Yau-Zaslow Formula

April 6 - 8, 2009

Conference on Geometry
Stefan Banach Intern. Math. Center
in honour of S.-T. Yau.

Naichung Conan Leung 梁迺聰
The Institute of Mathematical Sciences,
The Chinese University of Hong Kong

§ The Formula

X : K3 surface

$C \subset X$ holomorphic curve

$\rightsquigarrow [C] \in H^2(X, \mathbb{Z})$

$\rightsquigarrow \begin{cases} [C]\cdot[C] =: 2d-2 \\ r : \text{divisibility of } [C] \end{cases}$

i.e. $[C] = r\beta \quad \exists \beta \in H^2(X,\mathbb{Z})$
$[C] \neq r'\beta' \quad \forall\, r' > r > 0$

Define $N_g(d, r)$: # genus g curve in X
representing $[C]$

Yau-Zaslow conjectural formula

Yau, Zaslow, BPS states, string duality, nodal curves on K3, Nucl. Phys. B, 1996.

When genus $g = 0$,

$$\sum_{d \geq 0} N_0(d, r) q^d = \prod_{d \geq 1} \left(\frac{1}{1 - q^d} \right)^{24}$$

- Indep. of X, C.
- Indep. of r.
- quasi-modular form. $\frac{q}{\Delta(q)}$

Göttsche-Yau-Zaslow formula

<u>Conjecture</u>: X projective surface
C sufficiently ample divisor

Generating function for, $N_g(X,C)$,

genus g curves in $|C|$ w/ r point constraint:

$$B_1^{c_1^2(X)} \cdot B_2^{-C\cdot c_1(X)} \cdot (DG_2)^r \cdot \frac{D^2 G_2}{(\Delta (D^2 G_2))^{\chi(\mathcal{O}_X)/2}} \qquad \left(D = q\frac{d}{dq}\right)$$

where $G_2(q) = \frac{-1}{24} + \sum_{k>0} \left(\sum_{d|k} d\right) q^k$, $\Delta(q) = q \prod_{k>0} (1-q^k)^{24}$

$B_1(q) = 1 - q - 5q^2 + 39q^3 + \ldots$, $B_2(q) = 1 + 5q + 2q^2 + 35q^3 + \ldots$

- Existence of (non-explicit) formula for ALL surfaces : Universality Conjecture.

- Computing # of curves in specific cases :

When X K3, then G-Y-Z conj. says

$$\sum_d N_g(d, r) q^d$$

$$= q^{-1}\left(\prod_{m=1}^{\infty} \frac{1}{1-q^m}\right)^{24} \left(\sum_{k=1}^{\infty} k\,\sigma(k)\,q^k\right)^g$$

$$= \Delta^{-1}(D\,G_2)^g$$

Counting Number of Curves. Approaches:

(1) Classical (Schubert calculus)

(2) Localization (Mirror theorem. LLY, ...)

Klemm-Maulik-Pandharipande-Scheidegger ⤳ YZ.

(3) Symplectic Invariance (eg. matching, Bryan-L.)

(4) Degeneration Method (eg. Lee-Leung)

(5) Yau-Zaslow Method (eg YZ, Beauville, Li, Wu)

(6) Duality (eg BCOV, GW = SW, Liu)

§ Yau-Zaslow method

String theory considerations.

X : K3 $\Rightarrow$ Calabi-Yau 2-fold.

BPS state : $\begin{cases} \text{holomorphic curve} & C \subseteq X \\ \text{flat line bundle} & \mathbb{C} \to L \to C \end{cases}$

Moduli $\quad \mathcal{M}^{BPS} = \{ (C, L) \}$

$\rightsquigarrow$ Hilbert space $\quad H^{\bullet}(\mathcal{M}^{BPS})$.

Partition function $\quad Z_X = \chi(\mathcal{M}^{BPS})$

$$\chi(\mathcal{M}^{BPS}) = ?$$

- Physical arguments / Alg. geom. argument

Say $C\cdot C = 2d-2 \Rightarrow g(C) = d$

$$\mathcal{M}^{BPS} \underset{\text{birat}}{\sim} \mathrm{Sym}^{[d]} X \qquad \text{(hyperkähler)}$$

$\xRightarrow{\text{Kontsevich}}$ same Euler characteristics

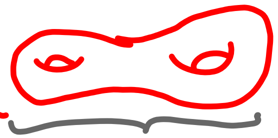
$$\sum_d \chi(\mathcal{M}^{BPS}_d)\, q^d = \sum_d \chi(\mathrm{Sym}^{[d]} X)\, q^d$$

$$= \prod_{m=1}^{\infty} \left(\frac{1}{1-q^m}\right)^{24} \qquad \text{(Göttsche)}$$

On the other hand,

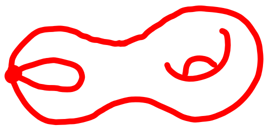
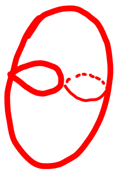
$$\begin{array}{ccc} (C,L) & \in & \mathcal{M}^{BPS} \\ \downarrow & & \downarrow \\ C & \in & |C| \simeq \mathbb{P}^d \end{array} \qquad \text{fiber} = \mathrm{Pic}^0(C)$$

- C smooth ($\Rightarrow g(C)=d$)

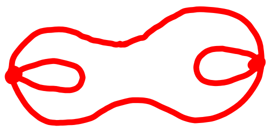
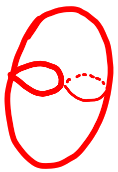
  $\mathrm{Pic}($  $)$ $\overset{\text{topo}}{=\!=}$ T^{2g} $\Rightarrow \chi = 0$

