_{k=0}^{n+1} C(k,n)U^k\Big) = \sum_{k=0}^{n+1} C(k,n) A_k \dim(k)$$

and

$$\dim(1)\dim(n) = \dim(U^1 \mathbin{\text{\textcircled{\scriptsize T}}} U^n) = \dim\Big(\bigoplus_{k=0}^{n+1} C(k,n)U^k\Big) = \sum_{k=0}^{n+1} C(k,n)\dim(k).$$

Combining these equations gives us

$$A_1 A_n \dim(1)\dim(n) \le \big(\max_{k\in\{0,\dots,n\}} A_k\big)\sum_{k=0}^{n} C(k,n)\dim(k) + C(n+1,n)A_{n+1}\dim(n+1)$$
$$= \big(\max_{k\in\{0,\dots,n\}} A_k\big)\big(\dim(1)\dim(n) - C(n+1,n)\dim(n+1)\big) + C(n+1,n)A_{n+1}\dim(n+1),$$

hence

$$A_{n+1} \ge \max_{k\in\{0,\dots,n\}} A_k + \big(A_1 A_n - \max_{k\in\{0,\dots,n\}} A_k\big)\tfrac{\dim(1)\dim(n)}{C(n+1,n)\dim(n+1)}.$$

The above inequality implies $A_{n+1} = \max_{p\in\{0,\dots,n+1\}} A_p$. Consequently, we can further write

$$A_{n+1} \ge A_n + A_n \tfrac{A_1 - 1}{\sup_{m\in\mathbb{Z}_+} C(m+1,m)} \tfrac{\dim(1)\dim(n)}{\dim(n+1)}.$$

Since U^{n+1} is a subrepresentation of $U^1 \textcircled{\scriptsize T}\, U^n$, we have $\dim(1)\dim(n) \ge \dim(n+1)$ and

$$A_{n+1} \ge A_n\big(1 + \tfrac{A_1 - 1}{\sup_{m\in\mathbb{Z}_+} C(m+1,m)}\big).$$

Write $c = 1 + \frac{A_1 - 1}{\sup_{m\in\mathbb{Z}_+} C(m+1,m)} > 1$. We have shown $A_{n+1} \ge cA_n$. Using $\dim_q(n) = A_n \dim(n)$ we arrive at

$$\dim_q(n) = A_n \dim(n) \ge c^{n-1} A_1 \dim(n) \quad (n\in\mathbb{N}).$$

□

In particular, the above lemma implies that $\mathscr{C}_{\mathbb{G}}$ is an abelian von Neumann algebra. Theorems 5.9 and 5.12 give us the following corollary (it follows from the fusion rules that the assumptions are satisfied).

Corollary 5.22. We have $\sum_{n=0}^{\infty}\sqrt{\frac{\dim(n)}{\dim_q(n)}} < +\infty$, hence the inclusion $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$ is quasi-split. Furthermore, $\mathscr{C}_{\mathbb{G}}$ is not a MASA.

5.6 Quantum unitary group U_F^+

Let F be an invertible matrix with complex entries and U_F^+ the associated compact quantum group (see Section 2.3.3). In this subsection we show that the sum condition

$$\sum_{\gamma\in\mathrm{Irr}(U_F^+)} \big(\tfrac{\dim(\gamma)}{\dim_q(\gamma)}\big)^{\frac{1}{2}} < +\infty \tag{5.10}$$

is satisfied provided U_F^+ is "sufficiently non-Kac" (see Proposition 5.25 for the precise statement). Consequently, in this case we obtain information about the inclusion $\mathscr{C}_{U_F^+} \subseteq \mathrm{L}^\infty(U_F^+)$.

The representation theory of U_F^+ was described by Banica in [4, Théorème 1]. Recall that $\mathrm{Irr}(U_F^+)$ can be identified with $\mathbb{Z}_+ \star \mathbb{Z}_+$ in such a way that the neutral element e corresponds to the trivial representation, the first generator α to the fundamental representation and the second one β to its contragradient (see Section 2.3.3). The fusion rule is given by

$$x \mathbin{\textcircled{\scriptsize T}} y \simeq \bigoplus_{\substack{a,b,c\in\mathbb{Z}_+\star\mathbb{Z}_+:\\ x=ac,\ y=\bar{c}b}} ab \quad (x, y \in \mathbb{Z}_+ \star \mathbb{Z}_+). \tag{5.11}$$

In order to efficiently calculate the sum (5.10), we need to single out a family of irreducible representations out of which all of $\mathrm{Irr}(U_F^+)$ is built. Observe that each non-trivial word $\gamma \in \mathrm{Irr}(U_F^+)\setminus\{e\}$ has a well defined beginning and an end $s(\gamma), t(\gamma) \in \{\alpha, \beta\}$. Let us define sets

$$I_{\alpha,\alpha} = \{\alpha(\beta\alpha)^n \,|\, n \in \mathbb{Z}_+\}, \qquad I_{\alpha,\beta} = \{(\alpha\beta)^n \,|\, n \in \mathbb{N}\},$$
$$I_{\beta,\alpha} = \{(\beta\alpha)^n \,|\, n \in \mathbb{N}\}, \qquad I_{\beta,\beta} = \{(\beta\alpha)^n\beta \,|\, n \in \mathbb{Z}_+\}.$$

The following observation was already made e.g. in [60]:

Lemma 5.23. *Every non-trivial word $\gamma \in \mathrm{Irr}(U_F^+) \setminus \{e\}$ can be uniquely written as*

$$\gamma = x_1 \cdots x_p = x_1 \mathbin{\textcircled{\scriptsize T}} \cdots \mathbin{\textcircled{\scriptsize T}} x_p$$

for some $p \in \mathbb{N}$, $\delta_1, \ldots, \delta_{p-1} \in \{\alpha, \beta\}$ and $x_1 \in I_{s(\gamma),\delta_1}, x_2 \in I_{\delta_1,\delta_2}, \ldots, x_p \in I_{\delta_{p-1},t(\gamma)}$.

The above result follows easily from the observation that if $\delta\alpha^n\delta'$ for some $\delta, \delta' \in \mathrm{Irr}(U_F^+)$ and $n \geq 2$ then (5.11) implies

$$\delta\alpha^n\delta' = \delta\alpha \mathbin{\textcircled{\scriptsize T}} \alpha^{n-1}\delta'$$

(and similarly for $\delta\beta^n\delta'$). It follows that in order to calculate the sum (5.10) we need to find the (quantum) dimension of representations from the sets $I_{\delta,\delta'}$.

Lemma 5.24. *Let d be the classical or the quantum dimension function. If $d(\alpha) = 2$, then*

$$d((\alpha\beta)^n) = d((\beta\alpha)^n) = 2n + 1,$$
$$d((\alpha\beta)^n\alpha) = d((\beta\alpha)^n\beta) = 2n + 2$$

for $n \in \mathbb{Z}_+$. If $d(\alpha) > 2$, then

$$d((\alpha\beta)^n) = d((\beta\alpha)^n) = \tfrac{(d(\alpha)+d')^{2n+1}-(d(\alpha)-d')^{2n+1}}{2^{2n+1}d'},$$
$$d((\alpha\beta)^n\alpha) = d((\beta\alpha)^n\beta) = \tfrac{(d(\alpha)+d')^{2n+2}-(d(\alpha)-d')^{2n+2}}{2^{2n+2}d'}$$

for $n \in \mathbb{Z}_+$, where $d' = \sqrt{d(\alpha)^2 - 4}$.

Proof. Fix $n \in \mathbb{Z}_+$. As

$$(\alpha\beta)^n\alpha \otimes \beta = (\alpha\beta)^{n+1} \oplus (\alpha\beta)^n,$$

we have

$$d(\alpha)d((\alpha\beta)^n\alpha) = d((\alpha\beta)^{n+1}) + d((\alpha\beta)^n). \tag{5.12}$$

Similarly,

$$(\alpha\beta)^n\alpha\beta \otimes \alpha = (\alpha\beta)^{n+1}\alpha \oplus (\alpha\beta)^n\alpha$$

and

$$d(\alpha)d((\alpha\beta)^{n+1}) = d((\alpha\beta)^{n+1}\alpha) + d((\alpha\beta)^n\alpha). \tag{5.13}$$

Equations (5.12), (5.13) imply

$$\begin{bmatrix} d((\alpha\beta)^{n+1}) \\ d((\alpha\beta)^{n+1}\alpha) \end{bmatrix} = \begin{bmatrix} -1 & d(\alpha) \\ -d(\alpha) & d(\alpha)^2 - 1 \end{bmatrix} \begin{bmatrix} d((\alpha\beta)^n) \\ d((\alpha\beta)^n\alpha) \end{bmatrix},$$

hence iterating the above equation gives us

$$\begin{bmatrix} d((\alpha\beta)^n) \\ d((\alpha\beta)^n\alpha) \end{bmatrix} = \begin{bmatrix} -1 & d(\alpha) \\ -d(\alpha) & d(\alpha)^2 - 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ d(\alpha) \end{bmatrix} \quad (n \in \mathbb{Z}_+).$$

Assume first that $d(\alpha) = 2$. One easily checks that

$$\begin{bmatrix} -1 & d(\alpha) \\ -d(\alpha) & d(\alpha)^2 - 1 \end{bmatrix}^n = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}^n = \begin{bmatrix} -2n+1 & 2n \\ -2n & 2n+1 \end{bmatrix},$$

hence

$$\begin{bmatrix} d((\alpha\beta)^n) \\ d((\alpha\beta)^n\alpha) \end{bmatrix} = \begin{bmatrix} -2n+1 & 2n \\ -2n & 2n+1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2n+1 \\ 2n+2 \end{bmatrix}.$$

Observe that

$$\alpha \otimes \beta\alpha(\beta\alpha)^n = \alpha(\beta\alpha)^{n+1} \oplus \alpha(\beta\alpha)^n,$$

hence

$$d((\beta\alpha)^{n+1}) = \tfrac{1}{d(\alpha)}\big(d(\alpha(\beta\alpha)^{n+1}) + d(\alpha(\beta\alpha)^n\big) = \tfrac{1}{2}(2n+4+2n+2) = 2n+3 = d((\alpha\beta)^{n+1}).$$

The last equation can be checked as follows

$$d((\beta\alpha)^n\beta) = d(\overline{(\alpha\beta)^n\alpha}) = d((\alpha\beta)^n\alpha). \tag{5.14}$$

Let us now consider the case $d(\alpha) > 2$. This time $d' = \sqrt{d(\alpha)^2 - 4} > 0$, the matrix $\begin{bmatrix} -1 & d(\alpha) \\ -d(\alpha) & d(\alpha)^2 - 1 \end{bmatrix}^n$ equals

$$\begin{bmatrix} \frac{(d(\alpha)-d')^{2n}(d'+d(\alpha))+(d(\alpha)+d')^{2n}(d'-d(\alpha))}{2^{2n+1}d'} & \frac{(d(\alpha)+d')^{2n}-(d(\alpha)-d')^{2n}}{2^{2n}d'} \\ -\frac{(d(\alpha)+d')^{2n}-(d(\alpha)-d')^{2n}}{2^{2n}d'} & \frac{(d(\alpha)-d')^{2n}(d'-d(\alpha))+(d(\alpha)+d')^{2n}(d'+d(\alpha))}{2^{2n+1}d'} \end{bmatrix},$$

hence $\begin{bmatrix} d((\alpha\beta)^n) \\ d((\alpha\beta)^n\alpha) \end{bmatrix}$ equals

$$\begin{bmatrix} \frac{(d(\alpha)-d')^{2n}(d'+d(\alpha))+(d(\alpha)+d')^{2n}(d'-d(\alpha))}{2^{2n+1}d'} & \frac{(d(\alpha)+d')^{2n}-(d(\alpha)-d')^{2n}}{2^{2n}d'} \\ -\frac{(d(\alpha)+d')^{2n}-(d(\alpha)-d')^{2n}}{2^{2n}d'} & \frac{(d(\alpha)-d')^{2n}(d'-d(\alpha))+(d(\alpha)+d')^{2n}(d'+d(\alpha))}{2^{2n+1}d'} \end{bmatrix} \begin{bmatrix} 1 \\ d(\alpha) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{(d(\alpha)+d')^{2n}(d'+d(\alpha))+(d(\alpha)-d')^{2n}(d'-d(\alpha))}{2^{2n+1}d'} \\ \frac{(d(\alpha)-d')^{2n}(d(\alpha)d'-d(\alpha)^2+2)+(d(\alpha)+d')^{2n}(d(\alpha)d'+d(\alpha)^2-2)}{2^{2n+1}d'} \end{bmatrix} = \begin{bmatrix} \frac{(d'+d(\alpha))^{2n+1}+(d'-d(\alpha))^{2n+1}}{2^{2n+1}d'} \\ \frac{(d'+d(\alpha))^{2n+2}-(d'-d(\alpha))^{2n+2}}{2^{2n+2}d'} \end{bmatrix}$$

for all $n \in \mathbb{Z}_+$. Next, as before we get

$$\begin{aligned} d((\beta\alpha)^{n+1}) &= \tfrac{1}{d(\alpha)}\big(d(\alpha(\beta\alpha)^{n+1}) + d(\alpha(\beta\alpha)^n)\big) \\ &= \tfrac{1}{d(\alpha)}\big(\tfrac{(d'+d(\alpha))^{2n+4}-(d'-d(\alpha))^{2n+4}}{2^{2n+4}d'} + \tfrac{(d'+d(\alpha))^{2n+2}-(d'-d(\alpha))^{2n+2}}{2^{2n+2}d'}\big) \\ &= \tfrac{(d'+d(\alpha))^{2n+2}((d'+d(\alpha))^2+4)-(d'-d(\alpha))^{2n+2}((d'-d(\alpha))^2+4)}{2^{2n+4}d(\alpha)d'} \\ &= \tfrac{(d'+d(\alpha))^{2n+2}(2d(\alpha)^2+2d(\alpha)d')-(d'-d(\alpha))^{2n+2}(2d(\alpha)^2-2d(\alpha)d')}{2^{2n+4}d(\alpha)d'} \\ &= \tfrac{(d'+d(\alpha))^{2n+3}+(d'-d(\alpha))^{2n+3}}{2^{2n+3}d'} = d((\alpha\beta)^{n+1}) \end{aligned}$$

and (5.14) gives $d((\beta\alpha)^n\beta) = d((\alpha\beta)^n\alpha)$ for all $n \in \mathbb{Z}_+$. □

Using the above result we can show that for "sufficiently non-Kac" quantum unitary groups, the sum condition (5.10) is satisfied.

