Integrability in Topological String Theory on Calabi-Yau Manifolds

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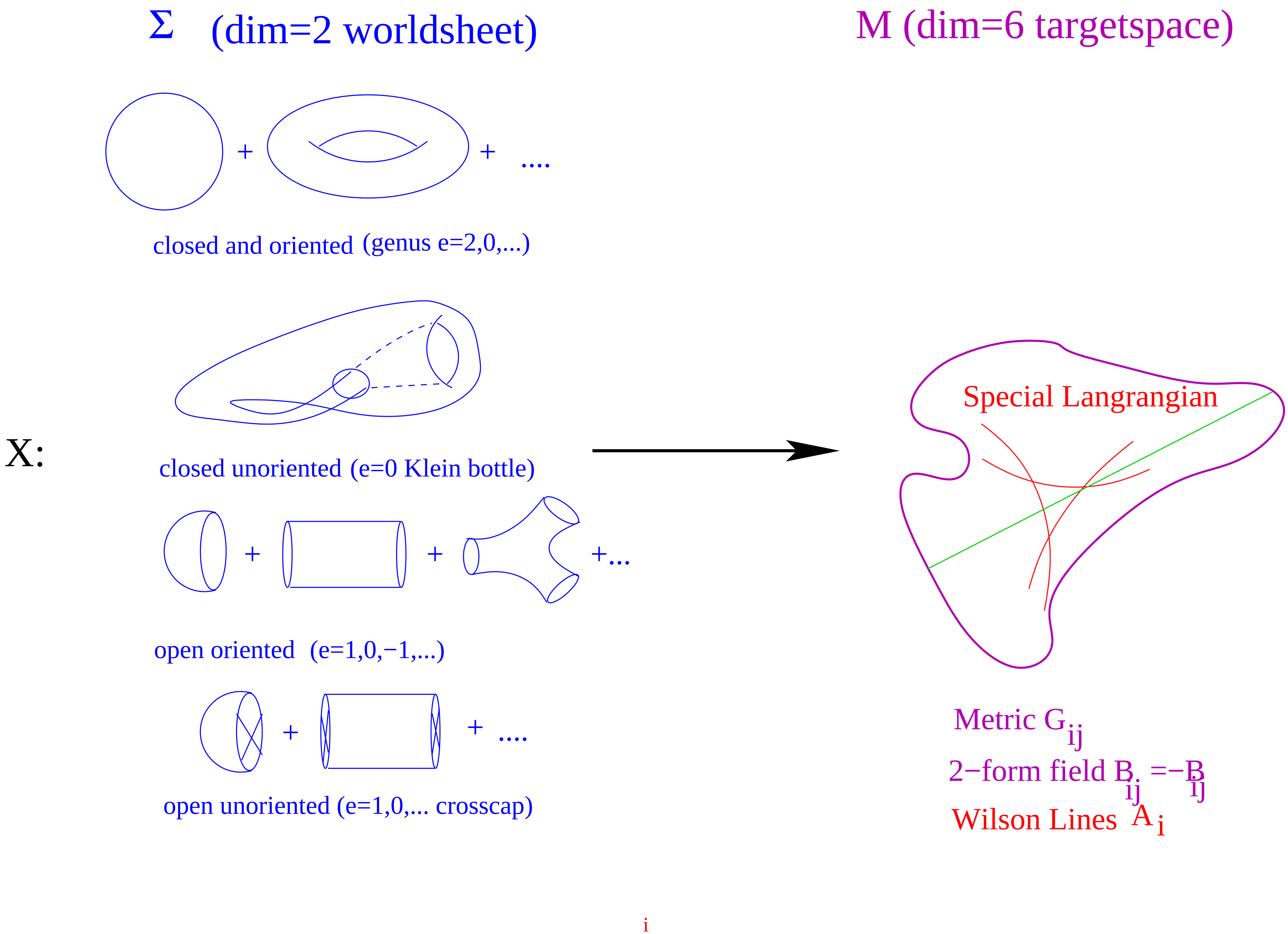

Σ (dim=2 worldsheet)
M (dim=6 targetspace)
+ +....
closed and oriented (genus e=2,0,...)
X:
closed unoriented (e=0 Klein bottle)
+ + +...
open oriented (e=1,0,−1,...)
+ + ...
open unoriented (e=1,0,... crosscap)
Special Langrangian
Metric G_{ij}
2−form field B_{ij} =−B_{ij}
Wilson Lines A_i
i

Topological String Theory

is a truncation to *cohomological string theory*, which eliminates the *oscillator modes* and turns the path integral in a mathematically well defined finite dimensional integral over the moduli space of holomorphic maps.

Consider e.g. the vacuum amplitude Z :

$$Z = \int \mathcal{D}X\mathcal{D}h e^{iS(h,X,G,\dots)},$$

where metric G of M is a background field.

