THERMODYNAMICAL FORMALISM AND DIMENSION

1. A panorama

1.1 Very familiar definitions. How can we best define the dimension of a closed bounded set X in $\mathbb{R}^n$, say?

First Definition. We can define the *Topological Dimension* $\dim_T(X)$ by induction. We say that X has zero dimension if for every point $x \in X$ every sufficiently small ball about x has boundary not intersecting X. We say that X has dimension d if for every point $x \in X$ every sufficiently small ball about x has boundary intersecting X in a set of dimension $d-1$.

Unfortunately, the topological dimension is always a whole number - making it less than subtle in distinguishing sizes of sets.

Second Definition. Given $\epsilon > 0$, let $N(\epsilon)$ be the smallest number of ϵ-balls needed to cover X. We can define the *Box dimension* to be

$$\dim_B(X) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

However, a more refined definition is the following.

Third Definition. We can define the *Hausdorff dimension* (or Hausdorff-Besicovitch dimension) as follows. Given X we can consider a cover $\mathcal{U} = \{U_i\}_i$ for X by open sets. For $\delta > 0$ we can define $H_\epsilon^\delta(X) = \inf_{\mathcal{U}}\{\sum_i \mathrm{diam}(U_i)^\delta\}$ where the infimum is taken over all open covers $\mathcal{U} = \{U_i\}$ such that $\mathrm{diam}(U_i) \leq \epsilon$. We define $H^\delta(X) = \lim_{\epsilon\to 0} H_\epsilon^\delta(X)$ and, finally,

$$\dim_H(X) = \inf\{\delta : H^\delta(X) = 0\}.$$

The following properties are standard.

Proposition 1.1.1.

(1) *For any countable set X we have that $dim_H(X) = 0$.*
(2) *The definitions are related by $dim_H(X) \leq dim_B(X)$.*
(3) *If $T : X \to X'$ is a surjective Lipschitz map then* $\dim_H(X') \leq \dim_H(X)$

Fortunately, we will only consider examples where the Hausdorff Dimension and Box Dimension are equal.

1.2 "Linear" Examples. We begin with two standard examples.

Example 1.2.1. Middle third Cantor set: Let X denote the middle third Cantor set. This is the set of closed set of points in the unit interval whose triadic expansion does not contain any occurrences of the the digit 1, i.e.,

$$X = \left\{ \sum_{k=1}^{\infty} \frac{i_k}{3^k} : i_k \in \{0,2\}, k \geq 1 \right\}$$

Proposition 1.2.1. *For the middle third Cantor set both the Box dimension and the Hausdorff dimension are* $\frac{\log 2}{\log 3} = 0.690....$

Remark. However a more interesting *non-linear* analogue if this is the set E_2: This is the set of points whose continued fraction expansion contains only the terms 1 and 2. Unlike the Middle third Cantor set, the dimension of this set is not explicitly known in a closed form and can only be numerically estimated to the desired level of accuracy, i.e.,

$$\dim_H(E_2) = 0.5312805062772051416244686\ldots$$

Example 1.2.3 :Sierpinski carpet. Let $X = \left\{\left(\sum_{n=1}^{\infty} \frac{i_n}{3^n}, \sum_{n=1}^{\infty} \frac{j_n}{3^n}\right) : (i_n, j_n) \in \mathcal{S}\right\}$ where $\mathcal{S} = \{0,1,2\} \times \{0,1,2\} - \{(1,1)\}$. This is a connected set without interior. We call X a Sierpinski carpet.

Proposition 1.2.2. *For the Sierpinski carpet both the Box dimension and the Hausdorff dimension are equal to* $\frac{\log 8}{\log 3} = 1.892...$

1.3 "Non-linear" examples. Things soon become more complicated.

Example 1.3.1. Julia sets: We shall specialize to quadratic polynomials. Consider a map $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by the polynomial function $T_c(z) = z^2 + c$, where $c \in \mathbb{C}$.

Definition. We define the *Julia set* J to be the closure of the repelling periodic points i.e.

$$J = \text{closure}\left(\left\{z \in \widehat{\mathbb{C}} : T^n(z) = z, \text{ for some } n \geq 1, \text{ and } |(T^n)'(z)| > 1\right\}\right).$$

The Julia set J is clearly a closed T-invariant set (i.e., $T(J) = J$).

Trivial Example. For $c = 0$ we have $J_0 = \{z \in \mathbb{C} : |z| = 1\}$, i.e., the unit circle.

We next consider the case of values of c of sufficiently small modulus, where the asymptotic behaviour of the limit set is well understood through a result of Ruelle:

Proposition 1.3.1 (Ruelle). *For* $|c| \neq 0$ *sufficiently small:*

(1) *the Julia set* J_c *for* $T_c(z) = z^2 + c$ *is still a Jordan circle, but it has* $dim_B(J_c) = dim_H(J_c) > 1$*; and*
(2) *the map* $c \mapsto dim_H(J_c)$ *is real analytic and we have the asymptotic*

$$dim_H(J_c) \sim 1 + \frac{|c|^2}{4\log 2}, \qquad as\ |c| \to 0.$$

This answered a question posed by Sullivan, and Ruelle's solution was inspired by Bowen's work on Quasi-circles (below). Ruelle's proof used (perhaps mysteriously) zeta functions.

For more specific choices for the parameter c we have to resort to numerical computation if we want to know the Hausdorff dimension of J_c.

(i) For $c = i/4$,say, we can estimate $\dim_H(J_{i/4}) = 1.02321992890309691....$
(ii) For $c = 1/100$, say, we can estimate $\dim_H(J_{1/100}) = 1.00003662....$

However, an important ingredient in the method of computation of these values is the following.

Definition. We say that the rational map is *hyperbolic* if there exist $\beta > 1$ and $C > 0$ such that for any $z \in \mathbb{C}$ we have $|(T^n)'(z)| \geq C\beta^n$, for all $n \geq 1$.

Remark (A cautionary tale: Parabolic Implosions). Douady studied the case as $c \to \frac{1}{4}$ (along the real axis). As c increases the dimension $\dim(J_c)$ increases monotonically, with derivative tending to infinity. However, as c increases past $\frac{1}{4}$ there is a discontinuity (where the hyperbolicity breaks down). Less is understood about what happens as $c < 0$ decreases.

Example 1.3.2. Kleinian Limit sets: The Limit sets of Kleinian groups often have similar features to those of Julia sets. In the 1970's Sullivan devised a wellknown "dictionary" describing many of the corresponding properties.

Let $\mathbb{H}^3 = \{z + jt \in \mathbb{C} \oplus j\mathbb{R} : t > 0\}$ be the three dimensional upper half space. We can equip this space with the Poincare metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$$

and with this metric the space has curvature $\kappa = -1$. We can identify the isometries for $\mathbb{H}^3$ and this metric with the (orientation preserving) transformations

$$(z, t) \mapsto \left(g(z) = \frac{az + b}{cz + d}, t + 2 \log |cz + d| \right),$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. In particular, the first component is a linear fractional transformation and we can identify the space of isometries with the matrices $G = SL(2, \mathbb{C})$.

Although the action of $g \in G$ is an isometry on $\mathbb{H}^3$, the action on the boundary is typically not an isometry. In particular, we can associate to each $g \in \Gamma$ its *isometric circle* $C(g) := \{z \in \mathbb{C} : |g'(z)| = 1\}$. This is a Euclidean circle in the complex plane $\mathbb{C}$.

Defintions. A Kleinian group $\Gamma < G$ is a finitely generated discrete group of isometries. Let Γ_0 be the generators of Γ.

We define the limit set $\Lambda = \Lambda_\Gamma \subset \mathbb{C} \cup \{\infty\}$ for Γ to be the set of all limit points (in the Euclidean metric) of the set of points $\{g(j) : g \in \Gamma\}$.

By way of clarification, we should explain that since Γ is a discrete group these limit points must necessarily be in the Euclidean boundary.

Example 1.3.2.1. Apollonian circle packing. Consider three circles C_1, C_2, C_3 in the euclidean plane that are pairwise tangent. Inscribe a fourth circle C_4 which is tangent to all three circles. Within the three triangular region whose sides consist of the new circle and pairs of the other circles inscribe three new circles. Proceed inductively. The limit set is call an Apollonian circle packing.

