Constant mean curvature surfaces with prescribed ideal boundary in negatively curved 3-manifolds

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Banach Center, 8 April 2009

The mean curvature of a surface Σ in a 3-manifold M is $\frac{1}{2}(\kappa_1 + \kappa_2)$, where κ_1 and κ_2 are the principal curvatures of Σ.

A surface with constant mean curvature equal to H shall be referred to as an H- surface.

An H-surface is a critical point of

Area - 2H Volume.

By volume we shall mean the oriented volume of a cone over Σ with respect to some point $o \in M$, which is assumed to be diffeomorphic via $\exp_o$ to $\mathbb{R}^3$. Indeed, from now on, M shall denote a Cartan-Hadamard manifold, i.e., a simply connected manifold of nonpositive sectional curvature. Mostly, M shall be $\mathbb{H}^3$.

Thus, for a map $u\colon \mathbb{B} \to M$,

$$V(u) := \int_{\mathbb{B}} \int_0^1 \lambda^2 u \cdot u_x \times u_s \, d\lambda \, dx \, ds$$

$$= \int_{\mathbb{B}} Q(u(x,s)) \cdot u_x \times u_s \, dx \, ds$$

where $\cdot$ and $\times$ are taken with respect to the metric at $\lambda u(x,s)$.

Yau (1975) If $\mathrm{sect}_M \leqslant -k < 0$ and $\Omega \subset M$ is a bounded domain, then

$$A(\partial\Omega) > (n-1)|V(\Omega)|.$$

So, if $\Sigma \subset M$ is a surface (with boundary), and $\text{sect}_M \leqslant -1$ and $|H| \leqslant 1$, then

$$I_{A,H}(\Sigma) := A(\Sigma) - 2HV(\Sigma) > 0.$$

Therefore, given a Jordan curve $\Gamma \subset M$, $\text{sect}_M \leqslant -1$ and $|H| \leqslant 1$, we expect to be able to span Γ by an H-surface. This was proved independently by Gulliver and Hildebrandt-Kaul in the early 70's. Cuschieri has written a more geometrically transparent version of this proof by exploiting a version of Yau's isopermetric inequality for surfaces with boundary — the wall of the cone makes no contribution to the area of the cone.

The approach here is similar to that of Courant for minimal surfaces in that $I_{A,H}$ gets replaced by

$$I_{E,H}(u) := E(u) - 2HV(u)$$

where

$$E(u) := \frac{1}{2}\int_B (|u_x|^2 + |u_s|^2)\, dx\, ds \geqslant A(u).$$

A parameterisation φ of Γ is fixed, and then a minimizer for $I_{E,H}$ among maps from $\mathbb{B}$ to M equal to φ on the boundary is an H-harmonic map u:

$$\tau(u) = -2Hu_x \times u_s\,.$$

φ is then varied so as to find u^* for which

$$I_{E,H}(u^*) \leqslant I_{E,H}(u)$$

for all $u \in W^{1,2}(\mathbb{B})$ such that $u|_{\partial\mathbb{B}}$ is a parameterisation of Γ. u^* is then conformal, and therefore its image is an H-surface.

What happens as $\Gamma \to \partial_\infty M$? Around 1980, Uhlenbeck and Wei produced complete area-minimizing surfaces in $\mathbb{H}^3$ with boundary in S^2_∞ which was not a round circle.

These examples showed that the direct analogue of the Bernstein theorem for area minimising surfaces in $\mathbb{R}^3$ does not hold in $\mathbb{H}^3$. In view of Yau's isoperimetric inequality, this is perhaps not so surprising — the area functional is, in a sense, too positive and allows for an abundance of minimal surfaces. Furthermore, Yau's isoperimetric inequality suggests that a Bernstein-type theorem might hold for stable 1-surfaces in $\mathbb{H}^3$. This was shown to be the case by da Silveira in 1987 who proved that a complete 1-surface in $\mathbb{H}^3$ which is stable for the functional Area - 2 Volume must be totally umbilic and therefore a horosphere.