Proposition 5.25. *If* $\frac{\dim(\alpha)}{\dim_q(\alpha)} \leq \frac{1}{15}$ *then* $\sum_{\gamma\in\mathrm{Irr}(U_F^+)} \sqrt{\frac{\dim(\gamma)}{\dim_q(\gamma)}} < +\infty$.

Proof. Lemma 5.23 shows that

$$\begin{aligned} \sum_{\gamma\in\mathrm{Irr}(U_F^+)} \big(\tfrac{\dim(\gamma)}{\dim_q(\gamma)}\big)^{\frac{1}{2}} &= 1 + \sum_{p\in\mathbb{N}} \sum_{\delta_0,\dots,\delta_p\in\{\alpha,\beta\}} \sum_{\gamma_1\in I_{\delta_0,\delta_1}} \cdots \sum_{\gamma_p\in I_{\delta_{p-1},\delta_p}} \big(\tfrac{\dim(\gamma_1 \,\textcircled{\scriptsize T}\, \cdots \,\textcircled{\scriptsize T}\, \gamma_p)}{\dim_q(\gamma_1 \,\textcircled{\scriptsize T}\, \cdots \,\textcircled{\scriptsize T}\, \gamma_p)}\big)^{\frac{1}{2}} \\ &= 1 + \sum_{p\in\mathbb{N}} \sum_{\delta_0,\dots,\delta_p\in\{\alpha,\beta\}} \Big(\sum_{\gamma_1\in I_{\delta_0,\delta_1}} \big(\tfrac{\dim(\gamma_1)}{\dim_q(\gamma_1)}\big)^{\frac{1}{2}}\Big) \cdots \Big(\sum_{\gamma_p\in I_{\delta_{p-1},\delta_p}} \big(\tfrac{\dim(\gamma_p)}{\dim_q(\gamma_p)}\big)^{\frac{1}{2}}\Big). \end{aligned} \tag{5.15}$$

Let us define $S_{\delta,\delta'} = \sum_{\gamma\in I_{\delta,\delta'}} \big(\frac{\dim(\gamma)}{\dim_q(\gamma)}\big)^{\frac{1}{2}}$ for $\delta, \delta' \in \{\alpha, \beta\}$. Lemma 5.24 implies that

$$S_{\alpha,\beta} = S_{\beta,\alpha}, \quad S_{\alpha,\alpha} = S_{\beta,\beta}.$$

We will now show that our assumption $\frac{\dim(\alpha)}{\dim_q(\alpha)} \leq \frac{1}{15}$ forces $\max(S_{\alpha,\alpha}, S_{\alpha,\beta}) < \frac{1}{2}$.

More generally, let us fix $2 < x < y$ such that $\frac{x}{y} \leq \frac{1}{15}$ and show

$$S_1(x,y) = \sum_{n=0}^{\infty} \Bigg(\frac{\big((x+\sqrt{x^2-4})^{2n+2}-(x-\sqrt{x^2-4})^{2n+2}\big)\sqrt{y^2-4}}{\big((y+\sqrt{y^2-4})^{2n+2}-(y-\sqrt{y^2-4})^{2n+2}\big)\sqrt{x^2-4}} \Bigg)^{\frac{1}{2}} \leq 0.499 \tag{5.16}$$

and

$$S_2(x,y)=\sum_{n=1}^{\infty}\left(\frac{\left((x+\sqrt{x^2-4})^{2n+1}-(x-\sqrt{x^2-4})^{2n+1}\right)\sqrt{y^2-4}}{\left((y+\sqrt{y^2-4})^{2n+1}-(y-\sqrt{y^2-4})^{2n+1}\right)\sqrt{x^2-4}}\right)^{\frac{1}{2}}\leq 0.499. \tag{5.17}$$

Clearly for $2<x=\dim(\alpha), y=\dim_q(\alpha)$, inequalities (5.16), (5.17) imply $\max(S_{\alpha,\alpha},S_{\alpha,\beta})<\frac{1}{2}$.

Observe that the mean value theorem applied to the function $]0,+\infty[\ni s\mapsto s^p\in(0,+\infty)$ gives us

$$(x+\sqrt{x^2-4})^p-(x-\sqrt{x^2-4})^p\leq 2p(x+\sqrt{x^2-4})^{p-1}\sqrt{x^2-4}\quad(x>2,p\in\mathbb{N}).$$

Consequently,

$$\begin{aligned}S_1(x,y)&\leq\sum_{n=0}^{\infty}\Big(\frac{2(2n+2)(x+\sqrt{x^2-4})^{2n+1}\sqrt{y^2-4}}{(y+\sqrt{y^2-4})^{2n+2}-(y-\sqrt{y^2-4})^{2n+2}}\Big)^{\frac{1}{2}}\\&=2(y^2-4)^{\frac{1}{4}}\sum_{n=0}^{\infty}\sqrt{n+1}\,\frac{(x+\sqrt{x^2-4})^{\frac{1}{2}}}{y+\sqrt{y^2-4}}\Big(\frac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}}\Big)^n\frac{1}{\Big(1-\big(\frac{y-\sqrt{y^2-4}}{y+\sqrt{y^2-4}}\big)^{2n+2}\Big)^{\frac{1}{2}}}\\&\leq 2(y^2-4)^{\frac{1}{4}}\sum_{n=0}^{\infty}(n+\tfrac{3}{2})^{\frac{1}{2}}\frac{(x+\sqrt{x^2-4})^{\frac{1}{2}}}{y+\sqrt{y^2-4}}\Big(\frac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}}\Big)^n\frac{1}{\Big(1-\big(\frac{y-\sqrt{y^2-4}}{y+\sqrt{y^2-4}}\big)^{2n+2}\Big)^{\frac{1}{2}}}=\star,\end{aligned}\tag{5.18}$$

where we replaced $n+1$ by $n+\frac{3}{2}$ to match the corresponding bound for $S_2(x,y)$. Similarly,

$$\begin{aligned}S_2(x,y)&\leq\sum_{n=1}^{\infty}\Big(\frac{2(2n+1)(x+\sqrt{x^2-4})^{2n}\sqrt{y^2-4}}{(y+\sqrt{y^2-4})^{2n+1}-(y-\sqrt{y^2-4})^{2n+1}}\Big)^{\frac{1}{2}}\\&=2(y^2-4)^{\frac{1}{4}}\sum_{n=1}^{\infty}(n+\tfrac{1}{2})^{\frac{1}{2}}\frac{(x+\sqrt{x^2-4})^{\frac{1}{2}}}{y+\sqrt{y^2-4}}\Big(\frac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}}\Big)^{n-\frac{1}{2}}\frac{1}{\Big(1-\big(\frac{y-\sqrt{y^2-4}}{y+\sqrt{y^2-4}}\big)^{2n+1}\Big)^{\frac{1}{2}}}\\&=2(y^2-4)^{\frac{1}{4}}\sum_{n=0}^{\infty}(n+\tfrac{3}{2})^{\frac{1}{2}}\frac{(x+\sqrt{x^2-4})^{\frac{1}{2}}}{y+\sqrt{y^2-4}}\Big(\frac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}}\Big)^{n+\frac{1}{2}}\frac{1}{\Big(1-\big(\frac{y-\sqrt{y^2-4}}{y+\sqrt{y^2-4}}\big)^{2n+3}\Big)^{\frac{1}{2}}}\\&\leq 2(y^2-4)^{\frac{1}{4}}\sum_{n=0}^{\infty}(n+\tfrac{3}{2})^{\frac{1}{2}}\frac{(x+\sqrt{x^2-4})^{\frac{1}{2}}}{y+\sqrt{y^2-4}}\Big(\frac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}}\Big)^n\frac{1}{\Big(1-\big(\frac{y-\sqrt{y^2-4}}{y+\sqrt{y^2-4}}\big)^{2n+2}\Big)^{\frac{1}{2}}}=\star.\end{aligned}$$

Thus, it is enough to bound the expression $\star$. Note that the expression $\big(1-(\frac{y-\sqrt{y^2-4}}{y+\sqrt{y^2-4}})^{2n+2}\big)$ is the smallest when $n=0$, and we will use this to obtain a crude upper bound for $\star$. We also have

$$(y+\sqrt{y^2-4})\left(1-\big(\tfrac{y-\sqrt{y^2-4}}{y+\sqrt{y^2-4}}\big)^2\right)^{\frac{1}{2}}=2\sqrt{y}(y^2-4)^{\frac{1}{4}},$$

so we obtain the following bound for $\star$ (using an elementary estimate $\sqrt{n+\frac{3}{2}} \leqslant \sqrt{\frac{5}{2}}n$ for $n \geqslant 1$):

$$\begin{aligned}\star &\leq 2(y^2-4)^{\frac{1}{4}}\frac{(x+\sqrt{x^2-4})^{\frac{1}{2}}}{2\sqrt{y}(y^2-4)^{\frac{1}{4}}}\Big(\sqrt{\tfrac{3}{2}}+\sqrt{\tfrac{5}{2}}\sum_{n=1}^{\infty} n\big(\tfrac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}}\big)^n\Big)\\ &= \frac{(x+\sqrt{x^2-4})^{\frac{1}{2}}}{\sqrt{y}}\Big(\sqrt{\tfrac{3}{2}}+\sqrt{\tfrac{5}{2}}\frac{\frac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}}}{(1-\frac{x+\sqrt{x^2-4}}{y+\sqrt{y^2-4}})^2}\Big).\end{aligned}$$

Now, let us introduce the variable $t := \frac{x}{y} \in \left]0, \frac{1}{15}\right]$. Recall that $2 < x = ty < y$, hence $\sqrt{1-4y^{-2}} \geq \sqrt{1-t^2}$. Using this observation and the fact that the function $f(z) = \frac{z}{(1-z)^2}$ is increasing on $]0,1[$, we get

$$\star \leq \sqrt{2t}\Big(\sqrt{\tfrac{3}{2}}+\sqrt{\tfrac{5}{2}}\frac{\frac{2t}{1+\sqrt{1-t^2}}}{(1-\frac{2t}{1+\sqrt{1-t^2}})^2}\Big).$$

Let us argue that the condition $t \leq \frac{1}{15}$ implies that the expression above is bounded by 0.499. As the function $h(z) = \frac{2z}{1+\sqrt{1-z^2}}$ defined on $]0,1[$ is increasing, we have $h(t) \leq h(\frac{1}{15}) \leq 0.067$. It follows that $f(h(t)) \leq f(0.067) \leq 0.077$. Putting this information together we arrive at

$$\star \leq \sqrt{2t}\Big(\sqrt{\tfrac{3}{2}}+\sqrt{\tfrac{5}{2}}f(h(t))\Big) \leq 0.4917 \leq 0.499.$$

Consequently, we have shown (5.16), (5.17) and $\max(S_{\alpha,\alpha}, S_{\alpha,\beta}) < \frac{1}{2}$ in the case $\dim(\alpha) \geq 3$. Let us now turn to the case $\dim(\alpha) = 2$. Using expressions from Lemma 5.24 we see

$$S_{\alpha,\alpha} = \sum_{\gamma\in I_{\alpha,\alpha}}\big(\tfrac{\dim(\gamma)}{\dim_q(\gamma)}\big)^{\frac{1}{2}} = \sum_{n=0}^{\infty}\Big(\frac{(2n+2)2^{2n+2}\sqrt{\dim_q(\alpha)^2-4}}{(\dim_q(\alpha)+\sqrt{\dim_q(\alpha)^2-4})^{2n+2}-(\dim_q(\alpha)-\sqrt{\dim_q(\alpha)^2-4})^{2n+2}}\Big)^{\frac{1}{2}} \tag{5.19}$$

and

$$S_{\alpha,\beta} = \sum_{\gamma\in I_{\alpha,\beta}}\big(\tfrac{\dim(\gamma)}{\dim_q(\gamma)}\big)^{\frac{1}{2}} = \sum_{n=0}^{\infty}\Big(\frac{(2n+1)2^{2n+1}\sqrt{\dim_q(\alpha)^2-4}}{(\dim_q(\alpha)+\sqrt{\dim_q(\alpha)^2-4})^{2n+1}-(\dim_q(\alpha)-\sqrt{\dim_q(\alpha)^2-4})^{2n+1}}\Big)^{\frac{1}{2}}. \tag{5.20}$$