Example 1.3.2.2. Schottky groups: Consider a Kleinian group Γ generated by a finite set $\Gamma_0 = \{g_1^{\pm 1}, \cdots, g_n^{\pm 1}\}$. If all of the isometric circles $\{C(g_i^{\pm 1})\}$ are disjoint then we call Γ a Schottky group. The limit set $\Lambda(\Gamma)$ is a (non-linear) Cantor set .

The following result is a little surprising.

Theorem 1.3.2 (Doyle). *There exists $\epsilon > 0$ such that for any Schottky group we have* $\dim_H(\Lambda) < 2 - \epsilon$.

Example 1.3.2.3. Fuchsian and Quasi-Fuchsian Groups: Let $K = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$. If each element g preserves K then Γ is a *Fuchsian group.* In this case the isometric circles for each element $g \in \Gamma$ meet K orthogonally. The Limit set of a non-cocompact convex cocompact Fuchsian group is either:

(1) a Cantor set lying in the unit circle; or
(2) the entire circle.

We can next consider a Kleinian group whose generators (and associated isometric circles) are close to that of a Fuchsian group. Such groups are called *quasi-Fuchsian.* In this case the limit set is still homeomorphic to a closed circle. This is called a *quasi-circle.*

However, although the quasi-circle is topologically a circle it can be quite different in terms of geometry.

Theorem 1.3.3 (Bowen). *The Hausdorff dimension of a quasi-circle is greater than or equal to* 1*, with equality only when it is actually a circle. (Moreover, the dimension changes analytically with the perturbation).*

This result was originally proved by Bowen, in one of two posthumous papers published after his death in 1978. There are generalizations to higher dimensions by C.-B. Yue.

Remark. It follows from the work of Sullivan that the Hausdorff Dimension of the limit set is closely related to the spectrum of the Laplacian on the space $\mathbb{H}^3/\Gamma$.

1.4. Iterated function schemes. A basic construction, common to most of our examples from the last section, is that of *iterated function schemes*

Let $U \subset \mathbb{R}^d$ be an open set. We say that $S : U \to U$ is a contraction if there exists $0 < \alpha < 1$ such that $||S(x) - S(y)|| \leq \alpha ||x - y||$ for all $x, y \in U$. (Here $|| \cdot ||$ denotes the usual Euclidean norm on $\mathbb{R}^d$.)

Definition. An *iterated function scheme* on an open set $U \subset \mathbb{R}^d$ consists of a family of contractions $T_1, \ldots, T_k : U \to U$.

In addition, we shall also want to make the following assumption: We say that a family of maps satisfies the *open set condition* if there exists an open set V such that the sets $T_1(V), \ldots, T_k(V)$ are pairwise disjoint.

We shall be particularly interested in the associated *limit set* Λ given in the following well known result.

Proposition 1.4.1. *Let $T_1, \ldots, T_k : U \to U$ be a finite family of contractions. There exists a unique smallest closed invariant set $\Lambda = \Lambda(T_1, \cdots, T_k)$ such that $\Lambda = \cup_{i=1}^k T_i \Lambda$.*

Equivalently, fix any point $z \in U$ then we define the limit set Λ by the set of all limit points of sequences:

$$\Lambda = \left\{ \lim_{n \to +\infty} T_{x_0} \circ T_{x_1} \circ \ldots \circ T_{x_n}(z) : x_0, x_1, \ldots \in \{1, \ldots, k\} \right\}.$$

A modified defintion. In fact, in some examples it is convenient to broaden slightly the definition of an iterated function scheme. More precisely, we might want to consider contractions $T_{ij} : U_i \to U_j$ which are only defined on part of the disjoint union $U = \coprod_{i=1}^k U_i$.

This second point of view in Proposition 2.1.1 has the additional advantage that every point is clearly coded by some infinite sequence. Given distinct sequences $\underline{x} = (x_n)_{n=0}^\infty, \underline{y} = (y_n)_{n=0}^\infty \in \{1, \dots, k\}^{\mathbb{Z}^+}$ we denote

$$n(\underline{x}, \underline{y}) = \min\{n \geq 0 : x_i = y_i \text{ for } 0 \leq i \leq k, \text{ but } x_k \neq y_k\}.$$

We then define the metric by

$$d(\underline{x}, \underline{y}) = \begin{cases} 2^{-n(\underline{x}, \underline{y})} & \text{if } \underline{x} \neq \underline{y} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that this is a metric. We can define a continuous map $\pi : \{1, \dots, k\}^{\mathbb{Z}^+} \to \mathbb{R}^d$ by $\pi(x) := \lim_{n\to+\infty} T_{x_0} \circ T_{x_1} \circ \dots \circ T_{x_n}(z)$. The map π is easily seen to be Hölder continuous (i.e., $\exists C > 0, \beta > 0$ such that $||\pi(\underline{x}) - \pi(\underline{y})|| \leq Cd(\underline{x}, \underline{y}))^\beta$ for any $\underline{x}, \underline{y}$.)

We shall assume that $T_1, \dots, T_k$ are *conformal*, i.e., the contraction is the same in each direction. Of course, for contractions on the line this is automatically satisfied, and is no restriction. In the one dimensional setting, such iterated function schemes are often called *cookie cutters*.

Proposition 1.4.2. *For conformal iterated function schemes satisfying the open set condition $dim_B(\Lambda) = dim_H(\Lambda)$.*

Example 1.4.1. For the set $E_2 \subset [0, 1]$ consisting of numbers whose continued fraction expansions contains only 1s or 2s, we can take $T : E_2 \to E_2$ to be $T(x) = \frac{1}{x} - [\frac{1}{x}]$. We can consider the local inverses $T_1 : [0, 1] \to [0, 1]$ and $T_2 : [0, 1] \to [0, 1]$ defined by $T_1(x) = 1/(1 + x)$ and $T_1(x) = 1/(2 + x)$. We can then view E_2 as the limit set $\Lambda = \Lambda(T_1, T_2)$.

1.5 Markov partitions and covers. Let $T : X \to X$ be a $C^{1+\alpha}$ locally expanding map on $X \subset \mathbb{R}^d$ In the present context we can assume that there exists $C > 0$ and $\lambda > 1$ such that

$$||D_x T^n(v)|| \geq C\lambda^n ||v||, \text{ for } n \geq 1.$$

The hypothesis that T is $C^{1+\alpha}$ means that the derivative DT is α-Hölder continuous, i.e.,

$$||DT||_\alpha := \sup_{x\neq y} \frac{||D_x T - D_y T||}{||x - y||} < +\infty.$$

Here the norm in the numerator on the Right Hand Side is the norm on linear maps from $\mathbb{R}^d$ to itself (or equivalently, on $d \times d$ matrices).

The contractions in an associated iterated function scheme are essentially the inverse branches to the expanding maps. To organize these preimages we want to introduce the idea of a Markov Partition.

Definition. We call a finite collection of closed subsets $\mathcal{P} = \{P_i\}_{i=1}^k$ a *Markov Partition* if it satisfies the following:

(1) Their union is X (i.e., $\cup_{i=1}^k P_i = X$);

(2) The sets are proper (i.e., each P_i is the closure of their interiors, relative to X);
(3) Each image TP_i, for $i = 1, \dots, k$, is the union of finitely many elements from $\mathcal{P}$ and $T : P_i \to TP_i$ is a local homeomorphism.

In many examples we consider, each image $TP_i = X$, for $i = 1, \dots, k$, in condition (iii). (Such partitions might more appropriately be called *Bernoulli Partitions*.)

Proposition 1.5.1. *For $T : X \to X$ a $C^{1+\alpha}$ locally expanding map, there exists a Markov Partition.*

If we write $T_i : X \to P_i$ for the local inverses then this describes an iterated function scheme. For each $n \geq 1$ we want to consider n-tuples $\underline{i} = (i_1, \dots, i_n) \in \{1, \dots, k\}^n$. We shall assume that $TP_{i_r} \supset P_{i_{r-1}}$, for $r = 2, \dots, n$. It is then an easy observation that

$$P_{\underline{i}} := T_{i_1} \cdots T_{i_{n-1}} P_{i_n}$$

is again a non-empty closed subset, and the union of such sets is equal to X.

Aim. We would like to estimate the dimension of X by making a cover using the sets $P_{\underline{i}}$, $|\underline{i}| = n$.

