# Geometry of Shrinking Ricci Solitons

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## Ricci Solitons

A complete Riemannian $(M^n, g_{ij})$ is called a *Ricci soliton* if there exists a smooth function $f$ on $M$ such that

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij} \tag{1}$$

for some constant $\lambda$. $f$ is called a *potential function* of the Ricci soliton.

$\lambda = 0$: *steady* soliton; $\lambda > 0$: *shrinking* soliton; $\lambda < 0$: *expanding* soliton; $f =$ Const.: Einstein metric.

*Shrinking* and *steady Ricci solitons* are

- natural extension of Einstein manifolds;
- self-similar solutions to the Ricci flow
- possible singularity models of the Ricci flow
- critical points of Perelman's $\lambda$-entropy and $\mu$-entropy.

Thus it is important to understand the geometry/topology of Ricci solitons and, if possible, obtain their classifications.

**Part I: Singularities of the Ricci Flow**

Given any complete Riemannian manifold $(M^n, g_0)$, the Ricci flow introduced by R. Hamilton in 1982 is

$$\frac{\partial g(t)}{\partial t} = -2Rc(t),$$

with $g(0) = g_0$. Here $Rc(t)$ denotes the Ricci tensor of $g(t)$.

- The Ricci flow is a system of *second order, nonlinear, weakly parabolic* partial differential equations;

- The Ricci flow is a natural analogue of the heat equation for metrics. Thus one expects that the initial metric could be improved under the Ricci flow and evolve into a more canonical metric, thereby leading to a better understanding of the topology of the underlying manifold.

## Exact Solutions I: Einstein Metrics

If $g_0$ is Einstein with $Rc_{g_0} = \lambda g_0$, then

$$g(t) = (1 - 2\lambda t) g_0.$$

- *Ricci flat metrics, such as a flat torus or Calabi-Yau metric on K3 surfaces, are stationary solutions*

$$g(t) = g_0.$$

- *Positive Einstein metrics shrink homothetically*

  When $\lambda = 1/2$,

$$g(t) = (1 - t) g_0$$

  exists for $-\infty < t < 1$, and shrinks homothetically as $t$ increases. Moreover, the curvature blows up like $1/(1-t)$ as $t \to 1$ (an example of Type I singularity). This is the case on a round sphere $\mathbb{S}^n$.

**Summary**: *under the Ricci flow, metrics expand in directions of negative Ricci curvature and shrink in directions of positive Ricci curvature.*

**Exact Solutions II: Ricci Solitons**

Suppose $g$ is a complete **gradient Ricci soliton**, either steady or shrinking, so that

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$$

for $\lambda = 0$, or $1/2$.

If $g$ is a steady Ricci soliton ($\lambda = 0$) and $V = \nabla f$ generates a one-parameter group of diffeomorphisms $\varphi_t$, then

$$g(t) = \phi_t^* g$$

is a solution to the Ricci flow.

If $g$ is a shrinking Ricci soliton, with $\lambda = 1/2$, and $V = \nabla f/(1-t)$ generates a one-parameter group of diffeomorphisms $\varphi_t$, then

$$g(t) = (1-t)\phi_t^* g$$

is also a solution to the Ricci flow which shrinks to a point in finite time.

**Some facts about Ricci solitions**

- Compact steady (and expanding) solitons are Einstein in any dimension $n$;

- Compact shrinking solitons in $n = 2$ and $n = 3$ must be of positive constant curvature (Hamilton, Ivey).

- No non-flat noncompact shrinking soliton in $n = 2$ (Hamilton).

- Three-dimensional complete noncompact non-flat shrinking gradient solitons are classified (Perelman, Ni-Wallach, Cao-Chen-Zhu).

- Ricci solitons exhibit rich geometric properties.

**Examples of gradient Shrinking Ricci Solitions**

- *Positive Einstein manifolds, such as space forms* $\mathbb{S}^n/\Gamma$
- *Gaussian shrinking solitons on* $\mathbb{R}^n$

  $(\mathbb{R}^n, g_0, f(x) = |x|^2/4)$ is a gradient shrinker:

$$\nabla^2 f = \frac{1}{2} g_0.$$

- *Round cylinders* $\mathbb{S}^{n-k} \times \mathbb{R}^k$ *or its quotients*

- *Compact Kähler shrinkers on* $\mathbb{C}P^2\#(-\mathbb{C}P^2)$

  In early 90's Koiso, and independently by myself, constructed a gradient shrinking metric on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$. It has $U(2)$ symmetry and $Rc > 0$.

- *Toric Kähler shrinkers on* $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$

  In 2004, Wang-Zhu found a gradient Kähler-Ricci soliton on $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ which has $U(1) \times U(1)$ symmetry.