Perturbative string theory has a genus expansion
$X : \Sigma_g \to M$

In critical dimension $\int \mathcal{D}h$ collapses

$$\int \mathcal{D}h \to \sum_g \int_{\overline{\mathcal{M}}_g} \mathrm{d}\mu_{\mathrm{g}}$$

to a sum of finite dimensional integrals over moduli space $\overline{\mathcal{M}}_g$ of Σ_g.

For M Kähler the DX integral localizes in the topological A-model to a finite dimensional integral over the moduli space of the holomorphic maps.

$$\int \mathcal{D}X\mathcal{D}h e^{iS(h,X,G,\dots)} \to \sum_g \sum_\beta \int_{\overline{\mathcal{M}}_g(X,\beta)} c^{vir}(g,\beta)\lambda^{2g-2}q^\beta.$$

This can be seen as a semi-classical approximation, which in the topological A-model is exact. The amplitudes in the topological A-model depend only on the complexified Kähler parameter of M: $\hat{t} = \int_{C_\beta} i\omega + B$ and $q := e^{2\pi i\hat{t}}$.

Formally one can write the Z as an expansion

$$Z(W, \hat{t}) = \exp(F(\lambda, \hat{t})), \quad F = \sum_{g=0} \lambda^{2g-2} F_g(\hat{t})$$

in the is the string coupling λ. However this is an asymptotic expansion in λ!

We can make a large radius expansion $\mathrm{Im}(\hat{t}) \to \infty$ and write a convergent series for the connected vacuum amplitudes

$$F_g(\hat{t}) = \sum_{\beta \in H_2(M,\mathbb{Z})} r_\beta^g e^{2\pi i \hat{t}\cdot\beta} \ ,$$

The finite dimensional integrals are topological in the sense that they depend only on the genus of the curve and the cohomology class of the image.

They are mathematically well defined

$$r_\beta^g = \int_{\mathcal{M}(\beta,g)} c_{vir}(g,\beta) \in \mathbb{Q}$$

and known as Gromov-Witten invariants.

Symplectic invariants closely related to integer invariants such as Donaldson-Thomas and Gopakumar-Vafa invariants $n_\beta^{(g)} \in \mathbb{Z}$.

$$Z(M,\hat{t}) \;= e^{\frac{c(t)}{\lambda^2}+l(t)} \exp\left(\sum_{g=0}^{\infty} \sum_{\beta\in H_2(M,\mathbb{Z})} \sum_{m=1}^{\infty} n_\beta^{(g)} \frac{1}{m} \left(2\sin\frac{m\lambda}{2}\right)^{2g-2} q^{\beta m} \right)$$

The critical Case: Grothendieck-Hirzebruch-Rieman-Roch

$$\dim \overline{\mathcal{M}}_g(M,\beta) = c_1(M)\cdot\beta + (\dim(M) - 3)(1-g) \geq 0$$

Special in this GHRR dimension formula are

- Calabi-Yau manifolds as $c_1(M) = 0$.
- complex 3-folds.
- the genus one amplitude.

as then $\dim \overline{\mathcal{M}}_g(M,\beta) = 0$ $\rightarrow r_g^\beta \neq 0$: a point counting problem sometimes solvable by localization with respect to torus action.

$r_g^\beta \neq 0$ Calabi Yau 4-folds relevant for M/F-theory compactifications

- GHRR $\rightarrow r_g^\beta \neq 0$ only for $g = 0, 1$. This sector is solved in `arXiv:math.ag/0702189` with R. Pandharipande and new integer meeting invariants defined.

Calabi-Yau 3-folds are the critical case.

- GHRR $\rightarrow r_g^\beta \neq 0$, $\forall g$

		non–compact CY(toric)	compact CY (AS toric)
A–model	localisation Pandharipande, Graber, Zaslow, Liu, Katz	✓	g=0 Kontsevich Giventhal,Yau,Lian g>0 ?
	large N duality Vertex Aganagic,Klemm Marino, Vafa	✓	?
	Relative G–W Pandharipande	✓	in principle g small Pandharipande Okounkov, Gathman
B–model	large N duality Matrix model Aganagic,Klemm Marino, Vafa	✓	?
	DT Okounkov, Maulik,Nekrasov, Pandharipande	✓	? Pandharipande, Thomas announced
	Holomorphic anomaly this talk	✓	g=0 Candelas della Ossa, Green,Parkes g small Bershadski,Cecotti, Ooguri,Vaf Katz, Klemm, Vafa g>0 this talk
	KS–H Action BCOV, Pestun, Witten	g =0,1	

Heterotic –II duality

K3–Fiber g=0 KLM, g=1, Harvey, Moore, all g: Gava, Narain, Taylor, Marino, Moore, Klemm, Kreuzer, Riegler, Scheidegger, Grimm Weiss 07
Maulik Pandharipande 07

New Developments:

- Direct integration of the closed sector. Huang, Bouchard, Grimm, Haghighat, Marino, Quakenbush, Rauch, Weiss, AK

- Solution of the open sector for small radius, e.g. at Orbifold point using matrix model. Bouchard, Pasquetti, Marino, AK

- Open string sector on compact Calabi Yau. Walcher, Krefl, Alim, Hecht, Mayr, Jockers, . . .

