Isometric embeddings of surfaces and quasilocal gravitational energy

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IMPAN
Warsaw, Poland
April 6, 2009

Plan of the talk

- Issues of energy and mass in general relativity. $E = mc^2$.
- What is the energy contained in a bounded region Ω^3 in spacetime?
- Closely related to isometric embedding of $\partial\Omega^3$ into $\mathbb{R}^{3,1}$.

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- Light cone $\{x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = \langle x, x\rangle = 0$

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- The set of all future timelike unit vector $\{\langle x, x\rangle = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = -1, x^4 > 0.\}$ forms the hyperbolic three-space in $\mathbb{R}^{3,1}$.

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- The minimum of E is the mass m. Achieved at v and $p = mv$ is the energy-momentum 4-vector of the particle.

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- Given a spacelike (i.e "Riemannian" with the induced metric) bounded region Ω.
- The energy intercepted by Ω as seen by the observer w is the flux integral

$$\int_{\Omega} T(w,n)$$

where n is the future timelike unit normal of Ω.

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- Determine by the boundary $\Sigma = \partial\Omega$, a spacelike 2-surface in $\mathbb{R}^{3,1}$.

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- Relation between gravitation field and matter fields.
- Question: Ω bounded spacelike region, what is the total energy intercepted by Ω as seen by an observer w, What is the total mass contained in Ω?

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- No constant (Killing) vector field w in general spacetime to make $T(w, \cdot)$ divergence free.
- Only accounts for the energy of matter fields.

$$Ric - \frac{1}{2}Rg = 0$$

Vacuum solution such as the Schwarzchild, still has gravitational energy.

Gravitation energy in general relativity

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- No density (equivalence principle). One can choose local coordinates around any $p \in M$ such that any derivative of g is zero at p
- Depends on underlying geometry. Gravitation binding energy depends on distance $m_1 + m_2 - \frac{m_1 m_2}{r}$.

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- The ADM energy momentum 4-vector is (E, P_1, P_2, P_3). Each component is given by the limit of a flux integral over coordinate sphere $S_r \subset \mathbb{R}^3$ as $r \to \infty$.

Positive energy theorem of Schoen-Yau

- Dominant energy condition: $T(w, \cdot)$ is past timelike for any future timelike w.
- Suppose the dominant energy condition holds along an asymptotically flat Ω, then (E, P_1, P_2, P_3) is a future-directed non-spacelike vector, i.e.

$$E \geq 0, -E^2 + P_1^2 + P_2^2 + P_3^2 \leq 0.$$

- In particular, the ADM mass $\sqrt{E^2 - P_1^2 - P_2^2 - P_3^2}$ is non-negative and $= 0$ if and only if the spacetime is flat along Ω.

Bounded region in general spacetime, strong field

- Recall in $\mathbb{R}^{3,1}$ the energy intercepted by Ω depends only on $\partial\Omega$.
- In, general given a $\Sigma = \partial\Omega$ a spacelike 2-surface, we want to attach to it a quasilocal energy-momentum 4 vector.

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- Recall particle case:

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- Given Σ a spacelike 2-surface in spacetime N,

$$(i, w) \Rightarrow E(i, w),\ \ m = \min E(i, w) \text{ and } p = mv$$

where $i : \Sigma \to \mathbb{R}^{3,1}$ an isometric embedding of Σ and $i \in \mathbb{R}^{3,1}$ is future timelike.

Why isometric embedding into $\mathbb{R}^{3,1}$?

- Hamilton-Jacobi analysis of gravitation action (whose Euler-Lagrange equation is the Einstein equation). The energy is the difference between physical Hamiltonian and reference Hamiltonian.
- Physical Hamiltonian=integral over $\Sigma \subset N$
- Reference Hamiltonian=integral over a reference surface.
- Isometric embedding gives reference surface in flat spacetime. Uniqueness?

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- However, there exist surfaces in $\mathbb{R}^{3,1}$ with strictly positive mass.

Physical surface Hamiltonian

- Derived by ADM, Brown-York, Hawking-Horowitz.
- Reduce to an integral on a spacelike 2-surface in spacetime N.
- Physical surface Hamiltonian

$$\int_{\Sigma} \mathfrak{H}(\bar{n}, \bar{w})$$

where $\bar{n}$ is a future directed timelike unit normal of Σ, $\bar{w}$ a future-directed timelike unit vector (observer) (Recall $\int_{\Omega} T(n, w)$).

The prescription of reference surface Hamiltonian (W-Yau)

- Given an isometric embedding $i : \Sigma \to \mathbb{R}^{3,1}$ and a constant future timlike unit vector w in $\mathbb{R}^{3,1}$, we define a reference Hamiltonian $\int_\Sigma \mathfrak{H}(n, w)$.
- (i, w) determines a canonical gauge n along $i(\Sigma)$. $\bar{n}$ and $\bar{w}$ are uniquely determined by matching conditions of Σ in N and $i(\Sigma)$ in $\mathbb{R}^{3,1}$.
- Quasilocal energy is

$$E(i, w) = \int_\Sigma \mathfrak{H}(n, w) - \int_\Sigma \mathfrak{H}(\bar{n}, \bar{w}).$$

- ADM mass on S_r.

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  where J (a future directed timelike normal vector field of Σ) is the reflection of the mean curvature vector field H of Σ along the light cone in the normal bundle.
- V is the tangent vector field on Σ that is dual to the connection one form determined by $\bar{n}$ (this is gauge dependent). Imagine $\bar{n}$ is the future timelike unit normal of a spacelike domain Ω that Σ bounds.

Observation (Gibbons)

Important property of $\mathfrak{H}(n, w)$, when Σ is in $\mathbb{R}^{3,1}$, w a constant future timelike unit vector in $\mathbb{R}^{3,1}$, there exists a canonical gauge n such that $\int_\Sigma \mathfrak{H}(n, w)$ is equal to the total mean curvature of $\hat{\Sigma}$, the projection of Σ onto the orthogonal complement of w.

Matching condition

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- $\Sigma \subset N$ and $i(\Sigma) \subset \mathbb{R}^{3,1}$ have the same induced metric.

Theorem

If N satisfies the dominant energy condition, and (i, w) is admissible, then

$$E(i, w) = \int_\Sigma \mathfrak{H}(n, w) - \int_\Sigma \mathfrak{H}(\bar{n}, \bar{w})$$

is non-negative.

1. variational properties of canonical gauges. 2. Solvability of Jang's equation (Schoen-Yau) to reduce to a Riemannian case. 3. A quasispherical construction (Bartnik) and positive mass theorem for manifolds with corners (Shi-Tam) 4. Relating the surface Hamiltonian in different ambient manifolds.

Isometric embeddings with convex shadows

- Admissible pair (i, w) is best described by the time function τ, the restriction of x^4 to $i(\Sigma)$.
- (i, w) is said to have convex shadow if the projection of $i(\Sigma)$ onto the orthogonal complement of w is a convex surface (always get an embedded surface).

Theorem

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Theorem

(W-Yau) Given an metric σ *on* $\Sigma \simeq S^2$ *suppose* τ *is a smooth function on* Σ *that satisfies*

$$K + (1 + |\nabla \tau|^2)^{-1} \det(\nabla^2 \tau) > 0$$

Then there exists a unique isometric embedding $i : \Sigma \to \mathbb{R}^{3,1}$ *with convex shadow and with* τ *as its time function.*

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