FM–representability and beyond

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Abstract. This work concerns representability of arithmetical notions in finite models. It follows the paper by Marcin Mostowski [8], where the notion of FM–*representability* has been defined. We discuss how far this notion captures the methodological idea of representing infinite sets in finite but potentially infinite domains.
We consider mainly some weakenings of the notion of FM–representability. We prove that relations *weakly* FM–*representable* are exactly those being Σ^0_2–definable. Another weakening of the notion, namely *statistical representability*, turns out to be equivalent to the original one. Additionally, we consider the complexity of sets of formulae naturally defined in finite models. We state that the set of sentences true in almost all finite arithmetical models is Σ^0_2–complete and that the set of formulae FM–representing some relations is Π^0_3–complete.

1 Introduction

This work concerns mainly the following problem.

> *Let us suppose that our world is finite, but not of a restricted size. It means that everytime it can be enlarged by a finite number of new entities. This assumption says, in Aristotelian words (see [1], Physics, book* 3*), that the world is finite but potentially infinite. Then, which infinite sets can be reasonably described in our language?*

For simplifying the problem we restrict our attention to sets (and relations) of natural numbers and we assume that our world contains only natural numbers.

Technically, the problem appears when one is trying to transfer some classical ideas into finite–models theoretic framework. It requires frequently a uniform representation for various infinite relations in finite models. As a rule, uniformity means that the representation of a relation is given by one formula. Of course in a single finite model only a finite approximation of any infinite relation can be defined. Therefore we have to consider representability in infinite classes of finite models — intuitively *finite but potentially infinite models.*[3] In the paper [8]

[3] In the context of foundations of mathematics a very similar approach to potential infitnity is presented by Jan Mycielski in [11].

an attempt to make the notion precise has been made and FM–representability theorem has been proved (see Theorem 5).[4]

Let R be a set of natural numbers. Then we say that R is FM–represented by a formula $\varphi(x)$ if for each initial segment I of natural numbers $\varphi(x)$ correctly describes R for all elements from I in all sufficiently large finite interpretations. Originally the notion was motivated by an attempt to transfer the Tarski's method of classifying concepts by means of truth definitions to finite models.[5] In this case we have to describe syntax of considered languages in finite models. Needed syntactical relations are essentially infinite. Therefore, the notion of FM–representability appeared as an answer to this problem.

In this paper we concentrate on the notion of FM–representability and some possible weakenings of it. We show that, in a sense, the notion captures strongly the idea of representing relations in finite models.

2 Basic Notions

We start with the crucial definition of FM–domain.

Definition 1. *Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a finite set of arithmetical relations on ω. By an $\mathcal{R}$–domain we mean the model $\mathcal{A} = (\omega, R_1, \ldots, R_k)$. We consider finite initial segments of these models. Namely, for $n \geq 1$, by $\mathcal{A}_n$ we denote the structure*

$$\mathcal{A}_n = (\{0, \ldots, n-1\}, R_1^n, \ldots, R_k^n),$$

where, for $i = 1, \ldots, k$, the relation R_i^n is the restriction of the relation R_i to the set $\{0, \ldots, n-1\}$. We treat n-ary functions as $n+1$-ary relations.

The FM*–domain of $\mathcal{A}$ (or* FM*–domain of $\mathcal{R}$), denoted by* FM$(\mathcal{A})$*, is the family $\{\mathcal{A}_n : n \in \omega\}$.*

Throughout this paper we are interested mainly in the family FM$(\mathbb{N})$, for $\mathbb{N} = (\omega, +, \times)$. By arithmetical formulae we mean first order formulae with addition and multiplication treated as ternary predicates. The standard ordering $x \leq y$ is definable by the formula $\exists z\, x + z = y$. Its strict version, $x < y$, is defined as $x \leq y \wedge x \neq y$. The constants 0 and MAX are defined respectively as $\leq$–smallest and $\leq$–greatest elements. For each $n \in \omega$, by $\bar{n}$ we mean the constant denoting the n-th element in the ordering $\leq$ counting from 0. If there is no such element we take $\bar{n} = \mathrm{MAX}$. We write $x|y$ for $\exists z \leq y(1 < z \wedge zx = y)$. It is known that all these notions are definable by bounded formulae. Thus, their interpretations conform to their intended meaning also in models from FM$(\mathbb{N})$.

