Boundedness and decay for the wave equation on Kerr and other stationary axisymmetric black hole backgrounds

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The dynamics of the vacuum equations

General relativity concerns four-dimensional Lorentzian manifolds $(\mathcal{M}, g)$ satisfying the *Einstein equations*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{1}$$

where the tensor $R_{\mu\nu}$ here denotes the Ricci curvature of g, and $T_{\mu\nu}$ is the so-called energy momentum tensor of matter. In the vacuum case $T_{\mu\nu} = 0$, we obtain the *Einstein vacuum equations*

$$R_{\mu\nu} = 0. \tag{2}$$

These equations are purely geometric and they are of hyperbolic character. The system (2) has a well-posed initial value problem and the unique solution $(\mathcal{M}, g)$ corresponding to Cauchy data $(\Sigma, \bar{g}, K)$ is known as the *maximal Cauchy development* of the data.

Special solutions I: Minkowski space

It was Minkowski's fundamental insight (c. 1908) that the physical content of Einstein's previous "special" relativity could be reformulated as the statement that the equations of physics should have a geometric expression on the four dimensional $\mathbb{R}^4$ with metric

$$-dt^2 + dx^2 + dy^2 + dz^2.$$

This was the beginning of the space-time concept. We now call the above spacetime *Minkowski space* $\mathbb{R}^{3+1}$, and it is of course the simplest special solution of (2).

Special solutions II: Schwarzschild

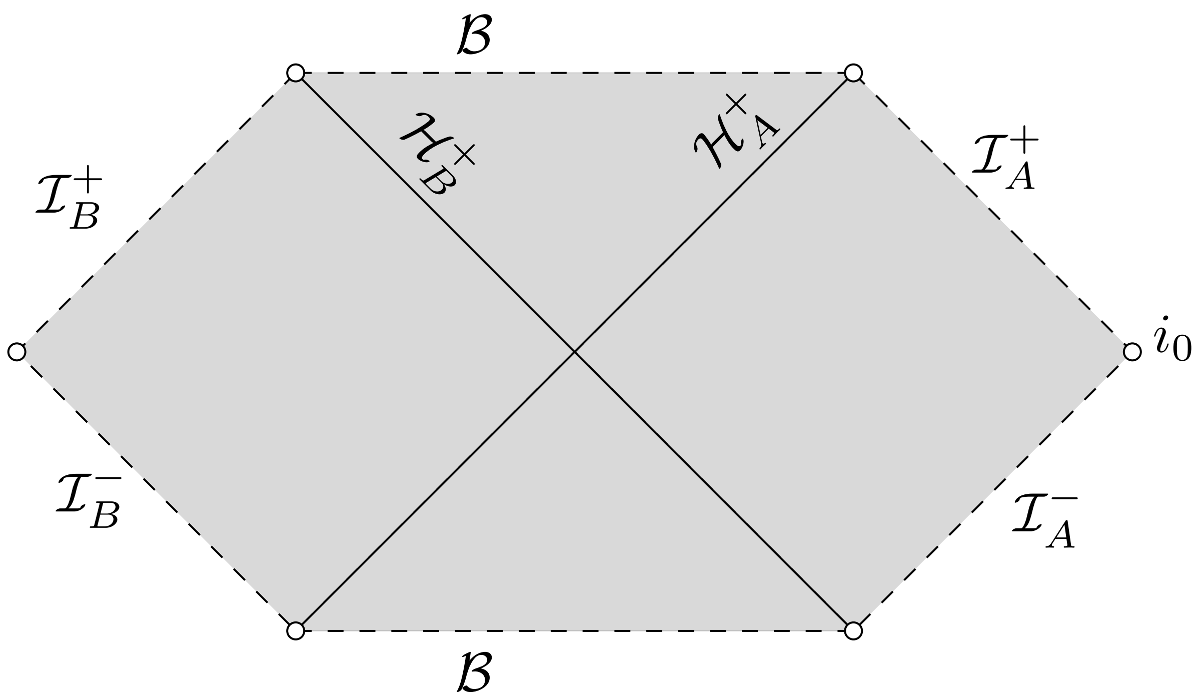
The *Schwarzschild solutions* form a one parameter family $(\mathcal{M}, g_M)$ of spherically symmetric, "static" solutions of the vacuum equations (2). They were discovered in 1916. In local coordinates, the metric element takes the form

$$-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2d\sigma_{\mathbb{S}^2}$$

where M is the parameter.