  C

- C 1-node ($g = d-1$)

  $\mathrm{Pic}^0($  $) \simeq$  $\times\, T^{2g-2}$ $\Rightarrow \chi = 0$

- C d-node ($g = 0$)

  $\mathrm{Pic}^0($  $) \simeq ($  $)^g$ $\Rightarrow \chi = 1$

"Suppose" all rational curves are nodal, (in particular $r = \mathrm{index}(C) = 1$), then

$$\chi(\mathcal{M}^{BPS}) = \sum_{C \in \mathbb{P}^d} \chi(\mathrm{Pic}^\circ(C))$$

$$= \sum_{g(C)=0} (+1)$$

$= \#$ rat$^{\ell}$ curves in X.

Hence,

$$\sum_d N_0(d,1)\, q^d = \left(\prod_{m=1}^{\infty} \frac{1}{1-q^m} \right)^{24}$$

X.Chen

§ Family GW for K3 (Bryan-L)

Calabi-Yau surface $\Rightarrow$ K3, Abelian surface
$\mathbb{C}^2/\mathbb{Z}^4$.

$SU(2) = Sp(1)$

i.e. Hyperkähler 4-manifolds.

X Kähler g w/ $Ricci = 0$

Kähler form ω_J, holomorphic volume form $\Omega_J = \omega_I - i\,\omega_K$

$\Rightarrow$ S^2-family of Kähler structures

$a\,\omega_I + b\,\omega_J + c\,\omega_K$ w/ $(a,b,c) \in S^2 \subseteq \mathbb{R}^3$

Twistor family

Generic complex structure on X has NO holomorphic curves!

$\forall\ c \in H^2(X, \mathbb{Z})$ w/ $c \cdot c \geqslant -2$

On each twistor family $\{J_t\}_{t \in S^2}$,

$\exists !\ J_{t_0}$ w/ J_{t_0}-holo. curve $C \subseteq X$

representing $[C] = c \in H^2(X, \mathbb{Z})$

$\Rightarrow$ GW for twistor family, GW^{family},

counts # of J_{t_0}-holo. curves in X.

$GW^{\text{family}}_{X,c,g}$ (pt./ curve/ descendent constraint)

- Indep. of choices of K3, X.

- Indep. of $c \in H^2(X,\mathbb{Z})$ besides

  $c \cdot c = 2d-2$ & $r = \text{index}(c)$
  (divisibility)

Reason: Global Torelli and Large Diffeo. group.

§ Matching method (Bryan-L.)

- Choose K3 X s.t.

$\forall$ holo. $f : \Sigma_g \longrightarrow X$ contributing to GW,

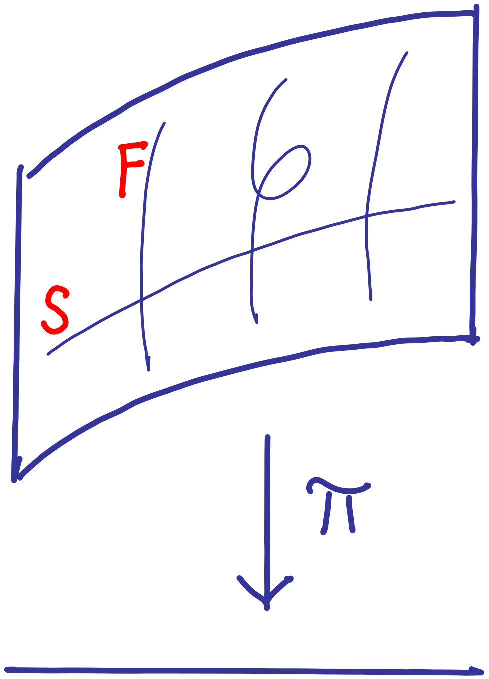
$f(\Sigma_g)$ can be easily identified
(but f can be complicated).

- $\{ f \}$ many components
big dimensions (expected = 0)

- Use invariance of GW to transfer to $\mathbb{P}^2$-blow ups, use Cremona transf.

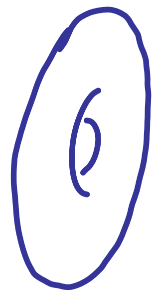
Model: Elliptic K3 w/ section

$$\begin{array}{c} X \\ \big\downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 \end{array}$$



smooth fiber

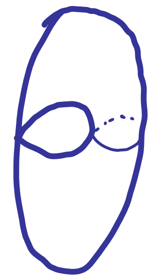
(g = 1)



singular fiber

(# = 24)

