Multigraphical Membrane Systems revisited

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Abstract. A concept of a (directed) multigraphical membrane system [20], akin to membrane systems in [22] and [19], for modeling complex systems in biology, evolving neural networks, perception, and brain function is recalled and its new inspiring examples are presented for linking it with object recognition in cortex and an idea of neocognitron for multidimensional geometry.

1 Introduction

Statecharts described in [16] and their wide applications, including applications in system biology, cf. [11], and the formal foundations for natural reasoning in a visual mode presented in [26] challenge a prejudice against visualizations in exact sciences that they are heuristic tools and not valid elements of mathematical proofs.

We recall from [20] a concept of a (directed) multigraphical membrane system to be applied for modelling complex systems in biology, evolving neural networks, perception, and brain function. A precise mathematical definition of this concept and its topological representation by Venn diagrams and the usual graph drawings constitute a kind of visual formalism related to that discussed in [16]. The concept of a multigraphical membrane system is some new variant of the notion of a membrane system in [22] and [19].

We extend [20] by presenting the new inspiring examples of the concept of multigraphical membrane system for linking it with multidimensional object recognition in cortex, an idea of neocognitron for multidimensional geometry, hierarchical networks, and even fractals. These new examples are based on the idea of drawing multidimensional hypercubes (Boolean n-cubes) due to Tamiko Thiel (cf. [27]) and the figures Fig. 3–6 recalling this idea in the present paper are also due to her.

2 Multigraphical membrane systems

Membrane system in [22] and [19] are simply finite trees with nodes labelled by multisets, where the finite trees have a natural visual presentation by Venn diagrams.

We introduce (*directed*) *multigraphical membrane systems* to be finite trees with nodes labelled by (directed) multigraphs.

We consider directed multigraphical membrane systems of a special feature described formally in the following way.

A *sketch-like membrane system* $\mathcal{S}$ is given by:

– its *underlying tree* $\mathbb{T}_{\mathcal{S}}$ which is a finite graph given by the set $V(\mathbb{T}_{\mathcal{S}})$ of *vertices*, the set $E(\mathbb{T}_{\mathcal{S}}) \subseteq V(\mathbb{T}_{\mathcal{S}}) \times V(\mathbb{T}_{\mathcal{S}})$ of *edges*, and the *root* r which is a distinguished vertex such that for every vertex v different from r there exists a unique path from v into r in $\mathbb{T}_{\mathcal{S}}$, where for every vertex v we define $\mathrm{rel}(v) = \{v' \mid (v', v) \in E(\mathbb{T}_{\mathcal{S}})\}$ which is the set of vertices *immediately related* to v;
– its family $(G_v \mid v \in V(\mathbb{T}_{\mathcal{S}}))$ of finite directed multigraphs for G_v given by the set $V(G_v)$ of *vertices*, the set $E(G_v)$ of *edges*, the *source function* $s_v : E(G_v) \to V(G_v)$, and the *target function* $t_v : E(G_v) \to V(G_v)$ such that the following conditions hold:
  1) $V(G_v) = \{v\} \cup \mathrm{rel}(v)$,
  2) $E(G_v)$ is empty for every *elementary* vertex v, i.e. such that $\mathrm{rel}(v)$ is empty,
  3) for every *non-elementary* vertex v, i.e. such that $\mathrm{rel}(v)$ is a non-empty set, we have
     (i) $G_v(v, v')$ is empty for every $v' \in V(G_v)$,
     (ii) $G_v(v', v)$ is a one-element set for every $v' \in \mathrm{rel}(v)$,
     where $G_v(v_1, v_2) = \{e \in E(G_v) \mid s_v(e) = v_1 \text{ and } t_v(e) = v_2\}$.

For every non-elementary vertex v of $\mathbb{T}_{\mathcal{S}}$ we define:

– the *v-diagram* $\mathrm{Dg}(v)$ to be that directed multigraph which is the *restriction* of G_v to $\mathrm{rel}(v)$, i.e. $E(\mathrm{Dg}(v)) = \big\{e \in E(G_v) \mid \{s_v(e), t_v(e)\} \subseteq \mathrm{rel}(v)\big\}$, $V(\mathrm{Dg}(v)) = \mathrm{rel}(v)$, and the source and target functions of $\mathrm{Dg}(v)$ are the obvious restrictions of s_v, t_v to $E(\mathrm{Dg}(v))$, respectively,
– the *v-cocone* to be a family $(e_{v'} \mid v' \in \mathrm{rel}(v))$ of edges of G_v such that $s_v(e_{v'}) = v'$ and $t_v(e_{v'}) = v$ for every $v' \in \mathrm{rel}(v)$.