Originally, the above metric was always interpreted as the vacuum region outside a star of "mass" M. Removing the star and extending the above metric to as large a spacetime as possible leads to the black hole concept.

We now use the term "Schwarzschild solution" to denote a maximal extension of the original metric, defined by Synge and Kruskal. The quotient $\mathcal{Q} = \mathcal{M}/\mathrm{SO}(3)$ can be represented by the Penrose diagram:



We call the region $\mathcal{Q} \setminus J^-(\mathcal{I}^+) = \mathcal{Q} \setminus J^-(\mathcal{I}_A^+ \cup \mathcal{I}_B^+)$ the *black hole region.*

Special solutions III: The Kerr solution

The *Kerr metric* is a 2-parameter family of metrics first discovered in 1963. The parameters are called *mass* M and specific angular momentum a, i.e. angular momentum per unit mass. In so-called Boyer-Lindquist local coordinates, the metric element takes the form:

$$-\left(1-\frac{2M}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right)dt^2+\frac{1+\frac{a^2\cos^2\theta}{r^2}}{1-\frac{2M}{r}+\frac{a^2}{r^2}}\,dr^2$$
$$+r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right)d\theta^2-4M\frac{a\sin^2\theta}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\,dt\,d\phi$$
$$+r^2\left(1+\frac{a^2}{r^2}+\left(\frac{2M}{r}\right)\frac{a^2\sin^2\theta}{r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right)\sin^2\theta\,d\phi^2.$$

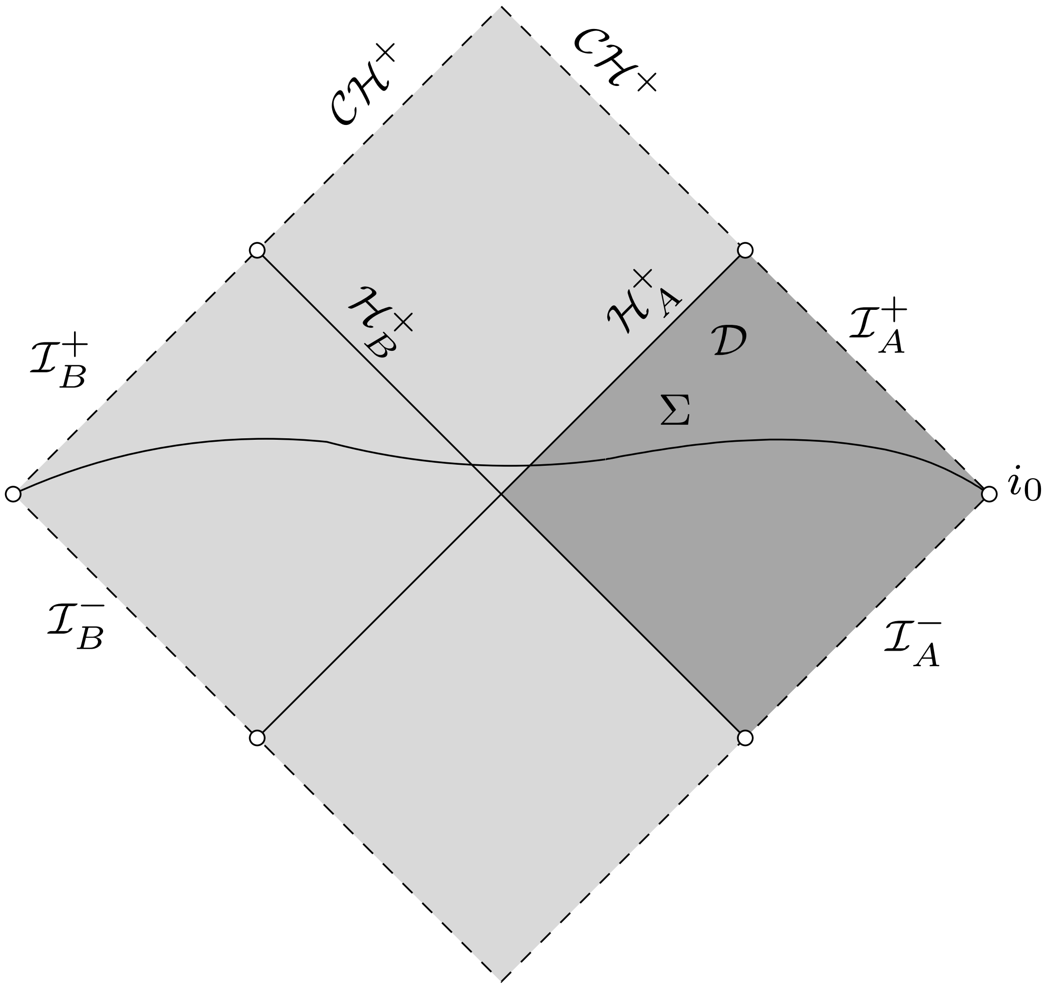
If $a=0$, the metric reduces to Schwarzschild.