Observe that

$$\lim_{x\to 2^+}\frac{(x+\sqrt{x^2-4})^p-(x-\sqrt{x^2-4})^p}{2^p\sqrt{x^2-4}} = p$$

for all $p \in \mathbb{N}$, hence Fatou's lemma and inequality (5.16) imply

$$\begin{aligned}S_{\alpha,\alpha} &= \sum_{n=0}^{\infty}\lim_{x\to 2^+}\Big(\frac{\big((x+\sqrt{x^2-4})^{2n+2}-(x-\sqrt{x^2-4})^{2n+2}\big)\sqrt{\dim_q(\alpha)^2-4}}{\big((\dim_q(\alpha)+\sqrt{\dim_q(\alpha)^2-4})^{2n+2}-(\dim_q(\alpha)-\sqrt{\dim_q(\alpha)^2-4})^{2n+2}\big)\sqrt{x^2-4}}\Big)^{\frac{1}{2}}\\ &\leq \liminf_{x\to 2^+}\sum_{n=0}^{\infty}\Big(\frac{\big((x+\sqrt{x^2-4})^{2n+2}-(x-\sqrt{x^2-4})^{2n+2}\big)\sqrt{\dim_q(\alpha)^2-4}}{\big((\dim_q(\alpha)+\sqrt{\dim_q(\alpha)^2-4})^{2n+2}-(\dim_q(\alpha)-\sqrt{\dim_q(\alpha)^2-4})^{2n+2}\big)\sqrt{x^2-4}}\Big)^{\frac{1}{2}} \leq 0.499.\end{aligned}$$

Similarly, we get $S_{\alpha,\beta} \leq 0.499$ using (5.20) and (5.17).
We have shown $\max(S_{\alpha,\alpha}, S_{\alpha,\beta}) < \frac{1}{2}$. Consequently, (5.15) gives us

$$\begin{aligned}\sum_{\gamma\in\mathrm{Irr}(U_F^+)} \left(\tfrac{\dim(\gamma)}{\dim_q(\gamma)}\right)^{\frac{1}{2}} &\leq 1 + \sum_{p\in\mathbb{N}} \sum_{\delta_0,\dots,\delta_p\in\{\alpha,\beta\}} \max(S_{\alpha,\alpha}, S_{\alpha,\beta})^p \\ &= 1 + \sum_{p=1}^{\infty} 2^{p+1} \max(S_{\alpha,\alpha}, S_{\alpha,\beta})^p < +\infty.\end{aligned}$$

□

Remark. Calculations in the proof of Proposition 5.25 are far from optimal, however it is clear that there are non-Kac type quantum unitary groups U_F^+ with $\sum_{\gamma\in\mathrm{Irr}(U_F^+)}\left(\frac{\dim(\gamma)}{\dim_q(\gamma)}\right)^{\frac{1}{2}} = +\infty$. Indeed, assume that $\min(\frac{\dim(\alpha)}{\dim_q(\alpha)}, \frac{\dim(\alpha\beta)}{\dim_q(\alpha\beta)}) \geq \frac{1}{4}$. Then

$$S_{\alpha,\alpha} = S_{\beta,\beta} \geq \tfrac{1}{2}, \quad S_{\alpha,\beta} = S_{\beta,\alpha} \geq \tfrac{1}{2}$$

and consequently

$$\sum_{\gamma\in\mathrm{Irr}(U_F^+)} \left(\tfrac{\dim(\gamma)}{\dim_q(\gamma)}\right)^{\frac{1}{2}} \geq 1 + \sum_{p=1}^{\infty} 2^{p+1} \min(S_{\alpha,\alpha}, S_{\alpha,\beta})^p = +\infty.$$

It follows from the rule (5.11) and Lemma 5.24 that non-trivial irreducible representations γ of U_F^+ have $\rho_\gamma \neq \mathbb{1}_\gamma$. Using Theorems 5.9 and 5.12 we get the following corollary.

Corollary 5.26. Assume that $\frac{\dim(\alpha)}{\dim_q(\alpha)} \leq \frac{1}{15}$. Then the inclusion $\mathscr{C}_{U_F^+} \subseteq \mathrm{L}^\infty(U_F^+)$ is quasi-split and the relative commutant $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+)$ is not contained in $\mathscr{C}_{U_F^+}$.

Proof. By [30, Theorem 33] $\mathrm{L}^\infty(U_F^+)$ is a type III factor. Therefore Proposition 5.4 applies and we know that $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+)$ is a type III algebra, hence cannot be contained in $\mathscr{C}_{U_F^+}$, which is a finite von Neumann algebra.

An alternative argument can go as follows. By [4, Théorème 1 (iii)] the character of the fundamental representation of U_F^+ has the same disribution (with respect to the Haar integral) as a circular variable[32], so we can conclude that $\mathscr{C}_{U_F^+}$ is isomorphic to the free group factor $\mathrm{L}(F_2)$, in particular it is a factor. If the relative commutant $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+)$ were contained in $\mathscr{C}_{U_F^+}$ then the center of $\mathrm{L}^\infty(U_F^+)$ would be contained in $\mathscr{C}_{U_F^+}$, so $\mathrm{L}^\infty(U_F^+)$ has to be a factor. Moreover, if $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+) \subseteq \mathscr{C}_{U_F^+}$ then $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+) = \mathscr{C}_{U_F^+} \cap \mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+) = \mathbb{C}\mathbb{1}$, i.e. the inclusion is irreducible. By [25, Corollary 1] a quasi-split inclusion of factors is actually split and it is easy to check that a proper split inclusion cannot be irreducible. □

[32] Recall that x is circular if $x = s_1 + is_2$, where s_1 and s_2 are freely independent semicircular variables.

6 Approximation properties of quantum groups and operator algebras

In this section we will look at approximation properties of a (mostly discrete) quantum group $\mathbb{G}$ and the associated operator algebras $\mathrm{C}_0(\widehat{\mathbb{G}}), \mathrm{L}^\infty(\widehat{\mathbb{G}})$, and see how these two are related. We will focus on amenability (on the quantum group side) and nuclearity/injectivity (on the operator algebraic side), however similar problems can be studied also for weaker properties: Haagerup property [28, 11, 12, 13, 30, 67] or weak amenability/CBAP [13, 40, 41].

Amenability of a quantum group is defined via the existence of an invariant mean.

Definition 6.1. A locally compact quantum group $\mathbb{G}$ is *amenable* if there exists a state $m \in \mathrm{L}^\infty(\mathbb{G})^*$ (called a *mean*) such that

$$m((\omega \otimes \mathrm{id})\Delta(x)) = m((\mathrm{id} \otimes \omega)\Delta(x)) = m(x)\omega(\mathbb{1}) \quad (x \in \mathrm{L}^\infty(\mathbb{G}), \omega \in \mathrm{L}^1(\mathbb{G})).$$

On the operator algebraic side we have nucleatity as the appropriate property of a C*-algebra. Rather then nuclearity per se, we will use an equivalent property – completely positive approximation property (CPAP).

Definition 6.2. A C*-algebra A has a CPAP if there exists a net $(\Phi_\iota)_{\iota\in\mathcal{I}}$ of finite rank CP maps $\Phi_\iota \colon A \to A$ such that $\Phi_\iota(x) \xrightarrow[\iota\in\mathcal{I}]{} x$ for all $x \in A$.

For von Neumann algebras we will consider injectivity:

Definition 6.3. A von Neumann algebra $\mathrm{M} \subseteq \mathrm{B}(\mathsf{H})$ is injective if there exists a conditional expectation $\mathrm{B}(\mathsf{H}) \to \mathrm{M}$, i.e. a UCP map $\mathbb{E} \colon \mathrm{B}(\mathsf{H}) \to \mathrm{M}$ such that[33] $\mathbb{E}(x) = x$ and $\mathbb{E}(xTy) = x\mathbb{E}(T)y$ for all $x, y \in \mathrm{M}, T \in \mathrm{B}(\mathsf{H})$.

By a fundamental result of Connes [23, Theorem 6] injectivity is equivalent to the weak* completely positive approximation property (w*-CPAP).

Definition 6.4. A von Neumann algebra M has a w*-CPAP if there exists a net $(\Phi_\iota)_{\iota\in\mathcal{I}}$ of finite rank normal UCP maps $\Phi_\iota \colon \mathrm{M} \to \mathrm{M}$ such that $\Phi_\iota(x) \xrightarrow[\iota\in\mathcal{I}]{\mathrm{w}^*} x$ for all $x \in \mathrm{M}$.

We refer the reader to the sources [13, 8] and [14] (as well as references therein) for an introduction to these notions and equivalent properties.

Amenability of $\mathbb{G}$ and CPAP of $\mathrm{C}_0(\widehat{\mathbb{G}})$ (injectivity of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$) are closely related, in some cases even equivalent. As a rule of thumb, it is typically easier to derive the implication (quantum group approximation property) $\Rightarrow$ (operator algebraic approximation property), hence this is where we will start.

[33] By a famous result of Tomiyama ([14, Theorem 1.5.10]), linear map $\mathbb{E} \colon \mathrm{B}(\mathsf{H}) \to \mathrm{M}$ is a conditional expectation if and only if it is a contraction and satisfies $\mathbb{E}(x) = x$ for all $x \in \mathrm{M}$.

Theorem 6.5 ([8, Theorem 3.3]). *If $\mathbb{G}$ is an amenable locally compact quantum group, then the* C^**-algebra* $\mathrm{C}_0(\widehat{\mathbb{G}})$ *is nuclear and the von Neumann algebra* $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ *is injective.*

Sketch of a partial proof. Let $\mathrm{V} \in \mathrm{L}^\infty(\widehat{\mathbb{G}})' \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ be the unitary operator implementing comultiplication via

$$\Delta_{\mathbb{G}}(x) = \mathrm{V}(x \otimes \mathbb{1})\mathrm{V}^* \quad (x \in \mathrm{L}^\infty(\mathbb{G}))$$

(see equation (2.5)). Next, define a linear map $\mathbb{E}\colon \mathrm{B}(\mathrm{L}^2(\mathbb{G})) \to \mathrm{B}(\mathrm{L}^2(\mathbb{G}))$ by

$$\omega(\mathbb{E}(T)) = m((\omega \otimes \mathrm{id})(\mathrm{V}(T \otimes \mathbb{1})\mathrm{V}^*)) \quad (T \in \mathrm{B}(\mathrm{L}^2(\mathbb{G})), \omega \in \mathrm{B}(\mathrm{L}^2(\mathbb{G}))_*),$$

where m is a fixed mean on $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. Clearly this map is well defined and has norm ≤ 1. It is not difficult to see that $\mathbb{E}(\mathrm{B}(\mathrm{L}^2(\mathbb{G}))) \subseteq \mathrm{L}^\infty(\widehat{\mathbb{G}})$ and $\mathbb{E}(x) = x$ for $x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$ (see [8] or [13] for details). $\square$

It is natural to wonder, whether the converse of Theorem 6.5 holds, i.e. whether injectivity of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ implies amenability of $\mathbb{G}$. Such an implication is not true, even for classical locally compact groups.

Proposition 6.6 ([23, Corollary 7], [14, Remark 2.6.10]). *If G is a connected, separable locally compact group then* $\mathrm{C}^*_r(G) = \mathrm{C}_0(\widehat{G})$ *is nuclear and* $\mathrm{L}(G) = \mathrm{L}^\infty(\widehat{G})$ *is injective. However, such a group G need not be amenable. An example is given by* $\mathrm{SL}(n, \mathbb{R})$ *for* $n \geq 2$.

The above result tells us that if we want to obtain some sort of a converse to Theorem 6.5, we need to restrict our attention to a smaller class of quantum groups or impose more conditions on the operator algebraic assumption.

First we will take the former approach and assume that $\mathbb{G}$ is a unimodular discrete quantum group (equivalently $\widehat{\mathbb{G}}$ is compact and of Kac type).

