Isoparametric Hypersurfaces in $\mathbb{S}^{n+1}$: The Chern Conjecture

Mike Scherfner

Institute of Mathematics, TU Berlin

Content

We present a short history of the Chern conjecture for isoparametric hypersurfaces in spheres and generalizations. Main results will be presented and we summarize the progress for this topic.

Basics

We are concerned with hypersurfaces, i. e. n-dimensional submanifolds in $(n+1)$-dimensional Riemannian manifolds.

The curvature of a hypersurface M in the ambient manifold is described by the second fundamental form h (or the associated shape operator A). The eigenvalues of h are the principal curvature functions λ_i, $i = 1...n$.

A hypersurface in a space of constant curvature c is called isoparametric, if all the λ_i are constant functions.

In $\mathbb{R}^{n+1}$ all the isoparametric hypersurfaces are hyperspheres, hyperplanes or (generalized) cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$. In the sphere $\mathbb{S}^{n+1}$ we could expect "much more" of such hypersurfaces.

A hypersurface in a space of constant curvature c is called isoparametric, if all the λ_i are constant functions.

In $\mathbb{R}^{n+1}$ all the isoparametric hypersurfaces are hyperspheres, hyperplanes or (generalized) cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$. In the sphere $\mathbb{S}^{n+1}$ we could expect "much more" of such hypersurfaces.

Important Quantities

The scalar curvature κ comes directly form the curvature tensor, describing the intrinsic curvature of M and we have

$$\kappa = c + \frac{1}{n(n-1)} \sum_{i \neq j} \lambda_i \lambda_j.$$

The mean curvature H of M is given by

$$H = \frac{1}{n} \sum_i \lambda_i,$$

and M is minimally immersed iff $H = 0$.

Important Quantities

The scalar curvature κ comes directly form the curvature tensor, describing the intrinsic curvature of M and we have

$$\kappa = c + \frac{1}{n(n-1)} \sum_{i \neq j} \lambda_i \lambda_j.$$

The mean curvature H of M is given by

$$H = \frac{1}{n} \sum_i \lambda_i,$$

and M is minimally immersed iff $H = 0$.

We also define

$$S := |h|^2 = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2$$

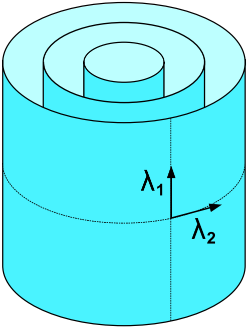
and for $r \geq 3$

$$f_r := \operatorname{tr}\left((h_{ij})^r\right).$$

with

$$f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki} = \sum_i \lambda_i^3, \quad f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li} = \sum_i \lambda_i^4.$$

Example



The Conjecture

The Chern conjecture for isoparametric hypersurfaces in spheres can be stated as follows:

Let M be a closed, minimally immersed hypersurface of the $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$ with constant scalar curvature. Then M is isoparametric.

It was originally proposed in a less strong version by Chern (1968) and in Chern, do Carmo and Kobayashi (1970), and finally by Verstraelen (1986) in the version presented above.

So far, no proof for the conjecture has been found, although partial results exist in particular for low dimensions (especially up to four) and with additional conditions for the curvature functions on M.

Its original version relates to the following theorem, first proved by Simons (1968):

Theorem

Let $M \subset \mathbb{S}^{n+1}$ be a closed, minimally immersed hypersurface and S the squared norm of its second fundamental form. Then

$$\int_M (S-n)S \geq 0.$$

In particular, for $S \leq n$ one has either $S = 0$ or $S = n$ identically on M.

Its original version relates to the following theorem, first proved by Simons (1968):

Theorem

Let $M \subset \mathbb{S}^{n+1}$ be a closed, minimally immersed hypersurface and S the squared norm of its second fundamental form. Then

$$\int_M (S-n)S \geq 0.$$

In particular, for $S \leq n$ one has either $S = 0$ or $S = n$ identically on M.

Since M is minimally immersed S is constant if and only if the scalar curvature κ is constant. In this case it follows that $S = 0$ or $S \geq n$, which led Chern to propose the following conjecture:

Consider closed minimal hypersurfaces $M \subset \mathbb{S}^{n+1}$ with constant scalar curvature κ. Then for each n the set of all possible values for κ (or equivalently S) is discrete.

The only known examples for minimal hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$ are isoparametric, i.e. all of their principal curvature functions are constant.

One obtains that S equals $(g-1)n$, where g is the number of pairwise distinct principal curvatures and can only take the values 1, 2, 3, 4 or 6, which establishes the conjecture in this case.

Results

The trivial case is given for $n = 2$:

$$\lambda_1 + \lambda_2 = 0$$
$$\lambda_1^2 + \lambda_2^2 = const$$

The first partial result was achieved by Peng and Terng, who gave further constraints for the possible values of S:

Theorem (Peng, Terng 1983)

For every $n \geq 3$ there exists a maximal $C(n)$ with the following property: Let $M \subset \mathbb{S}^{n+1}$ be a closed minimal hypersurface with constant $S > n$. Then it follows that $S \geq n + C(n)$ and one has $C(3) = 3$, $C(n) \geq \frac{1}{12n}$.