- *Noncompact Kähler shrinkers on* $\mathcal{O}(-1)$

  In 2003, Feldman-Ilmanen-Knopf found the first complete noncompact $U(n)$-invariant shrinking gradient Kähler-Ricci soliton on the tautological line bundle on $\mathcal{O}(-1)$ of $\mathbb{C}P^{n-1}$ $(n \geq 2)$ which is cone-like at infinity.

- Very recently, Dancer-Wang produced new examples of gradient shrinking Kähler solitons on bundles over the product of Fano Kähler-Einstein manifolds.

**Examples of Steady Ricci Solitons**

- *The cigar soliton* $\Sigma$

  In dimension $n = 2$, Hamilton discovered the *cigar soliton* $\Sigma = (\mathbb{R}^2, g_{ij})$, where the metric $g_{ij}$ is given by

  $$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

  with potential function

  $$f = -\log(1 + x^2 + y^2).$$

  The cigar has positive (Gaussian) curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at $\infty$.

- $\Sigma \times \mathbb{R}$: an 3-D steady Ricci soliton with nonnegative curvature.

- *The Bryant soliton on* $\mathbb{R}^n$

  In the Riemannian case, higher dimensional examples of noncompact gradient steady solitons were found by Robert Bryant on $\mathbb{R}^n$ ($n \geq 3$). They are rotationally symmetric and have positive sectional curvature. The volume of geodesic balls $B_r(0)$ grow on the order of $r^{(n+1)/2}$, and the curvature approaches zero like $1/s$ as $s \to \infty$.

- *Noncompact steady Kähler-Ricci soliton on* $\mathbb{C}^n$

  I found a complete $U(n)$-symmetric steady Ricci soliton on $\mathbb{C}^n$ ($n \geq 2$) with positive curvature. The volume of geodesic balls $B_r(0)$ grow on the order of $r^n$, $n$ being the complex dimension. Also, the curvature $R(x)$ decays like $1/r$.

- *Noncompact steady Kähler-Ricci soliton on* $\widehat{\mathbb{C}^n/\mathbb{Z}_n}$

  I also found a complete $U(n)$ symmetric steady Ricci soliton on the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at 0, the same underlying space that Eguchi-Hanson (n=2) and Calabi ($n \geq 2$) constructed ALE Hyper-Kähler metrics.

**Examples of 3-D Singularities in the Ricci flow.**

- *3-manifolds with $Rc > 0$.*

  According to Hamilton, any compact 3-manifold $(M^3, g_{ij})$ with $Rc > 0$ will shrink to a point in finite time and becomes round.

- *The Neck-pinching*

  If we take a dumbbell metric on topological $\mathbb{S}^3$ with a neck like $\mathbb{S}^2 \times I$, *as Yau pointed out to Hamilton in mid 80s*, we expect the neck will shrink under the Ricci flow because the positive curvature in the $\mathbb{S}^2$ direction will dominate the slightly negative curvature in the direction of interval $I$. We also expect the neck will pinch off in finite time. (In 2004, Angnents and Knopf confirmed the neck-pinching phenomenon in the rotationally symmetric case.)

- *The Degenerate Neck-pinching*

  One could also pinch off a small sphere from a big one. If we choose the size of the little to be just right, then we expect a degenerate neck-pinching: there is nothing left on the other side. (This picture is confirmed by X.-P. Zhu and his student Gu in the rotationally symmetric case in 2006)

## Singularities of the Ricci flow

In all dimensions, Hamilton showed that the solution $g(t)$ to the Ricci flow will exist on a maximal time interval [0, T), where either $T = \infty$, or $0 < T < \infty$ and $|Rm|_{\max}(t)$ becomes unbounded as $t$ tends to $T$. We call such a solution a *maximal solution.* If $T < \infty$ and $|Rm|_{\max}(t) \to \infty$ as $t \to T$, we say the maximal solution $g(t)$ *develops singularities* as $t$ tends to $T$ and $T$ is a *singular time.* Furthermore, Hamilton classified them into two types:

Type I: $\limsup_{t \to \infty} (T - t)\ |Rm|_{\max}(t) < \infty$

Type II: $\limsup_{t \to \infty} (T - t)\ |Rm|_{\max}(t) = \infty$

**Determine the structures of singularities**

Understanding the structures of singularities of the Ricci flow is the first essential step. The parabolic rescaling/blow-up method was developed by Hamilton since 1990s' and further developed by Perelman to understand the structure of singularities. We now briefly outline this method.