Theorem 6.7 ([13, Theorem 6.6]). *If $\mathbb{G}$ is a unimodular discrete quantum group and* $\mathrm{C}(\widehat{\mathbb{G}})$ *is nuclear or* $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ *is injective then $\mathbb{G}$ is amenable.*

Sketch of a proof. Since $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is injective it has w^*-CPAP and there exists a net $(\Phi_\iota)_{\iota \in \mathcal{I}}$ of normal UCP maps $\mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}})$ of finite rank such that $\Phi_\iota(x) \xrightarrow[\iota \in \mathcal{I}]{\mathrm{w}^*} x$ for all $x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$. Next, there exists a normal UCP map ([13, Section 7.1])

$$\Delta^\sharp_{\widehat{\mathbb{G}}}\colon \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{G}}) \ni U^\alpha_{i,j} \otimes U^\beta_{k,l} \mapsto \tfrac{\delta_{\alpha,\beta}\delta_{j,k}}{\dim_q(\alpha)} U^\alpha_{i,l} \in \mathrm{L}^\infty(\widehat{\mathbb{G}}).$$

Using these maps we define

$$\Psi_\iota\colon \mathrm{L}^\infty(\widehat{\mathbb{G}}) \ni x \mapsto \Delta^\sharp_{\widehat{\mathbb{G}}}(\Phi_\iota \otimes \mathrm{id})\Delta_{\widehat{\mathbb{G}}}(x) \in \mathrm{L}^\infty(\widehat{\mathbb{G}}) \quad (\iota \in \mathcal{I}).$$

These maps are still normal, UCP and converge to the identity map in the point-w^* topology, but they also take into account the structure of subspaces $V_\alpha = \mathrm{span}\{U^\alpha_{i,j} \,|\, i, j \in$

$\{1,\dots,\dim(\alpha)\}\}$, i.e. $\Psi_\iota(V_\alpha)\subseteq V_\alpha\,(\alpha\in\mathrm{Irr}(\mathbb{G}))$. One way to proceed from here is as follows: a direct calculation shows that

$$(\Psi_\iota\otimes\mathrm{id})\mathrm{W}^{\widehat{\mathbb{G}}}=(\mathbb{1}\otimes a_\iota)\mathrm{W}^{\widehat{\mathbb{G}}}\quad(\iota\in\mathcal{I}),$$

where $a_\iota\in\ell^\infty(\mathbb{G})$ is given by

$$a_\iota=(h\otimes\mathrm{id})((\Phi_\iota\otimes\mathrm{id})(\mathrm{W}^{\widehat{\mathbb{G}}})\mathrm{W}^{\widehat{\mathbb{G}}^*})\quad(\iota\in\mathcal{I}).$$

One can check that a_ι belong to the Fourier algebra $\mathrm{A}(\mathbb{G})=\lambda^{\widehat{\mathbb{G}}}(\mathrm{L}^\infty(\widehat{\mathbb{G}}))$ and form a bounded approximate identity. This implies amenability of $\mathbb{G}$. The C*-algebraic version of this result can be proved analogously (for details see [13]). □

Remark. One is tempted to try proving an extension of Theorem 6.7 to discrete quantum groups which are possibly non-unimodular. One immediate problem that appears (when using a similar strategy as presented above) stems from the fact that if $(\Phi_\iota)_{\iota\in\mathcal{I}}$ converges to the identity, then $(\Psi_\iota)_{\iota\in\mathcal{I}}$ will converge to $\Delta^\sharp_{\widehat{\mathbb{G}}}\circ\Delta_{\widehat{\mathbb{G}}}$ which acts via

$$\Delta^\sharp_{\widehat{\mathbb{G}}}\circ\Delta_{\widehat{\mathbb{G}}}\colon\,\mathrm{L}^\infty(\widehat{\mathbb{G}})\ni U^\alpha_{i,j}\mapsto\tfrac{\dim(\alpha)}{\dim_q(\alpha)}U^\alpha_{i,j}\in\mathrm{L}^\infty(\widehat{\mathbb{G}})\quad(\alpha\in\mathrm{Irr}(\mathbb{G}),i,j\in\{1,\dots,\dim(\alpha)\}),$$

hence it does not converge to the identity [13, Section 7.1]. One could try remedy this situation by composing with an inverse to $\Delta^\sharp_{\widehat{\mathbb{G}}}\circ\Delta_{\widehat{\mathbb{G}}}$ as follows. First, using the fact that injectivity of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ implies an existence of normal UCP maps $(\Phi_\iota)_{\iota\in\mathcal{I}}$ as above which factors through matrix algebras, we can assume that the image of Φ_ι is contained in $\mathrm{Pol}(\widehat{\mathbb{G}})$ (use [14, Proposition 1.5.12] and w*-density of $\mathrm{Pol}(\widehat{\mathbb{G}})$ in $\mathrm{L}^\infty(\widehat{\mathbb{G}})$). Next, define "corrected" maps

$$\tilde{\Psi}_\iota\colon\,\mathrm{L}^\infty(\widehat{\mathbb{G}})\ni x\mapsto\Delta^\sharp_{\widehat{\mathbb{G}}}((\Delta^\sharp_{\widehat{\mathbb{G}}}\circ\Delta_{\widehat{\mathbb{G}}})^{-1}\Phi_\iota\otimes\mathrm{id})\Delta_{\widehat{\mathbb{G}}}(x)\in\mathrm{L}^\infty(\widehat{\mathbb{G}})\quad(\iota\in\mathcal{I}).$$

These maps are well defined: even though $(\Delta^\sharp_{\widehat{\mathbb{G}}}\circ\Delta_{\widehat{\mathbb{G}}})^{-1}$ is an unbounded map, the above composition makes sense because the image of Φ_ι is a finite dimensional subspace in $\mathrm{Pol}(\widehat{\mathbb{G}})$. $(\Psi_\iota)_{\iota\in\mathcal{I}}$ will converge to the identity (on elements of $\mathrm{Pol}(\widehat{\mathbb{G}})$) in the point-w* topology. However, now it is not clear why $\tilde{\Psi}_\iota$ would be CP or even uniformly bounded in the CB norm.

As far as we know, it is an open question whether nuclearity of $\mathrm{C}(\widehat{\mathbb{G}})$ or injectivity of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ implies amenability of $\mathbb{G}$ for a discrete quantum group $\mathbb{G}$.

We can obtain a partial converse to Theorem 6.5 by imposing stronger assumptions on the operator algebraic side. One result of this kind was obtained by Sołtan and Viselter in [76].

Theorem 6.8 ([76, Theorem 3]). *Let $\mathbb{G}$ be a locally compact quantum group. The following conditions are equivalent:*

1) $\mathbb{G}$ is amenable,

2) there is a conditional expectation of $\mathrm{B}(\mathrm{L}^2(\mathbb{G}))$ *onto* $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ *that maps* $\mathrm{L}^\infty(\mathbb{G})$ *to* $\mathbb{C}\mathbb{1}$,

3) there is a conditional expectation of $\mathrm{B}(\mathrm{L}^2(\mathbb{G}))$ *onto* $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ *that maps* $\mathrm{L}^\infty(\mathbb{G})$ *to* $\mathcal{Z}(\mathrm{L}^\infty(\widehat{\mathbb{G}}))$.

Conditions 2. and 3. from the above result are strenghtening of the injectivity of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$, which also take into consideration algebra $\mathrm{L}^\infty(\mathbb{G})$. We will now present a result of similar kind: we will show that amenability is equivalent to a strenghtening of w*-CPAP of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ for discrete quantum groups $\mathbb{G}$. First we need to introduce two definitions.

Recall the following easy observation: if M is a von Neumann algebra with a faithful normal weight φ, and Φ is a UCP map $\mathrm{M}\to\mathrm{M}$ such that $\varphi\circ\Phi\le\varphi$, then there exists a L^2-implementation of Φ, i.e. a bounded map $T_\Phi\in\mathrm{B}(\mathsf{H}_\varphi)$ satisfying $T_\Phi\Lambda_\varphi(x)=\Lambda_\varphi(\Phi(x))\,(x\in\mathfrak{N}_\varphi)$.

Definition 6.9. Let M be a von Neumann algebra with a n.s.f weight φ. We say that (M,φ) has a w*-CPAP relative to (a von Neumann algebra) $\mathrm{N}\subseteq\mathrm{B}(\mathsf{H}_\varphi)$ if there exists a net $(\Phi_\iota)_{\iota\in\mathcal{I}}$ of finite rank, normal, UCP maps $\mathrm{M}\to\mathrm{M}$ such that $\varphi\circ\Phi_\iota\le\varphi$, $\Phi_\iota(x)\xrightarrow[\iota\in\mathcal{I}]{\mathrm{w}^*}x$ $(x\in\mathrm{M})$ and the L^2-implementations of Φ_ι belong to N.

Whenever it is clear from the context which weight on M we consider, we will simply say that M has a w*-CPAP relative to N. Observe that M has a w*-CPAP relative to $\mathrm{B}(\mathsf{H}_\varphi)$ if and only if M has a w*-CPAP.

A discrete quantum group $\mathbb{G}$ is amenable if, and only if there exists a bounded left approximate identity $(a_\iota)_{\iota\in\mathcal{I}}=(\widehat{\lambda}(\omega_\iota))_{\iota\in\mathcal{I}}$ of the Fourier algebra $\mathrm{A}(\mathbb{G})$ in $\mathrm{c}_c(\mathbb{G})$ consisting of completely positive definite functions, i.e. the maps $\mathrm{L}^\infty(\widehat{\mathbb{G}})\to\mathrm{L}^\infty(\widehat{\mathbb{G}})$ associated with a_ι are completely positive[34],

$$\sup_{\iota\in\mathcal{I}}\|a_\iota\|_{\mathrm{A}(\mathbb{G})}=\sup_{\iota\in\mathcal{I}}\|\omega_\iota\|<+\infty\quad\text{ and }\quad\lim_{\iota\in\mathcal{I}}a_\iota a=a\quad(a\in\mathrm{A}(\mathbb{G})).$$

Looking at amenability from this point of view, it is natural to introduce a central version of amenability[35]:

Definition 6.10 ([13, Definition 7.1], [41])**.** A discrete quantum group $\mathbb{G}$ is *centrally amenable* if there exists a bounded approximate identity $(a_\iota)_{\iota\in\mathcal{I}}=(\widehat{\lambda}(\omega_\iota))_{\iota\in\mathcal{I}}$ of $\mathrm{A}(\mathbb{G})$ in $\mathrm{c}_c(\mathbb{G})$ such that $\omega_\iota\ge0$ and $a_\iota\in\mathcal{Z}\ell^\infty(\mathbb{G})$ for all $\iota\in\mathcal{I}$.

[34]Equivalently $\omega_\iota\ge0$ – see [29, Theorem 15]. Furthermore, it is not difficult to see that here the condition $a_\iota\in\mathrm{c}_c(\mathbb{G})$ is superfluous.

[35]Equivalence of amenability and the existence of a bounded left approximate identity in $\mathrm{A}(\mathbb{G})$ consisting of completely positive definite functions uses the coamenability of $\widehat{\mathbb{G}}$ ([8, Theorem 3.1]). Arguably, it would therefore be more natural to introduce this condition as *central coamenability* – we will stick to the *central amenability* because amenability of $\mathbb{G}$ is equivalent to coamenability of $\widehat{\mathbb{G}}$ for discrete quantum groups by a famous result of Tomatsu [86].

Now we can state the advertised result which asserts the equivalence of a (central) amenability of $\mathbb{G}$ and a strenghtening of w*-CPAP of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$.

Theorem 6.11. *Let $\mathbb{G}$ be a discrete quantum group and h the Haar integral on $\widehat{\mathbb{G}}$. Consider the following conditions*

1a) $\mathbb{G}$ is centrally amenable,

1b) $(\mathrm{L}^\infty(\widehat{\mathbb{G}}), h)$ has a w-CPAP relative to $\mathcal{Z}\ell^\infty(\mathbb{G})$,*

2a) $\mathbb{G}$ is amenable,

2b) $(\mathrm{L}^\infty(\widehat{\mathbb{G}}), h)$ has a w-CPAP relative to $\ell^\infty(\mathbb{G})$,*

2c) $(\mathrm{L}^\infty(\widehat{\mathbb{G}}), h)$ has a w-CPAP relative to $\ell^\infty(\mathbb{G})'$.*

We have 1a) $\Leftrightarrow$ 1b) $\Rightarrow$ 2a) $\Leftrightarrow$ 2b) $\Leftrightarrow$ 2c).

Remark. Let us note that there are discrete quantum groups which are amenable but not centrally amenable, e.g. $\widehat{\mathrm{SU}_q(2)}$ for $q \in \,]-1,1[\, \setminus \{0\}$ [41]. In fact, we do not know any example of a non-unimodular, centrally amenable discrete quantum group.

It will be useful to prove first an auxiliary result, which we believe to be interesting on its own.

Proposition 6.12. *Let $\mathbb{G}$ be a locally compact quantum group with the left Haar integral φ and $\Phi\colon \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}})$ a normal UCP map satisfying $\Phi \circ \varphi \leq \varphi$. Let $\Phi_*\colon \mathrm{L}^1(\widehat{\mathbb{G}}) \to \mathrm{L}^1(\widehat{\mathbb{G}})$ be the predual of Φ and $T\colon \mathrm{L}^2(\mathbb{G}) \to \mathrm{L}^2(\mathbb{G})$ the L^2 implementation of Φ. We have*

1) $T \in \mathrm{L}^\infty(\mathbb{G})$ if and only if $\Phi_*(\omega \star \nu) = \Phi_*(\omega) \star \nu$ for all $\omega, \nu \in \mathrm{L}^1(\widehat{\mathbb{G}})$,

2) $T \in \mathrm{L}^\infty(\mathbb{G})'$ if and only if $\Phi_*(\omega \star \nu) = \omega \star \Phi_*(\nu)$ for all $\omega, \nu \in \mathrm{L}^1(\widehat{\mathbb{G}})$,

3) $T \in \mathcal{Z}(\mathrm{L}^\infty(\mathbb{G}))$ if and only if $\Phi_*(\omega \star \nu) = \Phi_*(\omega) \star \nu = \omega \star \Phi_*(\nu)$ for all $\omega, \nu \in \mathrm{L}^1(\widehat{\mathbb{G}})$.