- The holomorphic anomaly in topological string theory
  - Modularity in Topological String
  - Special Geometry
  - The holomorphic anomaly equation

- The holomorphic anomaly as modular anomaly
  - Ring of almost holomorphic functions
  - Direct integration of the holomophic anomaly equation

- Integrability of the holomorphic anomaly equation
  - The gap condition
  - Applications

Modularity in Topological String Theory

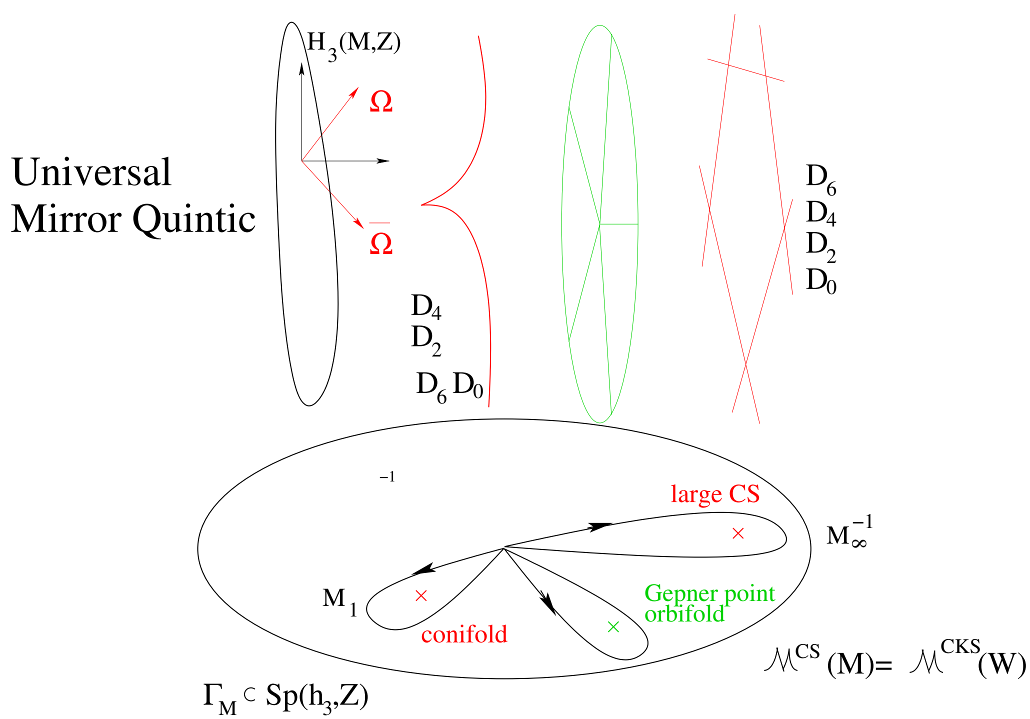
Some invariances $\hat{t} \to \hat{t} + 1$ are clear in this formulation, but full the global monodromy comes from mirror picture.

$$\boxed{Z(M, \hat{t}) = Z(W, t)}$$

Here t is the complex structure parameter of the mirror manifold W: $H^{p,q}(M) = H^{3-p,q}(W)$ and $\hat{t} = t + O(e^{2\pi i t})$ the mirror map.

E.g. for the family of mirror quintics (over $e^{-\frac{t}{5}} \in \mathbb{P}^1$)

$$W = \sum_{i=1}^{5} x_i^5 - e^{-\frac{t}{5}} \prod_{i=1}^{5} x_i = 0 \in \mathbb{P}^4,$$



the global monodromy is generated by

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ -8 & -5 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\infty^{-1} = \begin{pmatrix} -4 & 3 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ 8 & -5 & 0 & 1 \end{pmatrix}.$$

as a discrete subgroup of $\Gamma_M = \mathrm{Sp}(4, \mathbb{Z})$ acting on $H^3(W, \mathbb{Z})$, i.e. on the periods

$$\Pi(t) = \begin{pmatrix} \int_{A^i} \Omega = X^i \\ \int_{B_i} \Omega = P_i = \frac{\partial F_0}{\partial X^i} \end{pmatrix}$$

$\exists\ \Omega \in H^{3,0}(W, \mathbb{Z})$ is defining property of a Calabi-Yau space. T-duality $\Rightarrow Z(W, t)$ invariant under Γ.