By a *model* of a sketch-like membrane system $\mathcal{S}$ in a category $\mathbb{C}$ with finite colimits we mean a family of graph homomorphisms $h_v : G_v \to \mathbb{C}$ (v is a non-elementary vertex of $\mathbb{T}_{\mathcal{S}}$) such that $h_v(v)$ is a colimit of the diagram $h_v \restriction \mathrm{Dg}(v) : \mathrm{Dg}(v) \to \mathbb{C}$ and $(h_v(e_{v'}) \mid v' \in \mathrm{rel}(v))$ is a colimiting cocone for the v-cocone $(e_{v'} \mid v' \in \mathrm{rel}(v))$, where $h_v \restriction \mathrm{Dg}(v)$ is the restriction of h_v to $\mathrm{Dg}(v)$. For all categorical and sketch theoretical notions like graph homomorphism, colimit of the diagram, and colimiting cocone we refer the reader to [4].

The idea of a sketch-like membrane system and its categorical model is a special case of the concept of a sketch and its model described in [4] and [18], where one finds that sketches can serve as a visual presentation of some data structure and data type algebraic specifications. On the other hand the idea of a sketch-like membrane system is a generalization of the notion of ramification used in [8], [9], [10] to investigate hierarchical categories with hierarchies determined by iterated colimits understood as in [8]. Hierarchical categories with hierarchies determined by iterated colimits are applied in [2] and [9] to describe various emergence phenomena in biology and general system theory. The iterated colimits identified with binding of patterns in neural net systems are expected in [9] and [10] to be applied in the investigations of binding problems in vision systems (associated with perception and brain function) in [29] and [30], hence the notion of sketch-like membrane system is aimed to be a tool for these investigations.

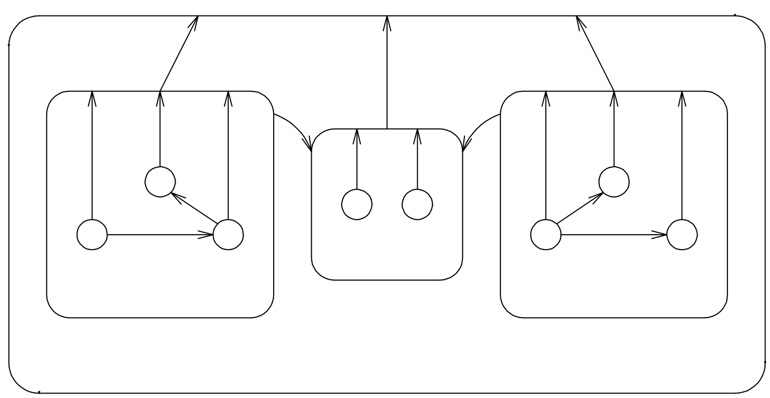
More precisely, sketch-like membrane systems are aimed to be presentations of objects of state categories of Memory Evolutive Systems in [8] and [9], where these state categories are hierarchical categories with hierarchies determined by iterated colimits. Hierarchical feature of sketch-like membrane systems and their categorical semantics reflect iterated colimit feature of objects of state categories of Memory Evolutive Systems [10].

If we drop condition 3) in the definition of a sketch-like membrane system, we obtain those directed multigraphical membrane systems which appear useful to describe alternating organization of living systems discussed in [3] with regard to nesting (represented by the underlying tree $\mathbb{T}_{\mathcal{S}}$) and interaction of levels of organization (represented by family of directed multigraphs G_v ($v \in V(\mathbb{T}_{\mathcal{S}})$)). According to [3] the edges in $G_v(v', v)$ describe integration, the edges in $G_v(v, v')$ describe regulation, and the edges of v-diagram $\mathrm{Dg}(v)$ describe interaction.

A directed multigraphical (a sketch-like) membrane system is illustrated in Fig. 1, whose semantics (model) in a hierarchical category is illustrated in Fig. 2.

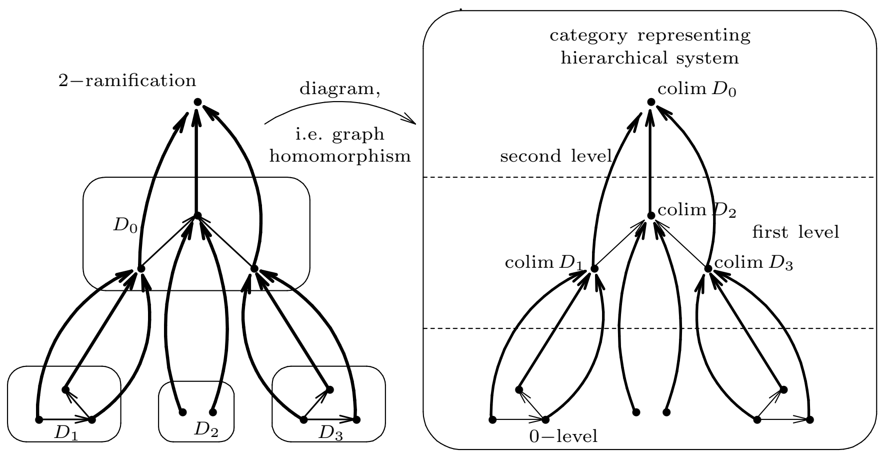
Fig. 1.