Kerr geometry of the exterior

For $0 < a < M$, the Killing fields are now spanned by ∂_t and ∂_ϕ. Thus, the Kerr solution is not spherically symmetric, only axisymmetric. Also, since ∂_t is not hypersurface orthogonal, we do not call it static but "stationary". The spacetime has again a black hole region, but the Killing field ∂_t is not everywhere timelike in the exterior: There is a region $\mathcal{E} = \{x : g(\partial_t, \partial_t) > 0\}$. This is the so-called *ergoregion*.

By considering a particle "process" where a particle coming from infinity splits in two in the region $\mathcal{E}$, Penrose suggested a mechanism where energy could be extracted from a black hole. Christodoulou later discovered a notion of irreversibility in this process. With later developments by Bekenstein, Carter and especially Hawking, this has led to a subject known as "black hole thermodynamics".

Penrose diagram for Kerr



Stability of Minkowski space

Special solutions are relevant only if they are stable!

Theorem. *(Christodoulou-Klainerman, 1993) Consider an asymptotically flat initial data set* $(\Sigma^3, \bar{g}, K)$ *for the Einstein vacuum equations* (2) *sufficiently close in a suitable sense to Minkowski initial data. Then the maximal Cauchy development* $(\mathcal{M}, g)$ *has qualitative behaviour similar to Minkowski space and tends to it in a suitable way. In particular, it is geodesically complete, a notion of* $\mathcal{I}^+$ *can be defined with associated Bondi mass,* $\mathcal{I}^+$ *is "complete" and the Bondi mass tends to zero along* $\mathcal{I}^+$*, and* $\mathcal{M} = J^-(\mathcal{I}^+)$*.*

New developments: *Proof with harmonic gauge*: Lindblad–Rodnianski, 2004; *Higher dimensions*: Chrusciel–Choquet-Bruhat–Loizelet, *Einstein–Maxwell*: Zipser, 2001; *Low regularity and weak asymptotics*: L. Bieri, 2007