[4] Some consequences of this idea are also discussed in [9], [6].

[5] The basics of the method of truth definitions in finite models were formulated in [7]. The paper [8] covers [7], giving additionally some refinement of the method. It was applied then in [9], and [4] for classifying finite order concepts in finite models. Some applications of the method for classifying computational complexity classes can be found in [3].

Let us mention, that in [9] a finite axiomatization ST has been presented which characterizes, up to isomorphism, the family $\mathrm{FM}(\mathbb{N})$ within the class of all finite models.

The other notions which we use here are fairly standard, one can consult e.g. [2] and [12] for model or recursion theoretic concepts, respectively. We write $\{e\}$ to denote the partial function computed by the Turing machine with the index e. $\{e\}(n)\uparrow$ means that the function $\{e\}$ is not defined on n, and $\{e\}(n)\downarrow$ means that $\{e\}(n)$ is defined. We put $W_e = \{n \in \omega : \{e\}(n)\downarrow\}$.

We consider the family of Σ_n^0 (Π_n^0) relations which are definable in $\mathbb{N}$ by Σ_n^0 (Π_n^0) formulae. Δ_n^0 are relations which are definable by Σ_n^0 and Π_n^0 formulae.

$R \subseteq \omega^r$ is many one reducible to $S \subseteq \omega^s$ ($R \leq_m S$) if there exists a total recursive function f such that for all $a_1, \ldots, a_r \in \omega$,

$$(a_1, \ldots, a_r) \in R \text{ if and only if } f(a_1, \ldots, a_r) \in S.$$

A relation S is complete for a class $\mathcal{K}$ if $S \in \mathcal{K}$ and for any other $R \in \mathcal{K}$, $R \leq_m S$.

We say that R is Turing reducible to S ($R \leq_T S$) if there is an oracle Turing machine which decides R using S as an oracle. R and S are Turing equivalent if $R \leq_T S$ and $S \leq_T R$. The degree of R, denoted by $\deg(R)$, is the class of all relations which are Turing equivalent to R. In particular, $\mathbf{0}'$ is the degree of any recursively enumerable (RE) complete set, and $\mathbf{0}''$ is the degree of any Σ_2^0–complete set.

We use bald characters, e.g. $\mathbf{a}$, for valuations in a given model $\mathcal{A}$. We write $|\mathcal{A}|$ for the universe of a model $\mathcal{A}$. If $\varphi(x_1, \ldots, x_k)$ is a formula in the vocabulary of $\mathcal{A}$ with all free variables between $x_1, \ldots, x_k$ we write $\mathcal{A} \models \varphi[a_1, \ldots, a_k]$, for $a_1, \ldots, a_k \in |\mathcal{A}|$, when φ holds in $\mathcal{A}$ under any valuation $\mathbf{a}$ for which $\mathbf{a}(x_i) = a_i$, for $i = 1, \ldots, k$.

Definition 2. *Let $\varphi(x_1, \ldots, x_r)$ be an arithmetical formula and $a_1, \ldots, a_r \in \omega$. We say that φ is true of $a_1, \ldots, a_r$ in all sufficiently large finite models ($\models_{\mathrm{sl}} \varphi[a_1, \ldots, a_r]$) if and only if $\exists k \forall n \geq k\ \mathbb{N}_n \models \varphi[a_1, \ldots, a_r]$ (or, in other words, if φ is true of $a_1, \ldots, a_r$ in almost all finite models from $\mathrm{FM}(\mathbb{N})$).*

For each unbounded family of finite models $\mathcal{K}$, by $\mathrm{sl}(\mathcal{K})$ we denote the set of formulae which are true in almost all models from $\mathcal{K}$. In particular, $\models_{\mathrm{sl}} \varphi$ means that $\varphi \in \mathrm{sl}(\mathrm{FM}(\mathbb{N}))$.

Definition 3. *We say that $R \subseteq \omega^r$ is FM–represented by a formula $\varphi(x_1, \ldots, x_r)$ if and only if for each $a_1, \ldots, a_r \in \omega$ both of the following conditions hold:*

(i) $\models_{sl} \varphi[a_1, \ldots, a_r]$ *if and only if* $R(a_1, \ldots, a_r)$.
(ii) $\models_{sl} \neg\varphi[a_1, \ldots, a_r]$ *if and only if* $\neg R(a_1, \ldots, a_r)$.