Multigraphical membrane system corresponding to 2-ramification:



nodes—membranes, edges—objects,
neurons—membranes, synapses—objects.

Fig. 2.



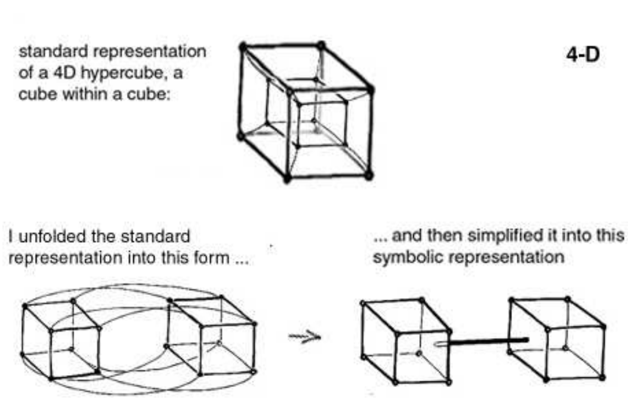
the fat arrows are colimiting injections,
i.e. the elements of colimiting cocons,
respectively

Concerning the underlying trees of multigraphical membrane systems we recommend to read [1] containing a discussion of advantages and disadvantages of using trees for visual presentation and an analysis of complex systems.

3 Inspiring examples

Following the idea of drawing hypercubes[a] from [27] recalled in Fig. 3–6 we show the examples of sketch-like multigraphical membrane systems which approach this idea in some formal way.

Fig. 3. 4th dimension of a hypercube



[a] for a notion of a hypercube see [21], [6], [25]

Fig. 4. 6th dimension of a hypercube

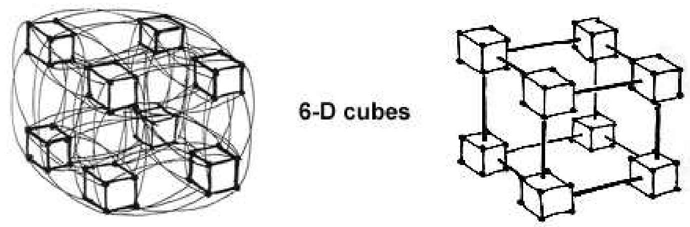


Fig. 5. 9th dimension of a hypercube

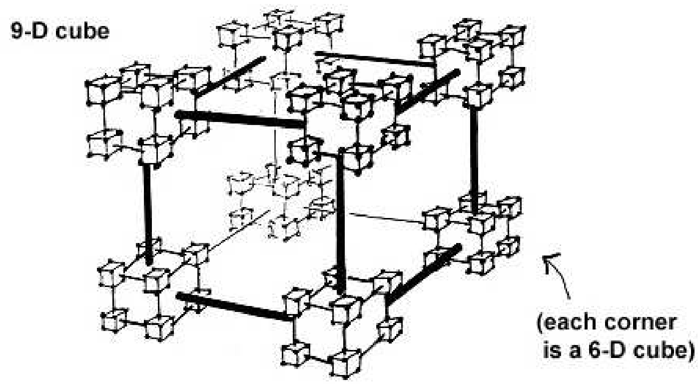
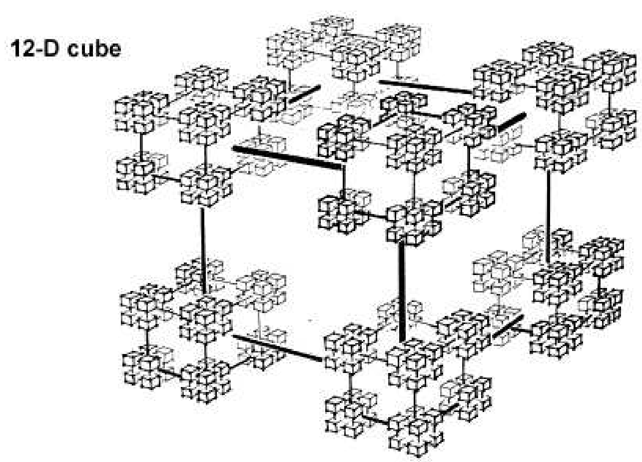