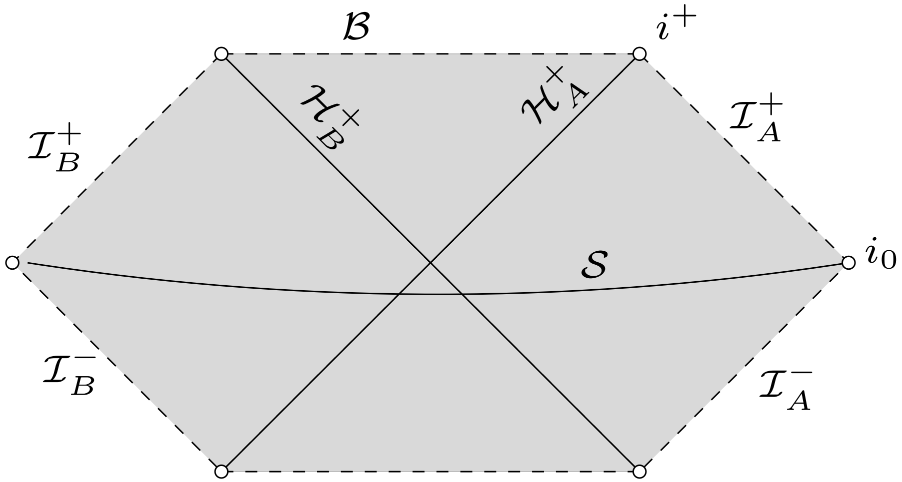
The nonlinear stability problem of Kerr

Conjecture. *Let* $(\Sigma, \bar{g}, K)$ *be a vacuum initial data set sufficiently close to the initial data on a Cauchy hypersurface in the Kerr solution* $(\mathcal{M}, g_{M_i,a_i})$ *for some parameters* $0 \leq |a_i| < M_i$*. Then the maximal vacuum development* $(\mathcal{M}, g)$ *possesses a complete null infinity* $\mathcal{I}^+$ *such that the metric restricted to* $J^-(\mathcal{I}^+)$ *approaches a Kerr solution* $(\mathcal{M}, g_{M_f,a_f})$ *in a uniform way with quantitative decay rates, where* M_f*,* a_f *are near* M_i*,* a_i *respectively.*

Note: $a_i = 0$ **will not imply** that $a_f = 0$, only that a_f is small. One thus cannot talk about the "stability of Schwarzschild"...

The above conjecture is a central unsolved problem in general relativity. There has been a lot of recent activity on the underlying linear theory necessary to address the above conjecture. The rest of this lecture will concern this problem.

Let $(\mathcal{M}, g)$ be Schwarzschild or Kerr or more generally, a spacetime **"near"** one of the above:

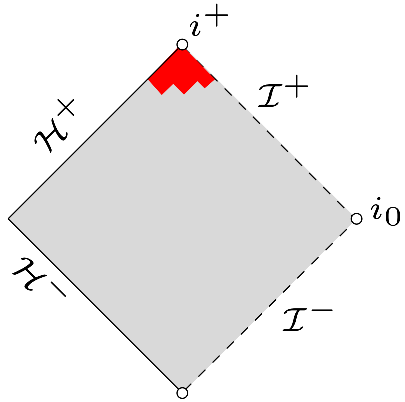


- $\mathcal{S}$ an arbitrary Cauchy surface for $(\mathcal{M}, g)$,
- We study ψ, the unique solution to the wave equation

$$\Box_g \psi = 0$$

on $(\mathcal{M}, g)$, with sufficiently regular initial data ψ, ψ' prescribed on $\mathcal{S}$; **no symmetry** assumed on ψ, ψ'.

The problem then is to understand boundedness and decay properties of ψ in $J^+(\mathcal{I}_A^-) \cap J^-(\mathcal{I}_A^+)$ near i^+.



We call $J^+(\mathcal{I}_A^-) \cap J^-(\mathcal{I}_A^+)$ a *domain of outer communications*. Let's denote its closure in $\mathcal{M}$ by $\mathcal{D}$.

Uniform boundedness on Schwarzschild

The first and most elementary result of the type to be described in this lecture is the following:

Theorem. *(Kay–Wald, 1986) Let* $(\mathcal{M}, g_M)$ *be Schwarzschild,* $\mathcal{D}$ *as above the closure of its domain of outer communications,* Σ *a Cauchy surface for* $\mathcal{M}$ *and* ψ *the unique solution of the wave equation on* $\mathcal{M}$ *with sufficiently regular initial data* ψ, ψ' *on* Σ*, decaying appropriately near* i^0*. Then there exists a* D *depending only on the data such that*

$$|\psi| \leq D$$

holds in $\mathcal{D}$*.*

Energy currents constructed from vector field multipliers

The proof of the above theorem and all future theorems in this talk exploit *energy currents* constructed from vector fields:

$$T_{\mu\nu}(\psi) = \psi_\mu \psi_\nu - \frac{1}{2} g_{\mu\nu} \psi^\alpha \psi_\alpha$$

$$J_\mu^V = T_{\mu\nu} V^\nu$$

$$\pi^{\mu\nu} = \frac{1}{2} V^{(\mu;\nu)}, K^V = 2\pi^{\mu\nu} T_{\mu\nu} = \nabla^\mu J_\mu$$

$$\int_{\Sigma_2} J_\mu^V n_{\Sigma_2}^\mu + \int_{\mathcal{B}} K^V = \int_{\Sigma_1} J_\mu^V n_{\Sigma_1}^\mu \tag{3}$$

Σ_1 homologous to Σ_2, bounding $\mathcal{B}$

Vector fields as commutators

One can also commute the wave equation with vector fields.