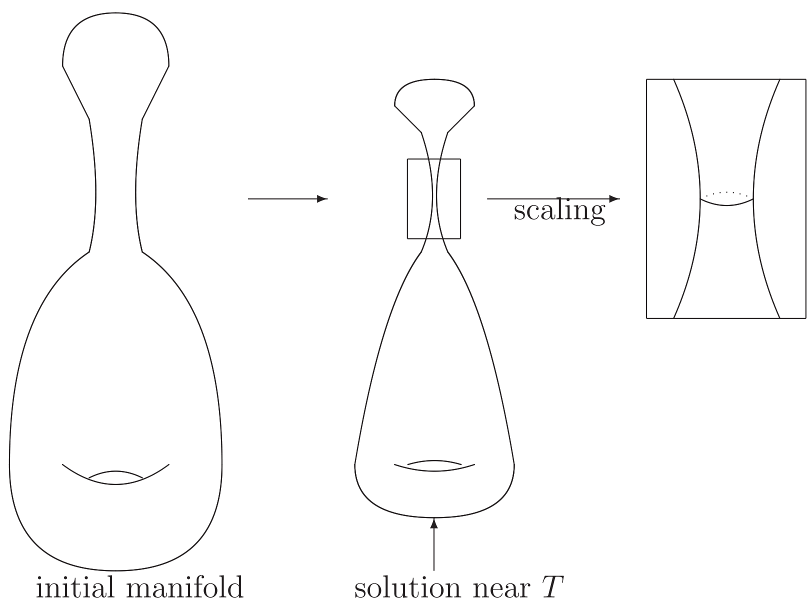


Figure 1: Rescaling

The Rescaling Argument:

- **Step 1**: Take a sequence of (almost) maximum curvature points $(x_k, t_k)$, where $t_k \to T$ and $x_k \in M$, such that for all $(x, t) \in M \times [0, t_k]$, we have

$$|Rm|(x,t) \leq CQ_k, \qquad Q_k = |Rm|(x_k, t_k).$$

- **Step 2**: rescale $g(t)$ around $(x_k, t_k)$ (by the factor $Q_k$ and shift $t_k$ to new time zero) to get the rescaled solution to the Ricci flow $\tilde{g}_k(t) = Q_k g(t_k + Q_k^{-1} t)$ for $t \in [-Q_k t_k, Q_k(T - t_k))$ with

$$|Rm|(x_k, 0) = 1, \quad \text{and} \quad |Rm|(x,t) \leq C$$

  on $M \times [-Q_k t_k, 0]$.

By Hamilton's **compactness theorem** and Perelman's **non-collapsing estimate**, rescaled solutions $(M^n, \tilde{g}_k(t), x_k)$ converges to $(\tilde{M}, \tilde{g}(t), \tilde{x})$, $-\infty < t < \Omega$, which is a complete *ancient solution* with bounded curvature and is *$\kappa$-noncollapsed on all scales.*

**Hamilton's Compactness Theorem:**

*For any sequence of marked solutions* $(M_k, g_k(t), x_k)$, $k = 1, 2, \ldots$, *to the Ricci flow on some time interval* $(A, \Omega]$, *if for all* $k$ *we have*

- $|Rm|_{g_k(t)} \leq C$, *and*
- *inj* $(M_k, x_k, g_k(0)) \geq \delta > 0$,

*then a subsequence of* $(M_k, g_k(t), x_k)$ *converges in the* $C^\infty_{\mathrm{loc}}$ *topology to a complete solution* $(M_\infty, g_\infty(t), x_\infty)$ *to the Ricci flow defined on the same time interval* $(A, \Omega]$.

**Remark**: In $n = 3$, by imposing an injectivity radius condition, Hamilton obtained the following structure results:

*Type I: spherical or necklike structures*;

*Type II: either a steady Ricci soliton with positive curvature or* $\Sigma \times \mathbb{R}$, *the product of the cigar soliton with the real line.*

**Perelman's No Local Collapsing Theorem**

*Given any solution $g(t)$ on $M^n \times [0,T)$, with $M$ compact and $T < \infty$, there exist constants $\kappa > 0$ and $\rho_0 > 0$ such that for any point $(x_0, t_0) \in M \times [0,T)$, $g(t)$ is $\kappa$-noncollapsed at $(x_0, t_0)$ on scales less than $\rho_0$ in the sense that, for any $0 < r < \rho_0$, whenever*

$$|Rm|(x,t) \leq r^{-2}$$

*on $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, we have*

$$Vol_{t_0}(B_{t_0}(x_0, r)) \geq \kappa r^n.$$

**Corollary**: *If $|Rm| \leq r^{-2}$ on $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, then*

$$inj(M, x_0, g(t_0) \geq \delta r$$

*for some positive constant $\delta$.*

**Remark**: There is also a stronger version: only require the scalar curvature $R \leq r^{-2}$ on $B_{t_0}(x_0, r)$.

## The Proof of Perelman's No Local Collapsing Theorems

Perelman proved two versions of "no local collapsing" property, one with a entropy functional, the $\mathcal{W}$-functional, and the other the reduced volume associated to a space-time distance function obtained by path integral *analogous to what Li-Yau did in 1986*.