The lowest dimension for which the Chern conjecture is non-trivial is $n = 3$. In this case, a more general theorem has been proven:

Theorem (Almeida, Brito 1990; Chang 1994)

Let $M \subset \mathbb{S}^4$ be a closed hypersurface with constant mean curvature H and constant scalar curvature κ. Then M is isoparametric.

The lowest dimension for which the Chern conjecture is non-trivial is $n = 3$. In this case, a more general theorem has been proven:

Theorem (Almeida, Brito 1990; Chang 1994)

Let $M \subset \mathbb{S}^4$ be a closed hypersurface with constant mean curvature H and constant scalar curvature κ. Then M is isoparametric.

Almeida and Brito initially showed this under the additional assumption that κ is non-negative. The approach of this proof has since been used to show a number of other results. Chang then completed the proof. He proved this separately for manifolds with three everywhere distinct principal curvatures and those where two principal curvatures coincide in a point, in the former case generalizing a proof given earlier by Peng and Terng for minimal hypersurfaces.

Instead of low dimensional manifolds, one can also consider those with a certain number g of pairwise different principal curvatures; $g = 3$ is the first non-trivial case, and one has the following result:

Theorem (CHANG 1994)

Let $M \subset \mathbb{S}^{n+1}$ be a closed hypersurface with constant mean and scalar curvatures which has exactly three pairwise distinct principal curvatures in every point. Then M is isoparametric.

For the case $n = 4$ a partial result has been proven under the additional assumption that M is a Willmore hypersurface, i.e. a critical point of the Willmore functional $W(M) := \int_M \rho^n$ with $\rho^2 = S - nH^2$.

H. Li computed the Euler-Lagrange equation for the Willmore functional and obtained the following characterization:

Theorem

Let $M^n \subset \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface. Then M^n is a Willmore hypersurface if and only if

$$0 = -\rho^{n-2}\Big(2HS - nH^3 - \sum_{i,j,k} h_{ij}h_{jk}h_{ki}\Big) + (n-1)\Delta(\rho^{n-2}H) - \sum_{i,j}(\rho^{n-2})_{ij}(nH\delta_{ij} - h_{ij}),$$

where $\rho^2 = S - nH^2$, Δ is the Laplacian and $(.)_{ij}$ is the covariant derivative with respect to the induced connection.

An immediate consequence is the following characterization of Willmore hypersurfaces in spheres with constant mean curvature and scalar curvature:

Theorem

Let $M^n \subset \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature and constant scalar curvature. Then M^n is a Willmore hypersurface if and only if

$$f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki} = 2HS - 4H^3.$$

In particular the Willmore condition for minimal hypersurfaces with constant scalar curvature is equivalent to the condition $f_3 \equiv 0$.

Theorem (Lusala, Scherfner, Sousa Jr. 2005)

Let $M \subset \mathbb{S}^5$ be a closed minimal Willmore hypersurface with constant non-negative scalar curvature. Then M is isoparametric.

Theorem (Scherfner, Sousa Jr. 2009)

Let $M \subset \mathbb{S}^5$ be a closed minimal Willmore hypersurface with constant scalar curvature. Then M is isoparametric.

Bryant conjecture

One obvious generalization is that on non-closed manifolds, i.e. a local version of the conjecture. This has in particular been proposed by Bryant for the case $n = 3$:

Let $M \subset \mathbb{S}^4$ be a minimal hypersurface with constant scalar curvature. Then M is isoparametric.

Summary

n	Chern Conjecture	Chern Conjecture ($H \neq 0$)	Chern Conjecture (locally)
2	Yes	Yes	Yes
3	Yes	Yes	If $S \leq 3$ or $g = 2$ in p
4	If $f_3 \equiv 0$, $S \leq 12$ or f_3 const., $S < \frac{20}{3}$ or $S < \frac{372}{61}$ or $g \equiv 3$	Yes (add. conditions)	Yes (add. conditions)
> 4	If f_3 const., $S < \frac{28}{15}n - \frac{4}{5}$ or $S < \frac{97}{61}n - \frac{16}{61}$ or $g \equiv 3$	Yes (add. conditions)	Yes (add. conditions)

This list is not complete, but exists...

Following Prof. S.-T. Yau we have the following situation:

The general philosophy of the Chern conjecture is that if we assume a hypersurface in the sphere to be minimal, it is very rigid. If we also make an extra assumption on the second fundamental form, such as constant length, it will force the manifold to be more rigid. And we expect such constants to be discrete and most likely to be finite. The relation with isoparametric hypersurfaces is interesting and perhaps typical.

Outlook

Theorem (Scherfner, Vrancken, Weiss 2009)

Let $M \subset \mathbb{S}^7$ be a closed hypersurface with $H = f_3 = f_5 = 0$, $f_4 = const$ and $\kappa \geq 0$. Then M is isoparametric.

More ideas:

- 4-dim. case for constant f_3 or K
- Related questions for submanifolds of higher codimension
- Analyze heat kernels
- ...