We say that R is FM–representable if there is an arithmetical formula φ which FM–represents R.

The notion of FM–representability has been defined in [8] in a slightly different way. We summarize various equivalent conditions in the following

Proposition 4. *Let $R \subseteq \omega^r$ and $\varphi(x_1, \ldots, x_r)$ be a formula in a vocabulary of* FM($\mathbb{N}$)*. The following conditions are equivalent:*

1. *$\varphi(x_1, \ldots, x_r)$ FM–represents R,*
2. *for each m there is k such that for all $a_1, \ldots, a_r \leq m$,*

$$R(a_1, \ldots, a_r) \text{ if and only if } \mathbb{N}_i \models \varphi[a_1, \ldots, a_r],$$

for all $i \geq k$.

The second condition expresses the intuition that φ FM–represents R in FM($\mathbb{N}$) if each finite fragment of R is correctly described by φ in all sufficiently large models from FM($\mathbb{N}$).

The main characterization of the notion of FM–representability is given by the following

Theorem 5 (FM**–representability theorem, see [8]).** *Let $R \subseteq \omega^r$. R is* FM*–representable if and only if R is of degree $\leq \mathbf{0}'$ (or, equivalently, is Δ^0_2–definable).*

The theorem does not depend on the strength of the underlying logic provided that the truth relation for this logic restricted to finite models is recursive and it contains first order logic. On the other hand, it is surprising that the theorem requires relatively weak arithmetical notions. In [5] it is proved that it holds in FM–domain of multiplication. It is improved in [10] to the divisibility relation.

3 Weak FM-representability

As the most natural weakening of the notion of the notion of FM–representability we consider the following:

Definition 6. *A relation $R \subseteq \omega^r$ is weakly* FM*–representable if there is a formula $\varphi(x_1, \ldots, x_r)$ with all free variables among $x_1, \ldots, x_r$ such that for all $a_1, \ldots, a_r \in \omega$,*

$$(a_1, \ldots, a_r) \in A \text{ if and only if } \models_{sl} \varphi[a_1, \ldots, a_r].$$

Since the definition of $\models_{\mathrm{sl}} \varphi$ can be expressed as an Σ^0_2–sentence the following holds.

Fact 7 *Let $R \subseteq \omega^r$. If R is weakly* FM*–representable, then $R \in \Sigma^0_2$.*

The reverse of the implication from Fact 7 will be proved after evaluating the degree of the theory sl(FM($\mathbb{N}$)).

As an analogue of the relation between FM–representability and weak FM–represetability we recall the relation between strong and weak representability in Peano arithmetic. We say that a relation $R \subseteq \omega^r$ is strongly PA–representable if there is a PA–formula $\varphi(x_1, \dots, x_r)$ with all free variables among $x_1, \dots, x_r$ such that for all $n_1, \dots, n_r \in \omega$,

$$\begin{aligned}(n_1, \dots, n_r) \in R &\iff \mathrm{PA} \vdash \varphi(\bar{n}_1, \dots, \bar{n}_r)\\ (n_1, \dots, n_r) \notin R &\iff \mathrm{PA} \vdash \neg\varphi(\bar{n}_1, \dots, \bar{n}_r).\end{aligned}$$

$R \subseteq \omega^r$ is weakly PA–representable if there is a PA–formula $\varphi(x_1, \dots, x_r)$ with all free variables among $x_1, \dots, x_r$ such that for all $n_1, \dots, n_r \in \omega$,

$$(n_1, \dots, n_r) \in R \iff \mathrm{PA} \vdash \varphi(\bar{n}_1, \dots, \bar{n}_r).$$

A relation R is strongly PA–representable if and only if it is decidable. R is weakly PA–representable if and only if R is recursively enumerable. If R and its complement are weakly PA–representable, then R is strongly PA–representable. We state the analogous fact for FM–representability and weak FM–representability. It follows easily from the known relations between the classes Σ^0_2 and Δ^0_2.

