An example illustrating the topic ideas of *Causal nets for geometrical Gandy–Păun–Rozenberg machines*

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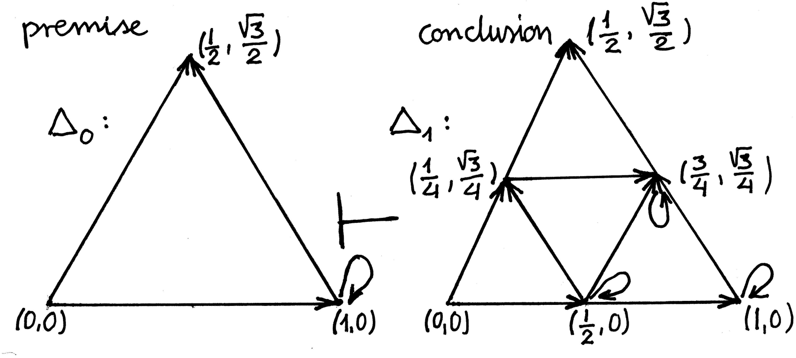
http://www.impan.pl/~adamo/causal-ex.pdf

Generation of contours

The example concerns a generation of the contours (frontiers) Δ_n (for a natural number $n \geq 0$) of Sierpiński gasket iterations by a geometrical Gandy–Păun–Rozenberg machine, where these contours Δ_n are finite directed graphs and the *instantaneous descriptions* of the machine.

Graph rewriting rule

The machine is equipped with exactly one graph rewriting rule of its instantaneous descriptions:



Places of application

The generation of the contours is here an inductive construction such that the graph Δ_{n+1} is constructed from Δ_n ($n \geq 0$) by using the graph rewriting rule $\Delta_0 \vdash \Delta_1$ and by a family of 3^n h-processors for the graph embeddings $h : \Delta_0 \rightarrowtail \Delta_n$, respectively, where these embeddings are called the *places of application of the rule in* Δ_n.

Graph replacement

Simultaneously the 3^n h-processors perform their tasks of graph replacement for every embedding $h : \Delta_0 \rightarrowtail \Delta_n$ according to the following program:

Program

Begin;

delete from Δ_n all edges of the image h in Δ_n;

scale (or contract, or diminish, or shrink) the conclusion Δ_1 of the rule by that factor by which h scales the premise Δ_0 to embed it in Δ_n

[comment: since this factor should be 2^{-n} in the case of an embedding $h : \Delta_0 \rightarrowtail \Delta_n$, we denote the result of scaling Δ_1 by $2^{-n}(\Delta_1)$];

translate the result of scaling to $h(0,0)$ according to translation formula $f_h(\vec{x}) = \vec{x} + h(0,0)$, where both $\vec{x}$ and $h(0,0)$ are ordered pairs of numbers and $+$ is the vector sum;

End.

Result

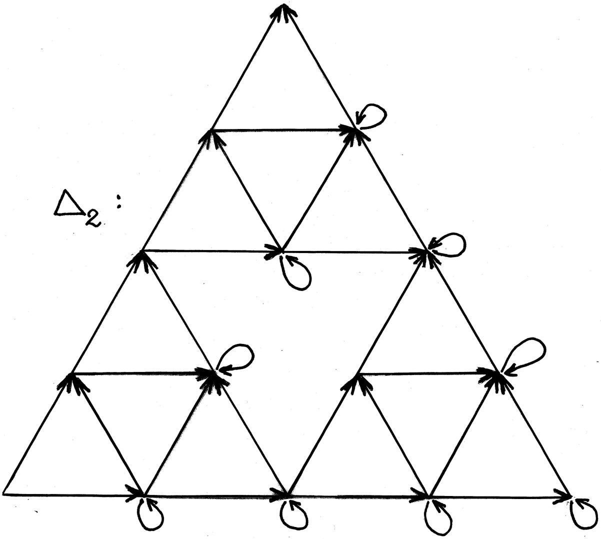
We denote the result of performance of the above program by $f_h(2^{-n}(\Delta_1))$ and call it the *result of the application of the rule in the place h in* Δ_n.
Thus the *global result of the application of the rule in* Δ_n is given by

$$\Delta_{n+1} := \bigcup_{\substack{h \text{ is an embedding} \\ \text{of } \Delta_0 \text{ in } \Delta_n}} f_h(2^{-n}(\Delta_1)).$$

Here the inductive steps of the construction of Δ_n $(n > 0)$ coincide with the steps of the Gandy–Păun–Rozenberg machine work.

Second step

After the second step we reach Δ_2:

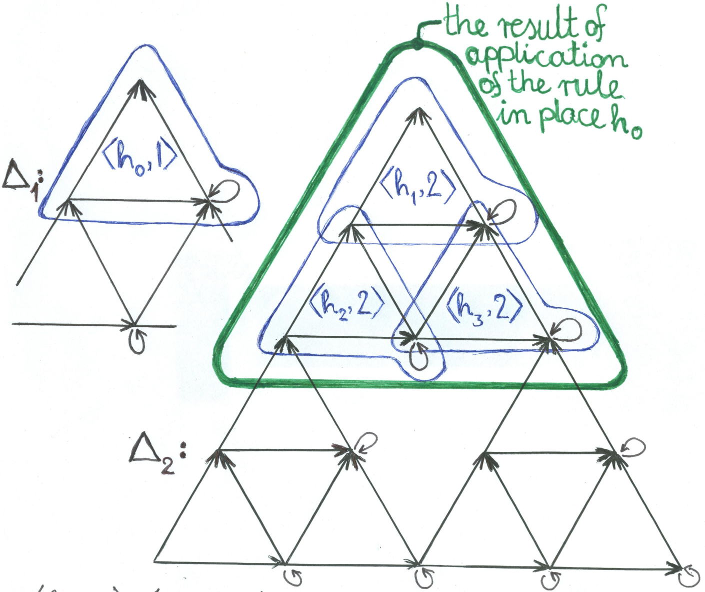


Event

Since a place $h : \Delta_0 \rightarrowtail \Delta_n$ of application of the machine rule determines unambigously a 'local' event of an appearance of the h-processor performing its program in the n-th 'global' step of the machine work, one can identify or represent this local event with the ordered pair $\langle h, n \rangle$ called simply an *event*. The *initial event* is $\langle \mathrm{id}_{\Delta_0}, 0 \rangle$, where id_{Δ_0} is the identity mapping.

Causal relation

The events are ordered by a *causal relation* $\prec$ *of events* defined by
$\langle h, i\rangle \prec \langle h', i'\rangle$ ($\langle h, i\rangle$ *is caused by* $\langle h', i'\rangle$)
iff $i = i' + 1$ and the image of h is a subgraph of the result of application of the rule in the place h' (i.e. the image of h is a subgraph of $f_{h'}(2^{-i}(\Delta_1))$).



$\langle h_1, 2\rangle, \langle h_2, 2\rangle, \langle h_3, 2\rangle$ are caused by $\langle h_0, 1\rangle$

Tree

The graph of the causal relation $\prec$ of events is a ternary tree (every node has exactly 3 children nodes) whose root is the initial event $\langle \mathrm{id}_{\Delta_0}, 0 \rangle$.

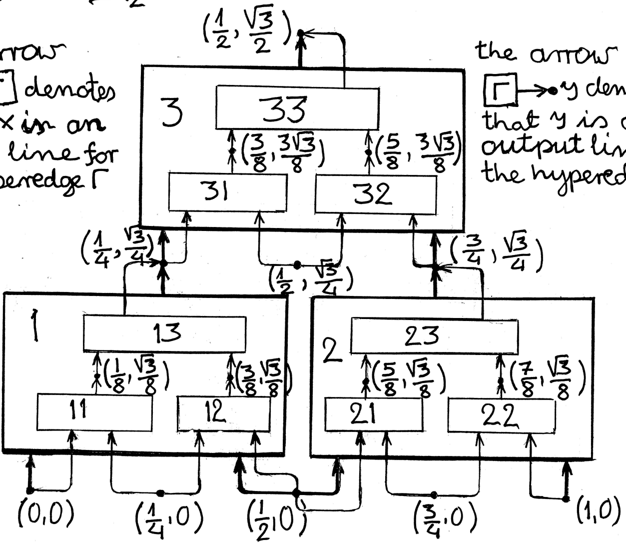
Self-similarity

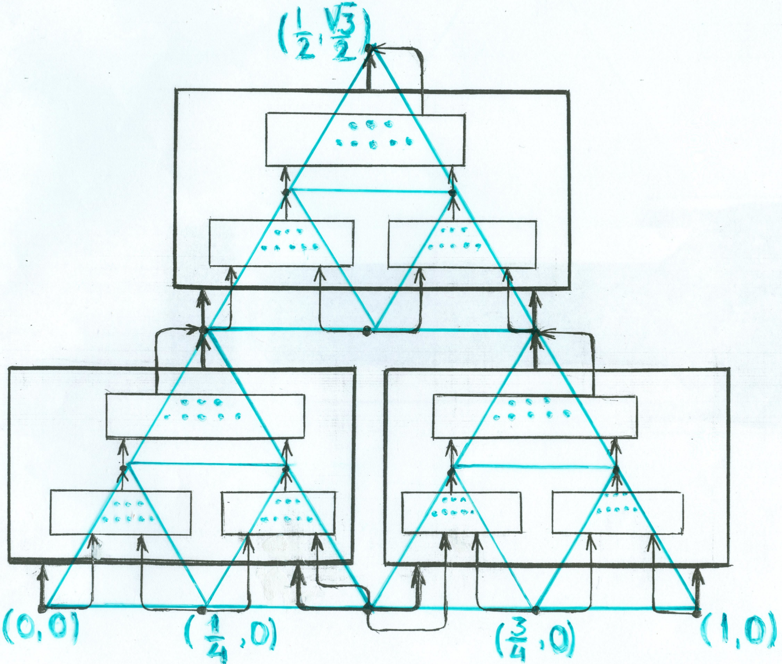
The causal relation of events coincides with the nesting relation of boxes presenting the hyperedges of multihyperedge membrane structures used to describe self-similarity structure of the iterations of Sierpiński gasket in *In search of a structure of fractals by using membranes as hyperedges* by A. Obtułowicz in the Proceedings of MCM14.

The multi-hyperedge membrane system S_2^{Sierp}:

the arrow $x \bullet\!\!\rightarrow \boxed{\Gamma}$ denotes that x is an input line for the hyperedge Γ

the arrow $\boxed{\Gamma} \rightarrow\!\!\bullet\, y$ denotes that y is an output line for the hyperedge Γ





$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
(0,0)
$(\frac{1}{4},0)$
$(\frac{3}{4},0)$
(1,0)

Process

Since the causal relation of events represents the process of computation of the machine, one can say that

structure is process

in the light of the coincidence of the causal relation of events and the nesting relation of boxes.