A slight technical difficulty is that these sets are closed, rather than open. The solution is rather easy: we simply make a cover by choosing open neighbourhoods $U_{\underline{i}} \supset P_{\underline{i}}$ which are slightly larger, and thus do form a cover for X. Let us assume that there is $0 < \theta < 1$ such that

$$\frac{\mathrm{diam}(U_{\underline{i}})}{\mathrm{diam}(P_{\underline{i}})} \leq 1 + O(\theta^n), \text{ for all } \underline{i}.$$

Let us define $T_{\underline{i}} : P_{i_1} \to P_{\underline{i}}$ by $T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n}$.

We can now obtain the following bounds.

Proposition 1.5.2. *There exist $C_1, C_2 > 0$ such that for all $\underline{i}$ and for all $x \in X$,*

$$C_1 \leq \frac{\mathrm{diam}(P_{\underline{i}})}{|T'_{\underline{i}}(x)|} \leq C_2.$$

In particular, for $t > 0$, there exist $C_1, C_2 > 0$ such that for any x and $n \geq 1$,

$$C_1 \leq \frac{\sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^t}{\sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|^t} \leq C_2.$$

The usefulness of this result is that we can now consider the local inverses $T_i : TP_i \to P_i$, i.e., $T \circ T_i(x) = x$ for $x \in TP_i$, (extended to suitable open neighbourhoods) to be an iterated function scheme for which X is the associated limit set.

Example 1.5.2. Hyperbolic Julia sets. Let $T : J \to J$ be a linear fractional transformation on the Julia set. Assume that the transformation $T : J \to J$ is hyperbolic (i.e., $\exists C > 0, \lambda > 1$ such that $|(T^n)'(x)| \geq C\lambda^n$, for all $x \in J$ and $n \geq 1$). Then Proposition 2.3.1 applies to give a Markov partition.

If we consider the particular case of a quadratic map $Tz = z^2 + c$, with $|c|$ small then we can define the local inverses by

$$T_1(z) = +\sqrt{z-c} \text{ and } T_2(z) = -\sqrt{z-c}$$

Of course, in order for these maps to be well defined, we need to define them on domains carefully chosen relative to the cut locus.

Example 1.5.3. Limit sets for Kleinian groups. We will mainly be concerned with the special case of Schottky groups. In this case, we have $2n$ pairs of disjoint disks $D_i^+, D_i{}^-$, with $1 \leq i \leq n$, whose boundaries are the isometric circles associated to the generators $g_1, \dots, g_n$ (and there inverses). In particular, we can define $T : \Lambda \to \Lambda$ by

$$T(z) = \begin{cases} g_i(z) & \text{if } z \in D_i^+ \\ g_i^{-1}(z) & \text{if } z \in D_i^- \end{cases}$$

If all of the closed disks are disjoint then $T : \Lambda \to \Lambda$ is expanding.

In particular, Proposition 1.4.2 now applies to two of our favorite examples.

Corollary 1.4.2.1. *For hyperbolic Julia sets and Schottky group limit sets the Hausdorff dimension and the Box dimension coincide.*

1.6 Transfer operators. It is not surprising that part of the approach to proving the Bowen-Ruelle result involves understanding the asymptotics of the expression

$$\sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^d \text{ as } n \to \infty$$

since this is intimately related to definition involving covers of the Hausdorff dimension of X. Moreover, the Propostion 1.5.2 tells us that it is an equivalent problem to understanding the behaviour of $\sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|^d$. Perhaps, at first sight, this doesn't seem to be an improvement. However, the key idea is to introduce a (Ruelle) transfer operator.

Definition. Let $C^\alpha(\mathcal{P})$ be the space of α-Hölder continuous functions on the disjoint union of the sets in $\mathcal{P}$. This is a Banach space with the norm $||f|| = ||f||_\infty + ||f||_\alpha$ where

$$||f||_\infty = \sup_x |f(x)| \text{ and } ||f||_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{||x-y||^\alpha}.$$

For each $t > 0$ we define a bounded linear operator $\mathcal{L}_t : C^\alpha(\mathcal{P}) \to C^\alpha(\mathcal{P})$ by

$$\mathcal{L}_t w(x) = \sum_i |T_i'(x)|^t w(T_i x).$$

To understand the role played by the transfer operator, we need only observe that iterates of the operator applied to the constant function 1 take the required form: for $x \in X$

$$\mathcal{L}_t^n 1(x) = \sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|^t,$$

i.e., the numerator in the last line of Proposition 2.4.1 (2). In particular, to understand what happens as n tends to infinity is now reduced to the behaviour of the operator $\mathcal{L}_t$.

Proposition 1.6.1 (Ruelle Operator Theorem).

(1) *The operator $\mathcal{L}_t$ has a simple maximal positive eigenvalue λ_t. Moreover the rest of the spectrum is contained in a disk of strictly smaller radius, i.e., we can choose $0 < \theta < 1$ and $C > 0$ such that $|\mathcal{L}_t^n 1 - \lambda_t^n| \le C\lambda_t^n \theta^n$, for $n \ge 1$.*

(2) *There exists a probability measure μ and $D_1, D_2 > 0$ such that for any $n \ge 1$ and $|\underline{i}| = n$ and $x \in X$:*

$$D_1 \lambda_t^n \le \frac{\mu(P_{\underline{i}})}{|T_{\underline{i}}'(x)|^t} \le D_2 \lambda_t^n.$$

(3) *The map $\lambda : \mathbb{R} \to \mathbb{R}$ given by $\lambda(t) = \lambda_t$ is real analytic and $\lambda'(t) < 0$ for all $t \in \mathbb{R}$.*

By Part (2) of Proposition 2.4.1 we see that for some $D_1, D_2 > 0$ and $0 \le t \le n$:

$$D_1 \lambda_t^n \le \sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^t \le D_2 \lambda_t^n, \text{ for } n \ge 1.$$

Recalling the definition of Hausdorff dimension we can bound

$$H_\epsilon^t(X) = \inf_{\mathcal{U}} \left\{ \sum_{U_i \in \mathcal{U}} \mathrm{diam}\ (U_i)^t \right\} \le \sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^t \le D_2 \lambda_t^n,$$

where the infimum is over open covers $\mathcal{U}$ whose elements have diameter at most $\epsilon > 0$, say, and n is chosen such that $\epsilon = \max_{|\underline{i}|=n}\{\mathrm{diam}(U_{\underline{i}})\}$.

Let us assume that $\lambda_d = 1$ (and $\lambda_t < 1$ when $t > d$; and $\lambda_t > 1$ when $t < d$)

We can therefore deduce that if $t > d$ then $\lambda_t < 1$ and thus $\lim_{\epsilon \to 0} H_\epsilon^t(X) = 0$. In particular, from the definition of Hausdorff dimension we see that $\dim_H(X) \le d$.

To obtain the lower bound for $\dim_H(X)$ we can use the mass distribution principle with the measure μ. In particular, for any $|\underline{i}| = n$ and $x \in X$ we can estimate

$$\begin{aligned} \mu(P_{\underline{i}}) &\le D_2 \lambda_d^n |T_{\underline{i}}'(x)|^d \\ &\le D_2 C_1^{-1} \lambda_d^n (\mathrm{diam}(P_{\underline{i}}))^d \end{aligned}$$

Given any $x \in X$ and any $\epsilon > 0$ we can choose n so that we can cover the ball $B(x, \epsilon)$ by a uniformly bounded number of sets $P_{\underline{i}}$ with $|\underline{i}| = n$.

In particular, since $\lambda_d = 1$ we can deduce that there exists $C > 0$ such that $\mu(B(x, \epsilon)) \le C\epsilon^d$ for $\epsilon > 0$. Thus, by the mass distribution we deduce that $\dim_H(X) \ge d$.

1.7 Pressure and Bowen-Ruelle Theorem. We collect together results from before. The main notational ingredient that we require is the following:

Definition. Given any continuous function $f : X \to \mathbb{R}$ we define its *pressure* $P(f)$ (with respect to T) as

$$P(f) := \limsup_{n \to +\infty} \frac{1}{n} \log \underbrace{\left(\sum_{\substack{T^n x = x \\ x \in X}} e^{f(x) + f(Tx) + \ldots + f(T^{n-1}x)} \right)}_{\text{Sum over periodic points}}$$

It is easy to see from the expressions for $\mathcal{L}_t^n 1(x)$ that we have the following well known corollary.

Corollary. *We can write* $P(-t\log|T'|) = \log \lambda_t$.