If $[W, \Box] = 0$ and $\Box\psi = 0$, then $\Box(W\psi) = 0$. Thus one can consider higher order energies

$$J^V_\mu(W_1 \cdots W_n\psi)$$

where the W_i are commutator vector fields.

Control of such expressions is important for obtaining pointwise bounds from energy bounds.

The Kay-Wald proof

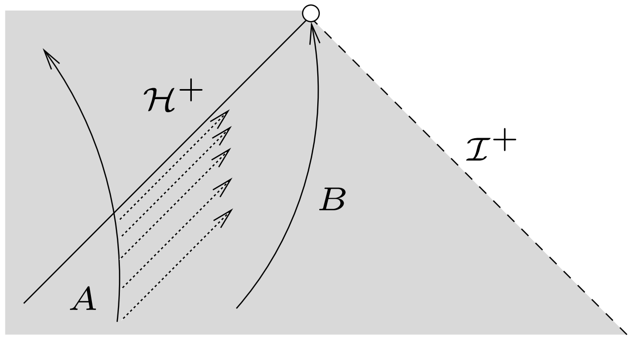
Uses only T as a multiplier.

Since T is everywhere causal in the closure of the domain of outer communications, and strictly timelike away from the horizon, the only non-trivial part of the problem is to obtain uniform bounds near $\mathcal{H}^+$.

Makes clever use of the staticity and discrete symmetries, and applies all the generators of spherical symmetries Ω_i as multipliers.

Red shift effect I

What is not sufficently exploited in the Kay-Wald theorem is the celebrated *red-shift* effect



There is also a local version of the redshift where B also crosses the horizon at a later advanced time from A. This depends on the positivity of the so-called *surface gravity* of the horizon.

Red shift effect II

Let ϕ_t denote the one-parameter flow of transformations generated by T.

Proposition 1. *(M.D.-I. Rodnianski) There exists a smooth vector field N, and two positive constants $0 < b < B$ such that N is timelike and ϕ_t-invariant such that*

$$bJ^N_\mu(\psi)N^\mu \leq K^N(\psi) \leq BJ^N_\mu(\psi)N^\mu,$$

along $\mathcal{H}^+$, for all solutions ψ of $\Box_g\psi = 0$.

This proposition captures "the red-shift" effect from the point of view of vector field multipliers.

Red shift effect III

Proposition 2. *(M.D.-I. Rodnianski) Under the assumptions of the above theorem, let* $Y = N - T$*, and extend* T, Y *to a null frame* T, Y, E_1, E_2 *on* $\mathcal{H}^+$*. If* ψ *satisfies* $\Box_g \psi = 0$*, then for all* $k \geq 1$*.*

$$\Box_g(Y^k \psi) = b_{k+1} Y^{k+1} \psi + \sum_{0 \leq |m| \leq k+1,\, 0 \leq m_4 < k+1} c_m E_1^{m_1} E_2^{m_2} T^{m_3} Y^{m_4} \psi \tag{4}$$

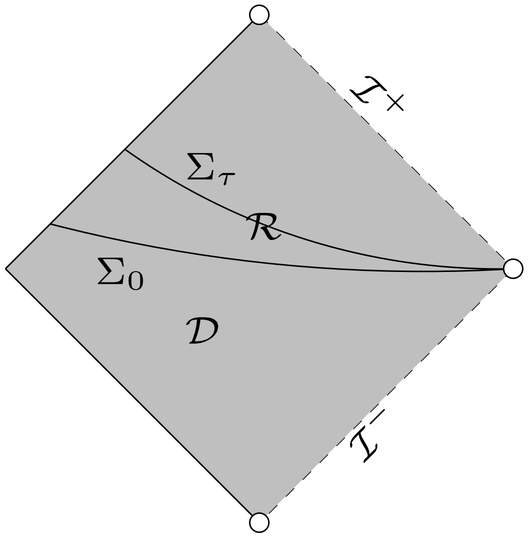
on $\mathcal{H}^+$*, where* $b_{k+1} > 0$*.*

The positivity of b_{k+1} captures "the red-shift" effect from the point of view of vector field commutators.