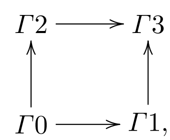
Fig. 6. 12th dimension of a hypercube



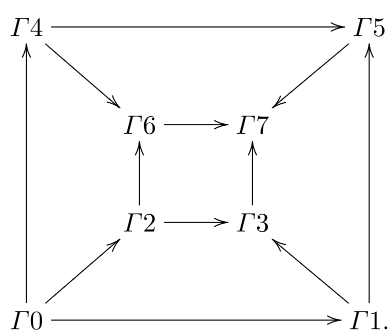
A large cube whose corners are smaller cubes can be treated as a large virtual membrane, where smaller cubes are treated as smaller virtual membranes contained in this large virtual membrane.

For natural numbers $n > 0$ and $i \in \{1, 2, 3\}$ we define sketch-like multigraphical membrane systems $\mathcal{S}_n^i$, the claimed examples, in the following way:

- the underlying tree $\mathbb{T}_n^i$ of $\mathcal{S}_n^i$ is such that
  - the set $V(\mathbb{T}_n^i)$ of vertices is the set of all strings (sequences) of length not greater than n of digits in $D^1 = \{0, 1\}$ for $i = 1$, in $D^2 = \{0, 1, 2, 3\}$ for $i = 2$, and in $D^3 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ for $i = 3$,
  - the set $E(\mathbb{T}_n^i)$ of edges of $\mathbb{T}_n^i$ is such that $E(\mathbb{T}_n^i) = \{(\Gamma j, \Gamma) \,|\, \{\Gamma j, \Gamma\} \subset V(\mathbb{T}_n^i)$ and $j \in D^i\}$ with source and target functions being the projections on the first and the second component, respectively, where Γj is the string obtained by juxtaposition a new digit j on the right end of Γ,
- the family $\big(G_\Gamma \,|\, \Gamma \in V(\mathbb{T}_n^i)\big)$ of directed graphs of $\mathcal{S}_n^i$ is such that for every non-elementary vertex $\Gamma \in V(\mathbb{T}_n^i)$ the Γ-diagram $\mathrm{Dg}(\Gamma)$ is determined in the following way:
  - for $i = 1$ the diagram $\mathrm{Dg}(\Gamma)$ is a graph consisting of a single edge $\Gamma 0 \to \Gamma 1$,
  - for $i = 2$ the diagram $\mathrm{Dg}(\Gamma)$ is the following square:



  - for $i = 3$ the diagram $\mathrm{Dg}(\Gamma)$ is the following cube:



The above sketch-like multigraphical membrane systems drawn by using Venn diagrams (with discs d_Γ corresponding to vertices Γ of $\mathbb{T}_n^i$ such that d_{Γ_j} is an immediate subset of d_Γ) coincide with the drawings shown in [27].

The following *interpretation* of $\mathcal{S}_n^i$ by an $i \cdot n$-dimensional hypercube $[\![\mathcal{S}_n^i]\!]$ ($n > 0$ and $i \in \{1, 2, 3\}$) completes the proposed formal approach to the idea of drawing hypercubes in [27].

We introduce the following notion to define hypercubes $[\![\mathcal{S}_n^i]\!]$. For a natural number $n \geq 0$ and a finite directed graph G whose vertices are natural numbers and the set $E(G)$ of edges of G is such that $E(G) \subseteq V(G) \times V(G)$ we define a new graph $G \uparrow n$, called the *translation of* G *to* n, by

$$V(G \uparrow n) = \{i + n \,|\, i \in V(G)\},$$
$$E(G \uparrow n) = \{(i + n, j + n) \,|\, (i, j) \in E(G)\}.$$

The hypercubes $[\![\mathcal{S}_n^i]\!]$ ($n > 0$, $i \in \{1, 2, 3\}$) are defined by induction on n in the following way:

- for every $i \in \{1, 2, 3\}$ the hypercube $[\![\mathcal{S}_1^i]\!]$ is the diagram $\mathrm{Dg}(\Lambda)$ of $\mathcal{S}_1^i$, where Λ is the empty string and the digits in $V(\mathrm{Dg}(\Lambda))$ are identified with corresponding natural numbers,
- for all $n > 0$ and $i \in \{1, 2, 3\}$ the hypercube $[\![\mathcal{S}_{n+1}^i]\!]$ is such that

$$V([\![\mathcal{S}_{n+1}^i]\!]) = \bigcup_{0 \le j < 2^i} V\big([\![\mathcal{S}_n^i]\!] \uparrow (j \cdot 2^{i \cdot n})\big),$$

$$E([\![\mathcal{S}_{n+1}^i]\!]) = \bigcup_{0 \le j < 2^i} E\big([\![\mathcal{S}_n^i]\!] \uparrow (j \cdot 2^{i \cdot n})\big) \cup \bigcup_{(k,m) \in E([\![\mathcal{S}_1^i]\!])} \big\{(j + k \cdot 2^{i \cdot n}, j + m \cdot 2^{i \cdot n}) \,|\, j \in V([\![\mathcal{S}_n^i]\!])\big\}.$$