Special Kähler Geometry: The moduli space is Kähler with potential K, i.e. $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ given by,

$$\boxed{\exp(K) = i \int \Omega \wedge \bar{\Omega} = -i(P_i \bar{X}^i - \bar{P}_i X^i) \ .}$$

Further we have

$$\boxed{C_{ijk} = \Omega \partial_i \partial_j \partial_k \Omega = D_i D_j D_k \mathcal{F}_0 \ .}$$

Compatibility ($P_i = \frac{\partial F_0}{\partial X^i}$, $\bar{C}_{\bar{l}}^{ij} = e^{2K} \bar{C}_{\bar{k}\bar{l}\bar{m}} G^{\bar{m}i} G^{\bar{n}j}$) implies

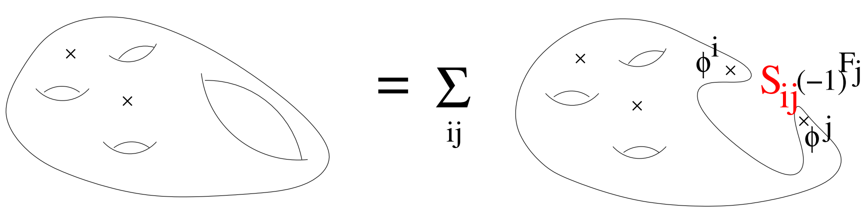
$$\partial_{\bar{l}} \Gamma^i_{km} = R^i_{k\bar{l}m} = \delta^i_k G_{\bar{l}m} + \delta^i_m G_{\bar{l}k} - C_{kmj} \bar{C}^{ij}_{\bar{l}} \ .$$

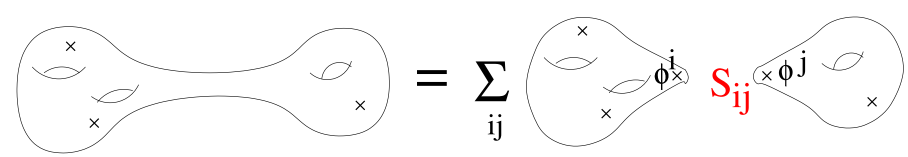
The holomorphic anomaly equations:

World-sheet analysis of Bershadski, Cecotti, Ooguri and Vafa

$$\begin{aligned}\bar\partial_{\bar t_{\bar k}} F_g &= \int_{\overline{\mathcal{M}}(g)} \partial\bar\partial\lambda \\ &= \tfrac{1}{2}\bar C^{ij}_{\bar k}\left(D_iD_jF_{g-1} + \textstyle\sum_{r=1}^{g-1} D_iF_rD_jF_{g-r}\right) .\end{aligned}$$

B-model Parameters are complex structur def. in $\mathcal{M}_{CS}(W)$ of mirror W





Equations come from factorization of *higher genus world-sheets*.

Note that the covariant derivatives are determined from the special Kähler metric, which follows from the genus zero prepotential $\mathcal{F}_0$.

Recursive equations in the genus but leave

- an *holomorphic ambiguity* (functions)
- *s-t modularity* $\longrightarrow$ *modular ambiguity* (discrete data)
- eventually fixed by *gap conditions*.

Implementation of interplay between world-sheet and space-time arguments requires

- an understanding of modular group Γ_M,
- control over the metaplectic transformation property of $Z(W, t, \bar{t})$ under Γ_M.

These ideas apply and are in fact easier explained in non-compact limits, e.g. $\mathcal{O}(-3) \to \mathbb{P}^2$ or $N = 2$ gauge theory limits of type II string compactifications on (M, W).

Geometrically these are decompatification limits of

(M, W), where the compact part of W reduces to a Riemann surface $\mathcal{C}$ and the holomophic $(3,0)$-form Ω reduces to a meromorphic one form λ on $\mathcal{C}$

Local non-compact geometry limit of W

$$v \cdot w = H(x, y, t) \ ,$$

where $v, w \in \mathbb{C}$ and $x, y \in \mathbb{C}^*$. The information about the complex structure is encoded in the periods