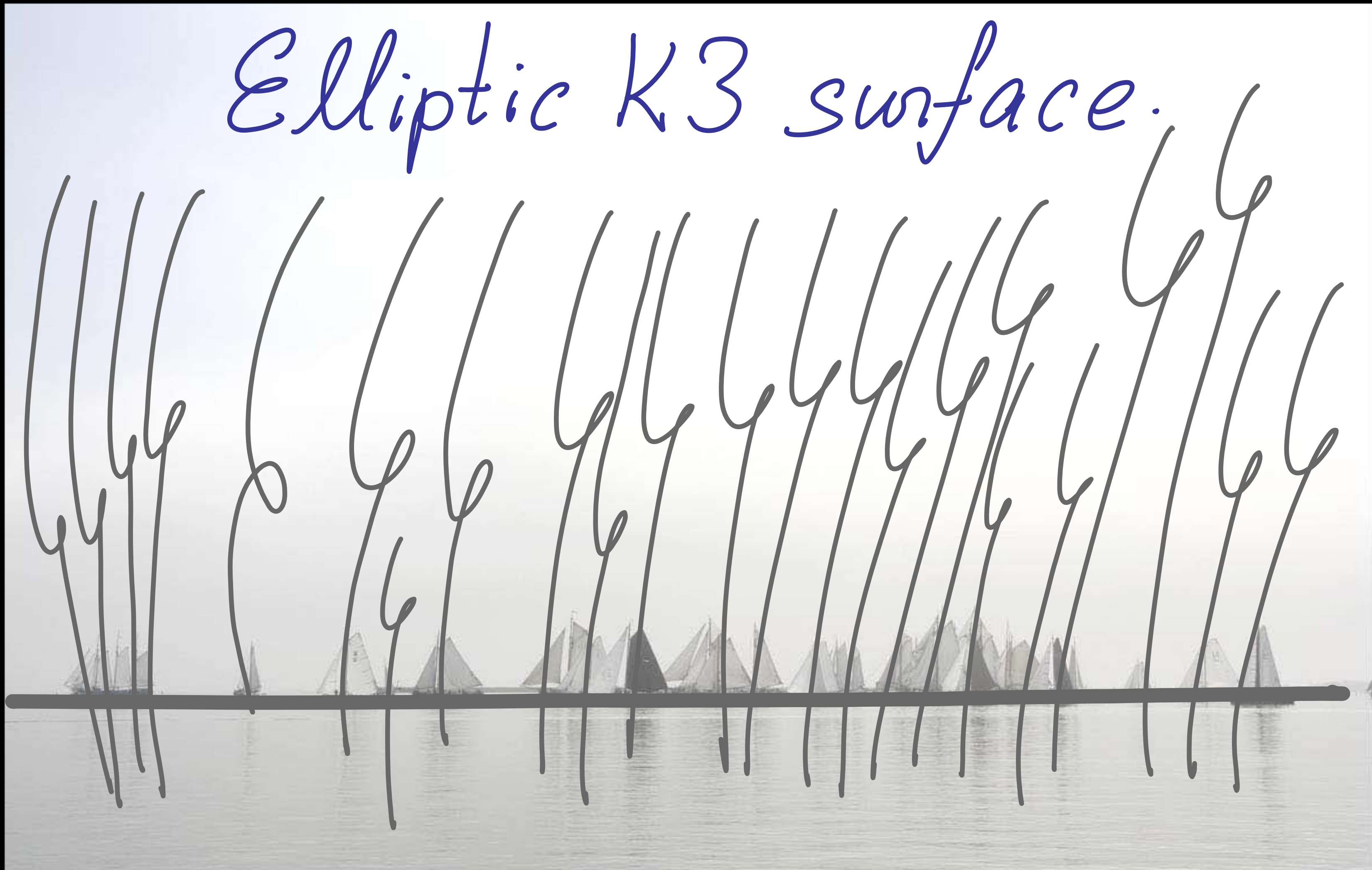
(g = 0)



$$C = S + dF$$

$$\Rightarrow \begin{cases} C \cdot C = 2d - 2 \\ r = \mathrm{index}(C) = 1 \end{cases}$$

$(\because S \cdot S = -2,\ S \cdot F = 1,\ F \cdot F = 0)$


Elliptic K3 surface.

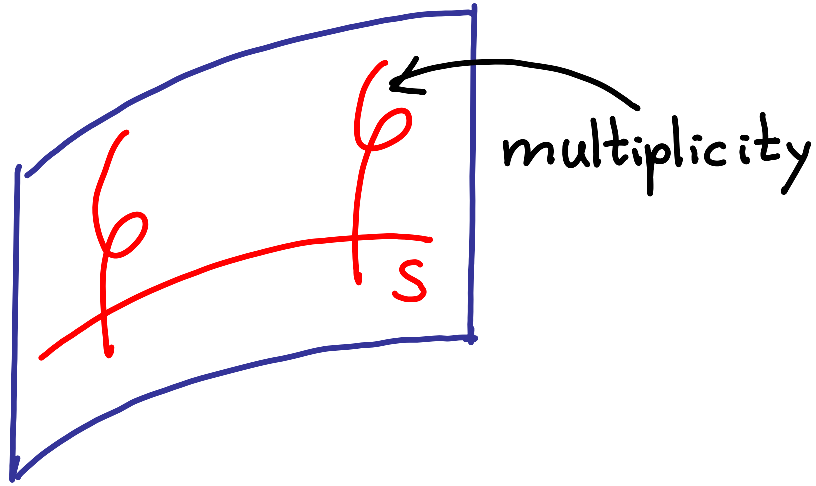
- $|C| \cong \mathbb{P}^d \ni S + F_1 + F_2 + \dots + F_d$

- Generic element $\rightsquigarrow$ genus $g = d$

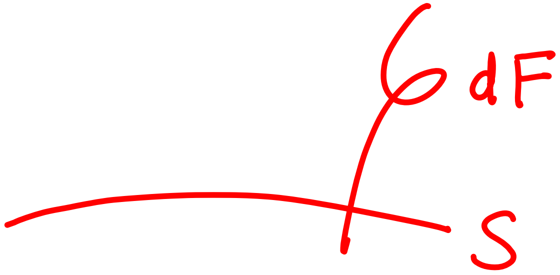
- $g = 0 \Rightarrow$ All F_i's are singular fiber.