In fact, the limit actually exists and so the "lim sup" can actually be replaced by a "lim". In practise, we shall mainly be interested in a family of functions $f_t(x) = -t\log|T'(x)|$, $x \in X$ and $0 \le t \le d$, so that the above function reduces to

$$\begin{aligned} &[0,d] \to \mathbb{R} \\ &t \mapsto P(f_t) = \limsup_{n\to+\infty} \frac{1}{n} \log\left(\sum_{\substack{T^n x = x \\ x\in X}} \frac{1}{|(T^n)'(x)|^t} \right) \end{aligned}$$

Lemma 1.7.1. *We can write* $P(f_t) := \log \lambda_t$

The following standard result is essentially due Bowen and Ruelle. Bowen showed the result in the context of quasi-circles and Ruelle developed the method for the case of hyperbolic Julia sets.

Theorem 1.7.2 (Bowen-Ruelle). *Let* $T : X \to X$ *be a* $C^{1+\alpha}$ *conformal expanding map. There is a unique solution* $0 \le s \le n$ *to*

$$P(-s\log|T'|) = 0,$$

which occurs precisely at $s = dim_H(X) (= dim_B(X))$.

Example 1.7.1 (Reduction to the case of linear contractions). In the case of linear iterated functions schemes this reduces to Moran's theorem. Let us assume that $T_i(x) = a_i x + d_i$ then we can write

$$\begin{aligned} \sum_{\substack{T^n x = x \\ x\in X}} \frac{1}{|(T^n)'(x)|^t} &= \sum_{i_1,\dots,i_n} |a_{i_1}|^t \cdots |a_{i_k}|^t \\ &= \left(|a_1|^t + \cdots + |a_n|^t\right)^n \end{aligned}$$

In particular, since one readily sees that this expression is monotone decreasing as a function of t, and we see from the definitions that the value s such that $P(-s\log|T'|) = 0$ is precisely the same as that for which $1 = \frac{1}{|a_1|^s} + \cdots + \frac{1}{|a_k|^s}$ (Moran's Theorem).

Finally, we observe that the function $t \mapsto P(f_t)$ has the following interesting proprties

(i) $P(0) = \log k$;
(ii) $t \mapsto P(f_t)$ is strictly monotone decreasing;
(iii) $t \mapsto P(f_t)$ is analytic on $[0,d]$.

Part (iii) comes from the interpretation of pressure in terms of the isolated eigenvalue λ_t of the operator $\mathcal{L}_t$ (and analytic perturbation theory).

One particularly nice application of the above theorem and properties of pressure is to showing the analyticity of dimension as we change the associated expanding map. More precisely:

Corollary 1.7.1.1. *Let T_λ, with $-\epsilon \leq \lambda \leq \epsilon$, be an analytic family of expanding maps. Then $\lambda \mapsto dim_H(\Lambda_\lambda)$ is analytic.*

This applies, in particular, to the examples of hyperbolic Julia sets and limit sets for Schottky groups.

Example: Quadratic maps again. . The map $T_c(z) = z^2 + c$ has a hyperbolic Julia set J_c provided $|c|$ is sufficiently small. Ruelle used the above method to show that $c \mapsto \dim(J_c)$ is analytic for $|c|$ sufficiently small. (He also gave the first few terms in the expansion for $\dim(J_c)$, as given in earlier).

In the next section we explain the details of the proof of Theorem 2.3.2.

2. Computing Hausdorff dimension

We now come to one of the main themes we want to discuss: *How can one compute the Hausdorff Dimension of a set*?

2.1 Algorithms. In some of the simpler examples, particularly those constructed by affine maps, it was possible to give explicit formulae for the Hausdorff dimension. In this part, we will consider some nonlinear examples.

The main hypotheses on the compact $X \subset \mathbb{R}^n$ is that there exists a transformation $T : X \to X$ such that:

(1) *Markov dynamics:* There is a Markov partition (to help describe the local inverses as an interted function scheme);
(2) *Hyperbolicity:* There exists some $\lambda > 1$ such that $|T'(x)| \geq \lambda$ for all $x \in X$;
(3) *Conformality:* T is a conformal map;
(4) *Local maximality:* For any sufficiently small open neighbourhood U of the invariant set X we have $X = \cap_{n=0}^{\infty} T^{-n}U$ (such an X is sometimes called a *repeller*).

Our two main examples remain the following:

Example 2.1.1. Consider a hyperbolic rational map $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$ and let J be the Julia set.

Example 2.1.2. Consider a Schottky group $\Gamma = \langle g_1, \cdots, g_n, g_{n+1} = g_1^{-1}, \cdots, g_{2n} = g_n^{-1}\rangle$ and let Λ be the limit set. We let $U = \cup_{i=1}^{2n} U_i$ be the union of the disjoint open sets $U_i = \{z \in \ : |g_i'(z)| > 1\}$ of isometric circles. We define $T : \Lambda \to \Lambda$ by $T(z) = g_i(z)$, for $z \in U_i \cap \Lambda$ and $i = 1, \dots, 2n$.

We now describe three different approaches to estimating Hausdorff dimension.

A first approach: Using the definition of pressure. The most direct approach is to try to estimate the pressure directly from its definition, and thus the dimension from the last chapter.

Lemma 2.1.1. *For each $n \geq 1$ we can choose $s_n > 0$ to be the unique solution to*

$$\frac{1}{n} \log \left(\sum_{T^n x = x} |(T^n)'(x)|^{-s_n} \right) = 1.$$

Then $s_n = \dim_H(X) + O\left(\frac{1}{n}\right)$.

In particular, in order to get an estimate with error of size $\epsilon > 0$, say, one expects to need the information on periodic points of period approximately $1/\epsilon$.

A second approach: Using the transfer operator on Hölder functions. McMullen (for example) observed that working with the transfer operator one can quite effectively compute the pressure and the dimension. In practise, the numerical competition uses the approximation of the operator by matrices. Some of the flavour is given by the following statement.

Proposition 2.1.2. *Given $x \in X$, and then for each $n \geq 1$ we can choose s_n to be the unique solution to $\sum_{T^n y = x} |(T^n)'(y)|^{-s_n} = 1$. Then $s_n = \dim_H(X) + O(\theta^n)$, for some $0 < \theta < 1$.*

For many practical purposes, this gives a pretty accurate approximation to the Hausdorff dimension of X. However, we now turn to the main method we want to discuss.

A third approach: Using the transfer operator on analytic functions. Finally, we want to consider an approach based on determinants of transfer operators. The advantage of this approach is that it gives very fast, super-exponential, convergence to the Hausdorff dimension of the compact set X. This is based on the map $T : X \to X$ satisfying the additional assumption:

(5) *Analyticity:* T is real-analytic.

We need to introduce some notation.

Definition. Let us define a sequence of functions of $s > 0$:

$$a_n(s) = \frac{1}{n} \sum_{T^n z = z} \frac{|(T^n)'(z)|^{-s}}{\det\left(I - |(T^n)'(z)|^{-1}\right)}, \text{ for } n \geq 1,$$

where the summation is over all periodic points of period n and $DT^n(z)$ denotes the derivative of T^n at the fixed point $z_{\underline{i}} = T_{\underline{i}}(z_{\underline{i}})$, and $|T^n(z)|$ denotes the modulus of the derivative. Next we can define a series

$$\Delta(s, z) := \exp\left(-\sum_{n=1}^{\infty} a_n(s) \frac{z^n}{n}\right)$$

Finally we define a sequence of functions corresponding to parts of the Taylor series at z:

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \frac{1}{n!} \frac{\partial^n \Delta(s, z)}{\partial z^n}|_{z=1}, \quad N \geq 1.$$

(Equivalently,

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1, \dots, n_m) \\ n_1 + \dots + n_m = n}} \frac{(-1)^m}{m!} a_{n_1} \dots a_{n_m},$$

where the second summation is over all ordered m-tuples of positive integers whose sum is n).

The main result relating these functions to the Hausdorff dimension of X is the following.

Theorem 2.1.3 (Jenkinson-Pollicott). *Let $X \subset \mathbb{R}^n$ and assume that $T : X \to X$ satisfies conditions (1)-(5). We can find $C > 0$ and $0 < \theta < 1$ such that if s_N is the largest real zero of Δ_N then*

$$|dim(X) - s_N| \leq C\theta^{N^{\left(1+\frac{1}{n}\right)}} \text{ for each } N \geq 1.$$

In the case of Cantor sets in an interval then we would take $d = 1$. In the case of Julia sets and Kleinian group limit sets we would take $d = 2$.

Practical points.