- The $\mathcal{W}$-functional and $\mu$-entropy:

  $$\mu(g,\tau) = \inf\left\{\mathcal{W}(g,f,\tau) \mid f \in C^\infty(M), \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1\right\},$$

  where $\mathcal{W}(g,f,\tau) = \int_M [\tau(R+|\nabla f|^2)+f-n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$.

- Perelman's reduced distance $l$ and reduced volume $\tilde{V}(\tau)$:

  For any space path $\sigma(s)$, $0 \le s \le \tau$, joining $p$ to $q$, define the action $\int_0^\tau \sqrt{s}(R(\sigma(s), t_0 - s) + |\dot{\sigma}(s)|^2_{g(t_0-s)})ds$, the $L$-length $L(q,\tau)$ from $(p,t_0)$ to $(q,0)$, $l(q,\tau) = \frac{1}{2\sqrt{\tau}}L(q,\tau)$, and

  $$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-l(q,\tau)} dV_\tau(q).$$

- Monotonicity of $\mu$ and $\tilde{V}$ under the Ricci flow:

  Perelman showed that under the Ricci flow $\partial g(\tau)/\partial\tau = 2Rc(\tau)$, $\tau = t_0 - t$, both $\mu(g(\tau),\tau)$ and $\tilde{V}(\tau)$ are nonincreasing in $\tau$.

**Remark**: $\mathbb{S}^1 \times \mathbb{R}$ is **NOT** $\kappa$-noncollapsed on large scales for any $\kappa > 0$ and neither is the cigar soliton $\Sigma$, or $\Sigma \times \mathbb{R}$. In particular, $\Sigma \times \mathbb{R}$ cannot occur in the limit of rescaling! (However, $\mathbb{S}^2 \times \mathbb{R}$ is $\kappa$-noncollapsed on all scales for some $\kappa > 0$.)

**A magic of 3-D Ricci flow: The Hamilton-Ivey pinching theorem**

In dimension $n = 3$, we can express the curvature operator $Rm : \Lambda^2(M) \to \Lambda^2(M)$ as

$$Rm = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix},$$

where $\lambda \geq \mu \geq \nu$ are the principal sectional curvatures and the scalar curvature $R = 2(\lambda + \mu + \nu)$.

**The Hamilton-Ivey pinching theorem** *Suppose we have a solution $g(t)$ to the Ricci flow on a three-manifold $M^3$ which is complete with bounded curvature for each $t \geq 0$. Assume at $t = 0$ the eigenvalues $\lambda \geq \mu \geq \nu$ of $Rm$ at each point are bounded below by $\nu \geq -1$. Then at all points and all times $t \geq 0$ we have the pinching estimate*

$$R \geq (-\nu)[\log(-\nu) + \log(1 + t) - 3]$$

*whenever $\nu < 0$.*

**Remark**: This means in 3-D if $|Rm|$ blows up, the positive sectional curvature blows up faster than the (absolute value of) negative sectional curvature. As a consequence, **any limit of parabolic dilations at an almost maximal singularity has** $Rm \geq 0$

**Ancient $\kappa$-Solution**

An **ancient $\kappa$-solution** is a *complete ancient solution with nonnegative and bounded curvature, and is $\kappa$-noncollapsed on all scales.*

**Recap**: Whenever a maximal solution $g(t)$ on a compact $M^n$ develop singularities, parabolic dilations around any (maximal) singularity converges to some limit ancient solution $(\tilde{M}, \tilde{g}(t), \tilde{x})$, which has *bounded curvature and is $\kappa$-noncollapsed.* Moreover, if $n = 3$, then the ancient solution has *nonnegative sectional curvature, thus an ancient $\kappa$-solution.*

**Crucial a priori estimate: Li-Yau-Hamilton inequality**

When $Rm \geq 0$, we have the following crucial a priori estimate for ancient solutions in all dimensions which is used repeatedly in understanding $\kappa$-solutions.

**Li-Yau-Hamilton Inequality** (Hamilton, 1993) *Let $g(t)$ be a complete ancient solution to the Ricci flow with bounded and nonnegative curvature operator. Then for any one-form $V_a$ we have*

$$\frac{\partial R}{\partial t} + 2\nabla_a R \cdot V_a + 2R_{ab}V_aV_b \geq 0.$$

**Corollary**

$$\frac{\partial R}{\partial t} > 0,$$

and $R(\cdot, t)$ is *pointwise nondecreasing in $t$.*

**Structure of Ancient $\kappa$-solutions in 3-D**

**Canonical Neighborhood Theorem** (Perelman):

$\forall\ \varepsilon > 0$, *every point* $(x_0, t_0)$ *on an orientable nonflat ancient* $\kappa$*-solution* $(\tilde{M}^3, \tilde{g}(t))$ *has an open neighborhood* $B$*, which falls into one of the following three categories:*