- Enough to count

$\Sigma_{g=0} \xrightarrow{f}$

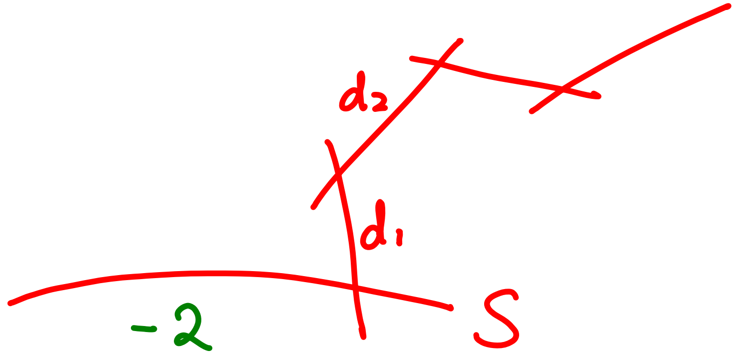


$$\underbrace{\Sigma_{g=0}}_{\substack{\Downarrow \\ \pi_1 = 0}} \xrightarrow{\quad f \quad}$$



$\Rightarrow$ can lift to universal cover

$$\Sigma_{g=0} \xrightarrow{\quad f \quad}$$



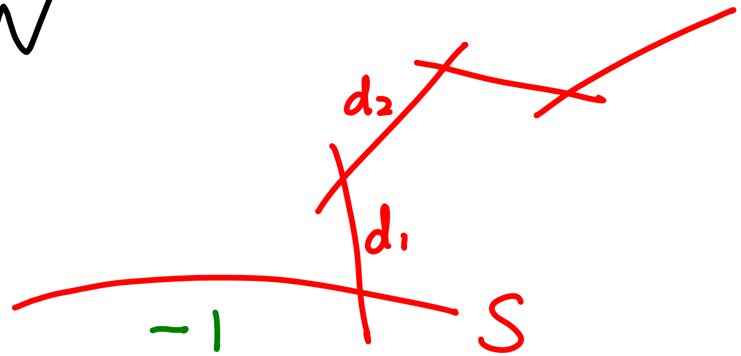
$\Sigma d_i = d$

family GW

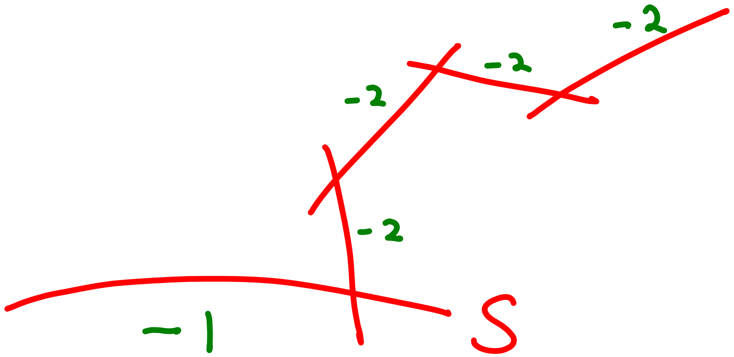
$\Downarrow$

usual GW

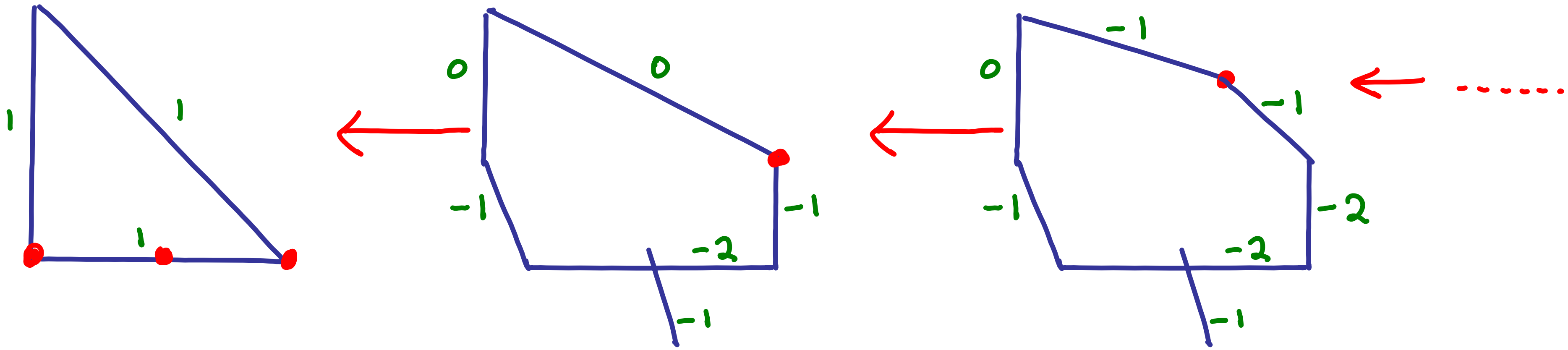
$$\Sigma_{g=0} \xrightarrow{\quad f \quad}$$