(1) In practise, we can get estimates for $C > 0$ and $0 < \theta < 1$ in terms of T. For example, θ is typically smaller for systems which are more hyperbolic.
(2) To implement this on a desktop computer, the main issue is amount memory required. In most examples it is difficult to get N larger than 18, say.

2.2 Examples. We can now revisit our favorite examples.

Example 2.2.1: E_2. We can consider the non-linear Cantor set

$$E_2 = \left\{ \frac{1}{i_1 + \frac{1}{i_2 + \frac{1}{i_3 + \dots}}} : i_n \in \{1, 2\} \right\}.$$

For $X = E_2$, we can define $Tx = \frac{1}{x} \pmod 1$. This forms a Cantor set in the line, contained in the interval $[\frac{1}{2}(\sqrt{3} - 1), \sqrt{3} - 1]$, of zero Lebesgue measure.

A number of authors have considered the problem of estimating the Hausdorff dimension $\dim_H(E_2)$ of the set E_2. In 1941, Good showed that $0.5194 \leq \dim_H(E_2) \leq 0.5433$. In 1982, Bumby improved these bounds to $0.5312 \leq \dim_H(E_2) \leq 0.5314$. In 1989 Hensley showed that $0.53128049 \leq \dim_H(E_2) \leq 0.53128051$. In 1996, he improved this estimate to 0.5312805062772051416.

We can apply Theorem 2.1.3 to estimate $\dim_H(E_2)$. In practice we can choose $N = 16$, say, and if we solve for $\Delta_{16}(s_{16}) = 0$ then we derive the approximation

$$\dim_H(E_2) = 0.5312805062772051416244686\dots$$

which is correct to the 25 decimal places given.

Example 2.2.2 : Julia sets. We can consider Julia sets for quadratic polynomials $T_c(z) = z^2 + c$ with different values of c.

For quadratic maps we know that $T'(z) = 2z$ and if $T^n(z) = z$ then by the chain rule

$$(T^n)'(z) = T'(T^{n-1}z) \cdots T'(Tz).T'(z) = 2^n(T^{n-1}z) \cdots (Tz).z$$

and so the coefficients in the expansions take a simpler form.

Example $c = -5$. For real values of c which are strictly less than -2, the Julia set J_c is a Cantor set completely contained in the real line. For such cases we have, by Corollary 3.1, the faster $O(\theta^{N^2})$ convergence rate to $\dim(J_c)$, as illustrated in the Table for the case $c = -5$.

N	N^{th} approximation to $\dim(J_{-5})$
1	0.4513993584764174609675959101241383349
2	0.4841518684194122992464635900326070715
3	0.4847979587486975778612282908975662571
4	0.4847982943561895699730717563576367090
5	0.4847982944381635057518511943420942957
6	0.4847982944381604305347487891271825909
7	0.4847982944381604305383984765793729512
8	0.4847982944381604305383984781726830747

TABLE. Successive approximations to $\dim_H(J_{-5})$

Example 2.2.3 Schottky groups Limit sets. Fix $2p$ disjoint closed discs $D_1, \dots, D_{2p}$ in the plane, and Möbius maps $g_1, \dots, g_p$ such that each g_i maps the interior of D_i to the exterior of D_{p+i}. The corresponding Schottky group is defined as the group generated by $g_1, \dots, g_p$. The associated limit set Λ is a Cantor subset of the union of the interiors of the discs $D_1, \dots, D_{2p}$. We define a map T on this union by $T|_{int(D_i)} = g_i$ and $T|_{int(D_{p+i})} = g_i^{-1}$. A *reflection group* is a Schottky group with $D_i = D_{p+i}$ for all $i = 1, \dots, p$.

Example 2.2.3 (a). The following family of reflection groups was considered by McMullen. Consider three circles $C_0, C_1, C_2 \subset \mathbb{C}$ of equal radius, arranged symmetrically around S^1, each intersecting the unit circle S^1 orthogonally, and meeting S^1 in an arc of length θ. We do not want the C_i to intersect each other, so we ask that $0 < \theta < 2\pi/3$. For definiteness let us suppose each C_i has radius $r = r_\theta = \tan \frac{\theta}{2}$, and that the circle centres are at the points $z_0 = a$, $z_1 = ae^{2\pi i/3}$ and $z_2 = ae^{-2\pi i/3}$ (where $a = a_\theta = \sqrt{1 + r^2} = \sec \frac{\theta}{2}$). The reflection $\rho_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in the circle C_i takes the explicit form

$$\rho_i(z) = \frac{r^2}{|z - z_i|^2}(z - z_i) + z_i.$$

Let $\Lambda_\theta \subset \mathbb{S}^1$ denote the limit set associated to the group Γ_θ of transformations given by reflection in these circles. For example, with the value $\theta = \pi/6$ we show that the dimension of the limit set $\Lambda_{\pi/6}$ is

$$\dim(\Lambda_{\pi/6}) = 0.18398306124833918694118127344474173288 \dots .$$

N	Largest zero of Δ_N
2	0.14633481296007741055454748401454596
3	0.18423440272351767688822531747382350
4	0.18399977929621235204864644797773486
5	0.18398305039516509087579859265399133
6	0.18398305988417009403195596234810316
7	0.18398306122261622100816402885866734
8	0.18398306124841998285455137338908131
9	0.18398306124833255797187772764544302
10	0.18398306124833929946685349025674957
11	0.18398306124833918404985469216386875
12	0.18398306124833918700689278881066430
13	0.18398306124833918693967757277042711
14	0.18398306124833918694121655021916395
15	0.18398306124833918694118046846226018
16	0.18398306124833918694118129222351397
17	0.18398306124833918694118127301338345
18	0.18398306124833918694118127345475071
19	0.18398306124833918694118127344451095
20	0.18398306124833918694118127344474707

TABLE . Successive approximations to $\dim(\Lambda_{\pi/6})$

2.3 Eigenvalues of the Laplacian and Kleinian groups. Given any Kleinian group Γ of isometries of 3-dimensional hyperbolic space $\mathbb{H}^n$ we can associate the quotient manifold $M = \mathbb{H}^3/\Gamma$. The Laplacian $\Delta_M : C^\infty(M) \to C^\infty(M)$ is a self-adjoint second order linear differential operator. This extends to a self-adjoint linear operator Δ_M on the Hilbert space $L^2(M)$. In particular, the spectrum of $-\Delta_M$ is contained in the interval $[\lambda_0, +\infty)$, where λ_0 is the smallest eigenvalue. If M is compact then the constant functions are an eigenfunction and so $\lambda_0 = 0$. More generally, we can have $\lambda_0 > 0$.

Perhaps surprisingly, λ_0 is related to the Hausdorff dimension $d = \dim_H(\Lambda)$ of the Limit set by the following result.

Theorem 2.3.1 (Sullivan's Theorem). $\lambda_0 = \min\{d(1-d), 1/4\}$

McMullen's Example. This problem is very closely related to the geometry of an associated surface of constant curvature $\kappa = -1$. Consider the unit disk

$$\mathbb{D}^2 = \{x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$$

with the Poincaré metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

then $(\mathbb{D}^2, ds^2)$ has constant curvature $\kappa = -1$. Let $C_1, C_2, C_3 \subset \mathbb{C}$ be the three similar circles in the complex plane which meet the unit circle orthogonally and enclose an arc of length θ. We can identify the reflections in these circles with isometries $R_1, R_2, R_3 \subset \mathrm{Isom}(\mathbb{D}^2)$ and then consider the Kleinian group Γ_θ they generate. We can then let $M = \mathbb{D}^2/\Gamma$ be the quotient manifold.

The Laplacian $\Delta_M : C^\infty(M) \to C^\infty(M)$ is given by

$$\Delta_M = (1 - x^2 - y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The smallest eigengvalue of $-\Delta_M$ is related to the dimension d of the boundary by Sullivan's Theorem. In particular, we have the following corollary.