We introduce the following constructs to prove the main theorems of the paper and to show the links of $\mathcal{S}_n^i$ ($i \in \{1, 2, 3\}$, $n > 0$) to Cantor set which is a known fractal, cf. [12].

For natural numbers k, n with $n > 0$ and $0 \le k < 2^n$ we define a binary vector $\mathrm{bin}^n(k)$ by induction on n:

$$\mathrm{bin}^1(k) = k,$$

$$\mathrm{bin}^{n+1}(k) = \begin{cases} [0, x_1, \ldots, x_n] & \text{if } k < 2^n \text{ and } [x_1, \ldots, x_n] = \mathrm{bin}^n(k), \\ [1, y_1, \ldots, y_n] & \text{if } k \ge 2^n \text{ and } [y_1, \ldots, y_n] = \mathrm{bin}^n(k - 2^n). \end{cases}$$

We propose some spatial organization (realization) of the diagrams G_Γ of $\mathcal{S}_n^i$ itself in the space $\mathbb{R}^i$, where $\mathbb{R}^i$ is a Cartesian product of i copies of the set $\mathbb{R}$ of real numbers. This spatial organization is determined by a graph $\mathrm{space}(\mathcal{S}_n^i)$ defined by induction on n. For $\Delta \in \{V, E\}$ we define

$$\Delta\big(\mathrm{space}(\mathcal{S}_1^i)\big) = \Delta\big([\![\mathcal{S}_1^i]\!]\big)$$

$$\Delta\big(\mathrm{space}(\mathcal{S}_{n+1}^i)\big) = \Delta\big(\tfrac{1}{3} \cdot \mathrm{space}(\mathcal{S}_n^i)\big) \cup \bigcup_{(k,m) \in E([\![\mathcal{S}_1^i]\!])} \Delta\big(\tfrac{1}{3} \cdot \mathrm{space}(\mathcal{S}_n^i) \uparrow (\tfrac{2}{3} \cdot \mathrm{bin}^i(m))\big),$$

where for a graph G with $E(G) \subseteq V(G) \times V(G)$ and $V(G) \subseteq \mathbb{R}^i$, for a real number α with $0 \le \alpha \le 1$, and a vector $[x_1, \ldots, x_i] \in \mathbb{R}^i$ we define *contraction* $\alpha \cdot G$ and *translation* $G \uparrow [x_1, \ldots, x_i]$ to be graphs given by

$$V(\alpha \cdot G) = \big\{\alpha \cdot \boldsymbol{v} \,|\, \boldsymbol{v} \in V(G)\big\}, \quad E(\alpha \cdot G) = \big\{(\alpha \cdot \boldsymbol{v}, \alpha \cdot \boldsymbol{v}') \,|\, (v, v') \in E(G)\big\},$$

$$V(G \uparrow [x_1, \ldots, x_i]) = \big\{\boldsymbol{v} + [x_1, \ldots, x_i] \,|\, \boldsymbol{v} \in V(G)\big\},$$

$$E(G \uparrow [x_1, \ldots, x_i]) = \big\{(\boldsymbol{v} + [x_1, \ldots, x_i], \boldsymbol{v}' + [x_1, \ldots, x_i]) \,|\, (\boldsymbol{v}, \boldsymbol{v}') \in E(G)\big\},$$

where $\cdot$ denotes scalar multiplication of a vector and $+$ denotes vector sum.

The correctness of the proposed formal approach to the drawing of hypercubes in [27] is provided by the following theorem.

Theorem 1. *For all natural numbers $n > 0$ and $i \in \{2, 3\}$*

- $[\![\mathcal{S}_n^1]\!]$ *is an n-dimensional hypercube,*
- $[\![\mathcal{S}_n^i]\!] = [\![\mathcal{S}_{i \cdot n}^1]\!]$.

Proof. The proof of the theorem is by induction on n. The graphs $[\![\mathcal{S}_n^1]\!]$ are identified with n-dimensional hypercubes (Boolean n-cubes) by identifying the numbers k in $V([\![\mathcal{S}_n^1]\!])$ with binary vectors $\mathrm{bin}^n(k) \in \mathbb{R}^n$, respectively.

