Codazzi-equivalent Riemannian metrics

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On a smooth manifold M of dimension $n \geq 2$ we denote tangent fields by $u, v, w, ...$, a Riemannian metric by g, its Levi-Civita connection by $\nabla := \nabla(g)$, and its sectional curvature with respect to a frame (e_i) by $\kappa(e_i, e_j)$. We introduce the concept of Codazzi-equivalent Riemannian metrics:
Two metrics g, g^* are called **Codazzi-equivalent** *if there exists a bijective operator L s.t. the pair $(\nabla(g), L)$ satisfies Codazzi type equations and $g^*(u,v) = g(Lu, Lv)$ for all u, v.*
In the talk we give examples for this situation and survey some results:

- **Curvature and metric**
  Let g and g^* be Codazzi-equivalent with operator L. Assume that L has an eigenbasis (e_i) corresponding to the eigenvalues (λ_i). Then the sectional curvatures satisfy the relation

  $$\kappa^*(e_i, e_j) = (\lambda_i \cdot \lambda_j)^{-1} \cdot \kappa(e_i, e_j).$$

  In particular, we get sufficient conditions that the sectional curvature determines the metric, in dimension $n \geq 3$ locally, for $n = 2$ globally.

*part. supp. DFG

- **Euclidean hypersurfaces**
  For a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ assume that the shape operator S has maximal rank. The three *fundamental forms*, $g := I$, II, $g^* := III$, are (semi)-Riemannian metrics; we denote their Levi-Civita connections by $\nabla(g) := \nabla(I)$, $\nabla(II)$, $\nabla(III) =: \nabla^*$, resp.

  (a) If $x, x^\sharp$ are $I-$isometric then $g^* = III$ and $g^{\sharp *} = III^\sharp$ are Codazzi-equivalent with $L := S^{-1} \cdot S^\sharp$ and $g^{*\sharp}(u,v) = g^*(Lu, Lv)$. Moreover, if one of the shape operators is (positive) definite then the operator L has a basis of eigenvectors, and in dimension $n = 2$ the operators $L, S, S^\sharp$ commute.

  (b) If $x, x^\sharp$ are $III-$isometric then $g = I$ and $g^\sharp = I^\sharp$ are Codazzi-equivalent with $L := S \cdot S^{\sharp\,-1}$ and $g^\sharp(u,v) = g(Lu, Lv)$. Moreover, if one of the shape operators is (positive) definite then the operator L has a basis of eigenvectors and in dimension $n = 2$ the operators $L, S, S^\sharp$ commute.

- **Local and global uniqueness theorems**
  We prove a series of local and global uniqueness results for Riemannian manifolds and hypersurfaces, in particular we give new proofs for classical uniqueness theorems for ovaloids of Minkowski and Cohn-Vossen type. For Cohn-Vossen's isometry theorem for ovaloids we give a proof using Monge-Ampère operators.

The concept of Codazzi-equivalence can be generalized from Riemannian metrics to affine connections.

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