Proposition 2.3.2. *When $\theta = \pi/6$ then we can estimate $\lambda_0 = 0.24922656...$*

Proof. In Chapter 3 we estimated that $\dim_H(\Lambda) = 0.4721891278821...$[1]. By applying Sullivan's Theorem, the result follows. $\square$

On can also study the asymptotic behavior of $\dim_H(\Lambda_\theta)$. McMullen showed the following:

Propositon 2.3.3. *The asymptotic behaviour of $dim_H(\Lambda_\theta)$ is described by the following result:*

(1)
$$dim_H(\Lambda_\theta) \sim \frac{1}{|\log \theta|} \text{ as } \theta \to 0;$$

(2)
$$dim_H(\Lambda_\theta) \sim 1 - \frac{1}{2}\left(\frac{2\pi}{3} - \theta\right) \text{ as } \theta \to \frac{2\pi}{3}.$$

(Equivalently, the associated smallest eigenvalue $\lambda_0(\theta)$ satisfies $\lambda_0(\theta) \sim \frac{1}{|\log \theta|}$ as $\theta \to 0$ and $\lambda_0(\theta) \sim \frac{1}{2}\left(\frac{2\pi}{3} - \theta\right)$ as $\theta \to \frac{2\pi}{3}$.)

2.4 Analytic functions and transfer operators. The proof of Theorem 2.1.3 is based on the (perhaps less familiar) study of the transfer operator on the smaller space of real analytic functions.

The strategy we shall follow is the following. The operators $\mathcal{L}_s$ are defined on analytic functions. This in turn allows us to define their Fredholm determinants $det(I - z\mathcal{L}_s)$. These are entire function of z which, in particular, have as a zero the value $z_s = 1/\lambda_s$. In this context we can get very good approximations to $det(I - z\mathcal{L}_s)$ using polynomials whose coefficients involve the traces $tr(\mathcal{L}_s^n)$. Finally, these expressions can be evaluated in terms of fixed points of the iterated function scheme, leading to the functions $\Delta_N(s)$ introduced above.

(i) Real Analytic Functions. We have a natural identification

$$\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times i\mathbb{R}^n = \mathbb{C}^n.$$

A function $f : U \to \mathbb{R}$ on an neighbourhood $U \subset \mathbb{R}^n$ is real analytic if it has a complex analytic extension to a function $f : D \to \mathbb{C}^n$, where $U \subset D \subset \mathbb{C}^n$ is an open set in $\mathbb{C}^d$.

[1]McMullen previously estimated $d = \dim_H(X) = 0.47218913...$

(ii) Expanding maps and Markov Partitions. We start from an expanding map $T : X \to X$ with a Markov Partition $\mathcal{P} = \{X_j\}$, say. For each $1 \leq j \leq k$, let us assume that U_j is an open neighbourhood of a element X_j of the Markov Partition.

(iii) A Hilbert space and a linear operator. For any open set $U \subset \mathbb{C}^n$, let $\mathcal{A}_2(U)$ denote the Hilbert space of square integrable holomorphic functions on U equipped with the norm

$$||f|| := ||f||_{\mathcal{A}_2(U)} = \sqrt{\int_U |f|^2 d(\mathrm{vol})}.$$

Given $w \in \mathcal{A}_2(U)$ (typically $w(z) = |T'(z)|^{-s}$) then define the bounded linear operator $\mathcal{L}_s : \mathcal{A}_2(U) \to \mathcal{A}_2(U)$ by

$$\mathcal{L}_s h(x) = \sum_i h(T_i x) w(T_i x)$$

for each $h \in \mathcal{A}_2(D)$.

(iv) Nuclear (trace class) operators. Consider a bounded linear operator $L : H \to H$ on a Hilbert space H.

Definition. A linear operator $L : H \to H$ on a Hilbert space H is called *nuclear* if there exist $u_n \in H$, $l_n \in H^*$ (with $||u_n|| = 1$ and $||l_n|| = 1$) and $\sum_{n=0}^{\infty} |\rho_n| < +\infty$ such that

$$L(v) = \sum_{n=0}^{\infty} \rho_n l_n(v) u_n, \qquad \text{for all } v \in H.$$

The following theorem is due to Ruelle.

Proposition 2.4.1. *The transfer operator $\mathcal{L} : \mathcal{A}_2(D) \to \mathcal{A}_2(D)$ is nuclear.*

Remark. It is here that looking at analytic functions helps. The spectra of these operators are much "better" than acting on Hölder functions.

(v) Determinants. We now associate to the transfer operators a function of a two complex variables.

Definition. For $s > 0$ and $z \in \mathbb{C}$ we define the *Fredholm determinant* $\det(I - z\mathcal{L}_s)$ of the transfer operator $\mathcal{L}_s$ by

$$\det(I - z\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}_s^n)\right)$$

This is similar to the way in which one associates to a matrix the determinant.

We can compute the traces explicitly.

The key to our method is the following explicit formula for the traces of the powers $\mathcal{L}_s^n$ in terms of the fixed points of our iterated function scheme.

Proposition 2.4.2. *If $\mathcal{L}_s : \mathcal{A}(U) \to \mathcal{A}(U)$ is the transfer operator then*

$$tr(\mathcal{L}_s^k) = \sum_{T^n z = z} \frac{|(T^k)'(z)|^{-s}}{(} I - |(T^k)'(z)|^{-1}).$$

This allows us to compute the determinant:

$$\Delta(z, s) := \det(I - z\mathcal{L}_s) = \exp\left(-\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{T^k z = z} \frac{|(T^k)'(z)|^{-s}}{I - |T^n(z)'|^{-1}}\right).$$

However, the most surprising (and useful) result is the following.

Proposition 2.4.3 (Grothendeick-Ruelle). *There exists $C > 0$ and $0 < \theta < 1$ such that*

$$\Delta(z, s) = 1 + \sum_{k=1}^{\infty} b_k(s) z^k$$

where $|b_k(s)| \leq C\theta^{k^{1+1/n}}$, for $k \geq 1$.

(vi) Pressure, Hausdorff Dimension and Determinants. We can now make the final connection with the Hausdorff dimension.

Proposition 2.4.4. *For any $s \in \mathbb{C}$, let $\lambda_r(s)$, $r = 1, 2, \ldots$ be an enumeration of the non-zero eigenvalues of $\mathcal{L}_s$, counted with algebraic multiplicities. Then*

$$\Delta(z, s) = \prod_{r=1}^{\infty} (1 - z\lambda_r(s)).$$

In particular, the set of zeros z of the Fredholm determinant $det(I - z\mathcal{L}_s)$, counted with algebraic multiplicities, is equal to the set of reciprocals of non-zero eigenvalues of $\mathcal{L}_s$, counted with algebraic multiplicities.

This brings us to the connection we want.

Corollary 2.4.4.1. *Given an iterated function scheme, the Hausdorff dimension $dim(\Lambda)$ of its limit set Λ is the largest real zero of the function $s \mapsto \Delta(s)$.*

3. Measures, Dimension and Multi-fractal analysis

3.1 Hausdorff dimension of measures. We define the Hausdorff dimension μ in terms of the *Hausdorff dimension* of subsets of Λ.

Definition. For a given probability measure μ we define the Hausdorff dimension of the measure by

$$\dim_H(\mu) = \inf\{\dim_H(X) : \mu(X) = 1\}.$$

We next want to define a local notion of dimension for a measure μ at a typical point $x \in X$.

Definition. The upper and lower pointwise dimensions of a measure μ are measurable functions $\overline{d}_\mu, \underline{d}_\mu : X \to \mathbb{R} \cup \{\infty\}$ defined by

$$\overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \text{ and } \underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}$$

where $B(x, r)$ is a ball of radius $r > 0$ about x.

There are interesting connections between these different notions of dimension for measures.

Theorem 3.1.1. *If $\underline{d}_\mu(x) \geq d$ for a.e. (μ) $x \in X$ then $\dim_H(\mu) \geq d$.*

In the opposite direction we have that a uniform bound on pointwise dimensions leads to an upper bound on the Hausdorff Dimension.

Theorem 3.1.2. *If $\overline{d}_\mu(x) \leq d$ for a.e. (μ) $x \in X$ then $\dim_H(\mu) \leq d$. Moreover, if there is a probability measure μ with $\mu(X) = 1$ and $\overline{d}_\mu(x) \leq d$ for* **every** *$x \in X$ then $\dim_H(X) \leq d$.*

Let us consider the particular case of iterated function schemes.

Example (Iterated Function Schemes and Bernoulli measures). For an iterated function scheme $T_1, \cdots, T_k : U \to U$ we can denote as before

$$\Sigma = \{\underline{x} = (x_m)_{m=0}^\infty : x_m \in \{1, \cdots, k\}\}$$

with the Tychonoff product topology. The shift map $\sigma : \Sigma \to \Sigma$ is a local homeomorphism defined by $(\sigma x)_m = x_{m+1}$. The nth level cylinder is defined by,

$$[i_0, \dots, i_{n-1}] = \{(x_m)_{m=0}^\infty \in \Sigma : i_m = x_m \text{ for } 0 \leq m \leq n-1\},$$

(i.e., all sequences which begin with $x_0, \dots, x_{k-1}$). We denote by $W_k = \{[x_0, \dots, x_{k-1}]\}$ the set of all kth level cylinders (of which there are precisely k^n).