(a) $B$ *is an* $\varepsilon$*-neck of radius* $r = R^{-1/2}(x_0, t_0)$*; (i.e., after scaling by the factor* $R(x_0, t_0)$*,* $B$ *is* $\epsilon$*-close, in* $C^{[\epsilon^{-1}]}$*-topology, to* $\mathbb{S}^2 \times [-\epsilon^{-1}, \epsilon^{-1}]$ *of scalar curvature 1.)*

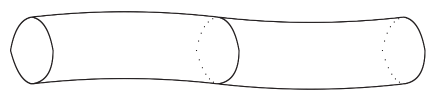

Figure 2: $\varepsilon$-neck

(b) $B$ *is an* $\varepsilon$*-cap; (i.e., a metric on* $\mathbb{B}^3$ *or* $\mathbb{RP}^3 \setminus \bar{\mathbb{B}}^3$ *and the region outside some suitable compact subset is an* $\varepsilon$*-neck).*

Figure 3: $\varepsilon$-cap

(c) $B$ *is compact (without boundary) with positive sectional curvature (hence diffeomorphic to the 3-sphere by Hamilton).*

**Part II: Classification of 3-D Complete Shrinking Ricci Solitons**

## 1. Classification of 3-D $\kappa$-shrinking solitons

- Classification of 3-d $\kappa$-shrinking solitons (Perelman): they are either round $\mathbb{S}^3/\Gamma$, or round cylinder $\mathbb{S}^2 \times \mathbb{R}$, or its $\mathbb{Z}_2$ quotients $\mathbb{S}^2 \times \mathbb{R}/\mathbb{Z}_2$.

  In particular, *there are no 3-d noncompact $\kappa$-noncollaping shrinking solitons with* $0 < Rm < C$.

*Sketch of the Proof*:

1) For $s$ sufficiently large

$$\left|\nabla f \cdot \dot{\gamma}(s) - \frac{s}{2}\right| \leq C,$$

and

$$\left|f(\gamma(s)) - \frac{s^2}{4}\right| \leq C \cdot (s+1).$$

In particular, $f$ has no critical points outside some large geodesic ball $B_{x_0}(s_0)$.

2) When $Rc > 0$,

$$\nabla R \cdot \nabla f = 2Rc(\nabla f, \nabla f) > 0$$

for $d(x, x_0) \geq s_0$. So outside $B_{x_0}(s_0)$, the scalar curvature $R$ is strictly increasing along the gradient curves of $f$. In particular

$$\bar{R} = \limsup_{d(x,x_0)\to+\infty} R(x) > 0.$$

3) At $\infty$, the soliton is asymptotic to a shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$ defined (at least) on $(-\infty, 1)$. Thus,

$$R(x) < 1$$

outside some $B_{x_0}(s_0)$.

4) the level surface $\Sigma_s = \{x \in M : f(x) = s\}$ is convex: The second fundamental form of $\Sigma_s$ is

$$h_{ij} = \nabla_i \nabla_j f / |\nabla f|, \qquad i, j = 1, 2.$$

But

$$\nabla_{e_i} \nabla_{e_j} f = \frac{1}{2}\delta_{ij} - Rc(e_i, e_j) \geq \frac{1}{2}(1 - R)\delta_{ij},$$

and

$$\frac{d}{ds}\mathrm{Area}\,(\Sigma_s) > \frac{1 - \bar{R}}{2\sqrt{s}}\mathrm{Area}\,(\Sigma_s) \geq 0$$

for $s \geq s_0$.

Here we used the fact that $Rm \geq 0$ *is equivalent to* $2R_{ij} \leq Rg_{ij}$ *when* $n = 3$.

5) Now Area $(\Sigma_s)$ is strictly increasing as $s$ increases, and

$$\log \text{Area}\,(\Sigma_s) > (1-\bar{R})\sqrt{s} - C,$$

for $s \geq s_0$. But Area $(\Sigma_s)$ is uniformly bounded from above by the area of the round sphere with scalar curvature one. Thus we conclude that $\bar{R} = 1$, and

$$\text{Area}\,(\Sigma_s) < 8\pi \qquad (*)$$

for $s$ large enough.

6) By using the Gauss equation and the soliton equation (1), the intrinsic curvature $K$ of the level surface $\Sigma_s$ can be computed and it turns out

$$K < \frac{1}{2} \qquad (**)$$

for $s$ sufficiently large. But (*) and (**) lead to a contradiction to the Gauss-Bonnet formula!