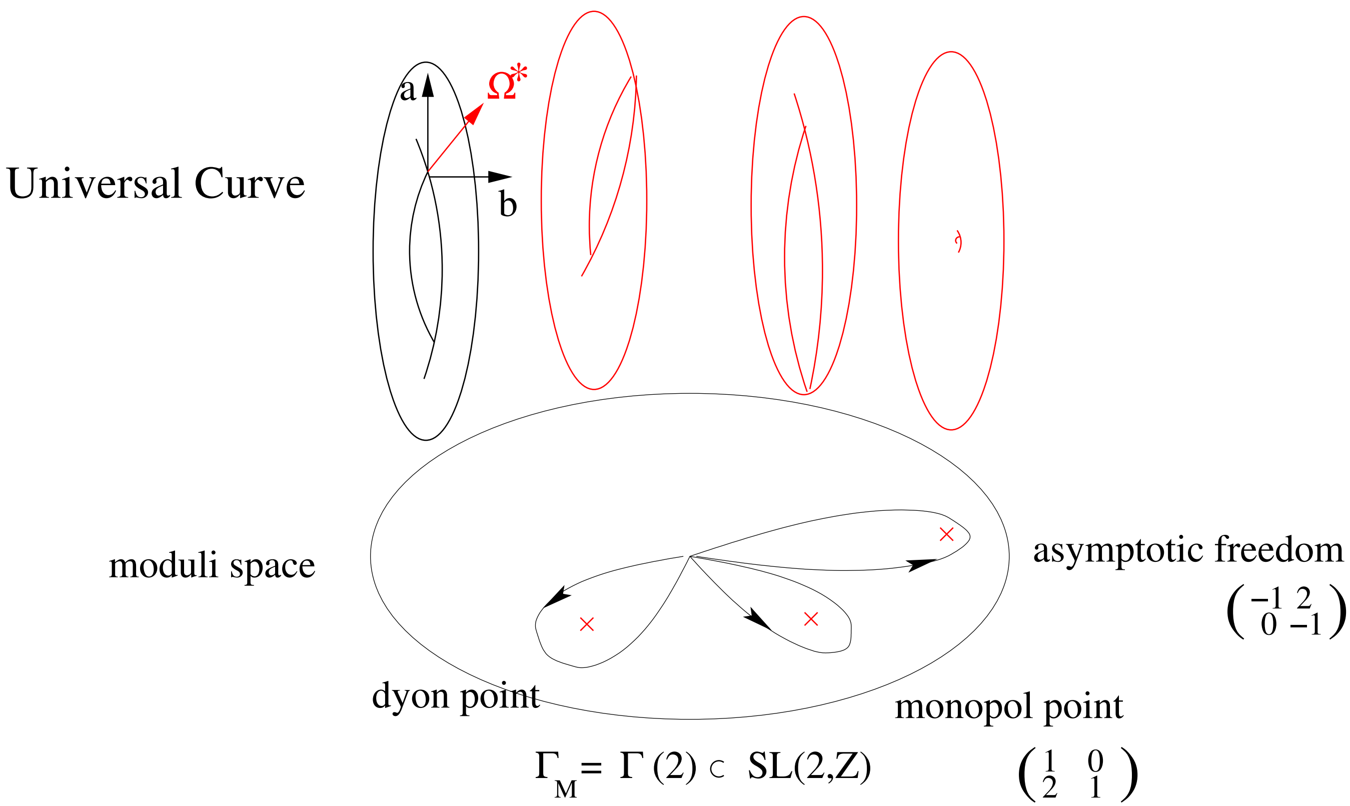
$$\begin{pmatrix} \int_{a^i} \lambda = x^i \\ \int_{b_i} \lambda = p_i = \frac{\partial F_0}{\partial x^i} \end{pmatrix}$$

of the Riemann surface

$$H(x, y, t) = 0$$

with $(a^i, b_i) \in H_1(\mathcal{C}, \mathbb{Z})$ a symplectic basis. $F_0(x^i)$ is the prepotential.

Simplest example for $H(x, y, t) = 0$ is the pure N=2 SU(2) curve. An elliptic curve with $\Gamma(2) \in \mathrm{SL}(2, \mathbb{Z})$ monodromy.

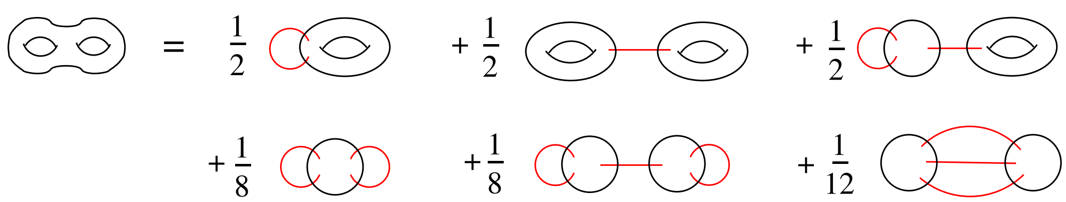


Modularity and WS degenerations:

- $F_g(\tau, \bar{\tau})$ invariant under $\Gamma_M = \Gamma(2)$, e.g.

$$F_1 = -\log(\sqrt{\mathrm{Im}(\tau)}\eta\bar{\eta})$$

- degenerations cap. by Feynmann rules:



- 'Propagator' transforms as form of weight 2 (derivative)

$$\text{———} = S = \frac{\partial}{\partial\tau} 2F_1 = \frac{1}{12}\left(E_2 - \frac{3}{\pi \mathrm{Im}\tau}\right) =: \hat{E}_2$$

- $F_g(\tau,\bar{\tau}) = \xi^{2g-2} \sum_{k=0}^{3(g-1)} \hat{E}_2^k(\tau,\bar{\tau}) c_k^{(g)}(\tau) =: \xi^{2g-2} f_g \;, x$

  where $\xi = \frac{\theta_2^2}{1728\theta_3^4\theta_4^4} = \frac{1}{F^{(0)}_{aaa}}$ is of weight -3.