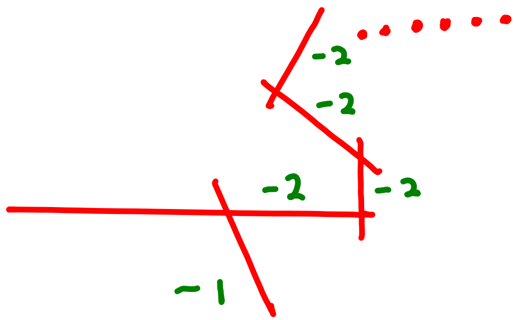
$\Sigma d_i = d$



can be realized in blowups of $\mathbb{P}^2$



$\Longrightarrow$

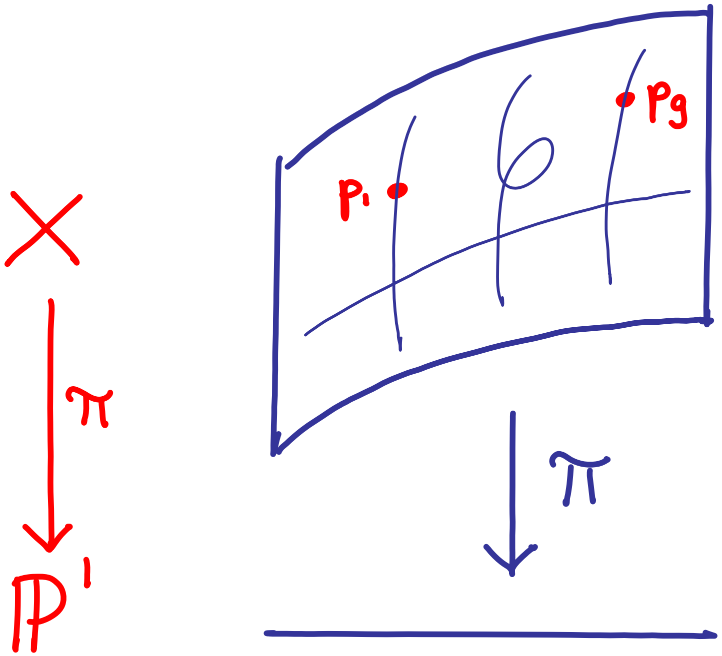


- Use Cremona transformations on $\widetilde{\mathbb{P}^2}$ to reduce to classical counts on $\mathbb{P}^2$ $\rightsquigarrow$ 1 or 0.
  (~ partition function)

$\Longrightarrow$ YZ formula for K3 with $r = 1$.

- Also ALL genus g count ✓

reason: genus g $\Longrightarrow$ impose g point constraints.



- Again "see" all $f(\Sigma_g)$.

- Multi-cover of elliptic curves by other elliptic curves,

$$\rightsquigarrow G_2(q) = \frac{-1}{24} + \sum_k \Big(\sum_{d \mid k} d \Big) q^k$$

Theorem (Bryan-L.) X K3

$C \subseteq X$ w/ $r = \mathrm{index}(C) = 1$

$$\Rightarrow \quad \sum_d N_g(d,1)\, q^d$$
$$= q^{-1}\left(\prod_{m=1}^{\infty} \frac{1}{1-q^m}\right)^{24} \left(\sum_{k=1}^{\infty} k\,\sigma(k)\, q^k\right)^g$$
$$= \Delta^{-1}(D\,G_2)^g$$

Remark: Also work for Abelian surfaces $\mathbb{C}^2/\Gamma$.

How about $r > 1$?

Issues :

(1) For $[C] = [2S + dF]$, we do not "see" all genus 0 curves.

(2) Multiple Cover Contributions

eg. Gathmann $C = 2(S + dF)$

$$GW_{g=0} - N_0 = \frac{1}{2^3} N_0(d, 1)$$

Each rational curve in class C contributes to GW in $[rC]$ by the amount r^{-3}.

§ Degeneration method (Lee-L.)

Degenerate elliptic K3 X

into normal crossing $X \underset{\mathbb{P}^1}{\cup} (\mathbb{P}^1 \times F)$.

elliptic curve

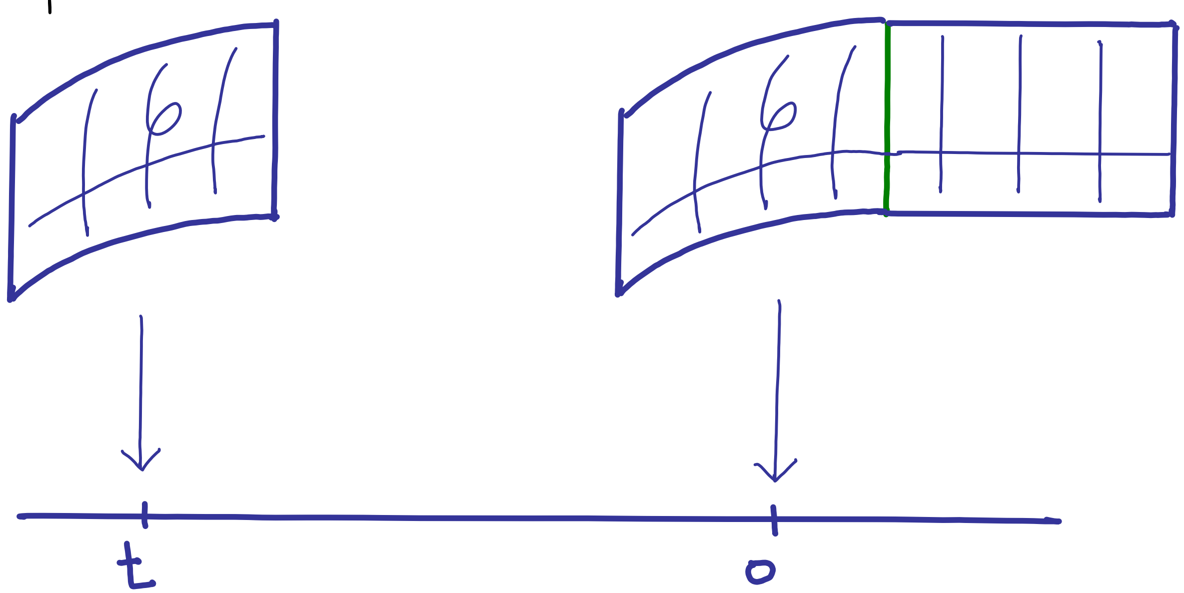
Equivalently, fiber connected sum $X \underset{F}{\#} (S^2 \times T^2)$.