## 2. Classification of shrinking solitons in 3-D

• Complete noncompact non-flat shrinking gradient soliton with $Rc \geq 0$ and with curvature growing at most as fast as $e^{ar(x)}$ are quotients of $\mathbb{S}^2 \times \mathbb{R}$. (Ni-Wallach, 2007)

*Sketch of the Proof.*

It suffices to show that if $(M^3, g_{ij}, f)$ is a complete with $Rc > 0$ and $|Rm|(x) \leq Ce^{ar(x)}$, then $(M^3, g_{ij})$ is a finite quotient of $\mathbb{S}^3$. Basic ideas of Ni-Wallach's proof:

1) Consider the identity

$$\Delta(\frac{|Rc|^2}{R^2}) = \nabla(\frac{|Rc|^2}{R^2})\cdot\nabla f+\frac{2}{R^4}|R\nabla Rc-\nabla RRc|^2-\frac{2}{R}\nabla(\frac{|Rc|^2}{R^2})\cdot\nabla R+\frac{P}{R^3}$$

satisfied by the soltion metric, where

$$P = \frac{1}{2}((\lambda+\mu-\nu)^2(\lambda-\mu)^2+(\mu+\nu-\lambda)^2(\mu-\nu)^2+(\nu+\lambda-\mu)^2(\nu-\lambda)^2)$$

and $\lambda \geq \mu \geq \nu$ are the eigenvalues of $Rc$. This is a special case of Hamilton's computation for any solution $g_{ij}(t)$ to the Ricci flow on 3-manifolds.

2) By multiplying $|Rc|^2 e^{-f}$ to the above identity and integration by parts, Ni-Wallach deduced

$$0 = \int_M (|\nabla(\frac{|Rc|^2}{R^2})|^2 R^2 + \frac{2|Rc|^2}{R^4}|R\nabla Rc - \nabla R Rc|^2 + \frac{P}{R^3}|Rc|^2)e^{-f}.$$

Thus:

(i) $\frac{|Rc|^2}{R^2} = constant$;

(ii) $R\nabla Rc - \nabla R Rc = 0$;

(iii) $P = 0$;

provided the integration by parts is legitimate. Moreover, $Rc > 0$ and $P = 0$ imply $\lambda = \mu = \nu$. Thus $R_{ij} = \frac{R}{3} g_{ij}$, implying $R$ is a (positive) constant and $(M^3, g_{ij})$ is a space form.

3) Finally, using the fact shown by Ni earlier that *if* $Rc \geq 0$ *then* $f$ *has a certain quadratic growth lower bound*, they argued that the integration by parts can be justified when the curvature bound $|Rm|(x) \leq Ce^{ar(x)}$ is satisfied.

• A complete noncompact non-flat shrinking gradient soliton is a quotient of $\mathbb{S}^2 \times \mathbb{R}$ (Cao-Chen-Zhu, 2007).

*Sketch of the Proof*:

1) $Rm \geq 0$ without any curvature bound assumption (B.-L. Chen);

2)$R(x) \leq C(r^2(x)+1)$: According to Hamilton, after adding a constant, $f$ satisfies

$$R + |\nabla f|^2 - f = 0.$$

Therefore,

$$0 \leq |\nabla f|^2 \leq f, \quad \text{or} \quad |\nabla \sqrt{f}| \leq \frac{1}{2}$$

whenever $f > 0$. Thus

$$|\sqrt{f(x)} - \sqrt{f(x_0)}| \leq Cr(x),$$

and

$$\sqrt{f(x)} \leq C(r(x)+1) \quad \text{or,} \quad f(x) \leq C'(r^2(x)+1).$$

This proves the upper estimate for $f$, from which it also follows that

$$|\nabla f|(x) \leq C(r(x)+1),$$

and

$$R(x) \leq C'(r^2(x)+1).$$

## 3. Further Extensions in 4-D

**4-D**:

- Any complete gradient shrinking soliton with $Rm \geq 0$ and positive isotropic curvature (PIC), and satisfying some additional assumptions, is either a quotient of $\mathbb{S}^4$ or a quotient of $\mathbb{S}^3 \times \mathbb{R}$. (Ni-Wallach, 2007)

- Any non-flat complete noncompact shrinking Ricci soliton with bounded curvature and $Rm \geq 0$ is a quotient of either $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}^2$. (Naber, 2007)

## Part III: Geometry of Complete Shrinking Ricci Solitons

### A. Asymptotic behavior of potential functions

**Theorem** (Cao-Zhou, 2009): *Let* $(M^n, g_{ij}, f)$ *be a complete noncompact gradient shrinking Ricci soliton, satisfying*

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}.$$

*Then,*

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2.$$

*Here* $r(x) = d(x_0, x)$ *for some* $x_0 \in M$, $c_1 > 0$ *and* $c_2 > 0$ *depending only on* $n$ *and the geometry of* $g_{ij}$ *on the unit ball* $B_{x_0}(1)$.