Proof. Using the biduality $\mathbb{G} = \widehat{\widehat{\mathbb{G}}}$ and [93, Definition 4.6] (see also Section 2.2), we deduce that the subspace

$$\mathcal{N} = \{\widehat{\lambda}(\omega) \,|\, \omega \in \mathrm{L}^1(\widehat{\mathbb{G}})\colon \exists_{\xi \in \mathrm{L}^2(\mathbb{G})} \forall_{x \in \mathfrak{N}_{\widehat{\varphi}}} \langle \Lambda_{\widehat{\varphi}}(x) \,|\, \xi \rangle = \omega(x^*)\}$$

is a core for Λ_φ and for $\widehat{\lambda}(\omega) \in \mathcal{N}$ we have $\Lambda_\varphi(\widehat{\lambda}(\omega)) = \xi$. Now, let us argue that

$$T^* \Lambda_\varphi(\widehat{\lambda}(\omega)) = \Lambda_\varphi(\widehat{\lambda}(\Phi_*(\omega))) \quad (\widehat{\lambda}(\omega) \in \mathcal{N}). \tag{6.1}$$

For $x \in \mathfrak{N}_{\widehat{\varphi}}$ we have

$$\begin{aligned}
&\langle \Lambda_{\widehat{\varphi}}(x) \,|\, T^* \Lambda_\varphi(\widehat{\lambda}(\omega)) \rangle = \langle T \Lambda_{\widehat{\varphi}}(x) \,|\, \Lambda_\varphi(\widehat{\lambda}(\omega)) \rangle = \langle \Lambda_{\widehat{\varphi}}(\Phi(x)) \,|\, \Lambda_\varphi(\widehat{\lambda}(\omega)) \rangle \\
&= \omega(\Phi(x)^*) = \omega(\Phi(x^*)) = \Phi_*(\omega)(x^*) = \langle \Lambda_{\widehat{\varphi}}(x) \,|\, \Lambda_\varphi(\widehat{\lambda}(\Phi_*(\omega))) \rangle,
\end{aligned}$$

which proves equation (6.1).

If $T \in \mathrm{L}^\infty(\mathbb{G})$ then (6.1) implies $T^*\widehat{\lambda}(\omega) = \widehat{\lambda}(\Phi_*(\omega))$ for all $\omega \in \mathrm{L}^1(\widehat{\mathbb{G}})$ such that $\widehat{\lambda}(\omega) \in \mathcal{N}$ and by density of such ω (Lemma 7.10) this equation holds for all $\omega \in \mathrm{L}^1(\widehat{\mathbb{G}})$. Consequently

$$\widehat{\lambda}(\Phi_*(\omega \star \nu)) = T^*\widehat{\lambda}(\omega \star \nu) = T^*\widehat{\lambda}(\omega)\widehat{\lambda}(\nu) = \widehat{\lambda}(\Phi_*(\omega))\widehat{\lambda}(\nu) = \widehat{\lambda}(\Phi_*(\omega) \star \nu)$$

and $\Phi_*(\omega \star \nu) = \Phi_*(\omega) \star \nu$ for all $\omega, \nu \in \mathrm{L}^1(\widehat{\mathbb{G}})$.

Using $\mathrm{W}^{\widehat{\mathbb{G}}} = \chi(\mathrm{W}^{\mathbb{G}})^*$ we get

$$\begin{aligned}
&\sigma_t^\varphi(\widehat{\lambda}(\omega)) = (\omega \otimes \mathrm{id})((\mathrm{id} \otimes \sigma_t^\varphi)\mathrm{W}^{\widehat{\mathbb{G}}}) = (\mathrm{id} \otimes \overline{\omega})((\sigma_t^\varphi \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}}))^* \\
&= (\mathrm{id} \otimes \overline{\omega})((\tau_t \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}})(\mathbb{1} \otimes \hat{\delta}^{it}))^* = (\mathrm{id} \otimes \overline{\omega})((\mathrm{id} \otimes \hat{\tau}_{-t})(\mathrm{W}^{\mathbb{G}})(\mathbb{1} \otimes \hat{\delta}^{it}))^* \\
&= (\mathrm{id} \otimes \omega)((\mathbb{1} \otimes \hat{\delta}^{-it})(\mathrm{id} \otimes \hat{\tau}_{-t})(\mathrm{W}^{\mathbb{G}})^*) = (\omega \otimes \mathrm{id})((\hat{\delta}^{-it} \otimes \mathbb{1})(\hat{\tau}_{-t} \otimes \mathrm{id})(\mathrm{W}^{\widehat{\mathbb{G}}})) \\
&= \hat{\lambda}((\omega\hat{\delta}^{-it}) \circ \hat{\tau}_{-t})
\end{aligned}$$

for $\omega \in \mathrm{L}^1(\widehat{\mathbb{G}}), t \in \mathbb{R}$ (see also equation (2.14), [93, Theorem 3.10, Proposition 5.15] and their proofs). If $\Phi_*(\omega \star \nu) = \Phi_*(\omega) \star \nu$ holds for all $\omega, \nu \in \mathrm{L}^1(\widehat{\mathbb{G}})$ then

$$\begin{aligned}
&\langle \Lambda_{\widehat{\varphi}}(x) \,|\, T^* J_\varphi \widehat{\lambda}(\omega)^* J_\varphi \Lambda_\varphi(\widehat{\lambda}(\nu))\rangle = \langle \Lambda_{\widehat{\varphi}}(x) \,|\, T^* \Lambda_\varphi(\widehat{\lambda}(\nu)\sigma^\varphi_{-i/2}(\widehat{\lambda}(\omega)))\rangle \\
&= \langle T\Lambda_{\widehat{\varphi}}(x) \,|\, \Lambda_\varphi(\widehat{\lambda}(\nu \star \rho))\rangle = (\nu \star \rho)(\Phi(x)^*) = \Phi_*(\nu \star \rho)(x^*) = (\Phi_*(\nu) \star \rho)(x^*) \\
&= \langle \Lambda_{\widehat{\varphi}}(x) \,|\, \Lambda_\varphi(\widehat{\lambda}(\Phi_*(\nu) \star \rho))\rangle = \langle \Lambda_{\widehat{\varphi}}(x) \,|\, \Lambda_\varphi(\widehat{\lambda}(\Phi_*(\nu))\sigma^\varphi_{-i/2}(\widehat{\lambda}(\omega)))\rangle \\
&= \langle \Lambda_{\widehat{\varphi}}(x) \,|\, J_\varphi \widehat{\lambda}(\omega)^* J_\varphi T^* \Lambda_\varphi(\widehat{\lambda}(\nu))\rangle
\end{aligned}$$

for $x \in \mathfrak{N}_{\widehat{\varphi}}, \widehat{\lambda}(\nu) \in \mathcal{N}$ and $\omega \in \mathrm{L}^1(\widehat{\mathbb{G}})$ such that $\widehat{\lambda}(\omega) \in \mathcal{N}$ and the map $\mathbb{R} \ni t \mapsto (\omega\hat{\delta}^{-it}) \circ \hat{\tau}_t \in \mathrm{L}^1(\widehat{\mathbb{G}})$ extends to an entire map $\mathbb{C} \to \mathrm{L}^1(\widehat{\mathbb{G}})$. We denote by $\rho \in \mathrm{L}^1(\widehat{\mathbb{G}})$ the value of the analitical continuation to $t = -i/2$. Density of appropriate vectors and operators (Lemma 7.10) gives us $T^* \in \mathrm{L}^\infty(\mathbb{G})'' = \mathrm{L}^\infty(\mathbb{G})$ and proves the first point. An alternative proof can be given using [13, Proposition 4.5].

Assume now that $T \in \mathrm{L}^\infty(\mathbb{G})'$. By equation (6.1) we have

$$\Lambda_\varphi(\widehat{\lambda}(\Phi_*(\omega \star \nu))) = T^*\Lambda_\varphi(\widehat{\lambda}(\omega \star \nu)) = \widehat{\lambda}(\omega)T^*\Lambda_\varphi(\widehat{\lambda}(\nu)) = \Lambda_\varphi(\widehat{\lambda}(\omega \star \Phi_*(\nu))),$$

hence $\Phi_*(\omega \star \nu) = \omega \star \Phi_*(\nu)$ for all functionals ω, ν such that $\widehat{\lambda}(\omega), \widehat{\lambda}(\nu)$ belongs to $\mathcal{N}$. Lemma 7.10 gives us the claim.

On the other hand, if $\Phi_*(\omega \star \nu) = \omega \star \Phi_*(\nu)$ for all $\omega, \nu \in \mathrm{L}^1(\widehat{\mathbb{G}})$ then

$$\widehat{\lambda}(\omega)T^*\Lambda_\varphi(\widehat{\lambda}(\nu)) = \Lambda_\varphi(\widehat{\lambda}(\omega \star \Phi_*(\nu))) = \Lambda_\varphi(\widehat{\lambda}(\Phi_*(\omega \star \nu))) = T^*\widehat{\lambda}(\omega)\Lambda_\varphi(\widehat{\lambda}(\nu))$$

for all $\widehat{\lambda}(\omega), \widehat{\lambda}(\nu) \in \mathcal{N}$ and consequently $T \in \mathrm{L}^\infty(\mathbb{G})'$.

The last point follows from a combination of 1) and 2). □

Proof of Theorem 6.11. First, assume that $\mathbb{G}$ is amenable and we have a bounded left approximate identity $(a_\iota)_{\iota\in\mathcal{I}} = (\widehat{\lambda}(\omega_\iota))_{\iota\in\mathcal{I}}$ of $\mathrm{A}(\mathbb{G})$ in $\mathrm{c}_c(\mathbb{G})$ with $\omega_\iota \geq 0\,(\iota\in\mathcal{I})$. Define

$$\Phi^\iota_*\colon\ \mathrm{L}^1(\widehat{\mathbb{G}}) \ni \omega \mapsto \omega_\iota \star \omega \in \mathrm{L}^1(\widehat{\mathbb{G}}) \quad (\iota\in\mathcal{I})$$

and correspondingly $\Phi^\iota = (\Phi^\iota_*)^*$. Clearly $\Phi^{\iota,\delta}$ are normal, CP and Proposition 6.12 implies that the L^2-implementation of Φ^ι are in $\ell^\infty(\widehat{\mathbb{G}})$ for all $\iota\in\mathcal{I}$. As $a_\iota\in\mathrm{c}_c(\mathbb{G})$, the image of Φ^ι is of finite rank. We need to make sure that our functions map $\mathbb{1}$ to $\mathbb{1}$. To do that, define

$$\tilde{\Phi}^\iota = \tfrac{\Phi^\iota}{\|\Phi^\iota\|} = \tfrac{\Phi^\iota}{\omega_\iota(\mathbb{1})} \quad (\iota\in\mathcal{I}).$$

It is clear that $(\tilde{\Phi}^\iota)_{\iota\in\mathcal{I}}$ converge in the point-w* topology to the identity, hence we have proved the implication $2a) \Rightarrow 2b)$.

If $\mathbb{G}$ is centrally amenable, we know additionally that $a_\iota \in \mathcal{Z}(\ell^\infty(\mathbb{G}))$, i.e. $\omega_\iota(U^\alpha_{i,j}) = \delta_{i,j}\omega_\iota(U^\alpha_{1,1})$ for all $\iota\in\mathcal{I},\alpha\in\mathrm{Irr}(\widehat{\mathbb{G}}), i,j\in\{1,\dots,\dim(\alpha)\}$. Then

$$(\omega_\iota\star\omega)(U^\alpha_{i,j}) = \sum_{k=1}^{\dim(\alpha)} \omega_\iota(U^\alpha_{i,k})\omega(U^\alpha_{k,j}) = \omega_\iota(U^\alpha_{i,i})\omega(U^\alpha_{i,j}) = \omega(U^\alpha_{i,j})\omega_\iota(U^\alpha_{j,j}) = (\omega\star\omega_\iota)(U^\alpha_{i,j})$$

and consequently Proposition 6.12 implies that the L^2-implementation of Φ^ι belongs to $\mathcal{Z}(\ell^\infty(\mathbb{G}))$. This proves $1a) \Rightarrow 1b)$.

Assume that $2b)$ holds with maps $(\Phi^\iota)_{\iota\in\mathcal{I}}$. Set $a_\iota = T^*_\iota \in \ell^\infty(\mathbb{G})$, where T_ι is the L^2-implementation of Φ^ι. In the proof of Proposition 6.12 we have showed that $a_\iota\widehat{\lambda}(\omega) = \widehat{\lambda}(\Phi^\iota_*(\omega))$ for $\omega\in\mathrm{L}^1(\widehat{\mathbb{G}})$. As Φ^ι is of finite rank and $T_\iota\in\ell^\infty(\mathbb{G})$, we have $a_\iota\in\mathrm{c}_c(\mathbb{G})\subseteq\mathrm{A}(\mathbb{G})$. Indeed, Proposition 6.12 implies $\Phi^\iota_*(\omega\star\nu) = \Phi^\iota_*(\omega)\star\nu$ for all $\omega,\nu\in\mathrm{L}^1(\widehat{\mathbb{G}})$ which gives us $(\Phi^\iota\otimes\mathrm{id})\circ\Delta_{\widehat{\mathbb{G}}} = \Delta_{\widehat{\mathbb{G}}}\circ\Phi^\iota$. This equation forces

$$\Phi^\iota(\mathrm{span}\{U^\alpha_{i,j}\,|\,i,j\in\{1,\dots,\dim(\alpha)\}\}) \subseteq \mathrm{span}\{U^\alpha_{i,j}\,|\,i,j\in\{1,\dots,\dim(\alpha)\}\}$$

for all $\alpha\in\mathrm{Irr}(\widehat{\mathbb{G}})$. Consequently each Φ^ι annihilates all but a finite number of subspaces $\mathrm{span}\{U^\alpha_{i,j}\,|\,i,j\in\{1,\dots,\dim(\alpha)\}\}$. We have $\|a_\iota\|_{\mathrm{A}(\mathbb{G})} = \|\Phi^\iota(\mathbb{1})\| = 1$. Clearly a_ι is completely definite positive (since Φ^ι is CP), and $(a_\iota)_{\iota\in\mathcal{I}}$ form a bounded left approximate identity of $\mathrm{A}(\mathbb{G})$ consisting of elements in $\mathrm{c}_c(\mathbb{G})$. This shows $2a)$.