Fact 8 *Let $R \subseteq \omega^r$. R and $\omega^r - R$ are weakly* FM*–representable if and only if R is* FM*–representable.*

Below, we prove the stronger fact that weakly FM–representable relations are exactly the Σ^0_2 relations.

Firstly, we consider some properties of coding computations and the formula $\mathrm{Comp}(e, c)$ which says that c is a finished computation of the machine e. (Here and in what follows by a Turing machine we mean a deterministic Turing machine.) We construct $\mathrm{Comp}(e, c)$ using Kleene predicate $\mathrm{T}(e, x, c)$, which means that c is a finished e–computation with the input x. It is known that this predicate is definable by an arithmetical formula with all quantifiers bounded by c. Moreover, if $\mathrm{T}(e, x, c)$ then $e < c$ and $x < c$. We define $\mathrm{Comp}(e, c)$ as $\exists x < c\, \mathrm{T}(e, x, c)$.

Now let us state a few facts about the formula $\mathrm{Comp}(e, c)$. All quantifiers in $\mathrm{Comp}(e, c)$ are bounded c. It follows that the truth value of $\mathrm{Comp}(e, c)$ in a given model M does not depend on the elements in M greater than c and that $\mathrm{Comp}(e, c)$ will hold in a given model $M \in \mathrm{FM}(\mathbb{N})$ as soon as the code of the computation appears in M.

Now, we state the lemma summarizing these considerations.

Lemma 9. *There is a formula* $\mathrm{Comp}(x, y)$ *such that*

$$\begin{gathered}\forall e\, \forall c\, \forall M \in \mathrm{FM}(\mathbb{N})(\mathrm{card}(M) > c \Rightarrow\\ (c \textit{ is a finished computation of } e \iff M \models \mathrm{Comp}[e, c]))).\end{gathered}$$

Definition 10. *By* Fin *we mean the set of indices of Turing Machines having finite domains, i. e.*

$$\mathrm{Fin} = \{e \in \omega : \exists n \in \omega\ \mathrm{card}(W_e) = n\}.$$

By a well known result from recursion theory (see e.g. [12]) Fin is Σ^0_2–complete.

Lemma 11. Fin *is weakly* FM*–representable.*

Proof. Let $\varphi(x)$ be the formula $\neg\mathrm{Comp}(x, \mathrm{MAX})$, where $\mathrm{Comp}(x, y)$ is the formula from the last lemma. By properties of Comp stated there, for all e,

$$e \in \mathrm{Fin} \text{ if and only if } \models_{sl} \varphi[e].$$

If $e \in \mathrm{Fin}$ then there are only finitely many finished computations of e. (Here, we use the assumption that all machines are deterministic.) In this case φ is true of e in all models in which MAX is greater than the biggest computation of e. On the other hand, if $\models_{\mathrm{sl}} \varphi[e]$, then there are only finitely many finished computations of e. Thus, the domain of e is also finite.

Thus, Fin is weakly FM–representable. □

We have the following lemma.

Lemma 12. *The family of weakly* FM*–representable relations is closed on many–one reductions.*

Proof. For simplicity we consider only sets $A, B \subseteq \omega$. Let $f : \omega \longrightarrow \omega$ be a reduction from A to B that is for all z,

$$z \in A \text{ if and only if } f(z) \in B$$

and let $\varphi_B(x)$ weakly FM–represent B. Additionally, let $\psi_f(x, y)$ FM–represent the graph of f. Now, the following formula $\varphi_A(x)$ FM–represents A,

$$\exists y\, (\psi_f(x, y) \wedge \forall y' < y \neg\psi_f(x, y') \wedge \varphi_B(y)).$$

Here, we need to add the conjunct $\forall y' < y \neg\psi_f(x, y')$ to force the uniqness of y. □

As a corollary from Lemmas 11 and 12 we obtain the following characterization of weak FM–representability.

Theorem 13. *Let $R \subseteq \omega^r$. R is weakly* FM*–representable if and only if A is Σ^0_2.*

Now, we are in a position to solve some questions which were put, explicitly or implicitly, in [8]. Let us recall that $\mathrm{sl}(\mathrm{FM}(\mathbb{N})) = \{\varphi : \models_{\mathrm{sl}} \varphi\}$. So, $\mathrm{sl}(\mathrm{FM}(\mathbb{N}))$ is the theory of almost all finite models from $\mathrm{FM}(\mathbb{N})$. By the definition of $\models_{\mathrm{sl}}$ the above set is in Σ^0_2.