**Remark**: In view of the Gaussian shrinker $(\mathbb{R}^n, g_0)$ with the potential function $|x|^2/4$, the leading term $\frac{1}{4}r^2(x)$ for the lower and upper bounds on $f$ is optimal.

**Remark**: $\rho(x) = 2\sqrt{f(x)}$ defines a distance-like function:

$$r(x) - c \leq \rho(x) \leq r(x) + c,$$

and

$$|\nabla \rho| = \frac{|\nabla f|}{\sqrt{f}} \leq 1.$$

*Sketch of the Proof*

1) The upper bound on $f$:

By a result of B.-L.Chen we have $R \geq 0$. Also, according to Hamilton,

$$R + |\nabla f|^2 - f = C_0$$

for some constant $C_0$. So, by adding $C_0$ to $f$, we can normalize $f$ so that

$$R + |\nabla f|^2 - f = 0.$$

Therefore,

$$0 \leq |\nabla f|^2 \leq f, \quad \text{or} \quad |\nabla \sqrt{f}| \leq \frac{1}{2}$$

whenever $f > 0$. Thus

$$|\sqrt{f(x)} - \sqrt{f(x_0)}| \leq \frac{1}{2} r(x).$$

Hence

$$\sqrt{f(x)} \leq \frac{1}{2} r(x) + \sqrt{f(x_0)},$$

or

$$f(x) \leq \frac{1}{4}(r(x) + 2\sqrt{f(x_0)})^2.$$

This proves the upper estimate for $f$, from which it also follows that

$$|\nabla f|(x) \leq \frac{1}{2} r(x) + \sqrt{f(x_0)},$$

and

$$R(x) \leq \frac{1}{4}(r(x) + 2\sqrt{f(x_0)})^2.$$

2) The lower bound on $f$:

Consider any minimizing normal geodesic $\gamma(s)$, $0 \leq s \leq s_0$ for $s_0 > 0$ large, with $\gamma(0) = x_0$. Denote by $X(s) = \dot{\gamma}(s)$ the unit tangent vector along $\gamma$. Then, by the second variation of arc length, we have

$$\int_0^{s_0} \phi^2 Rc(X,X)ds \leq (n-1)\int_0^{s_0} |\dot{\phi}(s)|^2 ds \qquad (2)$$

for every $\phi(s) \geq 0$ defined on the interval $[0, s_0]$. Now, following Hamilton, we choose $\phi(s)$ by

$$\phi(s) = \begin{cases} s, & s \in [0,1], \\ 1, & s \in [1, s_0 - 1], \\ s_0 - s, & s \in [s_0 - 1, s_0]. \end{cases}$$

Then

$$\begin{aligned} \int_0^{s_0} Rc(X,X)ds &= \int_0^{s_0} \phi^2 Rc(X,X)ds + \int_0^{s_0} (1-\phi^2)Rc(X,X)ds \\ &\leq (n-1)\int_0^{s_0} |\dot{\phi}(s)|^2 ds + \int_0^{s_0} (1-\phi^2)Rc(X,X)ds. \end{aligned}$$

On the other hand, using the Ricci soliton equation, we have

$$\nabla_X \dot{f} = \nabla_X \nabla_X f = \frac{1}{2} - Rc(X, X). \tag{3}$$

Integrating (3) along $\gamma$ from 0 to $s_0$, we get

$$\dot{f}(\gamma(s_0)) - \dot{f}(\gamma(0)) = \frac{1}{2}s_0 - \int_0^{s_0} Rc(X, X)ds.$$

Now if $|Rc| \leq C$, as in Perelman's case, then

$$\int_0^{s_0} Rc(X, X)ds \leq 2(n-1) + \max_{B_{x_0}(1)} |Rc| + \max_{B_{\gamma(s_0)}(1)} |Rc|.$$

Hence

$$\begin{aligned} \dot{f}(\gamma(s_0)) &\geq \frac{s_0}{2} + \dot{f}(\gamma(0)) - 2(n-1) - \max_{B_{x_0}(1)} |Rc| - \max_{B_{\gamma(s_0)}(1)} |Rc| \\ &\geq \frac{1}{2}s_0 - \dot{f}(\gamma(0)) - 2(n-1) - 2C = \frac{1}{2}(s_0 - c), \end{aligned}$$

and

$$f(\gamma(s_0)) \geq \frac{1}{4}(s_0 - c)^2 - f(x_0) - \frac{c^2}{4}.$$

However, since we do not assume any curvature bound, we have to modify the above argument.