One sees that the edges of Γ-diagrams $\mathrm{Dg}(\Gamma)$ of $\mathcal{S}_n^i$ are the results of compression or binding the edges linking appropriate disjoint subhypercubes of $[\![\mathcal{S}_n^i]\!]$, where the idea of this compression or binding is fundamental for drawing hypercubes in [27]. The elements of cocones for $\mathcal{S}_n^i$ correspond to the embeddings between appropriate subhypercubes of $[\![\mathcal{S}_n^i]\!]$.

The following theorem shows the links between hypercubes, the sketch-like multigraphical membrane systems $\mathcal{S}_n^i$ ($i \in \{1, 2, 3\}$, $n > 0$) and Cantor set $\mathbb{C}$.

Theorem 2. *For all natural numbers $i \in \{1, 2, 3\}$ and $n > 0$ the following conditions hold*:

- *there exists an embedding, i.e. a graph homomorphism which is an injection of* $\mathrm{space}(\mathcal{S}_n^i)$ *into* $[\![\mathcal{S}_n^i]\!]$ *such that the image of this embedding is* $[\![\mathcal{S}_n^i]\!]$ *excluding all compressed edges, i.e. those belonging for $n > 1$ to*

$$\bigcup_{0<q<n} \ \bigcup_{0 \leq p < 2^{i \cdot (n-q-1)}} \ \bigcup_{(k,m) \in E([\![\mathcal{S}_1^i]\!])} \big\{ (j + k \cdot 2^{i \cdot q}, j + m \cdot 2^{i \cdot q}) \,|\, j \in V_{p,q}^i \big\},$$

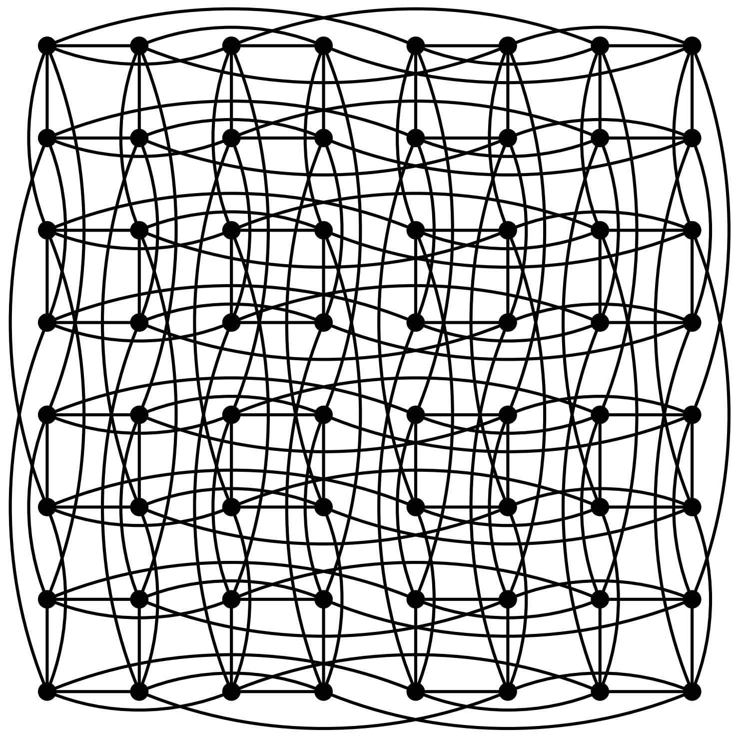
$$\text{where } V_{p,q}^i = V\big([\![\mathcal{S}_q^i]\!] \uparrow (p \cdot 2^{i \cdot (q+1)})\big),$$

- *the undirected connectedness components of* $\mathrm{space}(\mathcal{S}_n^i)$ *coincide in a one to one correspondence with connectedness components of the Cartesian product $\mathbb{C}_n^i$ of i copies of the n-th iteration $\mathbb{C}_n = \frac{\mathbb{C}_{n-1}}{3} \cup (\frac{2}{3}, \frac{\mathbb{C}_{n-1}}{3})$ of the Cantor set.*

Proof. The proof of the theorem is by induction on n. The connectedness components of $\mathbb{C}_n^i$ are intervals of $\mathbb{R}$ for $i = 1$, the squares with their interiors in $\mathbb{R}^2$ for $i = 2$, the cubes with their interiors in $\mathbb{R}^3$ for $i = 3$. For $i > 1$ the edges of these squares and cubes are the intervals laying on the straight lines connecting the vertices $\boldsymbol{v}, \boldsymbol{v}'$ of the pair $(\boldsymbol{v}, \boldsymbol{v}') \in E(\mathrm{space}(\mathcal{S}_i^n))$ and these vertices are the ends of the intervals, respectively. Thus one obtains a one to one correspondence between connectedness components of $\mathrm{space}(\mathcal{S}_n^i)$ and $\mathbb{C}_n^i$. The small cubes in Fig. 5, 6 illustrate both the connectedness components of $\mathrm{space}(\mathcal{S}_n^3)$ and $\mathbb{C}_n^3$. The connectedness components of some iteration of 3D Cantor set $\mathbb{C}^3$ are shown also as small cubes in [31].