In [8], it was proven by the method of undefinability of truth, that

$$\mathbf{0}' < deg(\mathrm{sl}(\mathrm{FM}(\mathbb{N}))) \leq \mathbf{0}''.$$

Here we strengthen this result by the following,

Theorem 14. $\mathrm{sl}(\mathrm{FM}(\mathbb{N}))$ *is* Σ_2^0*–complete, so its degree is* $\mathbf{0}''$.

Proof. We know that $\mathrm{sl}(\mathrm{FM}(\mathbb{N}))$ is Σ_2^0. It is Σ_2^0–complete by the procedure from the proof of Lemma 11 which reduces Fin to $\mathrm{sl}(\mathrm{FM}(\mathbb{N}))$. We put $f(e) = \ulcorner\neg\mathrm{Comp}(\bar{e}, \mathrm{MAX})\urcorner$. By properties of $\mathrm{Comp}(x, y)$ we obtain:

$$e \in \mathrm{Fin} \text{ if and only if } f(e) \in \mathrm{sl}(\mathrm{FM}(\mathbb{N})).$$

Since Fin is Σ_2^0–complete, $\mathrm{sl}(\mathrm{FM}(\mathbb{N}))$ is too. □

Let us observe that the degree of $\mathrm{sl}(\mathrm{FM}(\mathbb{N}))$ does not depend on the underlining logic provided it has decidable "*truth in a finite model*" relation and contains first order logic.

Now, let us consider the complexity of the question whether a given formula $\varphi(x_1, \ldots, x_k)$ with free variables $x_1, \ldots, x_k$ FM–represents some relation in $\mathrm{FM}(\mathbb{N})$. Let us define the set

$$\begin{aligned} F_{Det} = \{\varphi(x_1, \ldots, x_k) :& \\ &\forall n_1 \ldots n_k \in \omega \models_{sl} \varphi[n_1, \ldots, n_k] \text{ or } \models_{sl} \neg\varphi[n_1, \ldots, n_k]\}. \end{aligned}$$

F_{Det} is the set of formulae which are determined for all substitutions of constant numerical terms for their free variables. In other words, this is the set of formulae which FM–represent some concepts.

We have the following theorem characterizing the degree of F_{Det}.

Theorem 15. F_{Det} *is* Π_3^0*–complete.*

Proof. F_{Det} has a Π_3^0 definition so it is a Π_3^0 relation. Now, let $A \subseteq \omega^k$ be a Π_3^0–relation. We show a many–one reduction from A to F_{Det}.

There is a recursive relation R such that for all $n_1, \ldots, n_k \in \omega$,

$$(n_1, \ldots, n_k) \in A \text{ if and only if } \forall x \exists y \forall z R(n_1, \ldots, n_k, x, y, z).$$

Since Fin is Σ_2^0–complete, we have a total recursive function $g : \omega^{k+1} \longrightarrow \omega$ such that for all $n_1, \ldots, n_k \in \omega$,

$$\forall x \exists y \forall z R(n_1, \ldots, n_k, x, y, z) \text{ if and only if } \forall x\, g(n_1, \ldots, n_k, x) \in \mathrm{Fin}.$$

Now, let $\psi_g(x_1, \ldots, x_k, x, y)$ FM–represent the graph of g and let $\varphi(x_1, \ldots, x_k, x)$ be the following formula

$$\exists y(\psi_g(x_1, \ldots, x_k, x, y) \wedge \forall z < y \neg\psi_g(x_1, \ldots, x_k, x, z) \wedge \neg\mathrm{Comp}(y, \mathrm{MAX})),$$

where $\mathrm{Comp}(x, y)$ is the formula from Lemma 9. Because we consider only deterministic Turing machines $\mathrm{Comp}(y, \mathrm{MAX})$ can be determined only negatively. Thus, for all $n_1, \ldots, n_k \in \omega$,

$$\begin{aligned} (n_1, \ldots, n_k) \in A &\text{ if and only if } \forall m \in \omega \models_{\mathrm{sl}} \varphi(\bar{n}_1, \ldots, \bar{n}_k, \bar{m}) \\ &\text{ if and only if } \varphi(\bar{n}_1, \ldots, \bar{n}_k, x) \in F_{Det}. \end{aligned}$$

Thus, we obtained a reduction from A to F_{Det}. □

4 Statistical Representability

In this section we present another weakening of the original concept of FM–representabiliy.[6] Now, we do not require that for all $a_1, \ldots, a_k$ a given formula correctly describes a given relation R on $a_1, \ldots, a_k$. We only want that the description is more likely to be correct than incorrect.