$$\begin{array}{ccc} \mathcal{X} & \supset & X_t \\ \downarrow & & \downarrow \\ D & \ni & t \\ & & |t| < 1 \end{array}$$

$X_t \approx X$

elliptic K3 w/ section

$X_0 \simeq X \underset{F}{\#} (S^2 \times T^2)$



Idea: Use (1) gluing formula and (2) Topological Recursion Relations (TRR) to obtain O.D.E..

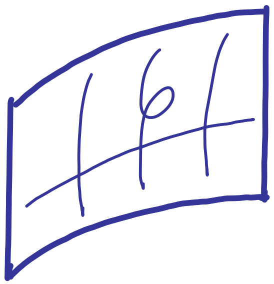
Gluing formula

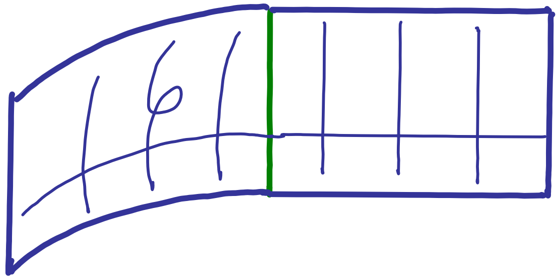
Elliptic K3 w/ section:

$F \;,\; S \subseteq X$

fiber   section

$X \Longrightarrow X \#_f (S^2 \times T^2)$





2 different gluings:

(i) $(s, f) = (2S, F)$

(ii) $(s, f) = (S - 3F, 2F)$ (symplectic non-algebraic)

$*$ $(s + d f) \cdot f = 2$

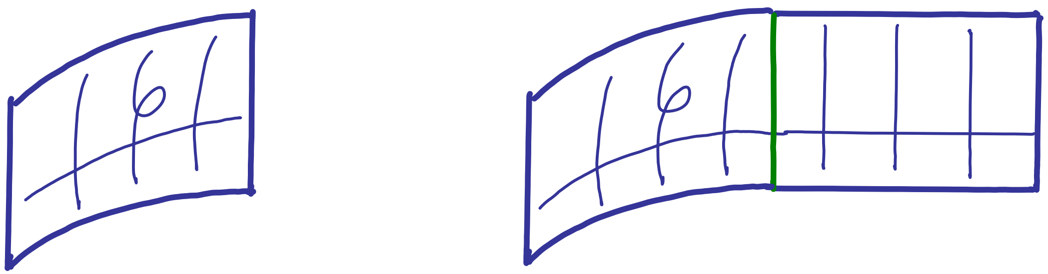
Gluing Formula (for family GW)

(relate Absolute GW to Relative GW)

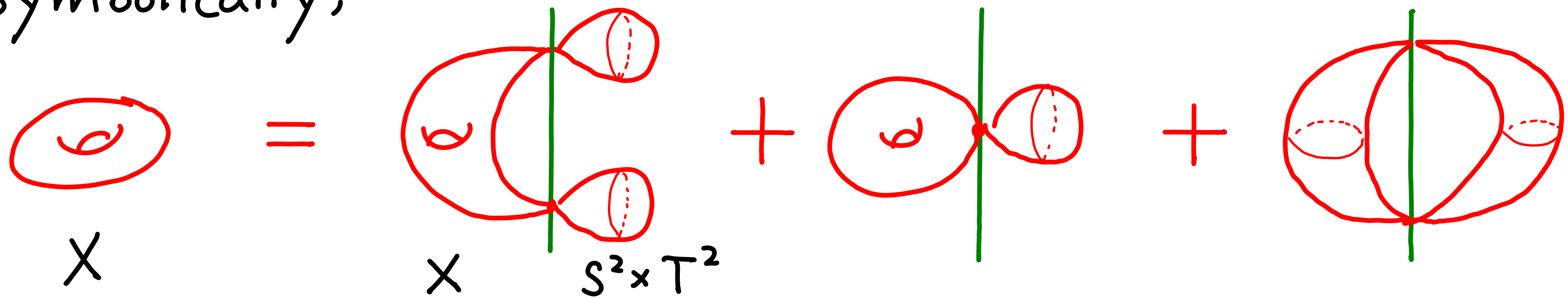
$$GW^{family/X}_{2S+dF,\,g}\left(\tau(F)^k,\ pt.^{g-k}\right)$$

$$= \sum \frac{|S|}{m!}\, GW^{family/X,\ rel.}_{2S+d_1F,\ g_1,\ S}(C_{\gamma_m}) \times GW^{S^2\times T^2,\ rel.}_{2S+d_2F,\ \chi_2,\ S}\left(C_{\gamma_m}^{*};\ \tau(F)^k,\ pt^{g-k}\right)$$