Remark 1. Thus the sketch-like multigraphical membrane systems $\mathcal{S}_n^i$ represent some internal fractal-like structure of hypercubes $[\![\mathcal{S}_n^i]\!]$ which was not visible at first glance, e.g. in the drawing of 6-dimensional hypercube in Figure 1 in [25], presented in Fig. 7 of the present paper.

Fig. 7. Shown here is a two-dimensional projection of a six-dimensional hypercube, or binary 6-cube, which corresponds to a 64-node machine

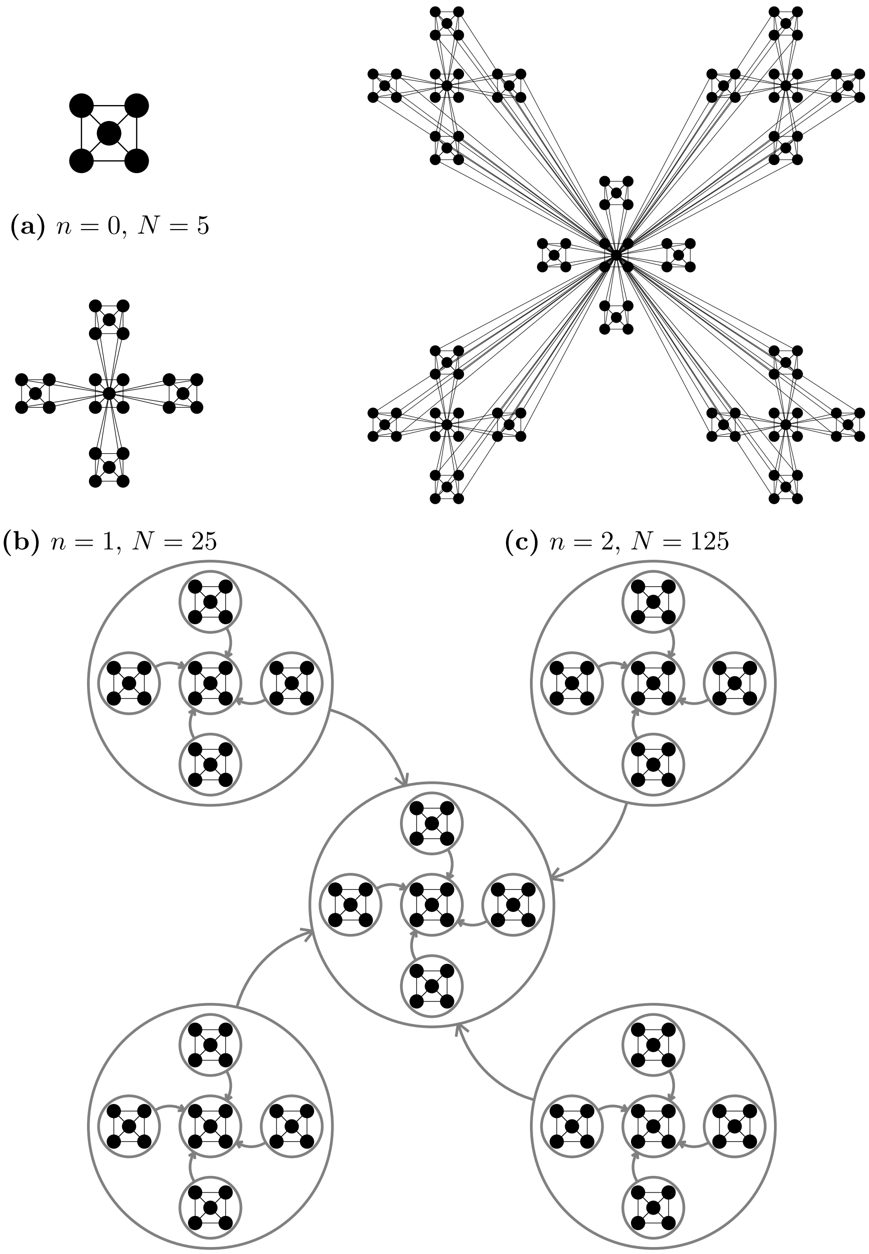


The internal fractal-like structure represented by $\mathcal{S}_n^i$ can be described and explained by the following two representations.