Definition 16. *Let $\varphi(x_1, \ldots, x_k)$ be a formula and $\mathbf{a}$ be a valuation in $\mathbb{N}$. By $\mu_n(\varphi, \mathbf{a})$ we denote*

$$\mu_n(\varphi, \mathbf{a}) = \frac{\mathrm{card}\{\mathcal{A} \in \mathrm{FM}(\mathbb{N}) : \max_{1 \le i \le k}\{\mathbf{a}(x_i)\} \le \mathrm{card}(\mathcal{A}) \le n \wedge \mathcal{A} \models \varphi[\mathbf{a}]\}}{n}.$$

By $\mu(\varphi, \mathbf{a})$ we denote the limit value of μ_n for $n \to \infty$, if it exists.

$$\mu(\varphi, \mathbf{a}) = \lim_{n \to \infty} \mu_n(\varphi, \mathbf{a}).$$

Since, $\mu(\varphi(x_1, \ldots, x_k), \mathbf{a})$ is determined by values $\mathbf{a}$ on the free variables of φ we write also $\mu(\varphi, a_1, \ldots, a_k)$ with the obvious meaning. If φ is a sentence then the value of $\mu(\varphi, \mathbf{a})$ does not depend on $\mathbf{a}$. In this case we write $\mu(\varphi)$.

Definition 17. *The relation $R \subseteq \omega^r$ is statistically representable if there is a formula $\varphi(x_1, \ldots, x_r)$ with all free variables among $x_1, \ldots, x_r$ such that for all $a_1, \ldots, a_r \in \omega$,*

- *$\mu(\varphi, a_1, \ldots, a_r)$ exists,*
- *if $(a_1, \ldots, a_r) \in R$ then $\mu(\varphi, a_1, \ldots, a_r) > 1/2$*
- *if $(a_1, \ldots, a_r) \notin R$ then $\mu(\varphi, a_1, \ldots, a_r) < 1/2$.*

Theorem 18. *Let $R \subseteq \omega^r$. Then, R is statistically representable if and only if R is* FM*–representable.*

Proof. The implication from right to left is obvious. To prove the converse let us assume that $R \subseteq \omega^r$ is statistically represented by $\varphi(x_1, \ldots, x_r)$. We will give a Σ^0_2 definition of R. Then, since the set of statistically representable relations is obviously closed on the complement, we get that R has to be Δ^0_2. We have the following: for all $a_1, \ldots, a_r \in \omega$,

$$(a_1, \ldots, a_r) \in R \iff \exists N \forall n \ge N \mu_n(\varphi, a_1, \ldots, a_r) > \frac{1}{2}. \qquad (*)$$

The formula on the right side of (*) is Σ^0_2 so it remains to show that it gives a good description of R.

If the right side of (*) holds then of course $\mu(\varphi, a_1, \ldots, a_r)$ is greater or equal $\frac{1}{2}$. But, by the definition of statistical representability, $\mu(\varphi, a_1, \ldots, a_r)$ cannot be equal to $\frac{1}{2}$. Thus,

$$\mu(\varphi, a_1, \ldots, a_r) > \frac{1}{2} \text{ and } (a_1, \ldots, a_r) \in R.$$

[6] The results contained in this section are based on [13]

On the other hand, if $(a_1,\ldots,a_r)\in R$ then $\mu(\varphi,a_1,\ldots,a_r)=\frac{1}{2}+\varepsilon$, for some $\varepsilon>0$. Now, if we choose N in such a way that for all $n\geq N$,

$$|\mu(\varphi,a_1,\ldots,a_r)-\mu_n(\varphi,a_1,\ldots,a_r)|<\frac{\varepsilon}{2}$$

then the right side of (*) holds. □

Definition 19. *The relation $R\subseteq\omega^r$ is weakly statistically representable if there is a formula $\varphi(x_1,\ldots,x_r)$ such that for all $a_1,\ldots,a_r\in\omega$,*

$(a_1,\ldots,a_r)\in R$ if and only if the value $\mu(\varphi,a_1,\ldots,a_r)$ exists and equals 1.