Notation. For a sequence $\underline{i} \in \Sigma$ and a symbol $r \in \{1, \dots, n\}$ we denote by $k_r(\underline{i}) = \text{card}\{0 \leq m \leq k-1 : i_m = r\}$ the number of occurrences of r in the first k terms of $\underline{i}$.

Consider a probability vector $\underline{p} = (p_1, \dots, p_n)$ and define the Bernoulli measure of any kth level cylinder to be,

$$\mu([i_0, \dots, i_{k-1}]) = {p_1}^{k_1(\underline{i})} {p_2}^{k_2(\underline{i})} \cdots {p_n}^{k_n(\underline{i})}.$$

A probability measure μ on σ is said to be invariant under the shift map if for any Borel set $B \subset X$, $\mu(B) = \mu(\sigma^{-1}(B))$. We say that μ is ergodic if any Borel set $B \subseteq \Sigma$ such that $\sigma^{-1}(B) = B$ satisfies $\mu(B) = 0$ or $\mu(B) = 1$. A Bernoulli measure is both invariant and ergodic.

Definition. For any ergodic and invariant measure μ on Σ the entropy of μ is defined to be the value

$$h_\mu(\sigma) = \lim_{n\to\infty} -\frac{1}{n} \sum_{\omega \in W_n} \mu(\omega) \log(\mu(\omega)).$$

In particular, for a Bernoulli measure μ associated to a probability vector $\underline{p} = (p_1, \dots, p_n)$ the entropy can easily seen to be simply

$$h_\mu(\sigma) = -\sum_{i=1}^{k} p_i \log p_i.$$

An important classical result for entropy is the following.

Proposition 3.1.3 (Shannon-McMillan-Brieman Theorem). *Let μ be an ergodic σ-invariant probability measure on Σ. For μ almost all $\underline{i} \in \Sigma$,*

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu([i_0, \dots, i_{n-1}]) = h_\mu(\sigma).$$

We can define a continuous map $\Pi : \Sigma \to \Lambda$ by $\Pi(\underline{i}) = \lim_{n\to\infty} T_{i_0} \cdots T_{i_n}(0)$. We can associated to a probability measure μ on Σ a measure ν on Λ defined by $\nu = \mu \circ \Pi^{-1}$.

In the case where all the contractions $T_1, \dots, T_k$ are similarities it is possible to use the Shannon-Mcmillan-Brieman Theorem to get an upper bound on the Hausdorff dimension of ν. Let T_i have contraction ratio $|T_i'| = r_i < 1$, say, and let

$$\chi = \sum_{i=0}^{n-1} p_i \log r_i < 0$$

be the Lyapunov exponent of ν.

Proposition 3.1.4. *Consider a conformal affine iterated function scheme $T_1, \cdots, T_k$ satisfying the open set condition. Let ν be the image of a Bernoulli measure. Then*

$$\dim_H(\nu) = \frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i} \left(= \frac{h_\mu(\sigma)}{|\chi|} \right)$$

Without the open set condition we still get an inequality $\leq$.

Proof. For two distinct sequences $\omega, \tau \in \Sigma$ we denote by $|\omega \wedge \tau| = \min\{m : \omega_m \neq \tau_m\}$ the first term in which the two sequences differ. For two sequences $\omega, \tau \in \Sigma$ we denote by $|\omega \wedge \tau| = \min\{m : \omega_m \neq \tau_m\}$ the first term in which the two sequences differ. Given $\omega, \tau \in \Sigma$ let $m = |\omega \wedge \tau|$, then we define a metric by

$$d(\omega, \tau) = \prod_{i=0}^{k-1} r_i^{m_i(\omega)} \left(= \prod_{i=0}^{k-1} r_i^{m_i(\tau)} \right).$$

A useful property of this metric d is that the diameter of any cylinder in the shift space is (essentially) the same as the diameter of the projection of the cylinder in $\mathbb{R}^n$. Fix $\tau \in \Sigma_n$ and let $x = \Pi^{-1}\tau$. For $r > 0$ there exists $k(r)$ such that,

$$[i_1, \dots, i_{k(r)}, i_{k(r)+1}] \leq 2r \leq [i_1, \dots, i_{k(r)}]$$

and $k(r) \to \infty$ as $r \to 0$. Hence

$$\lim_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} = \lim_{n\to\infty} \frac{\log(\mu([\tau_0, \dots, \tau_{n-1}]))}{\log(\mathrm{diam}([\tau_0, \dots, \tau_{n-1}]))}.$$

(Without the open set condition $\nu(B(x,r))$ can be much bigger than $\mu([\tau_1, \dots, \tau_{k(r)-1}])$.)

By the Shannon-McMillan-Brieman Theorem we have that,

$$\lim_{n\to\infty} \frac{1}{n} \log(\mu([\tau_0, \dots, \tau_{n-1}])) \to \sum_{i=0}^{n-1} p_i \log p_i = h_\mu(\sigma)$$

for μ almost all τ and by the Birkhoff Ergodic theorem we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{diam}[\tau_0, \dots, \tau_{n-1}] \to \sum_{i=0}^{n-1} p_i \log r_i = \chi$$

for μ almost all τ. Hence for μ almost all τ where $x = \Pi\tau$ (or equivalently, ν almost all x)

$$\lim_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} = \frac{h_\mu(\sigma)}{|\chi|}.$$

$\square$.

It is follows from the proof that we still get an upper bound $\dim_H(\nu)$ if we replace μ by any other ergodic σ-invariant measure on Σ or if we don't assume the Open Set Condition.

A more general statement (with much the same proof) is the following:

3.2 Multifractal Analysis. For a measure μ on a limit set Λ we can ask about the set of points x for which the limit

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

exists. Let $X_\alpha = \{x : \text{ the limit } d_\mu(x) = \alpha\}$ be the set for which the limit exists, and equals α. There is a natural decomposition of the set X by "level sets":

$$\Lambda = \bigcup_{-\infty<\alpha<\infty} \Lambda_\alpha \cup \{x \in \Lambda \,|\, d_\mu(x) \text{ does not exist}\}.$$

To study this decomposition one defines the following:

Definition. The *dimension spectrum* is a function $f_\mu : \mathbb{R} \to [0,d]$ given by $f_\mu(\alpha) = \dim_H(\Lambda_\alpha)$, i.e., the Hausdorff dimension of the set Λ_α.

The "multifractal analysis" of the measure μ describes the size of the sets Λ_α through the behaviour of the function f_μ.

Example 3.2.1: A special case. Let us consider an iterated function scheme $T_1, \ldots, T_k$ with similarities satisfying the open set condition. Consider the Bernoulli measure μ associated with the vector $(p_1, \ldots, p_k)$. We have already seen that:

(1) $d_\mu(x)$ exists for a.e. (μ) x and is equal to $\dim_H(\mu)$. (In this particular case, this limit is equal to $\frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i}$).

We claim that the following is also true.

(2) Except in the very special case $p_i = r_i^{\dim_H(\Lambda)}$, for $i = 1, \ldots, k$, there is an interval (a,b) containing $\dim_H(\Lambda)$ such that $f_\mu : (a,b) \to \mathbb{R}$ is analytic.