If $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ has a w*-CPAP relative to $\mathcal{Z}(\ell^\infty(\mathbb{G}))$ then we additionally know that $a_\iota\in\mathcal{Z}(\ell^\infty(\mathbb{G}))$, i.e. $\mathbb{G}$ is centrally amenable.

We are left to show the equivalence of $2b)$ and $2c)$. Let $(\Phi^\iota)_{\iota\in\mathcal{I}}$ be a net given by w*-CPAP of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ relative to $\ell^\infty(\mathbb{G})$, i.e. $2b)$. Define $\Psi^\iota = R^{\widehat{\mathbb{G}}}\circ\Phi^\iota\circ R^{\widehat{\mathbb{G}}}\colon\ \mathrm{L}^\infty(\widehat{\mathbb{G}})\to\mathrm{L}^\infty(\widehat{\mathbb{G}})$. It is not difficult to see that these maps are also normal, UCP, of finite rank and converge to id in the point-w* topology (to see that Ψ^ι is CP, one can use the Stinespring representation). Furthermore, the L^2-implementation of Ψ^ι is in $\ell^\infty(\widehat{\mathbb{G}})$. Indeed, take $\omega,\nu\in\mathrm{L}^1(\widehat{\mathbb{G}})$ and

$x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$. Then using Proposition 6.12 and equation (2.8)

$$
\begin{aligned}
&\langle \Psi^\iota_*(\omega \star \nu), x\rangle = \langle \omega\otimes\nu, \Delta_{\widehat{\mathbb{G}}}(\Psi^\iota(x))\rangle = \langle \omega\otimes\nu, \Delta_{\widehat{\mathbb{G}}}\circ R^{\widehat{\mathbb{G}}}\circ \Phi^\iota(R^{\widehat{\mathbb{G}}}(x))\rangle\\
&= \langle \nu\otimes\omega, (R^{\widehat{\mathbb{G}}}\otimes R^{\widehat{\mathbb{G}}})\circ\Delta_{\widehat{\mathbb{G}}}\circ\Phi^\iota(R^{\widehat{\mathbb{G}}}(x))\rangle = \langle \Phi^\iota_*((\nu\circ R^{\widehat{\mathbb{G}}})\star(\omega\circ R^{\widehat{\mathbb{G}}})), R^{\widehat{\mathbb{G}}}(x)\rangle\\
&= \langle \Phi^\iota_*(\nu\circ R^{\widehat{\mathbb{G}}})\star(\omega\circ R^{\widehat{\mathbb{G}}}), R^{\widehat{\mathbb{G}}}(x)\rangle = \langle (\nu\circ R^{\widehat{\mathbb{G}}})\otimes(\omega\circ R^{\widehat{\mathbb{G}}}), (\Phi^\iota\otimes\mathrm{id})\Delta_{\widehat{\mathbb{G}}}(R^{\widehat{\mathbb{G}}}(x))\rangle\\
&= \langle \omega\otimes\nu, (\mathrm{id}\otimes\Psi^\iota)\Delta_{\widehat{\mathbb{G}}}(x)\rangle = \langle \omega\star\Phi^\iota_*(\nu), x\rangle
\end{aligned}
$$

which thanks to Proposition 6.12 proves that the L^2-implementation of Ψ^ι belongs to $\ell^\infty(\widehat{\mathbb{G}})'$. Implication $2c) \Rightarrow 2b)$ can be showed analogously. □

Remark. Roughly speaking, if one replaces the conditions of being finite rank (or in $\mathrm{A}(\mathbb{G})\cap \mathrm{c}_c(\mathbb{G})$) in the definitions w*-CPAP (or amenability) by having compact L^2 implementations (or belonging to $\mathrm{c}_0(\widehat{\mathbb{G}})$), one obtains the Haagerup property (see [13, Section 5.1] and [28, 19]). One can define the relative Haagerup property of a von Neumann algebra in a similar spirit to Definition 6.9 and prove an analog of Theorem 6.11 using pretty much the same reasoning.

Let us end this section with another result of the type (operator algebraic approximation property + additional condition) ⇒ (quantum group approximation property). This time we work in the C*-algebraic setting.

Theorem 6.13. *Let $\mathbb{G}$ be a discrete quantum group such that the reduced* C**-algebra* $\mathrm{C}(\widehat{\mathbb{G}})$ *is nuclear and admits a tracial state. Then $\mathbb{G}$ is amenable.*

We are not aware of this implication being recorded in a literature before. However, let us mention here a couple of results in a similar spirit.

- First, Caspers and Skalski [20, Proposition 2.5] proved that a discrete quantum group $\mathbb{G}$ is amenable provided there exists a finite dimensional representation of $\mathrm{C}(\widehat{\mathbb{G}})$.
- In a classical setting, Ng proved ([65, Theorem 8]) a similar result for locally compact groups.
- For a locally compact quantum groups $\mathbb{G}$, it is known that nuclearity of $\mathrm{C}_0(\widehat{\mathbb{G}})$ and existence of a tracial state in $\mathrm{C}_0(\widehat{\mathbb{G}})^*$ implies amenability of $\mathbb{G}$, provided the scaling group is trivial [66, Theorem 3.2].

Proof. Our proof will use the notion of *Kac quotient* $\widehat{\mathbb{G}}_{KAC}$ introduced by Sołtan in [74]. Define an ideal

$$\mathcal{J} = \{a\in \mathrm{C}(\widehat{\mathbb{G}})\,|\, \tau(a^*a)=0\ \forall\ \text{tracial state } \tau\in \mathrm{C}(\widehat{\mathbb{G}})^*\}.$$

Clearly $\mathcal{J}\neq \mathrm{C}(\widehat{\mathbb{G}})$, hence $A=\mathrm{C}(\widehat{\mathbb{G}})/\mathcal{J}$ is a non-zero unital C*-algebra. A is nuclear, as quotients of nuclear C*-algebras are nuclear ([14, Theorem 10.1.4]). Let $\pi\colon \mathrm{C}(\widehat{\mathbb{G}})\to A$ be the quotient mapping. Define

$$\mathrm{C}(\widehat{\mathbb{G}}_{KAC}) = A,\quad \Delta_{\widehat{\mathbb{G}}_{KAC}}(\pi(x)) = (\pi\otimes\pi)\Delta_{\widehat{\mathbb{G}}}(x)\quad (x\in \mathrm{C}(\widehat{\mathbb{G}})).$$

Then $\widehat{\mathbb{G}}_{KAC} = (\mathrm{C}(\widehat{\mathbb{G}}_{KAC}), \Delta_{\widehat{\mathbb{G}}_{KAC}})$ is a compact quantum group of Kac type, see [74, Proposition 5.1].

Since $\mathrm{C}(\widehat{\mathbb{G}}_{KAC})$ is nuclear and $\widehat{\mathbb{G}}_{KAC}$ is of Kac type, $\widehat{\mathbb{G}}_{KAC}$ is coamenable (see Theorem 6.7 and [86]) – consequently $\mathrm{C}(\widehat{\mathbb{G}}_{KAC})$ is universal and admits a character $\varepsilon \in \mathrm{C}(\widehat{\mathbb{G}}_{KAC})^*$ [8, Theorem 3.1]. It follows that $\varepsilon \circ \pi$ is a character on $\mathrm{C}(\widehat{\mathbb{G}})$ and consequently $\widehat{\mathbb{G}}$ is coamenable [7, Theorem 2.8]. $\square$

Remark. Even though, generally speaking, nuclear C*-algebras are "well behaved", there exist nuclear C*-algebras without tracial states. Examples are given by the Cuntz algebras $\mathcal{O}_n$ $(n \geq 2)$ ([10, IV.3.5.3]).

7 Appendix

7.1 Direct integrals

This subsection is devoted to a brief introduction to the technical tool of direct integrals. We advise the reader to think of them as a continuous (or rather measurable) version of direct sums. For more details see e.g. [34, 33, 39].

Throughout this subsection, $(\Omega, \mathfrak{M})$ is a *measurable space*, i.e. a set Ω together with a choice of a σ-algebra of subsets $\mathfrak{M} \subseteq \mathcal{P}(\Omega)$. Sometimes we will additionally assume that $(\Omega, \mathfrak{M})$ is *standard* which means that there is a Polish topology on Ω (i.e. a topology which is separable and completely metrizable) such that $\mathfrak{M}$ is the corresponding family of Borel sets. Measure μ on $(\Omega, \mathfrak{M})$ is *standard* if there exists a μ-null set $N \subseteq \Omega$ such that $\Omega \setminus N$ (with the corresponding σ-algebra) is a standard measurable space.

To begin with, we will introduce a measurable field of Hilbert spaces $(\mathsf{K}_x)_{x\in\Omega}$ and the associated Hilbert space $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$. Roughly speaking, this space should be the set of (classes of) vector fields $(\xi_x)_{x\in\Omega}$ which are measurable and satisfy the integrability condition $\int_\Omega \|\xi_x\|^2 \,\mathrm{d}\mu(x) < +\infty$. However, because there is no way to compare ξ_x with $\xi_{x'}$ for different $x, x' \in \Omega$, it is not clear what "measurable" means here. One way to resolve this mistery, is to choose a family of vector fields $e^i \colon \Omega \ni x \mapsto e^i_x \in \mathsf{K}_x$ $(i \in \mathbb{N})$ which will play a role of exemplary measurable fields, relative to which we define measurability of an arbitrary field[36] $(\xi_x)_{x\in\Omega}$.

To avoid unnecessary technical difficulties, we will often make some separability assumptions.

Definition 7.1. A *measurable field of Hilbert spaces* is a family of separable Hilbert spaces $(\mathsf{K}_x)_{x\in\Omega}$ together with a countable family of vector fields $\{e^i\}_{i\in\mathbb{N}}$ such that

- for all $i, j \in \mathbb{N}$, the function $\Omega \ni x \mapsto \langle e^i_x \,|\, e^j_x\rangle \in \mathbb{C}$ is measurable,
- for each $x \in \Omega$, the set $\{e^i_x \,|\, i \in \mathbb{N}\}$ is linearly dense in K_x.

Vector fields $\{e^i\}_{i\in\mathbb{N}}$ are called *fundamental.*

It is straightforward to check that if $(\mathsf{K}_x)_{x\in\Omega}, (\mathsf{L}_x)_{x\in\Omega}$ are measurable fields of Hilbert spaces with fundamental vector fields $\{e^i\}_{i\in\mathbb{N}}$, $\{f^j\}_{j\in\mathbb{N}}$, then $(\overline{\mathsf{K}_x})_{x\in\Omega}$ and $(\mathsf{K}_x \otimes \mathsf{L}_x)_{x\in\Omega}$ also are measurable when equipped with fundamental vector fields $\{\overline{e^i}\}_{i\in\mathbb{N}}$ and $\{e^i \otimes f^j\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ ($\overline{e^i}$ and $e^i \otimes f^j$ are to be understood in the obvious manner).

Now we can say what it means for a vector field $(\xi_x)_{x\in\Omega}$ to be measurable:

Definition 7.2. Let $(\mathsf{K}_x)_{x\in\Omega}$ be a measurable field of Hilbert spaces with fundamental vector fields $\{e^i\}_{i\in\mathbb{N}}$. We say that a vector field $(\xi_x)_{x\in\Omega}$ is *measurable* if the function $\Omega \ni x \mapsto \langle e^i_x \,|\, \xi_x\rangle \in \mathbb{C}$ is measurable for all $i \in \mathbb{N}$.

[36] Whenever we say that $(\xi_x)_{x\in\Omega}$ is a vector field, we mean that $\xi_x \in \mathsf{K}_x$ for all $x \in \Omega$. A similar note applies also to different fields, e.g. the field of operators, etc.

If it is clear from the context, we will often follow the usual convention of not mentioning fundamental vector fields and saying simply that $(\mathsf{K}_x)_{x\in\Omega}$ is a measurable field of Hilbert spaces – though one should keep in mind that it is necessary to choose them, as for example different choice of fundamental fields can give a different family of measurable vector fields!

It is easy to check that if $(\xi_x)_{x\in\Omega}, (\eta_x)_{x\in\Omega}$ are measurable vector fields, then the function $\Omega \ni x \mapsto \langle \xi_x \,|\, \eta_x\rangle \in \mathbb{C}$ is also measurable. Using this observation we can finally define the direct integral Hilbert space $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$:

Definition 7.3. Let $(\mathsf{K}_x)_{x\in\Omega}$ be a measurable field of Hilbert spaces, assume moreover that we have a measure μ on Ω. Let $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$ be the set of (classes of[37]) measurable vector fields $(\xi_x)_{x\in\Omega}$ satisfying $\int_\Omega \|\xi_x\|^2 \,\mathrm{d}\mu(x) < +\infty$. It becomes a Hilbert space with the inner product given by

$$\langle (\xi_x)_{x\in\Omega} \,|\, (\eta_x)_{x\in\Omega}\rangle = \int_\Omega \langle \xi_x \,|\, \eta_x\rangle \,\mathrm{d}\mu(x) \quad ((\xi_x)_{x\in\Omega}, (\eta_x)_{x\in\Omega} \in \int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)).$$

If the measure μ is standard, then $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$ is separable [33, Appendix A73]. Let us mention here a couple of examples:

- if μ is the counting measure on Ω, then $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$ is the direct sum $\bigoplus_{x\in\Omega} \mathsf{K}_x$,
- if $\mathsf{K}_x = \mathbb{C}$ for all $x \in \Omega$, then $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x) = \mathrm{L}^2(\Omega, \mu)$,
- if A is a separable C*-algebra of type I, then its spectrum $\mathrm{Irr}(A)$ is a standard measurable space when equipped with the Borel σ-algebra (which is equal to the Mackey Borel structure) [33, Proposition 4.6.1]. There exists a measurable field of Hilbert spaces $(\mathsf{K}_x)_{x\in\mathrm{Irr}(A)}$, called *the canonical measurable field of Hilbert spaces,* such that $\mathsf{K}_x = \mathbb{C}^{\dim(x)}$ for all $x \in \mathrm{Irr}(A)$ [33, Section 8.6.1].