The underlying trees $\mathbb{T}_n^i$ of $\mathcal{S}_n^i$ represent that *hierarchical organization* of both space($\mathcal{S}_n^i$) and $\mathbb{C}_n^i$ which is determined by the scales corresponding to fractions $(\frac{1}{3})^k$ $(0 \leq k \leq n)$. Moreover, the trees $\mathbb{T}_n^i$ have some common features with the trees generated by some iteration function systems, cf. [7], for fractals $\mathbb{C}^i$ being iD Cantor sets, where $\mathbb{C}_n^i$ are the iterations of $\mathbb{C}^i$ $(i \in \{1, 2, 3\})$.

For $n > 2$ and $i \in \{2, 3\}$ the diagrams G_Ω of $\mathcal{S}_n^i$ with $V(G_\Omega)$ being a set of non-elementary vertices in $\mathcal{S}_n^i$, called *spatial arrangement diagrams* of $\mathcal{S}_n^i$, represent some uniform spatial arrangement of subgraphs of space($\mathcal{S}_n^i$) in $\mathbb{R}^i$. Namely, for every spatial arrangement diagram G_Ω of $\mathbb{R}^i$ the virtual membrane Ω of space($\mathcal{S}_n^i$) (illustrated in Fig. 6 and corresponding to the real membrane Ω of $\mathcal{S}_n^i$) contains those 2^i different translations of the contraction $(\frac{1}{3})^{l(\Omega)+1} \cdot$ space($\mathcal{S}_{n-l(\Omega)-1}^i$) which are mutually related (arranged) according to the edges of $E(G_\Omega)$, where $l(\Omega)$ denotes the length of a string Ω. For instance, for $i = 2$ if Ω is empty word Λ, then $(\frac{1}{3}) \cdot \text{space}(\mathcal{S}_{n-1}^2) \uparrow \frac{2}{3} \cdot (\text{bin}^2(3))$ is located above $(\frac{1}{3}) \cdot \text{space}(\mathcal{S}_{n-1}^2) \uparrow \frac{2}{3} \cdot (\text{bin}^2(1))$ with distance $\frac{1}{3}$ according to $(1, 3) \in E(G_\Lambda)$. The iterations $\mathbb{C}_n^i$ have an analogous spatial arrangement represented by spatial arrangement diagrams of $\mathcal{S}_n^i$.

Fig. 8.



(a) $n = 0$, $N = 5$

(b) $n = 1$, $N = 25$

(c) $n = 2$, $N = 125$

(d)

Remark 2. The presentation of multidimensional hypercubes by sketch-like multigraphical membrane systems $\mathcal{S}_n^i$ with their interpretations $[\![\mathcal{S}_n^i]\!]$, respectively, suggest a similar presentation of hierarchical networks in [23] (see Fig. 1

in [23]) and [5] by applying sketch-like multigraphical membrane systems, which is outlined in Fig. 8 of the present paper, where Fig. 8(a)–(c) is Fig. 1 in [23].

The arcs (links) from the peripherical nodes of each cluster to the central node of the original cluster (in Fig. 8(c)) are compressed to the arcs between non-elementary membranes (in Fig. 8(d)) corresponding to the clusters. The skin membrane (root) is omitted in Fig. 8(d).

Conclusion

The sketch-like multigraphical membrane systems play a dual role in object recognition and visual processing realized in brain neural networks and by artificial neural network of neocognitron [14]. Namely, they present the "objective" multilevel features[b] to be represented neuronally (at best by embedding) in "subjective" multilayer brain neural networks[c], cf. e.g. [13], [28], and in artificial neural networks of neocognitron.

The idea of drawing multidimensional hypercubes outlined in [27] together with its formal treatment by sketch-like multigraphical membrane systems shown in Section 3 propose a new approach to feature recognition and visual processing of multidimensional objects by information compression[d], may be different from that proposed in [17]. Thus one can ask for reliability of processes of feature recognition of multidimensional objects by neocognitron in the manner of [15] and according to this new approach.

Remarks 1 and 2 suggest the new applications of sketch-like multigraphical membrane systems for representation of fractal iterations and for presentation of hierarchical networks.

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[b] with respect to e.g. natural abstraction levels: pixel level, local feature level, structure-level, object-level, object-set-level, and scene characteriztion, or with respect to the levels of subhypercubes (faces) of a multidimensional hypercube.

[c] like in a classical model of visual processing in cortex which is hierarchy of increasingly sophisticated representations extending in natural way the model of simple to complex cells (neurons) of Hubel and Wisel, cf. [24].

[d] realized e.g. by binding some links between subhypercubes of a given multidimensional hypercube.

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