Yau's isoperimetric inequality also suggests that it should be possible to span a Jordan curve $\Gamma \subset S^2_\infty$ by an H-surface in $\mathbb{H}^3$ for any $H \in (-1, 1)$. Using GMT methods, Anderson (1982, $H = 0$) and Tonegawa (1996, $-1 < H < 1$) showed this to be the case. Using PDE methods, Nelli-Spruck (1996) and Guan-Spruck (2000) constructed H-graphs over mean convex domains and starlike domains respectively. Using volume-preserving mean curvature flow, Biao Wang (2008) was able to foliate an almost fuchsian 3-manifold by H-surfaces, H running from (but not including) -1 to 1.

It is difficult to control the topology of a surface produced by GMT methods. It is therefore desirable to ask what happens to the solution of Gulliver and Hildebrandt-Kaul as $\Gamma \to \Gamma_\infty \subset S^2_\infty$.

It is also desirable to consider the existence of H-annuli and, more generally, H-surfaces of prescribed higher topological type.

Theorem (Cuschieri). Suppose $\Gamma_i \subset \mathbb{H}^3$ is a sequence of Jordan curves converging to $\Gamma_\infty \subset S^2_\infty$. Suppose $u_i \colon \mathbb{B} \to \mathbb{H}^3$ is a sequence of conformal H-harmonic maps such that $u_i|_{\partial\mathbb{B}}$ is a parameterisation of Γ_i. Then, a subsequence converges uniformly on compact subsets of $\mathbb{B}$ to a conformal H-harmonic map such that $\partial(u(\mathbb{B})) = \Gamma_\infty$.

We now consider $\Gamma = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^3$, Γ_1 and Γ_2 both being Jordan curves, and we wish to span Γ by an H-annulus. As far as I am aware, the only cases dealt with in the literature require Γ_1 and Γ_2 to be circles. Γ is said to satisfy a Douglas-H condition if there exists $u\colon \mathcal{A} \to \mathbb{H}^3$ such that $u|_{\partial_i \mathcal{A}}$ is a parameterisation of Γ_i and

$$I_{E,H}(u) < \inf I_{E,H,1}(v) + \inf I_{E,H,2}(w)$$

where $v, w\colon \mathbb{B} \to \mathbb{H}^3$, $v|_{\partial \mathbb{B}}$ is a parameterisation of Γ_1 and $w|_{\partial \mathbb{B}}$ is a parameterisation of Γ_2.

Theorem (Cuschieri, M). Suppose $\Gamma = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^3$ satisfies a Douglas-H condition, $H \in [-1, 1]$, then Γ can be spanned by an H-annulus.

As is the case for minimal annuli in $\mathbb{R}^3$, if Γ_1 and Γ_2 are too far apart in terms of a convex hull distance, then Γ cannot be spanned by an H-annulus.

Suitable versions of these theorems continue to hold for $\Gamma \subset S^2_\infty$.

Continuous dependence of conformal H-harmonic maps on data at infinity.

Alexakis and Mazzeo (2008) have studied the moduli space of embedded minimal hypersurfaces in hyperbolic 3-manifolds and renormalised area. Using similar techniques,

Theorem (Cuschieri). Given $\varphi\colon \partial\mathbb{B} \to S^2_\infty$ such that φ is of class $C^{3,\alpha}$ and φ' never vanishes, the Li-Tam harmonic extension u_φ of φ depends continuously on φ.

The conformal distortion of the harmonic map u_φ is measured by the Hopf differential,

$$\Psi_\varphi := \frac{\partial u_\varphi}{\partial z} \cdot \frac{\partial u_\varphi}{\partial z} dz^2, \qquad z = x + is,$$

which is holomorphic.

Let $\mathfrak{B}$ be the space of $\varphi\colon \partial\mathbb{B} \to S^2_\infty$ such that φ' never vanishes, and let $\mathfrak{D}$ be the space of diffeomorphisms of S^1 satisfying a 3-point condition.

Define a conformality operator k on $\mathfrak{B} \times \mathfrak{D}$ by

$$k(\varphi, \omega) := \frac{\partial u_{\varphi\circ\omega}}{\partial z} \cdot \frac{\partial u_{\varphi\circ\omega}}{\partial z}.$$

Theorem (Cuschieri, M). $D_\omega k$ is Fredholm.

$\varphi \in \mathfrak{B}$ is nondegenerate if $(D_\omega k)(\varphi, \text{identity})$ has no kernel.

The space of conformal harmonic maps $u\colon \mathbb{B} \to \mathbb{H}^3$ with nondegenerate boundary values is a manifold. Similar results hold for conformal H-harmonic maps.