Since the statistical representability coincides with FM–representability one could expect that relations which are weakly statisticaly representable are exactly relations which are weakly FM–representable. On the other hand, the quantifier prefix in the expression $\mu(\varphi)=1$ suggests that these relations are exactly relations which are Π^0_3 in the arithmetical hierarchy. The second guess is correct.

Before we present the theorem we define some auxiliary notions. We write $\sqrt{\mathrm{MAX}}<x$ for the formula $\forall z(xx\neq z)$. We write $\mathrm{Input}(c)=n$ for $\exists e<c\,\mathrm{T}(e,n,c)$ and $x\in W_e$ for $\exists c\,T(e,x,c)$.

Theorem 20. *The family of relations which are weakly statistically representable is exactly the family of Π^0_3 relations in the arithmetical hierarchy.*

Proof. By the method from the proof of Lemma 12, It may be easily shown that the family of weakly statistically representable relations is closed on many one reductions. Thus, it suffices to show that a Π^0_3–complete set is in this family. We take the Π^0_3–complete set coInf of Turing machines with coinfinite domain:

$$\mathrm{coInf}=\{e:\omega\setminus W_e \text{ is infinite}\}.$$

Now, we write the formula $\varphi(z):=$

$$\forall n\forall c[\{\sqrt{\mathrm{MAX}}<c\wedge n=\mathrm{Input}(c)\wedge\forall c_1(\sqrt{\mathrm{MAX}}<c_1\Rightarrow n\leq\mathrm{Input}(c_1))\}\Rightarrow$$
$$\forall x\{([(x\notin W_z\wedge x<n)\vee x=1]\wedge\forall y((y\notin W_z\wedge y<n)\Rightarrow y\leq x))\Rightarrow\neg(x|\mathrm{MAX})\}]$$

with the property that for all $e\in\omega$,

$$e\in\mathrm{coInf}\text{ if and only if }\mu(\varphi,e)=1. \qquad (**)$$

The formula φ in a model on $\{0,\ldots,m-1\}$ looks for a computation c greater than $\sqrt{m-1}$ with the smallest imput n. Then, it takes the greatest $x<n$ which is not an input of any e–computation in the model (or it takes 1 if there is no such a x) and forces its own density close to $1-1/x$. If there is no such a computation c then φ is simply true. Now, we show (**).

Let us assume that W_e is coinfinite and let $\varepsilon>1/k$ such that $k\notin W_e$. Let $N=\max\{c^2:\mathrm{Input}(c)\leq k\}+1$. We show that for all $m>N$, $|1-\mu_m(\varphi,e)|<\varepsilon$. In the model $\mathbb{N}_m$ there is no computation c such that $\sqrt{m-1}<c$ and $\mathrm{Input}(c)<k$. Thus, φ forces its density at least to $1-1/k$ in models greater than N.

Now, let us assume that W_e is cofinite and let $k=\max(\omega\setminus W_e)$. Let us fix arbitrary large N and $c_0=\max\{c:\mathrm{Input}(c)\leq N\}$. Starting from $\mathbb{N}_{c_0+1}$ up to $\mathbb{N}_{c_0^2}$, φ forces its density to $1-1/k$. In follows that $|1-\mu_{c_0^2}(\varphi,e)|\geq 1/2k$. □

5 Conclusions

We have investigated some variants and weakenings of the notion of FM–representability. Summarizing we observe that:

1. The notion of FM–representability has a natural characterization in terms of arithmetical definability.
2. It captures in a natural way the idea of a relation which can be in a meaningfull way described in finite but potentially infinite domains.
3. FM–representing formulae can be considered as computing devices finitely deciding some relations. So the notion of FM–representability behaves similarly to recursive decidability. The main difference is that in the former case the halting condition – being still finite – cannot be determined in a single finite model. Let us observe that weak FM–representability corresponds – in this sense – to recursive enumerability.

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