$$X \Longrightarrow X \#_{f} (S^2\times T^2)$$



Symbolically,

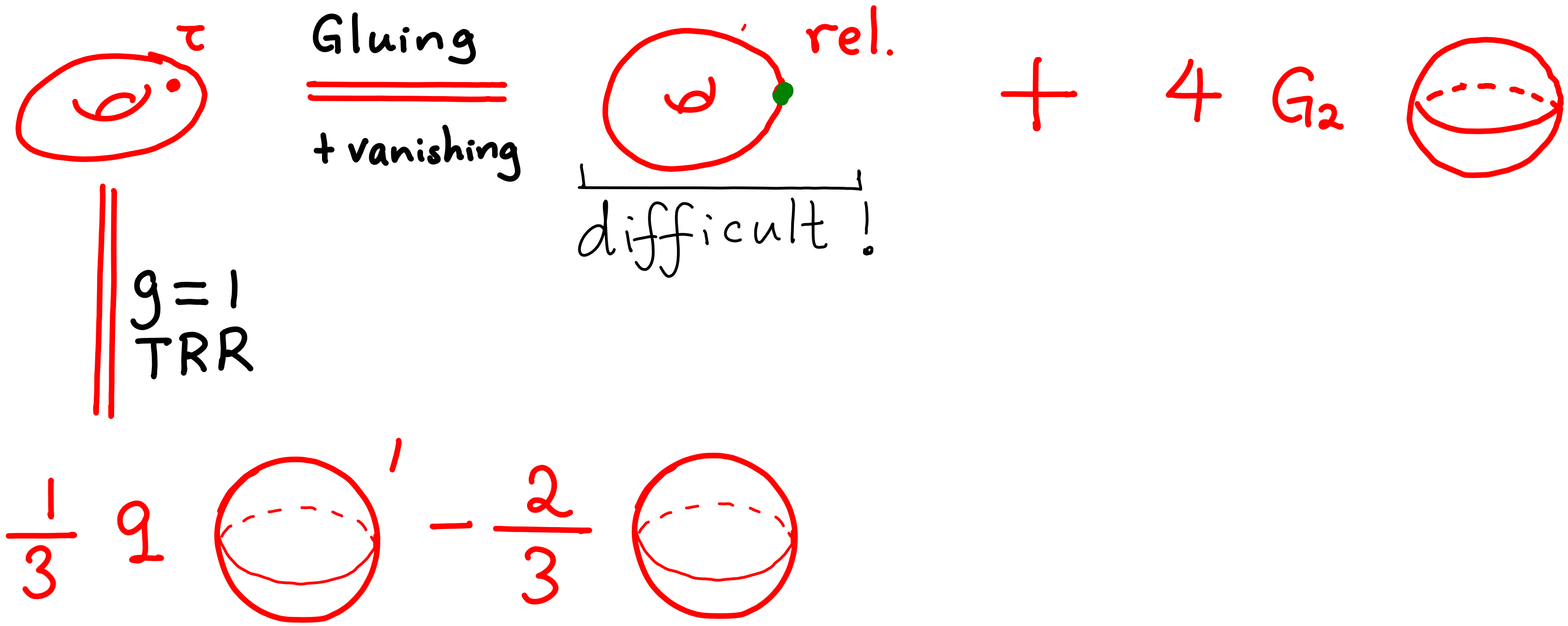


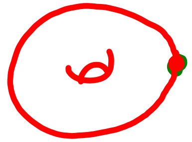
Need

- Point constraints, circle constraints.
- Need descendant constraints.
- Absolute vs relative invariants.
- Various vanishing arguments.

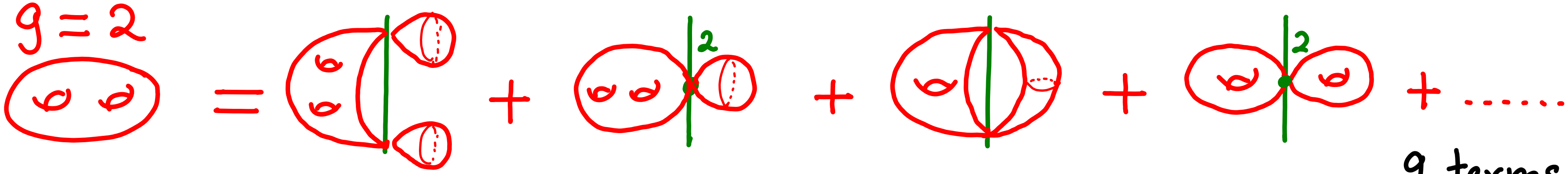
Consider $\sum_d GW^{\text{family}}_{s+df}(\tau)\, q^d$

with (s, f) being (i) $(2S, F)$, (ii) $(S-3F, 2F)$.



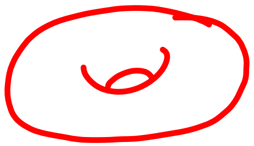
To take care of  $^{\text{rel.}}$, consider $g=2$

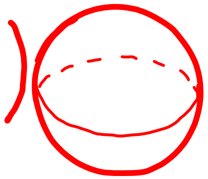
$g=2$

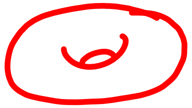
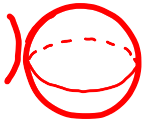


9 terms

• Gluing for  & TRR $\Longrightarrow$

 $-\ 2$ 

$= \frac{20}{3} G_2 \cdot 9$  $' - \left(64 G_2^2 + \frac{40}{3} G_2 - 89 G_2'\right)$ 

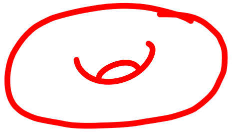
Recall  -2  $= \frac{20}{3} G_2 \cdot q$ $'$ $- \left(64 G_2^2 + \frac{40}{3} G_2 - 8 q G_2'\right)$ 

with

$'$ $= \sum_d GW^{\text{family}}_{s+df,0} \, q^d =: \begin{cases} M(t) & \text{(i) } (s,f) = (2S, F) \\ P(t) & \text{(ii) } (s,f) = (S-3F, 2F) \end{cases}$

$$s + df = \begin{cases} 2S + dF \\ (S-3F) + 2dF \end{cases}$$ (primitive)

$\ast$ Both primitive when $d \in 2\mathbb{Z}+1$

$\xRightarrow{\text{Bryan-L.}}$ $\Big($  -2  $\Big)_{\text{odd}}$ same for (i), (ii).