- Invariance means mathematically

$$\boxed{f_g \in \hat{\mathcal{M}}_{6(g-1)}(\hat{E}_2, \Delta, h)}$$

the *ring* of *almost holomorphic functions* of $\Gamma(2)$ of weight $6(g-1)$ *finitely generated* by

$$(\hat{E}_2, h = \theta_2^4 + 2\theta_4^4, \Delta = \theta_3^4\theta_4^4)\ .$$

Modular origin of the homolomorphic anomaly

Example $\Gamma = PSL(2,\mathbb{Z})$. Ring of modular forms $\mathcal{M}[E_4, E_6]$ generated by E_4 and E_6.

$$\tau \to \tau_\gamma = \frac{a\tau + b}{c\tau + d}$$

$$E_k = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq 0}} \frac{1}{(m\tau+n)^k}$$

$$E_k(\tau_\Gamma) = (c\tau+d)^k E_k(\tau)$$

Converges for $k > 2$. However we need a ring on which we can differentiate. It is easy to see that the differential operator $\frac{d}{d\tau}$ is of weight 2.

$k = 2$ is a borderline case as far as convergence is concerned, which can be regularized

$$E_2 = \frac{1}{2}\sum_{n\neq 0}\frac{1}{2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}$$

Breaks the symmetry

$$E_2(\tau_\Gamma) = (c\tau + d)^2 E_2(\tau) - \pi i c(c\tau + d) \ .$$

But it can be restored by defining

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)} \ .$$

Now $\mathcal{M}[\hat{E}_2, E_4, E_6]$ is a ring of almost holomorphic forms on which we can differentiate!

Direct integration:

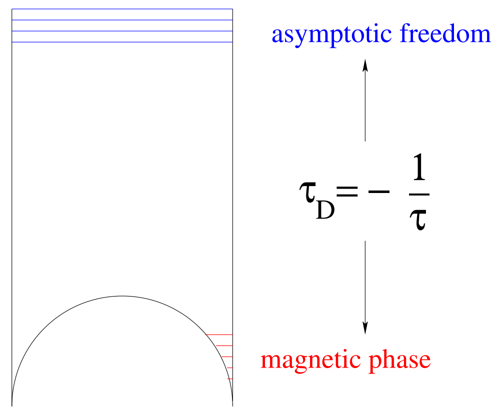
The only antiholomorphic dependence is in the $S \propto \hat{E}_2$:

$\frac{\partial}{\partial\bar\tau} \longrightarrow \frac{\partial}{\hat{E}_2}$:

$$\begin{aligned} \frac{1}{24^2}\frac{\mathrm{d}}{\mathrm{d}\hat{\mathrm{E}}_2} f_g &= d_\xi^2 f_{g-1} + \frac{1}{3}\frac{(\partial_\tau \xi)}{\xi} d_\xi f_{g-1} + \sum_{r=1}^{g-1} d_\xi f_r d_\xi f_{g-r}, \\ \text{with} \quad d_\xi f_k &= \partial_\tau f_k + \frac{k}{3}\frac{(\partial_\tau \xi)}{\xi} f_k \quad \text{Serre operator} \end{aligned}$$

- Only the degree 0 part in $\hat{E}_2$ remains undetermined. Ambiguity is a holomorphic modular form $c_0^{(g)}(\tau) \in \mathcal{M}_{6(g-1)}(\Delta, h)$.
- $\dim\left(\mathcal{M}_{6(g-1)}(h,\Delta)\right) = \left[\frac{3g}{2}\right]$ number of required boundary conditions

Global properties:



$$\mathbb{F}(\Gamma(2))$$

$$F_g^D(\tau_D, \bar{\tau}_D) = F_g(-\frac{1}{\tau_D}, -\frac{1}{\bar{\tau}_D})$$

- ST-instanton expansion

$$\mathcal{F}_g(\tau(a)) = lim_{\bar{\tau}\to\infty} F_g(\tau, \bar{\tau})$$

- Strong-coupling expansion

$$\mathcal{F}_g^D(\tau_D(a_D)) = lim_{\bar{\tau}_D\to\infty} F_g^D(\tau_D, \bar{\tau}_D)$$

Can be seen as metaplectic transformation on $\Psi = Z$

The strong coupling gap :

$$\mathcal{F}_g^D \;=\frac{B_{2g}}{2g(2g-2)a_D^{2g-2}}+\ldots+k_1^{(g)}a_D+\mathcal{O}(a_D^2)$$

$$\uparrow$$

$$2g-2 \text{ independent vanishing conditions}$$

$$\boxed{2g-2>\left[\frac{3g}{2}\right]}$$

- theory completly solved

Why the Gap ?