Sketch proof of (2). For each α, we can write

$$\Lambda_\alpha = \Pi \left\{ \underline{x} \in \Sigma : \lim_{n\to+\infty} \frac{\sum_{j=1}^n \log p_{x_j}}{\sum_{j=1}^n \log r_{x_j}} = \alpha \right\}.$$

For each $q \in \mathbb{R}$, we can choose $T(q) \in \mathbb{R}$ such that $P(-T(q)\log|r_{x_0}| + q\log p_{x_0}) = 0$. There exists an Bernoulli measure ν_q associated to the probability vector $(p_1^{(q)}, \cdots, p_k^{(q)})$, where $p_i^{(q)} = r_i^{-T(q)} p_i^q$ for $i = 1, \cdots, k$, and constants $C_1, C_2 > 0$ such that

$$C_1 \le \frac{\nu_q([i_0, \cdots, i_{n-1}])}{\prod_{i=0}^{n-1} \exp\left(-T(q)\log r_{x_i} + q\log p_{x_i}\right)} \le C_2.$$

Furthermore, we associate to q the particular value

$$\alpha(q) = \frac{\int \log p_{x_0} d\nu_q}{\int \log r_{x_0} d\nu_q}.$$

For a.e. (ν_q) $x \in X_{\alpha(q)}$ we have that $d_\nu(x) = \alpha(q)$ by the Birkhoff ergodic theorem and the definition of X_α. In particular, $\nu_q(X_\alpha) = 1$.[2] If $(r_1^d, \ldots, r_k^d) \neq (p_1, \ldots, p_k)$ then $f_\nu(\alpha)$ and $T(q)$ are strictly convex (and are Legendre transforms of each other).

We then claim that:

(a) $\alpha(q)$ is analytic
(b) $f_\nu(\alpha(q)) = (dim_H X_{\alpha(q)}) = T(q) + q\alpha(q)$.

and then (2) follows.

For part (a) observe that since $P(\cdot)$ is analytic, we deduce from the Implicit Function Theorem that the function $T(q)$ is analytic as a function of q. Observe that $T(0) = \dim_H X$. We can check by direct computation that $T'(q) \leq 0$ and $T''(q) \geq 0$.

Part (b) follows from the observation that $d_{\nu_q}(x) = T(q) + q\alpha(q)$ for a.e. $x \in \Lambda_\alpha$ and $\bar{d}_{\nu_q}(x) = T(q) + q\alpha(q)$ for all $x \in \Lambda_\alpha$ by (5.1). $\square$

Remark. The set of points for which the pointwise dimension doesn't converge has full dimnsion. enddemo

Example 3.2.2: Expanding maps. Let $T\colon I \to I$ be an expanding transformation on the unit interval I. Let μ be a T-invariant ergodic probability measure. We say that μ is a *Gibbs measure* if $\phi(x) = \log \frac{d\mu T}{d\mu}$ is piecewise C^1 (or merely Hölder continuous would suffice). The most familiar example of a Gibbs measure is given by the following.

Proposition 3.2.1 ('Folklore Lemma'). *There is a unique absolutely continuous invariant probability measure ν (i.e., we can write $d\nu(x) = \rho(x)dx$).*

The main result is the following.

Theorem 3.2.2. *Assume that μ is a Gibbs measure (but not ν):*

(1) *The pointwise dimension $d_\mu(x)$ exists for μ-almost every $x \in I$. Moreover, $d_\mu(x) = d_\mu \equiv h_\mu(T)/\int_X \log|T'|\, d\mu$ for μ-almost every $x \in I$.*
(2) *The function $f_\mu(\alpha)$ is smooth and strictly convex on some interval $(\alpha_{\min}, \alpha_{\max})$ containing d_μ.*

Let ψ be a positive function defined by $\log\psi = \phi - P(\phi)$, where $P(\phi)$ denotes the pressure of ϕ. Clearly ψ is a Hölder continuous function on I such that $P(\log\psi) = 0$ and μ is also the equilibrium state for $\log\psi$. We define the two parameter family of Hölder continuous functions $\phi_{q,t} = -t\log|T'| + q\log\psi$. Define the function $t(q)$ by requiring that $P(\phi_{q,t(q)}) = 0$ and let μ_q be the equilibrium state for $\phi_{q,t(q)}$

3.3 Computing Lyapunov exponents. In many examples, the Lyapunov exponents $\int \log|T'(x)| d\mu(x)$ can be computed in much the same way that Hausdorff dimension was. More precisely, this integral can be approximated by periodic orbit estimates. In the interests of definiteness, consider the absolutely continuous T-invariant measure ν.

[2]We can also identify $\alpha(q) = -T'(q)$, then it has a range $[\alpha_1, \alpha_2] \subset \mathbb{R}^+$.

For definiteness, let us consider the case of the absolutely continuous invariant measure ν. We construct the family of approximating measures by a more elaborate regrouping of the periodic points to define new invariant probability measures. Let λ_n be the sequence of numbers given by

$$\lambda_n = \frac{\sum_{\substack{\underline{k}=(k_1,\dots,k_m)\\ k_1+\dots+k_m\le n}} \frac{(-1)^m r(\underline{k})}{m!} \left(\sum_{\substack{i=1,\dots,m\\ x\in \mathrm{Fix}(T^{k_i})}} k_i \log |T'(x)| \right)}{\sum_{\substack{\underline{k}=(k_1,\dots,k_m)\\ k_1+\dots+k_m\le n}} \frac{(-1)^m r(\underline{k})}{m!} \left(\sum_{\substack{i=1,\dots,m\\ x\in \mathrm{Fix}(T^{k_i})}} k_i \right)}$$

where we write

$$r(\underline{k}) = \prod_{j=1}^{m} \sum_{z\in \mathrm{Fix}(T^{k_j})} \frac{1}{k_j|(T^{k_j})'(z)-1|}.$$

and $\mathrm{Fix}(T^n) = \{x \in [0,1] : T^n x = x\}$.
We have the following superexponentially converging estimate.

Theorem 3.3.1. *If $T : [0,1] \to [0,1]$ is a C^ω piecewise expanding Markov map with absolutely continuous invariant measure μ then there exists $C > 0$ and $0 < \theta < 1$ with $|\lambda_n - \int \log |T'|\, d\nu| \le C\theta^{n^2}$*

Example 5.3.1. Consider the family $T_{\frac{1}{4\pi}} : [0,1] \to [0,1]$ defined by

$$T_{\frac{1}{4\pi}}(x) = 2x + \varepsilon \sin 2\pi x \ (\mathrm{mod}\ 1),$$

for $-\frac{1}{2\pi} < \varepsilon < \frac{1}{2\pi}$. We can estimate the Lyapunov exponent $\int \log |T'_{1/4\pi}|\, d\nu$ in terms of the estimates

$$\lambda_n \to \int \log |T'_{1/4\pi}|\, d\nu \ \text{[super-exponential rate]}$$

n	using λ_n
6	0.6837719
7	0.68377196
8	0.68377196024
9	0.6837719602421451
10	0.6837719602421451396
11	0.683771960242145139619160
12	0.68377196024214513961916071

3.4 Large deviations.

Typically, large deviation results describe how a set of values differs from their average. An example of this might be to consider one of our examples $T : X \to X$ of an expanding map and to consider the distribution of preimages $\{y : T^n y = x\}$.

There exists a probability measure μ such that for any Hölder continuous function $f : X \to \mathbb{R}$ we have that

$$\frac{\sum_{T^n y = x} f^n(y)/n}{\mathrm{Card}\{T^n y = x\}} \to \int f d\mu \text{ where } n \to +\infty.$$

where $f^n(x) = f(x) + f(\sigma x) + \cdots + f(\sigma^{n-1}x)$. (In fact, μ is the unique measure of maximal entropy.) A basic large deviation result would be that for any $\epsilon > 0$ we can choose $C > 0$ and $0 < \theta < 1$ such that

$$\frac{\mathrm{Card}\left\{y \in T^{-n}x : \left|f^n(y)/n - \int f d\mu\right| \geq \epsilon\right\}}{\mathrm{Card}\{T^n y = x\}} \leq C\theta^n.$$

If we take $f(x) = \log T'(x)$ then we see that if y lies in an interval $I_n(y)$ (corresponding to the image of cylinder of length n) then there exists $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{\mathrm{diam}(I_n(y))}{|(T^n)'(y)|^{-1}} \leq C_2$$

In particular, we see that

$$\left|\frac{\log \mathrm{diam}(I_n(y))}{n} - \frac{\log |(T^n)'(y)|}{n}\right| = O\left(\frac{1}{n}\right)$$

Combining these estimates we get that

$$\frac{\mathrm{Card}\left\{y \in T^{-n}x : \left|\mathrm{diam}(I_n(y))/n - \int f d\mu\right| \geq \epsilon\right\}}{\mathrm{Card}\{T^n y = x\}} \leq C\theta^n$$

which give more information on the size of the intervals in the basic cover. The value of $0 < \theta < 1$ can be characterized by

$$\theta = \exp\left(-\sup\left\{h(T) - h_m(T) : m = T\text{-invariant measure}, \left|\int \log|T'|dm - \int \log|T'|d\mu\right| \geq \epsilon\right\}\right).$$