Now we will introduce two classes of operators on $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$ which respect the structure of direct integral.

Definition 7.4. Let $(\mathsf{K}_x)_{x\in\Omega}$ be a measurable field of Hilbert spaces and $(T_x)_{x\in\Omega}$ a field of operators, i.e. for each $x \in \Omega$ we have $T_x \in \mathrm{B}(\mathsf{K}_x)$.

- We say that $(T_x)_{x\in\Omega}$ is *measurable* if for all measurable vector fields $(\xi_x)_{x\in\Omega}$, the vector field $(T_x\xi_x)_{x\in\Omega}$ is measurable.
- A measurable field of operators $(T_x)_{x\in\Omega}$ is *essentially bounded* if $\sup_{x\in\Omega} \|T_x\| < +\infty$. In such case we can define a bounded operator

  $$\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x) \colon \int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x) \ni \int_\Omega^\oplus \xi_x \,\mathrm{d}\mu(x) \mapsto \int_\Omega^\oplus T_x\xi_x \,\mathrm{d}\mu(x) \in \int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x).$$

  Operators of this form are called *decomposable.* The set of decomposable operators is denoted by $\mathrm{Dec}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$.

[37] We identify two measurable vector fields $(\xi_x)_{x\in\Omega}$ and $(\eta_x)_{x\in\Omega}$ if $\xi_x = \eta_x$ for almost all $x \in \Omega$.

- A decomposable operator $\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x)$ is *diagonalisable* if $T_x \in \mathbb{C}\mathbb{1}_{\mathsf{H}_x}$ for almost all $x \in \Omega$. The set of diagonalisable operators is denoted by $\mathrm{Diag}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$.

It is easy to check that $\mathrm{Dec}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$ and $\mathrm{Diag}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$ are von Neumann subalgebras of $\mathrm{B}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$. Furthermore, we have $\mathrm{Diag}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))' = \mathrm{Dec}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$. In fact, these two are special cases of direct integrals of von Neumann algebras:

Definition 7.5. Let $(\mathsf{K}_x)_{x\in\Omega}$ be a measurable field of Hilbert spaces and $(\mathrm{M}_x)_{x\in\Omega}$ a field of von Neumann algebras, i.e. for each $x \in \Omega$, M_x is a von Neumann subalgebra of $\mathrm{B}(\mathsf{K}_x)$.

- We say that the field $(\mathrm{M}_x)_{x\in\Omega}$ is *measurable* if there exists a countable collection $\{(T_x^i)_{x\in\Omega}\}_{i\in\mathbb{N}}$ of measurable fields of operators, such that for almost all $x \in \Omega$, M_x is the von Neumann algebra generated by $\{T_x^i \,|\, i \in \mathbb{N}\}$.
- Assume that $(\mathrm{M}_x)_{x\in\Omega}$ is measurable. We define $\int_\Omega^\oplus \mathrm{M}_x \,\mathrm{d}\mu(x)$ as the set of decomposable operators $\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x)$ such that $T_x \in \mathrm{M}_x$ for almost all $x \in \Omega$. It is a von Neumann algebra – von Neumann algebras arising from this construction are called *decomposable*.

Examples of decomposable von Neumann algebras are given by $\mathrm{Diag}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$ and $\mathrm{Dec}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$ – in the first case we have $\mathrm{M}_x = \mathbb{C}\mathbb{1}_{\mathsf{K}_x}\,(x \in \Omega)$ and in the second $\mathrm{M}_x = \mathrm{B}(\mathsf{K}_x)\,(x \in \Omega)$. In general, whenever $(\mathrm{M}_x)_{x\in\Omega}$ is a measurable field of von Neumann algebras we have

$$\mathrm{Diag}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)) \subseteq \int_\Omega^\oplus \mathrm{M}_x \,\mathrm{d}\mu(x) \subseteq \mathrm{Dec}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)).$$

Furthermore, if the measure μ is standard, then the field $(\mathrm{M}_x')_{x\in\Omega}$ is also measurable and

$$(\int_\Omega^\oplus \mathrm{M}_x \,\mathrm{d}\mu(x))' = \int_\Omega^\oplus \mathrm{M}_x' \,\mathrm{d}\mu(x).$$

The above properties and definitions are taken mainly from [34, Part II]. We end this part of the appendix with two notions: the direct integral of unbounded operators [58] and the direct integral of weights [79].

Definition 7.6. Let $(\mathsf{K}_x)_{x\in\Omega}$ be a measurable field of Hilbert spaces and $(T_x)_{x\in\Omega}$ a field of closed, densely defined (unbounded) operators. We say that $(T_x)_{x\in\Omega}$ is measurable if

- for any measurable vector field $(\xi_x)_{x\in\Omega}$ such that $\xi_x \in \mathrm{Dom}(T_x)$ $(x \in \Omega)$, vector field $(T_x\xi_x)_{x\in\Omega}$ is also measurable,
- there is a countable collection $\{\xi^i\}_{i\in\mathbb{N}}$ of measurable vector fields such that for all $x \in \Omega$ we have $\xi_x^i \in \mathrm{Dom}(T_x)\,(i \in \mathbb{N})$ and

  $$\{(\xi_x^i, T_x\xi_x^i) \,|\, i \in \mathbb{N}\}$$

  is total in $\mathrm{Graph}(T_x)$.

If $(T_x)_{x\in\Omega}$ is measurable and μ is a measure on Ω, then we define an (unbounded) operator $\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x)$ on $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$ with domain $\mathrm{Dom}(\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x))$ consisting of those $\int_\Omega^\oplus \xi_x \,\mathrm{d}\mu(x) \in \int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$ for which $\xi_x \in \mathrm{Dom}(T_x)$ for almost all $x \in \Omega$ and the integral $\int_\Omega \|T_x\xi_x\|^2 \,\mathrm{d}\mu(x)$ is finite. For such $\int_\Omega^\oplus \xi_x \,\mathrm{d}\mu(x)$ we define

$$\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x) \colon \int_\Omega^\oplus \xi_x \,\mathrm{d}\mu(x) \mapsto \int_\Omega^\oplus T_x\xi_x \,\mathrm{d}\mu(x).$$

As previously, operators of this form will be called *decomposable*. Such defined operator $\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x)$ is densely defined, closed and we have

$$\Big(\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x)\Big)^* = \int_\Omega^\oplus T_x^* \,\mathrm{d}\mu(x). \tag{7.1}$$

If almost all T_x $(x \in \Omega)$ are self-adjoint and $f\colon \mathbb{R} \to \mathbb{C}$ is measurable, then

$$f\Big(\int_\Omega^\oplus T_x \,\mathrm{d}\mu(x)\Big) = \int_\Omega^\oplus f(T_x) \,\mathrm{d}\mu(x) \tag{7.2}$$

(in particular, the direct integrals on the right hand side in equations (7.1), (7.2) are well defined).

We also have the following useful result: a closed, densely defined operator T on $\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x)$ is decomposable if and only if it is affiliated with $\mathrm{Dec}(\int_\Omega^\oplus \mathsf{K}_x \,\mathrm{d}\mu(x))$.

Definition 7.7. Let $(\mathsf{K}_x)_{x\in\Omega}$ be a measurable field of Hilbert spaces, $(\mathrm{M}_x)_{x\in\Omega}$ measurable field of von Neumann algebras and for each $x \in \Omega$, let φ_x be a weight on M_x. Assume furthermore that each M_x $(x \in \Omega)$ has a separable predual. We say that the field of weights $(\varphi_x)_{x\in\Omega}$ is weakly measurable if:

- there exists a sequence $\{a^i\}_{i\in\mathbb{N}}$ of measurable fields of operators such that for all $x \in \Omega$ we have $a^i_x \in \mathfrak{N}_{\varphi_x}$ $(i \in \mathbb{N})$ and $\{a^i_x \,|\, i \in \mathbb{N}\}$ is w*-dense in $\mathfrak{N}_{\varphi_x}$,
- whenever $(a_x)_{x\in\Omega}$ is a measurable field of operators with $a_x \in \mathrm{M}_x^+$ for almost all $x \in \Omega$, then $\Omega \ni x \mapsto \varphi_x(a_x) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is also measurable.

Whenever we have a weakly measurable field of weights $(\varphi_x)_{x\in\Omega}$, we can define its direct integral weight via

$$\int_\Omega^\oplus \varphi_x \,\mathrm{d}\mu(x) \colon \Big(\int_\Omega^\oplus \mathrm{M}_x \,\mathrm{d}\mu(x)\Big)^+ \ni \int_\Omega^\oplus a_x \,\mathrm{d}\mu(x) \mapsto \int_\Omega \varphi_x(a_x) \,\mathrm{d}\mu(x) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

If for all $x \in \Omega$, φ_x is a n.s.f. weight on M_x, then $\int_\Omega^\oplus \varphi_x \,\mathrm{d}\mu(x)$ is also n.s.f.

7.2 Lemmas

Lemma 7.8. *Let* H *be a Hilbert space and let*

$$J\colon \mathsf{H}\otimes\overline{\mathsf{H}} \ni \xi\otimes\overline{\eta} \mapsto \eta\otimes\overline{\xi} \in \mathsf{H}\otimes\overline{\mathsf{H}}.$$

Let $(a_t)_{t\in\mathbb{R}}$ *and* $(b_t)_{t\in\mathbb{R}}$ *be strongly continuous one-parameter groups of unitary operators on* H *and assume that* $J(a_t\otimes b_t^\top) = (a_t\otimes b_t^\top)J$ *for all* $t\in\mathbb{R}$*. Then for all* t *we have* $a_t = b_{-t}$*.*

Proof. On one hand we have

$$J(a_t\otimes b_t^\top)J = a_t\otimes b_t^\top, \tag{7.3}$$

so for any $r,s\in\mathbb{R}$ and any $x\in \mathrm{B}(\mathsf{H})$

$$\begin{aligned}
&(a_s\otimes b_s^\top)J(a_t\otimes b_t^\top)J(x\otimes \mathbb{1}_{\overline{\mathsf{H}}})J(a_{-t}\otimes b_{-t}^\top)J(a_{-s}\otimes b_{-s}^\top)\\
&= (a_s\otimes b_s^\top)(a_t\otimes b_t^\top)(x\otimes \mathbb{1}_{\overline{\mathsf{H}}})(a_{-t}\otimes b_{-t}^\top)(a_{-s}\otimes b_{-s}^\top)\\
&= (a_t\otimes b_t^\top)(a_s\otimes b_s^\top)(x\otimes \mathbb{1}_{\overline{\mathsf{H}}})(a_{-s}\otimes b_{-s}^\top)(a_{-t}\otimes b_{-t}^\top)\\
&= J(a_t\otimes b_t^\top)J(a_s\otimes b_s^\top)(x\otimes \mathbb{1}_{\overline{\mathsf{H}}})(a_{-s}\otimes b_{-s}^\top)J(a_{-t}\otimes b_{-t}^\top)J.
\end{aligned} \tag{7.4}$$

On the other hand

$$J(a_t\otimes b_t^\top)J = b_{-t}\otimes a_{-t}^\top \quad (t\in\mathbb{R}), \tag{7.5}$$

so (7.4) reads

$$\begin{aligned}
&(a_s\otimes b_s^\top)(b_{-t}\otimes a_{-t}^\top)(x\otimes \mathbb{1}_{\overline{\mathsf{H}}})(b_t\otimes a_t^\top)(a_{-s}\otimes b_{-s}^\top)\\
&= (b_{-t}\otimes a_{-t}^\top)(a_s\otimes b_s^\top)(x\otimes \mathbb{1}_{\overline{\mathsf{H}}})(a_{-s}\otimes b_{-s}^\top)(b_t\otimes a_t^\top).
\end{aligned}$$

Thus for any x and all s,t we have $a_sb_{-t}xb_ta_{-s} = b_{-t}a_sxa_{-s}b_t$, i.e. $a_{-s}b_ta_sb_{-t}$ commutes with all $x\in\mathrm{B}(\mathsf{H})$.

Therefore there exists a continuous family $\{\lambda_{t,s}\}_{t,s\in\mathbb{R}}$ of complex numbers of absolute value 1 such that

$$a_{-s}b_ta_sb_{-t} = \lambda_{t,s}\mathbb{1}_{\mathsf{H}} \quad (t,s\in\mathbb{R}). \tag{7.6}$$

Note now that in view of the canonical isomorphism $\mathrm{B}(\mathsf{H}\otimes\overline{\mathsf{H}})\cong\mathrm{B}(\mathrm{HS}(\mathsf{H}))$ given by

$$\mathrm{B}(\mathsf{H}\otimes\overline{\mathsf{H}}) \ni x\otimes y^\top \longmapsto \bigl(S\mapsto xSy\bigr)\in\mathrm{B}\bigl(\mathrm{HS}(\mathsf{H})\bigr)$$

equations (7.3) and (7.4) mean that

$$a_tSb_t = b_{-t}Sa_{-t} \quad (t\in\mathbb{R},\ S\in\mathrm{HS}(\mathsf{H}))$$

which by strong density of $\mathrm{HS}(\mathsf{H})$ in $\mathrm{B}(\mathsf{H})$ gives

$$a_tb_t = b_{-t}a_{-t} \quad (t\in\mathbb{R}).$$

In particular for each t the operator a_tb_t is self-adjoint, and taking adjoints of this for $-t$ instead of t we see that also b_ta_t is self-adjoint for all t.