$\Longrightarrow$ Consider odd terms only,

$$0 = \frac{20}{3} G_{odd}\, q\, (M_{even} - P_{even})' - \left(128\, G_{even} G_{odd} + \frac{40}{3} G_{odd} - 8q\, G'_{even}\right)(M_{even} - P_{even})$$

$$= \frac{20}{3} G_{odd}\, q\, (M \quad - P \quad)' - \left(128\, G_{even} G_{odd} + \frac{40}{3} G_{odd} - 8q\, G'_{even}\right)(M \quad - P \quad)$$

Since $M_{odd} = P_{odd}$ by Bryan-L.

$\rightsquigarrow$ O.D.E.

Matching initial conditions

$\Longrightarrow$ YZ formula w/ $r = 2$. (Lee-L.)

We also obtain $g=1$ formula
by using $g=2$ TRR
and $g=2$ gluing formula.
But more complicated situation.

For $g \geqslant 2$ or $r \geqslant 2$, we need
higher genus TRR and a more clever
way to organize these terms.

§ CY 3-folds method. Klemm-Maulik-Pandharipande-Scheidegger

Proof of YZ formula via Mirror Symmetry.

Royal Łazienki Park, Warsaw.

$M^3 \subset X^4_\Delta$ $\qquad CY^3 \subset$ Fano toric

Mirror theorem (localization) $\Longrightarrow GW^M$ ✓

Assume $M \longrightarrow \mathbb{P}^1$ K3 fibration

$\rightsquigarrow \quad \mathbb{P}^1 \longrightarrow \{K3's\} \simeq \Gamma \backslash O(N,2)/O(N)O(2)$

Fix $[C] \in H^2(X, \mathbb{Z})$. K3 fiber

For the K3 fibration
$$\begin{array}{ccc} M & \longrightarrow & \mathbb{P}^1 \\ \cup & & \rotatebox[origin=c]{90}{$\in$} \\ X_t & \longmapsto & t \end{array}$$

Only finite $t \in \mathbb{P}^1$ w/ $[C] \in H^{1,1}(X_t)$

(for holo. curves to exist on X_t)

Can be determined: Noether-Lefschetz #.

$$\underset{K3}{X_t^2} \longrightarrow M^3_{CY} \longrightarrow \mathbb{P}^1, \quad [C] \text{ fiberwise class.}$$

$$GW^{M_{CY^3}}_{[C],\, g=0} = (\text{Noe-Lef.}\#) \times GW^{X_{K3},\, \mathcal{H}}_{[C],\, g=0}$$

- Need: Singular fibers X_t are mild.
- For $g > 0$, $\sim GW^{X,\mathcal{H}}_g(\lambda_g)$. ← Hodge class.

Explicit example: STU Calabi-Yau 3-fold.

$$\xRightarrow{\text{mirror thm.}} GW^{X,\mathcal{H}}_{[C],\, g=0} \xRightarrow[\text{Harvey-Moore formula (Zagier)}]{\text{express as } \Delta^{-1}} \text{YZ formula}$$

§ Further discussions.

When X K3, then G-Y-Z conj. says

$$\sum_d N_g(d,r)\, q^d = q^{-1}\left(\prod_{m=1}^{\infty}\frac{1}{1-q^m}\right)^{24}\left(\sum_{k=1}^{\infty} k\,\sigma(k)\, q^k\right)^g$$

YZ conj. 96' $g=0$, Göthsche 97' all g.

Bryan-L. 98' $r=1$; all g.

Lee-Leung 05', 06' $r=2$; $g=0,1$.

Beauville 99' + Chen 02' $r=1$, $g=0$.

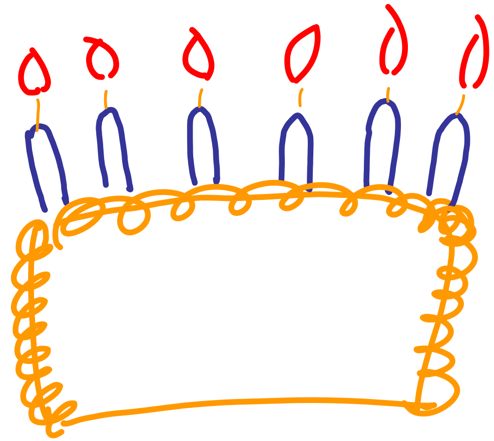
Li-Wu. 07' $r\leq 3$; $g=0$ under nodal assumption

Klemm + Scheidegger + Maulik + Pandharipande 08' All r, $g=0$ i.e. YZ formula.

Liu 00' studies universality conj. using SW=GW.


Happy Birthday to Yau
SALA KONGRESOWA
RESTAURACJA

Happy Birthday
to
You.




Workshop on Geometry, Poland
in honour of Yau's 60th birchday