In fact, the above propositions depend only on the fact that $\mathcal{H}^+$ is a Killing horizon with positive "surface gravity".

How to measure energy

Let Σ be a Cauchy surface of $(\mathcal{M}, g)$ such that $\Sigma \cap \mathcal{H}^- = \emptyset$, let Σ_0 denote $\Sigma \cap \mathcal{D}$. Let ϕ_t denote the one-parameter family of transformations generated by T, and let $\Sigma_\tau = \phi_\tau(\Sigma_0)$, $\mathcal{R} = \cup_{\tau \geq 0} \Sigma_\tau$, and let $n^\mu_{\Sigma_\tau}$ denote the normal of Σ_τ.



A stronger boundedness theorem

Theorem. *(M.D.-I. Rodnianski) Let* $(\mathcal{M}, g_M)$ *be Schwarzschild and* Σ_τ *as above. For all* $m \geq 0$ *there exists a constant* C_m *depending only on* M, Σ_0 *such that for all* ψ *satisfying* $\Box_g \psi = 0$, *the following holds:*

$$|n_{\Sigma_\tau}\psi|_{H^m(\Sigma_\tau)} + |\nabla_{\Sigma_\tau}\psi|_{H^m(\Sigma_\tau)} \leq C_m(|n_{\Sigma_0}\psi|_{H^m(\Sigma_0)} + |\nabla_{\Sigma_0}\psi|_{H^m(\Sigma_0)}).$$

Moreover, for all $m \geq 0$, *the m'th order pointwise bounds*

$$\sum_{0 \leq m_1 + m_2 \leq m} |\nabla^{(m_1)}_{\Sigma_\tau} n^{(m_2)}_{\Sigma_\tau} \psi| \leq C_m \, \mathbf{Q}_m$$

hold in $\mathcal{R}$, *where* $\mathbf{Q}_m$ *is an appropriate norm on initial data.*

Perturbing the metric?

The above proof now is much more robust. In fact, it can be perturbed to nearby metrics as long as one retains $\mathcal{H}^+$ as a null boundary and T as Killing and causal:

Theorem. *(M.D.-I. Rodnianski) Let $\mathcal{R}$, T be as before, and let g be a metric on $\mathcal{R}$ sufficiently close to Schwarzschild such that T is Killing and causal on $\mathcal{R}$, and $\mathcal{H}^+$ is null with respect to g. Then the statement of the previous theorem applies verbatim.*

In fact, one need not assume that T is Killing, merely that $\pi^T_{\mu\nu}$ decays appropriately. Thus the theorem applies to spacetimes "settling down" to Schwarzschild at a sufficiently fast rate.

What about Kerr?

Unfortunately, for all $a \neq 0$, the vector field T is no longer timelike in the interior of $\mathcal{D}$! The part of $\mathcal{D}$ where it is spacelike is precisely the ergoregion $\mathcal{E}$ discussed above; the associated behaviour of waves is known as *superradiance*.

(Recall that the test-particle manifestation of this fact is the celebrated *Penrose process*.)

Unlike the Schwarzschild case, there is no trivial statement which can be proven away from the horizon.

The above suggests that it may be difficult to prove boundedness alone, and one of necessity must try to prove more than boundedness, i.e. **decay**.