Therefore inserting $s=-t$ in (7.6) gives $a_tb_t = \lambda_{t,-t}b_ta_t$ and since a_tb_t and b_ta_t are self-adjoint, $t\mapsto\lambda_{t,-t}$ is continuous and $\lambda_{0,0}=1$, we obtain $\lambda_{t,-t}=1$ for all t.

Consequently $a_tb_t = b_ta_t = (b_ta_t)^* = a_{-t}b_{-t}$, so that $b_{2t} = a_{-2t}$ for all $t\in\mathbb{R}$. ☐

Lemma 7.9. *Let U be a unitary, finite dimensional representation of a compact quantum group $\mathbb{G}$ on a Hilbert space H_U and let χ_U be its character. We have*

$$\operatorname{span}\{U_{\xi,\eta}\,|\,\xi,\eta\in\mathsf{H}_U\}=(\mathbb{C}\chi_U)\oplus\operatorname{span}\{U_{\xi,\eta}\,|\,\xi,\eta\in\mathsf{H}_U\colon \xi\perp\rho_U^{-1}\eta\},\tag{7.7}$$

where the above orthogonal direct sum corresponds to the scalar product induced by the Haar integral.

Proof. Using orthogonality relations [64, Theorem 1.4.3] we have

$$h(\chi_U^*U_{\xi,\eta})=\sum_{i=1}^{\dim(U)}h(U^*_{\xi_i,\xi_i}U_{\xi,\eta})=\sum_{i=1}^{\dim(U)}\tfrac{\langle\xi\,|\,\rho_U^{-1}\xi_i\rangle\langle\xi_i\,|\,\eta\rangle}{\dim_q(U)}=\tfrac{\langle\xi\,|\,\rho_U^{-1}\eta\rangle}{\dim_q(U)}=0$$

for all $\xi,\eta\in\mathsf{H}_U$ such that $\xi\perp\rho_U^{-1}\eta$, hence we indeed have an orthogonal direct sum on the right hand side of equation (7.7). Consequently, it is enough to show that

$$\dim(\operatorname{span}\{U_{\xi,\eta}\,|\,\xi,\eta\in\mathsf{H}_U\colon\xi\perp\rho_U^{-1}\eta\})=\dim(U)^2-1.\tag{7.8}$$

Let us show that

$$(\mathbb{C}\mathbb{1}_U)^{\perp}=\operatorname{span}\{|\xi\rangle\langle\eta|\,|\,\xi,\eta\in\mathsf{H}_U\colon\xi\perp\eta\}\tag{7.9}$$

in $\mathrm{HS}(\mathsf{H}_U)$. The inclusion $\supseteq$ is clear, therefore we need to argue that

$$(\mathbb{C}\mathbb{1}_U)+\operatorname{span}\{|\xi\rangle\langle\eta|\,|\,\xi,\eta\in\mathsf{H}_U\colon\xi\perp\eta\}=\mathrm{HS}(\mathsf{H}_U)=\mathsf{H}_U\otimes\overline{\mathsf{H}_U}.\tag{7.10}$$

Let $\{\xi_i\}_{i=1}^{\dim(U)}$ be an orthonormal basis in H_U. Clearly for $i\neq j$, the operator $|\xi_i\rangle\langle\xi_j|$ belongs to the left hand side of (7.10). Furthermore,

$$\begin{aligned}|\xi_i\rangle\langle\xi_i|&=\tfrac{1}{\dim(U)}\sum_{j=1}^{\dim(U)}\bigl(|\xi_i-\xi_j\rangle\langle\xi_i+\xi_j|-|\xi_i\rangle\langle\xi_j|+|\xi_j\rangle\langle\xi_i|+|\xi_j\rangle\langle\xi_j|\bigr)\\&=\tfrac{1}{\dim(U)}\mathbb{1}_U+\tfrac{1}{\dim(U)}\sum_{\substack{i=1\\i\neq j}}^{\dim(U)}\bigl(|\xi_i-\xi_j\rangle\langle\xi_i+\xi_j|-|\xi_i\rangle\langle\xi_j|+|\xi_j\rangle\langle\xi_i|\bigr)\end{aligned}$$

which also belongs to the right hand side of (7.10). This shows equation (7.10) and consequently (7.9). Consider now the linear map

$$T\colon\ \mathrm{HS}(\mathsf{H}_U)=\mathsf{H}_U\otimes\overline{\mathsf{H}_U}\ni\eta\otimes\overline{\xi}\mapsto U_{\xi,\rho_U\eta}\in V_U,$$

where V_U is the finite dimensional Hilbert space

$$V_U=\operatorname{span}\{U_{\xi,\eta}\,|\,\xi,\eta\in\mathsf{H}_U\}.$$

As

$$U_{\xi_i,\xi_j}=T(\rho_U^{-1}\xi_j\otimes\xi_i)$$

for all $i,j\in\{1,\dots,\dim(U)\}$, T is a surjection. It is easy to see using the orthogonality relations that T has trivial kernel, hence it is a vector space isomorphism (it is not, however, an isometry). Using T we can show (7.8):

$$\begin{aligned}\dim(\operatorname{span}\{U_{\xi,\eta}\,|\,\xi,\eta\in\mathsf{H}_U\colon \xi\perp \rho_U^{-1}\eta\}) &= \dim T(\operatorname{span}\{\eta\otimes\overline{\xi}\,|\,\xi,\eta\in\mathsf{H}_U\colon \xi\perp\eta\})\\ &= \dim(\mathbb{C}\mathbb{1}_U)^{\perp} = \dim(U)^2-1,\end{aligned}$$

which ends the proof. □

Let $\mathbb{G}$ be a locally compact quantum group with the left Haar integral φ. Recall that while defining the left Haar integral $\widehat{\varphi}$ on $\widehat{\mathbb{G}}$, one introduces

$$\widehat{\mathscr{N}}=\{\lambda(\omega)\,|\,\omega\in \mathrm{L}^1(\mathbb{G})\colon \exists_{\xi\in\mathrm{L}^2(\mathbb{G})}\forall_{x\in\mathfrak{N}_\varphi}\langle\Lambda_\varphi(x)\,|\,\xi\rangle=\omega(x^*)\}.$$

Then $\widehat{\mathscr{N}}$ is a (σ-SOT$\times$norm) core for $\Lambda_{\widehat{\varphi}}$ and $\Lambda_{\widehat{\varphi}}(\lambda(\omega))=\xi$ (see Section 2.2, [93, Definition 4.6] and [57, Proposition 2.6]). In Section 6 we needed a refinment of this density result: it was desirable to work with functionals having nice analytical properties. The next lemma asserts density of such functionals.

Lemma 7.10. *Let us introduce a subspace*

$$\begin{aligned}\mathscr{I}=\{\omega\in\mathrm{L}^1(\mathbb{G})\,|\,&\lambda(\omega)\in\widehat{\mathscr{N}},\\ &\mathbb{R}\ni t\mapsto(\omega\delta^{-it})\circ\tau_{-t}\in\mathrm{L}^1(\mathbb{G}) \text{ extends to an entire map } \mathbb{C}\to\mathrm{L}^1(\mathbb{G})\}\end{aligned}$$

Then $\mathscr{I}$ is dense in $\mathrm{L}^1(\mathbb{G})$, $\lambda(\mathscr{I})$ is σ-SOT*-dense in $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ and $\Lambda_{\widehat{\varphi}}(\lambda(\mathscr{I}))$ is dense in $\mathrm{L}^2(\mathbb{G})$.

Proof. Since we already know that $\mathcal{I}=\{\omega\in\mathrm{L}^1(\mathbb{G})\,|\,\lambda(\omega)\in\widehat{\mathscr{N}}\}$ is dense in $\mathrm{L}^1(\mathbb{G})$, $\lambda(\mathcal{I})=\widehat{\mathscr{N}}$ is σ-SOT*-dense in $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ and $\Lambda_{\widehat{\varphi}}(\widehat{\mathscr{N}})$ is dense in $\mathrm{L}^2(\mathbb{G})$ (see e.g. [93, Lemma 4.7]), it is enough to show that for each $\omega\in\mathcal{I}$, we can find a sequence $(\omega_n)_{n\in\mathbb{N}}$ in $\mathscr{I}$ such that

$$\omega_n\xrightarrow[n\to\infty]{}\omega,\quad \lambda(\omega_n)\xrightarrow[n\to\infty]{\sigma\text{-SOT}^*}\lambda(\omega),\quad \Lambda_{\widehat{\varphi}}(\lambda(\omega_n))\xrightarrow[n\to\infty]{}\Lambda_{\widehat{\varphi}}(\lambda(\omega))$$

and for each $n\in\mathbb{N}$, the map $\mathbb{R}\ni t\mapsto(\omega_n\delta^{-it})\circ\tau_{-t}\in\mathrm{L}^1(\mathbb{G})$ extends to an entire map $\mathbb{C}\to\mathrm{L}^1(\mathbb{G})$.

Fix $\omega\in\mathcal{I}$ and define

$$\omega_n=\sqrt{\tfrac{n}{\pi}}\int_{\mathbb{R}}e^{-ns^2}(\omega\delta^{is})\circ\tau_s\,\mathrm{d}s\in\mathrm{L}^1(\mathbb{G})\quad(n\in\mathbb{N})$$

(the above integral converges in the weak topology). First, let us show that $\omega_n\in\mathcal{I}$: take

$x \in \mathfrak{N}_\varphi$. We have

$$\begin{aligned}
\omega_n(x^*) &= \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \langle \omega, \delta^{is} \tau_s(x^*) \rangle \,\mathrm{d}s = \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \langle \omega, (\tau_s(x)\delta^{-is})^* \rangle \,\mathrm{d}s \\
&= \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \langle \Lambda_\varphi(\tau_s(x)\delta^{-is}) \,|\, \Lambda_{\widehat{\varphi}}(\lambda(\omega)) \rangle \,\mathrm{d}s \\
&= \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \langle J_\varphi \sigma^\varphi_{i/2}(\delta^{-is})^* J_\varphi \nu^{-\frac{s}{2}} P^{is} \Lambda_\varphi(x) \,|\, \Lambda_{\widehat{\varphi}}(\lambda(\omega)) \rangle \,\mathrm{d}s \\
&= \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \langle J_\varphi \delta^{is} J_\varphi P^{is} \Lambda_\varphi(x) \,|\, \Lambda_{\widehat{\varphi}}(\lambda(\omega)) \rangle \,\mathrm{d}s \\
&= \big\langle \Lambda_\varphi(x) \big| \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} P^{-is} J_\varphi \delta^{-is} J_\varphi \Lambda_{\widehat{\varphi}}(\lambda(\omega)) \,\mathrm{d}s \big\rangle
\end{aligned}$$

hence $\omega_n \in \mathcal{I}$ and $\Lambda_{\widehat{\varphi}}(\lambda(\omega_n)) = \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} P^{-is} J_\varphi \delta^{-is} J_\varphi \Lambda_{\widehat{\varphi}}(\lambda(\omega)) \,\mathrm{d}s$ (this integral converges in the weak topology). Using the fact that $\mathrm{L}^\infty(\mathbb{G}) \subseteq \mathrm{B}(\mathrm{L}^2(\mathbb{G}))$ is represented in the standard way, it is not difficult to show $\omega_n \xrightarrow[n\to\infty]{} \omega$ and consequently $\lambda(\omega_n) \xrightarrow[n\to\infty]{\sigma\text{-}\textsc{sot}^*} \lambda(\omega)$. Equation $\Lambda_{\widehat{\varphi}}(\lambda(\omega_n)) = \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} P^{-is} J_\varphi \delta^{-is} J_\varphi \Lambda_{\widehat{\varphi}}(\lambda(\omega)) \,\mathrm{d}s$ implies that $\Lambda_{\widehat{\varphi}}(\lambda(\omega_n)) \xrightarrow[n\to\infty]{} \Lambda_{\widehat{\varphi}}(\lambda(\omega))$. Furthermore we have

$$\begin{aligned}
(\omega_n \delta^{-it}) \circ \tau_{-t}(x) &= \omega_n(\delta^{-it} \tau_{-t}(x)) = \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \langle \omega, \nu^{-ist} \delta^{i(s-t)} \tau_{s-t}(x) \rangle \,\mathrm{d}s \\
&= \big\langle \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s+t)^2} \nu^{-i(s+t)t} (\omega \delta^{is}) \circ \tau_s \,\mathrm{d}s, x \big\rangle,
\end{aligned}$$

hence $\mathbb{R} \ni t \mapsto (\omega_n \delta^{-it}) \circ \tau_{-t} \in \mathrm{L}^1(\mathbb{G})$ extends to the entire map

$$\mathbb{C} \ni z \mapsto \sqrt{\tfrac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s+z)^2} \nu^{-i(s+z)z} (\omega \delta^{is}) \circ \tau_s \,\mathrm{d}s \in \mathrm{L}^1(\mathbb{G}).$$

□

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