- Dijkgraaf & Vafa: SW is described by a matrix model: Typical in MM is a pole $\frac{1}{s^{2g-2}}$ from the measure followed by a regular perturbative expansion.

- String LEEA explanation: $F(\lambda, t)$ graviphoton couplings given by Schwinger-Loop calculation Antoniadis, Gava, Narain, Taylor, Gopakumar, Vafa. For one HM at conifold Strominger t_D mass of HM

$$F(\lambda, t_D) = \int_{\epsilon}^{\infty} \frac{\mathrm{d}s}{s} \frac{e^{-st_D}}{4\sin^2(s\lambda/2)} = \sum_{g=2}^{\infty} \left(\frac{\lambda}{t_D}\right)^{2g-2} \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \; .$$

Compact Calabi-Yau HKQ

$$W = \sum_{i=1}^{5} x_i^5 - j_q^{\frac{1}{5}} \prod_{i=1}^{5} x_i = 0 \in \mathbb{P}^4,$$

Properties of Γ_M, even if of finite index unknown, but we can build modular objects using the periods $\Pi(z) = \int_\Gamma \Omega(z)$ fullfilling

$$[\theta^4 - 5 j_q^{-1} \prod_{i=1}^{4} (\theta + i)] \, \Pi(z) = 0, \qquad \theta := -j_q \frac{\mathrm{d}}{\mathrm{dj_q}} \; .$$

E.g. from the mirror map an analog of j-function,

$$q = \exp(\int_C \omega) = \exp(\Pi_1(j_q)/\Pi_0(j_q))$$

$$\begin{aligned} j_q &= \tfrac{1}{q} + 770 + 421375\, q + 274007500\, q^2 + 236982309375\, q^3 + \ldots \\ (j_e &= \tfrac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \ldots) \end{aligned}$$

The generators of the ring of almost holomorphic modular (tensor) forms of Γ_M for Calabi-Yau are not known, but Yau, Yamaguchi `hep-th/0406078`, showed following BCOV,KKV that $P_g = \xi^{g-1} F_g$, where $\xi = \frac{j_q}{1-j_q} = j_q X$ can be written as polynomials in 3

an-holomophic and one holomorphic generator

$$A_p \;:= \frac{(j\partial_j)^p G_{j,\bar{j}}}{G_{j\bar{j}}}, \quad B_p := \frac{(j\partial_j)^p e^{-K}}{e^{-K}}, \quad p = 1, \ldots$$
$$C : \; = C_{jjj} \dot{j}^3, \quad X = \frac{1}{1-j}$$

- Special geometry & Picard-Fuchs eq. truncate to A_1, B_1, B_2, B_3, X.

- One combination does not appear in $P_g = C^{g-1} F_g$. $B_1 = u$, $A_1 = v_1 - 1 - 2u$, $B_2 = v_2 + uv_1$, $B_3 = v_3 - uv_2 + uv_1 X - c_1 uX$

- The P_g are degree $3g - 3$ weighted inhomogeneous polynomials in v_1, v_2, v_3, X,

- hol. anom. eq.

$$(\partial_{v_1} - u\partial_{v_2} - u(u+X)\partial_{v_3})\, P_g = -\frac{1}{2}\left(P^{(2)}_{g-1} + \sum_{r=1}^{g-1} P^{(1)}_r P^{(1)}_{g-r}\right)$$

From regularity at the Gepner point, the *leading singular behaviour* of the F_g at the conifold $j_q = 1$ and regularity at the large CS, we conclude that the ansatz for the *holomorphic* and *modular ambiguity* is given by

$$\boxed{c_0^{(g)} = \sum_{i=0}^{3g-3} a_i X^i}$$

Boundary conditions:

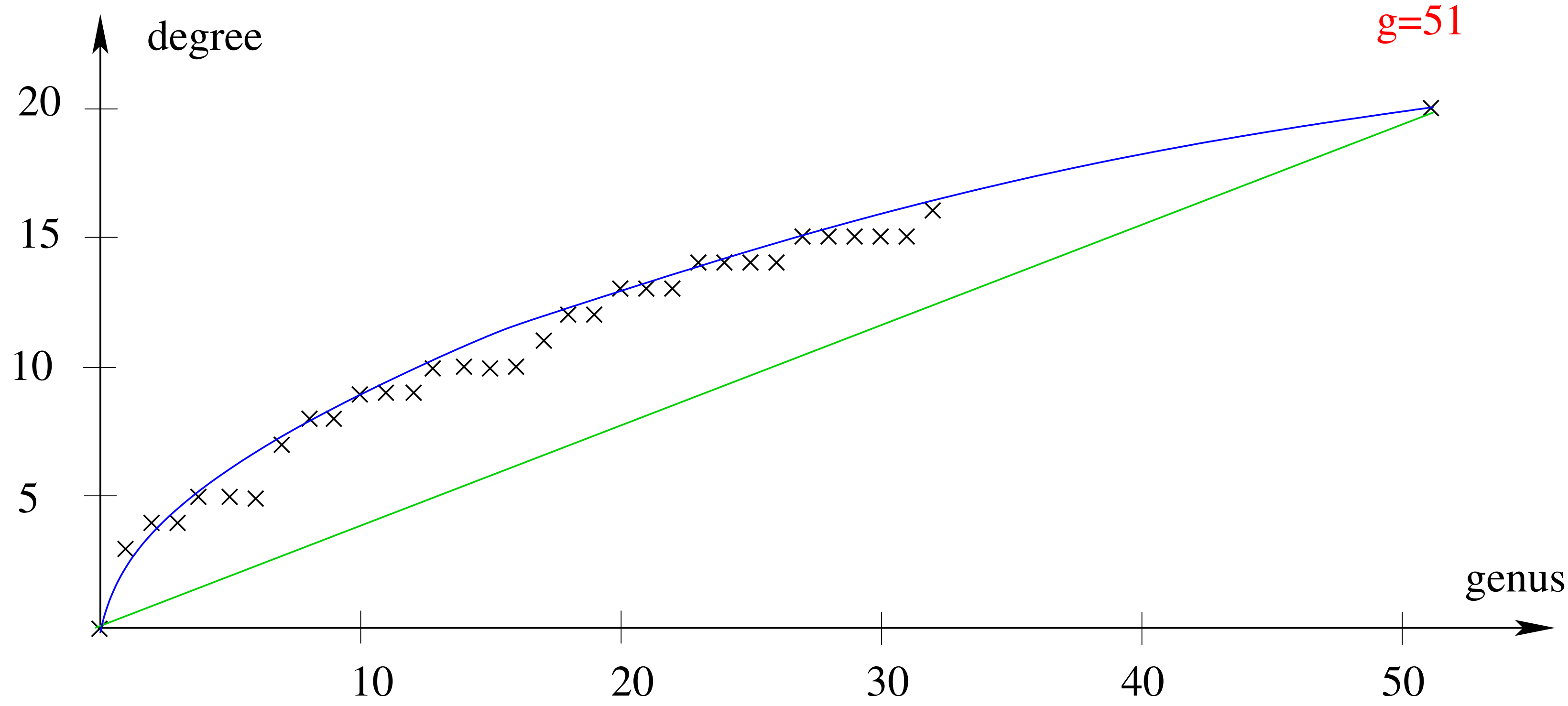
- Gap at the conifold $j = 1$

$$\mathcal{F}_g^D = \frac{B_{2g}}{2g(2g-2)t_D^{2g-2}} + k_g^1 + \mathcal{O}(t_D)$$

provides $2g - 2$ conditions.

- Regularity at Gepner point $j = 0$ provides $\left[\frac{3(g-1)}{5}\right]$ conditions $\rightarrow \left[\frac{2(g-1)}{5}\right]$ unkowns.

- Castelnouvo's bound for GV invariants at large radius. From aqjunction formula in $\mathbb{P}^4$ ones find there are no genus g curves for $d \leq \sqrt{g}$



genus	degree=18
0	144519433563613558831955702896560953425168536
1	491072999366775380563679351560645501635639768
2	826174252151264912119312534610591771196950790
3	866926806132431852753964702674971915498281822
4	615435297199681525899637421881792737142210818
5	306990865721034647278623907242165669760227036
6	109595627988957833331561270319881002336580306
7	28194037369451582477359532618813777554049181
8	5218039400008253051676616144507889426439522
9	688420182008315508949294448691625391986722
10	63643238054805218781380099115461663133366
11	4014173958414661941560901089814730394394
12	166042973567223836846220100958626775040
13	4251016225583560366557404369102516880
14	61866623134961248577174813332459314
15	451921104578426954609500841974284
16	1376282769657332936819380514604
17	1186440856873180536456549027
18	2671678502308714457564208
19	-59940727111744696730418
20	1071660810859451933436
21	-13279442359884883893
22	101088966935254518
23	-372702765685392
24	338860808028
25	23305068
26	-120186
27	-5220
28	-90
29	0

Summary

- The holomorphic anomaly equations, modularity and suitable boundary conditions allow to solve:
  - closed topological string on non-compact Calabi-Yau completly. Application: Geometrical engineering of supersymmetric gauge theories
  - closed topological string on compact Calabi-Yau to very high genus. Application: Black hole microstate counting

- Another advantage of the approach is that is gives analytic expressions for amplitudes everywhere in the moduli space: Large radius $\sim$ *symplectic invariants*, Orbifold point $\sim$ *marginal deformation of Gepner model*, Conifold point $\sim c = 1$ *string*, other singularities $\sim \ldots$