That it may be possible to prove decay on Kerr is suggested by beautiful work of Yau and collaborators:

Theorem (Finster, Kamran, Smoller, Yau)**.** *Let* ψ *a smooth solution of the wave equation on the domain of outer communications of Kerr for* $0 \leq a < M$*. If* ψ *is supported away from* $\mathcal{H}^+ \cap \mathcal{H}^-$ *and* i^0*, then for all* $m \in \mathbb{Z}$*,*

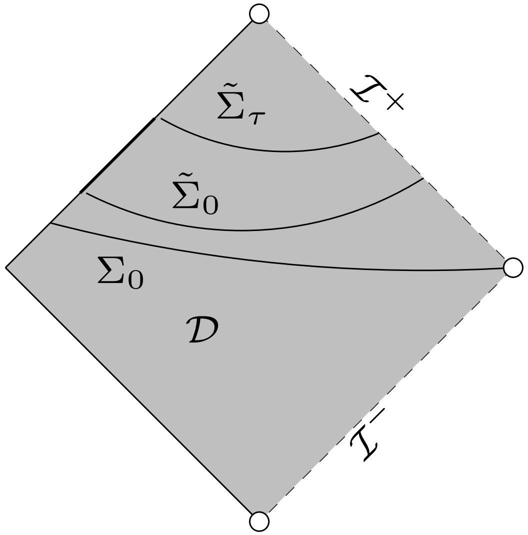
$$\lim_{t\to\infty} \psi_m(r,t) \to 0 \tag{5}$$

for $r > r_+$*, where* ψ_m *denotes the projection of* ψ *to the* m*'th eigenspace of the axisymmetric Killing field.*

Pretty contour integration methods, ode techniques. Unfortunately, (5) is a non-quantitative statement, so one cannot sum over m to obtain a statement for ψ itself, not even boundedness.

Quantitative decay

For quantitative results, energy is fundamental. To talk about energy decay, one must introduce a different type of foliation.



Let Σ be the Cauchy hypersurface as before, and let $\tilde{\Sigma}$ now be a hypersurface with $\tilde{\Sigma} \subset J^+(\Sigma)$ such that $\tilde{\Sigma} \cap \mathcal{H}^+ \neq \emptyset$, and $\tilde{\Sigma}$ meets $\mathcal{I}^+$ appropriately, $\tilde{\Sigma}_0 = \tilde{\Sigma} \cap \mathcal{D}$, $\tilde{\Sigma}_\tau = \phi_\tau(\tilde{\Sigma}_0)$ for $\tau \geq 1$.

Decay for slowly rotating Kerr

Theorem. *(M.D.-I. Rodnianski, 2008) Let* $(\mathcal{M}, g_{M,a})$ *be exactly Kerr for* $|a| \ll M$, Σ, $\tilde{\Sigma}$, $\mathcal{D}$ *and Killing field* T *as before. Then there exists a constant* C *depending only on* M, Σ *and* $\tilde{\Sigma}$, *and a* δ *depending on* $|a|$ *with* $\delta \to 0$ *as* $a \to 0$, *such that for all* ψ *satisfying* $\Box_g \psi = 0$, *the following holds:*

$$|n_{\tilde{\Sigma}_\tau}\psi|_{L^2(\tilde{\Sigma}_\tau)} + |\nabla_{\tilde{\Sigma}_\tau}\psi|_{L^2(\tilde{\Sigma}_\tau)} \leq C\tau^{-1+\delta}\sum_{m\leq 2}\left(r|n_\Sigma T^m\psi|_{L^2(\Sigma_0)} + r|\nabla_\Sigma T^m \psi|_{L^2(\Sigma_0)}\right).$$

Moreover, the pointwise decay rates

$$|r^{1/2}\psi| \leq C\,\mathbf{Q}\,\tau^{-1+\delta}, \qquad |r\psi| \leq C\mathbf{Q}\,\tau^{(-1+\delta)/2}$$

hold, where $\mathbf{Q}$ *is an appropriate norm on initial data.*

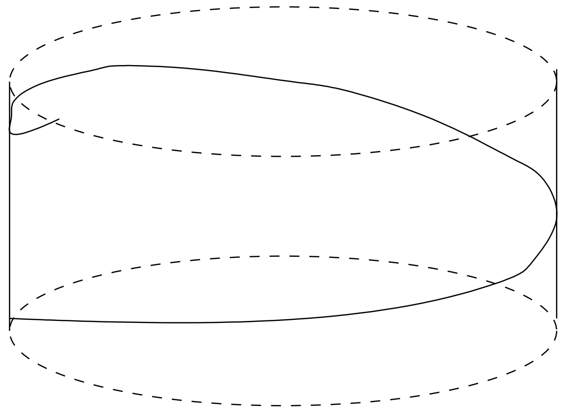
The trapping problem

It is known a priori from general results on the wave equation that a quantitative energy decay estimate through the foliation $\tilde{\Sigma}_\tau$ has to "lose" derivatives, that is to say, one needs control of more derivatives initially on Σ to estimate the energy later on $\tilde{\Sigma}_\tau$.