Integrating along $\gamma$ from $s = 1$ to $s = s_0 - 1$ instead, we have

$$\begin{aligned}
\dot{f}(\gamma(s_0 - 1)) - \dot{f}(\gamma(1)) &= \int_1^{s_0 - 1} \nabla_X \dot{f}(\gamma(s)) ds \\
&= \frac{1}{2}(s_0 - 2) - \int_1^{s_0 - 1} Rc(X, X) ds \\
&= \frac{1}{2}(s_0 - 2) - \int_1^{s_0 - 1} \phi^2(s) Rc(X, X) ds \\
&\geq \frac{s_0}{2} - 2n + 1 - \max_{B_{x_0}(1)} |Rc| + \int_{s_0 - 1}^{s_0} \phi^2 Rc(X, X) ds.
\end{aligned}$$

Next, using equation (3) one more time and integration by parts, we obtain

$$\begin{aligned}
\int_{s_0 - 1}^{s_0} \phi^2 Rc(X, X) ds &= \frac{1}{2} \int_{s_0 - 1}^{s_0} \phi^2(s) ds - \int_{s_0 - 1}^{s_0} \phi^2(s) \nabla_X \dot{f}(\gamma(s)) ds \\
&= \frac{1}{6} + \dot{f}(\gamma(s_0 - 1)) - 2 \int_{s_0 - 1}^{s_0} \phi(s) \dot{f}(\gamma(s)) ds.
\end{aligned}$$

Therefore,

$$2 \int_{s_0 - 1}^{s_0} \phi(s) \dot{f}(\gamma(s)) ds \geq \frac{s_0}{2} - 2n + \frac{7}{6} - \max_{B_{x_0}(1)} |Rc| + \dot{f}(\gamma(1)).$$

$$2\int_{s_0-1}^{s_0}\phi(s)\dot{f}(\gamma(s))ds \geq \frac{s_0}{2} - 2n + \frac{7}{6} - \max_{B_{x_0}(1)}|Rc| + \dot{f}(\gamma(1)).$$

Furthermore, we claim

$$\sqrt{f(\gamma(s_0))} \geq \max_{s_0-1\leq s\leq s_0}|\dot{f}(\gamma(s))| - \frac{1}{2}.$$

Indeed,

$$|\dot{f}(\gamma(s))| \leq \sqrt{f(\gamma(s))},$$

and

$$|\sqrt{f(\gamma(s))} - \sqrt{f(\gamma(s_0))}| \leq \frac{1}{2}(s_0 - s) \leq \frac{1}{2}$$

for $s_0 - 1 \leq s \leq s_0$. Thus,

$$\max_{s_0-1\leq s\leq s_0}|\dot{f}(\gamma(s))| \leq \sqrt{f(\gamma(s_0))} + \frac{1}{2}.$$

Combining the above two inequality and noting $2\int_{s_0-1}^{s_0}\phi(s)ds = 1$, we conclude that

$$\sqrt{f(\gamma(s_0))} \geq \frac{1}{2}(s_0 - c_1)$$

for some constant $c_1$ depending only on $n$ and the geometry of $g_{ij}$ on the unit ball $B_{x_0}(1)$.

**B. Volume Growth Lower Estimate**

**Theorem** (Cao-Zhu, 2008): *Let* $(M^n, g_{ij}, f)$ *be a complete noncompact gradient*

*shrinking Ricci soliton. Then* $(M^n, g_{ij})$ *has infinite volume. More specifically, there exists some positive constant* $C_3 > 0$ *such that*

$$\mathrm{Vol}(B_{x_0}(r)) \geq C_3 \log r$$

*for* $r > 0$ *sufficiently large.*

**Remark**: A theorem of Yau (and Calabi) states that on a complete Riemannian manifolds with $Rc \geq 0$,

$$\mathrm{Vol}(B_{x_0}(r)) \geq Cr.$$

We believe an analogous result for complete shrinking soliton should be true.

### C. Volume Growth Upper Estimate

**Theorem** (Cao-Zhou, 2009): *Let $(M^n, g_{ij}, f)$ be a complete noncompact gradient shrinking Ricci soliton and suppose the scalar curvature $R$ is bounded above by*

$$R(x) \leq \alpha r^2(x) + A(r(x) + 1)$$

*for any $0 \leq \alpha < \frac{1}{4}$ and $A > 0$. Then, there exists some positive constant $C_4 > 0$ such that*

$$\mathrm{Vol}(B_{x_0}(r)) \leq C_4 r^n$$

*for $r > 0$ sufficiently large.*

**Remark**: In general we have $R(x) \leq \frac{1}{4}(r(x) + c)^2$. Moreover, observed that our argument in fact does not need the assumption on $R$.

**Remark**: The theorem can be regarded as an analog of Bishop's theorem for complete Riemannian manifolds with $Rc \geq 0$.

**Remark**: The noncompact Kähler shrinker of Feldman-Ilmanen-Knopf has Euclidean volume growth, with $Rc$ changing signs and $R$ decaying to zero. This shows that the volume growth rate in the above theorem is optimal.