This is an essential aspect of the problem and has to do with **trapping**, i.e. the fact that there are null geodesics neither crossing the event horizon nor approaching null infinity.

The Schwarzschild photon sphere

In the Schwarzschild case, trapped null geodesics all asymptote to the so-called *photon sphere* at $r = 3M$:



(which is itself spanned by null geodesics).

In the general Kerr case, the codimensionality of the set of null geodesics is only apparent in phase space.

Trapping and decay in Schwarzschild

- A first breakthrough in understanding trapping came with the pioneering work of Laba-Soffer for Schroedinger equation in the Schwarzschild case. They sketched how to construct a Morawetz vector field X such that K^X is non-negative definite, with X vanishing at $3M$ (clarified by M.D.–Rodnianski, Blue–Soffer, Alinhac). This yields a version of "integrated decay".
- To turn this into energy decay away from the horizon, one constructs a weighted Morawetz-type current J^Z, where Z is a generalization of inverted time translations (M.D.–Rodnianski, independently by Blue–Sterbenz).
- Decay of local observer's energy and uniform pointwise decay rates up to the horizon using J^N (M.D.–Rodnianski).

Trapping and decay in Kerr

To construct an analogue of J^X, one must choose X “one frequency at a time” in the dangerous frequency range. This frequency decomposition can be done geometrically using the separability of the wave equation and geodesic flow, originally discovered by Carter.

Once one has a positive definite K^X, one can make up for the fact that T is not timelike by adding to J^T a small amount of J^N

(This idea has also been taken up by Tohaneanu and collaborators, who replace the geometric separation with the machinery of pseudodifferential calculus. The separability of geodesic flow is essential here as well.)

It is somewhat disappointing that one needs separability of geodesic flow in order to get a hold on trapping and prove quantitative decay.

It turns out, however, that one can prove just the boundedness statement with much less assumptions on the geometry, in particular, without separability assumptions.

Uniform boundedness on axisymmetric stationary black hole exteriors

Theorem. *(M.D.–Rodnianski, 2008) Let $\mathcal{R}$ be as before, g be a metric defined on $\mathcal{R}$, and let T and $\Phi = \Omega_1$ be Schwarzschild Killing fields. Assume*

1. *g is close to Schwarzschild in an appropriate sense*
2. *T and Φ are Killing with respect to g*
3. *$\mathcal{H}^+$ is null with respect to g, and T and Φ together span the null generator of $\mathcal{H}^+$.*

Then the statement of the Schwarzschild boundedness Theorem holds.

In particular, the theorem applies to Kerr for $|a| \ll M$, Kerr-Newman for $|a| \ll M$, $Q \ll M$, etc.

Proof:

Heuristic idea: If we associate frequencies ω, m to the Killing fields T and Φ, we can decompose

$$\psi = \psi_\sharp + \psi_\flat$$

where $\psi_\flat$ is supported in $\omega^2 \leq cm^2$ and $\psi_\sharp$ is supported in $\omega^2 \geq cm^2$.

It turns out that, for c small enough, and for g close enough to Schwarzschild, there is no superradiance for $\psi_\sharp$, and there is no trapping for $\psi_\flat$.

So, use just T and N for $\psi_\sharp$ as in the boundedness proof, and these in addition to a variant of X from the decay proof (which is now stable to construct in view of the absence of trapping) for $\psi_\flat$...

Open problems

Even before attempting the non-linear stability problem, there are a host of linear problems that remain to be understood.

1. Large $a < M$? Even boundedness is open.
2. Extremal case $a = M$?
3. Decay under minimal assumptions on the geometry.
4. Higher spin wave equations
5. Kerr-dS and Kerr-AdS (G. Holzegel, 2009)
6. Klein-Gordon: Exponentially growing solutions?

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